Dynamics of PDE, Vol.12, No.4, 321-342, 2015

On the locally self-similar singular solutions for the incompressible Euler equations

Liutang Xue

Communicated by Dong Li, received May 22, 2015.

Contents

Abstract. In this paper we consider the locally backward self-similar solutions for the Euler system, and specially focus on the case that the possible nontrivial velocity profiles have non-decaying spatial asymptotics. We derive the representation formula of the pressure profile in terms of the velocity profiles in such a situation, and by using it and the local energy inequality of the profiles, we prove some nonexistence results and show the energy behavior concerning these possible velocity profiles.

1. Introduction

Perfect incompressible fluids are governed by the well-known Euler system

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \nabla\cdot v = 0, \\ v|_{t=0} = v_0(x), \end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^N\times\mathbb{R}^+$, $N=2,3,\cdots$ is the spatial dimension, $v=(v_1,v_2,\cdots,v_N)$ is the velocity vector field of $\mathbb{R}^N$ and p is the scalar-valued pressure field. Assume $v_0\in H^s(\mathbb{R}^N)$, $s>\frac{N}{2}+1$, it has been known for decades (e.g. [**15**]) that there is a unique local-in-time smooth solution $v\in C([0,T[;H^s(\mathbb{R}^N))$ and the pressure can

1991 *Mathematics Subject Classification.* 76B35, 35Q35, 35L67, 35B99.

Key words and phrases. Euler equations, backward self-similar solutions, nonexistence, energy behaviour.

be expressed up to a constant by $p = -\mathrm{div}\,\mathrm{div}\Delta^{-1}(v\otimes v)$, that is,

$$p(x,t) = -\frac{1}{N}|v(x,t)|^2 + \mathrm{p.v.}\int_{\mathbb{R}^N} K_{ij}(x-y)v_i(y,t)v_j(y,t)\,\mathrm{d}y, \tag{1.2}$$

where $K_{ij}(y) = \frac{1}{|\mathbb{S}^{N-1}|}\frac{Ny_iy_j-|y|^2\delta_{ij}}{|y|^{N+2}}$ $(i,j=1,2,\cdots,N)$ is the Calderón-Zygmund kernel (for the formula of K_{ij} cf. [**3**]) and the Einstein convention on repeated indices is used here and thereafter. However, for $N\geq 3$, whether such smooth solutions are globally well-posed or they form finite-time blowup remains a challenging open problem.

In this paper we address the problem of existence or not of the locally backward self-similar solutions for the Euler system. More precisely, we consider the solutions that develop a finite-time self-similar singularity on a spacetime domain $B_\rho(x_0)\times]0,T[$ of the form

$$v(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{\alpha+1}}}u\left(\frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}}\right), \tag{1.3}$$

and

$$p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}}q\left(\frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}}\right) + d(t), \tag{1.4}$$

where (u,q) are stationary profiles, $T>0$, $\alpha>-1$, $x_0\in\mathbb{R}^N$, $\rho>0$, and the solutions v, p remain regular outside the ball $B_\rho(x_0)$. If $\rho=\infty$, i.e. $B_\rho(x_0)=\mathbb{R}^N$, this corresponds to the "globally" self-similar solutions; while if $\rho<\infty$, these are the "locally" self-similar solutions, and $d(t)$ is a function depending only on t. For the locally self-similar solutions, from (1.2) and (1.3), it seems not obvious to get the expression (1.4), but which under some suitable assumptions can indeed be justified by Lemma 2.1 below. In terms of (u,q), we formally have

$$\begin{cases} \frac{\alpha}{\alpha+1}u + \frac{1}{\alpha+1}y\cdot\nabla u + u\cdot\nabla u + \nabla q = 0,\\ \mathrm{div}\,u = 0, \end{cases} \tag{1.5}$$

where $y\in\mathbb{R}^N$ and q up to a harmonic polynomial is given by $\Delta q = -\mathrm{divdiv}(u\otimes u)$. In Lemma 2.1, we shall also show that q up to a constant is given by a more precise formula (2.3) according to the value of α and the spatial asymptotic assumptions of u.

The self-similar ansatz (1.3)-(1.4) for the Euler system (1.1) is widely used in the numerical simulations, and through studying the vortex filament models or high-symmetric flows, there were much work suggesting that such backward self-similar blowup may happen at a finite time (see e.g. [**2, 16, 17, 19**], and the very recent work, [**13, 14**]).

The self-similar singular solutions of Euler equations are also analytically studied in the literature. X. He in [**11**] constructed the non-trivial solutions to the 3D Euler equations (1.5) with $\alpha=1$ on the exterior domain $\mathbb{R}^3\setminus B_1(0)$, and the asymptotic decay of such solutions are $|u(y)|\lesssim\frac{1}{|y|}$ and $|\nabla u(y)|\lesssim\frac{1}{|y|^2}$. Besides that, there are some noticeable works on the self-similar solutions from the viewpoint of nonexistence. D. Chae in [**4**] considered the globally self-similar solutions to the 3D Euler system and proved that if $u\in C^1(\mathbb{R}^3)$ and $\omega=\nabla\times u$ belongs to $\cap_{0<r<r_0}L^r(\mathbb{R}^3)$ with some $r_0>0$, then $\omega\equiv 0$ for all $\alpha>-1$. R. Takada [**22**] treated the strong solutions of the self-similar Euler equations (1.5) and show $u \equiv 0$ under the condition $u \in C^1_{\text{loc}} \cap X^{2,\infty} \cap L^p$ with $p \in [\frac{3N}{N-1}, \frac{4N}{N-2}]$ and $X^{2,\infty} = \{f \in L^2_{\text{loc}} : \sup_R \int_{R \le |y| \le 2R} |f(y)|^2 \mathrm{d}y < \infty\}$. See also [**10, 20**] for other similar nonexistence results. For the locally self-similar solutions (1.3)-(1.4) with $\rho > 0$, D. Chae and R. Shvydkoy [**5**] proved that if $u \in C^1_{\text{loc}} \cap L^r$ with $r \in [3, \infty]$, then $u \equiv 0$ for all $-1 < \alpha < \frac{N}{r}$ and $\alpha > \frac{N}{2}$. They also improved the result of [**4**] to get $u \equiv const$ for all $\alpha > -1$ under the assumptions $u \in C^1_{\text{loc}}(\mathbb{R}^N)$, $\omega = \nabla \times u \in L^p(\mathbb{R}^N)$ for some $p \in]0, \frac{N}{1+\alpha}[$ and $|\nabla u(y)| = o(1)$ as $|y| \to \infty$. Recently, A. Bronzi and R. Shvydkoy in [**3**] rigorously justified the formula of pressure (1.4) for the locally self-similar solutions at the case $\alpha > 0$ and $\rho > 0$, and under the assumptions $u \in C^3_{\text{loc}}(\mathbb{R}^N)$ and

$$\text{for some } p \ge 3, \gamma < p-2, \quad \int_{|y| \approx L} |u(y)|^p \, \mathrm{d}y \lesssim L^\gamma, \quad \forall L \gg 1,$$

they proved at the case $0 < \alpha < N/2$ either $u \equiv 0$ or the velocity profiles u behave as (1.12).

In this article we deal with the locally backward self-similar solutions (1.3)-(1.4) of Euler equations (1.1) to show some nonexistence results, and we specially consider the scenario that the velocity profiles u have non-decaying spatial asymptotics, e.g. for some $\delta \in]0, 1[$,

$$1 \lesssim |u(y)| \lesssim |y|^\delta, \quad \forall |y| \gg 1. \tag{1.6}$$

This case is not much addressed in the literature (except for the implicitly related nonexistence results based on the vorticity profile, e.g. [**5**]), but it is motivated by the direct numerical simulations (e.g. the recent work [**13, 14**]) and also by several works on the 1D models of Euler equations: the 1D Burgers equation, the 1D CCF model ([**7**]), the 1D CKY model ([**6**]) and so on. The blowup issue of all these 1D equations is clear: the Burgers equation develops shock singularity at finite time, while it is proved in [**7**] and [**6**] respectively that the CCF equation and the CKY equations form finite-time singularity for some smooth data. The further study ([**9**] for Burgers, [**8**] for CCF, and [**12**] for CKY) shows that the finite-time singularities of these equations are of locally self-similar type with some index $\alpha \in]-1, 0[$ and the corresponding velocity profiles have growing spacial asymptotics. Based on these motivations, it deserves much to consider such a scenario for the Euler equations (1.1).

In order to do so, we have to derive a meaningful representation formula of the pressure profile in terms of the velocity profiles, since the usual one

$$q(y) = -\frac{1}{N}|u(y)|^2 + \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(x-y) u_i(y) u_j(y) \, \mathrm{d}y, \tag{1.7}$$

works for the velocity profiles with suitable decaying asymptotics, e.g. $u \in L^p(\mathbb{R}^N)$ for some $p \in]2, \infty[$, but it does not make sense for the profiles satisfying (1.6). We justify the needed formula in Lemma 2.1, which can respectively be stated as follows: if $1 \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [0, \frac{1}{2}[$,

$$\begin{aligned} q(y) = &-\frac{1}{N}|u(y)|^2 + \text{p.v.} \int_{|z| \le M} K_{ij}(y-z) u_i(z) u_j(z) \, \mathrm{d}z + \\ &+ \int_{|z| \ge M} \big(K_{ij}(y-z) - K_{ij}(z)\big) u_i(z) u_j(z) \, \mathrm{d}z, \end{aligned} \tag{1.8}$$

and if $|z|^{\frac{1}{2}} \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [\frac{1}{2}, 1[$,

$$
\begin{aligned}
q(y) = & -\frac{1}{N}|u(y)|^2 + \text{p.v.}\int_{|z|\leq M} K_{ij}(y-z)u_i(z)u_j(z)\,\mathrm{d}z + \\
& + \int_{|z|\geq M} \big(K_{ij}(y-z) - K_{ij}(z) - y\cdot\nabla K_{ij}(z)\big)u_i(z)u_j(z)\,\mathrm{d}z + A\cdot y,
\end{aligned} \tag{1.9}
$$

where $A \in \mathbb{R}^N$ is some fixed constant vector and $M > 0$ is an absolute constant so that (1.6) holds for all $|y| \geq M$. The formula (1.8)-(1.9) also have the decomposition (2.6) (and its variant), which can be used to show that $q(y)$ belongs to $C^2_{\text{loc}}(\mathbb{R}^N)$ as long as $u \in C^3_{\text{loc}}(\mathbb{R}^N)$ (see Lemma 2.1). Note that (1.8) is reminiscent of a similar formula of the pressure in terms of the velocity field when considering the local Leray solutions of the 3D Navier-Stokes equations (cf. [**18**, Chapter 32]).

Our main results read as follows.

Theorem 1.1. *Suppose that $v \in C([0,T[; H^s(\mathbb{R}^N))$, $s > \frac{N}{2}+1$ is a locally self-similar solution for the Euler system which satisfies* (1.3) *on the spacetime domain $B_\rho(x_0)\times]0,T[$ with profiles $u \in C^3_{\text{loc}}(\mathbb{R}^N)$ and $\alpha > -1$. In addition, we assume that u satisfies that for some $\delta \in]0,1[$,*

$$|u(y)| \lesssim |y|^\delta, \quad \forall |y| \gg 1. \tag{1.10}$$

We have the following statements.

(1) If additionally there is a small number $\epsilon_0 \ll 1$ so that $0 < \epsilon_0 < \delta$ and

$$|u(y)| \gtrsim |y|^{\epsilon_0}, \quad \forall |y| \gg 1, \tag{1.11}$$

then the possible scope of α to admit nontrivial self-similar velocity profiles is $-\delta \leq \alpha \leq -\epsilon_0$, and for every such an α, the corresponding profiles satisfy that

$$\int_{|y|\leq L} |u(y)|^2\,\mathrm{d}y \approx L^{N-2\alpha}, \quad \forall L \gg 1. \tag{1.12}$$

(2) If $\delta < \frac{1}{2}$, $\alpha > -\frac{1}{2}$, and additionally

$$|u(y)| \gtrsim 1, \quad \forall |y| \gg 1, \tag{1.13}$$

then the possible range of α to admit nontrivial self-similar profiles is $-\delta \leq \alpha \leq 0$, and the velocity profiles corresponding to each $\alpha \in [-\delta, 0]$ satisfy (1.12).

For the proof of Theorem 1.1, we first show Lemma 2.1 which states that under the assumptions of Theorem 1.1, the pressure p can be expressed as (1.4) on the domain $B_\rho(x_0)\times]0,T[$ with the pressure profile q has an explicit formula (1.8)-(1.9) in terms of u; then we start with the following local energy inequality of the profiles (u,q) (cf. [**5**]) for $0 < l_1 < l_2$ and the standard test function ϕ,

$$
\begin{aligned}
& \Big|\frac{1}{l_2^{N-2\alpha}}\int_{|y|\leq l_2} |u(y)|^2\phi\Big(\frac{y}{l_2}\Big)\mathrm{d}y - \frac{1}{l_1^{N-2\alpha}}\int_{|y|\leq l_1}|u(y)|^2\phi\Big(\frac{y}{l_1}\Big)\mathrm{d}y\Big| \leq \\
& \leq C\int_{l_1/2\leq|y|\leq l_2} \frac{|u(y)|^3 + |q(y)||u(y)|}{|y|^{N-2\alpha+1}}\mathrm{d}y,
\end{aligned} \tag{1.14}
$$

and by applying the bootstrapping method and a careful analysis according to the values of (α,δ), we prove the main results, which is placed in the whole Section 3. In this process, the treating of the term containing the pressure profile q is technical and is repeatedly used, and we specially tackle with it in Lemma 2.2 of Section 2.

REMARK 1.2. From (1.12) for every $-1 < \alpha < \frac{N}{2}$ (cf. [**3**] for the case $0 < \alpha < \frac{N}{2}$ and cf. Theorem 1.1 for the case $-1 < \alpha \leq 0$), we can expect the "typical" possible velocity profiles are that

$$|u(y)| \approx |y|^{-\alpha} + l.o.t., \quad \forall |y| \gg 1, \tag{1.15}$$

where *l.o.t.* is the abbreviation of the lower order terms. By scaling, we can also expect the typical vorticity profiles are

$$|\nabla \times u(y)| \approx |y|^{-\alpha-1} + l.o.t., \quad \forall |y| \gg 1, \tag{1.16}$$

which is compatible with the nonexistence result based on the vorticity profile of [**5**]. Furthermore, in the considered blowup scenario and using (1.16), we have that for all $(t, x) \in]0, T[\times \big(B_\rho(x_0) \setminus \{x_0\}\big)$,

$$|\nabla \times v(x,t)| = \frac{1}{T-t} \left| \nabla \times u\left(\frac{x - x_0}{(T-t)^{\frac{1}{1+\alpha}}} \right) \right| \approx \frac{1}{|x - x_0|^{1+\alpha}}. \tag{1.17}$$

This typical self-similar blowup case is consistent with the Beale-Kato-Majda criterion (cf. [**1**]) since for all $\alpha > -1$,

$$\int_0^T \|\nabla \times v\|_{L^\infty} \, \mathrm{d}t \approx T \sup_{0 < |x - x_0| \leq \rho} |x - x_0|^{-(1+\alpha)} = \infty.$$

On the other hand, the bound $\int_0^T \|\nabla \times v\|_{L^r} \, \mathrm{d}t < \infty$ with some $1 \leq r < \infty$ is not sufficient to get rid of such typical blowup scenario (1.15)-(1.16) for all $\alpha > -1$; indeed, this typical blowup scenario still may happen at the range $-1 < \alpha < -1 + N/r$.

REMARK 1.3. Under the assumptions of Theorem 1.1, we have the energy inequality of the velocity field $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} \lesssim 1$ for all $t < T$, which can naturally lead to a part of the estimate (1.12):

$$\int_{|y| \leq L} |u(y)|^2 \, \mathrm{d}y \lesssim L^{N-2\alpha}, \quad \text{with } L = (T-t)^{-\frac{1}{1+\alpha}} \gg 1;$$

indeed, this can be seen from plugging the self-similar scenario (1.3) into the inequality $\int_{B_\rho(x_0)} |v(x,t)|^2 \, \mathrm{d}x \lesssim 1$ and then changing of variables. But we still resolve this estimate in Section 3 based on the local energy inequality (1.14) of the profiles (u, q), which may have its own interest (due to that the assumptions on u and the formula of q truly take part in the proof).

Throughout this paper, C stands for a constant which may be different from line to line, $X \lesssim Y$ means that there is a harmless constant C such that $X \leq CY$, and $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$ simultaneously. Denote $B_r(x) := \{y \in \mathbb{R}^N : |y - x| \leq r\}$ the ball of $\mathbb{R}^N$ and $B_r^c(x) := \mathbb{R}^N \setminus B_r(x)$ its complement set. For a number $a \in \mathbb{R}$, denote $[a]$ by the integer part of a.

2. Auxiliary lemmas concerning the pressure profile

We collect two useful auxiliary lemmas in this section: one is about the justification of the representation formula of the pressure profile, and the other is about the estimation of the term containing the pressure profile.

Lemma 2.1. *Suppose $v \in C([0,T[;H^s(\mathbb{R}^N))$, $s > \frac{N}{2}+1$ is a locally self-similar solution* (1.3) *to the Euler equations and $\alpha > -1$. Assume that $u \in C^3_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies that for some $\delta \in [0,1[$,*

$$|u(y)| \lesssim |y|^\delta, \quad \forall |y| \geq M, \tag{2.1}$$

with $M > 0$ a pure number, then the corresponding pressure on the ball $B_\rho(x_0)$ for all t near T is expressed as

$$p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} q\left(\frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}}\right) + d(t), \tag{2.2}$$

where $d(t)$ is a function depending only on t (satisfying (2.23) *below), and $q(y)$ is a C^2_{loc}-smooth scalar function defined by*

$$\begin{aligned} q(y) = &-\frac{1}{N}|u(y)|^2 + \mathrm{p.v.}\int_{|z|\leq M} K_{ij}(y-z)u_i(z)u_j(z)\,\mathrm{d}z + \\ &+ \int_{|z|\geq M} \widetilde{K}_{ij}(y,z)u_i(z)u_j(z)\,\mathrm{d}z + A\cdot y \end{aligned} \tag{2.3}$$

with $A \in \mathbb{R}^N$ some fixed constant vector (especially, A equals 0 at the case $\{\alpha > -\frac{1}{2}, \delta < 1/2\}$ or $\{\alpha > -\frac{1}{2}, u \in L^p, p \in]2,\infty[\}$) and the kernel function $\widetilde{K}_{ij}$ given by

$$\widetilde{K}_{ij}(y,z) = \begin{cases} K_{ij}(y-z), & \text{if } u \in L^p(\mathbb{R}^N), p \in]2,\infty[, \\ K_{ij}(y-z) - K_{ij}(z), & \text{if } 1 \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [0,\frac{1}{2}[, \\ K_{ij}(y-z) - K_{ij}(z) - y\cdot\nabla K_{ij}(z), & \text{if } |z|^{\frac{1}{2}} \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [\frac{1}{2},1[. \end{cases} \tag{2.4}$$

Proof of Lemma 2.1. We mainly adapt the strategy in the proof of [**3**, Lemma 2.1] with suitable modification. We first introduce a function $I(y)$, which is a part of (2.3), and prove that it is meaningfully defined and is a tempered distribution, and also pointwisely solves the equation $\Delta I = -\mathrm{divdiv}(u\otimes u)$. Then we find a tempered distribution $q(y)$ solving the first equation of (1.5). Since q also solves the same Poisson equation, the difference between I and q is a harmonic polynomial, and at last we prove that the order of this polynomial is at most one and conclude the formula (2.3).

First define the quantity contained in (2.3) as

$$\begin{aligned} I(y) := &-\frac{1}{N}|u(y)|^2 + \mathrm{p.v.}\int_{|z|\leq M} K_{ij}(y-z)u_i(z)u_j(z)\,\mathrm{d}z + \\ &+ \int_{|z|\geq M} \widetilde{K}_{ij}(y,z)u_iu_j(z)\,\mathrm{d}z \end{aligned} \tag{2.5}$$

and we show that $I(y)$ is meaningful and is a tempered distribution. Let $\phi \in C^\infty_c(\mathbb{R}^N)$ be a test function supported on $B_1(0)$ such that $\phi \equiv 1$ on $B_{1/2}(0)$ and $0 \leq \phi \leq 1$. For any $L \geq M$, set $\phi_L(z) = \phi(\frac{z}{L})$, then we have the following decomposition

$$\begin{aligned} I(y) = & -\frac{1}{N}|u(y)|^2 + \text{p.v.}\int_{\mathbb{R}^N} K_{ij}(y-z)\phi_{4L}(z)u_iu_j(z)\,\mathrm{d}z + \\ & + \int_{|z|\geq M} \big(\widetilde{K}_{ij}(y,z) - K_{ij}(y-z)\big)\phi_{4L}(z)u_iu_j(z)\,\mathrm{d}z + \\ & + \int_{\mathbb{R}^N} \widetilde{K}_{ij}(y,z)(1-\phi_{4L}(z))u_iu_j(z)\,\mathrm{d}z \\ := & -\frac{1}{N}|u(y)|^2 + I_1(y,L) + I_2(y,L) + I_3(y,L). \end{aligned} \tag{2.6}$$

Since $u \in C^3_{\mathrm{loc}}(\mathbb{R}^3)$, from the Besov embedding, we infer that $I_1(y,L) \in C^\beta$ for all $\beta < 3$. For $I_2(y,L)$, due to that

$$\widetilde{K}_{ij}(y,z) - K_{ij}(y-z) = \begin{cases} 0, & \text{if } u \in L^p(\mathbb{R}^N), p \in]2,\infty[, \\ -K_{ij}(z), & \text{if } 1 \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [0,\frac{1}{2}[, \\ -K_{ij}(z) - y\cdot\nabla K_{ij}(z), & \text{if } |z|^{\frac{1}{2}} \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [\frac{1}{2},1[, \end{cases}$$

we deduce that for all $y \in B_L(0)$ and $1 \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [0,1[$,

$$I_2(y,L) \lesssim L^{2\delta}. \tag{2.7}$$

Similarly, for $s = 1,2$, we see that all the terms $\nabla^s_y(I_2(y,L))$ vanishes except for $\nabla^1_y(I(y,L))$ at the case $|z|^{1/2} \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [1/2,1[$, and for which we get

$$|\nabla^1_y(I_2(y,L))| \lesssim \int_{M\leq|z|\leq 4L} |\nabla K_{ij}(z)||u(z)|^2\,\mathrm{d}z \lesssim L^{2\delta-1}.$$

We next consider $I_3(y,L)$ acting on the ball $B_L(0)$, and if $u \in L^p(\mathbb{R}^N)$ for some $p \in]2,\infty[$, then

$$\begin{aligned} I_3(y,L) & \lesssim \sum_{k=0}^{\infty}\int_{2^kL\leq|z|\leq 2^{k+1}L} \frac{1}{|z|^N}|u(z)|^2\mathrm{d}z \\ & \lesssim \sum_{k=0}^{\infty}(2^kL)^{-N+N(1-2/p)}\|u\|^{2/p}_{L^p} \lesssim L^{-2N/p}; \end{aligned} \tag{2.8}$$

and if $1 \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in]0,\frac{1}{2}[$ for all $|z| \geq M$, then

$$I_3(y,L) \lesssim \int_{|z|\geq 2L} \frac{|y|}{|z|^{N+1}}|u(z)|^2\,\mathrm{d}z \lesssim L^{2\delta}; \tag{2.9}$$

and if $|z|^{1/2} \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [1/2,1[$ for all $|z| \geq M$, then

$$I_3(y,L) \lesssim \int_{|z|\geq 2L} \frac{|y|^2}{|z|^{N+2}}|u(z)|^2\,\mathrm{d}z \lesssim L^{2\delta}. \tag{2.10}$$

For $s = 1,2$ and for all $y \in B_L(0)$, we also get that if $u \in L^p(\mathbb{R}^N)$, $p \in]2,\infty[$,

$$\nabla^s_y(I_3(y,L)) \lesssim \sum_{k=0}^{\infty}\int_{2^kL\leq|z|\leq 2^{k+1}L} \frac{1}{|z|^{N+s}}|u(z)|^2\,\mathrm{d}z \lesssim L^{-s-\frac{2N}{p}},$$

and if $1 \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in]0,1/2[$, $\forall |z| \geq M$,

$$\begin{aligned}\nabla_y^s\big(I_3(y,L)\big) &\leq \nabla_y^s\left(\int_{\mathbb{R}^N}\int_0^1 y\cdot\nabla K_{ij}(\tau y - z)(1-\phi_{4L}(z))u_i u_j(z)\,\mathrm{d}\tau\mathrm{d}z\right)\\ &\lesssim \int_{|z|\geq 2L}\frac{|y|}{|z|^{N+1+s}}|u(z)|^2\,\mathrm{d}z \lesssim L^{-s+2\delta},\end{aligned}$$

and if $|z|^{1/2} \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [1/2,1[$, $\forall |z| \geq M$,

$$\begin{aligned}&\nabla^s\big(I_3(y,L)\big) \leq\\ &\leq \nabla_y^s\left(\int_{|z|\geq 4L}\int_0^1\int_0^1 \left(y\cdot\nabla^2 K_{ij}(\tau\theta y - z)\cdot y\right)(1-\phi_{4L}(z))u_i u_j(z)\tau\,\mathrm{d}\theta\mathrm{d}\,\tau\mathrm{d}z\right)\\ &\lesssim \int_{|z|\geq 2L}\frac{|y|^2}{|z|^{N+2+s}}|u(z)|^2\,\mathrm{d}z \lesssim L^{-s+2\delta}.\end{aligned}$$

Hence the scalar function $I(y)$ defined by (2.5) is C^2-smooth on $B_L(0)$. Moreover, for all $y \in B_L(0)$, we have

$$\begin{aligned}\Delta I &= \Delta\left(-\frac{1}{N}|u|^2\phi_{4L} + I_1\right) + \Delta\left(I_2 + I_3\right)\\ &= -\mathrm{div}\,\mathrm{div}\big(u\sqrt{\phi_{4L}}\otimes u\sqrt{\phi_{4L}}\big) = -\mathrm{div}\,\mathrm{div}\big(u\otimes u\big),\end{aligned}$$

where in the second line $\Delta(I_2+I_3) = 0$ due to that the term $\widetilde{K}_{ij}(y,z)$ is harmonic in the y-variable for all $y \in B_L(0)$ and $z \in B_{2L}^c(0)$. Besides, it is not hard to show that I is a tempered distribution, which can be seen from the following computation, that is, if $1 \lesssim |u(z)| \lesssim |z|^\delta$, $\delta \in [0,1[$, $\forall |z| \geq M$, then for $L \gg 1$ and some $r > 2$,

$$\int_{|y|\leq L}|I_1(y,L)|^{\frac{r}{2}}\,\mathrm{d}y \lesssim \int_{|z|\leq 4L}|u(z)|^r\,\mathrm{d}z \lesssim L^{N+r\delta},$$

and by (2.7), (2.9)-(2.10),

$$\int_{|y|\leq L}|I_2(y,L) + I_3(y,L)|^{\frac{r}{2}}\,\mathrm{d}y \lesssim L^{N+r\delta};$$

while if $u \in L^p(\mathbb{R}^N)$, $p \in]2,\infty[$, then $\|I(y)\|_{L^{\frac{p}{2}}} \lesssim \|u\|_{L^p}^2$ from the Calderón-Zygmund theorem.

Next we intend to find a distributional pressure profile solving the first equation of (1.5), i.e.,

$$\frac{\alpha}{\alpha+1}u + \frac{1}{\alpha+1}y\cdot\nabla u + u\cdot\nabla u + \nabla q = 0. \tag{2.11}$$

Applying the ansatz (1.3) to the Euler equations (1.1), and by setting

$$y := \frac{x - x_0}{(T-t)^{\frac{1}{1+\alpha}}}, \quad p(x,t) := \bar{p}(y,t), \tag{2.12}$$

we obtain that for all $|y| \leq \rho(T-t)^{-\frac{1}{1+\alpha}}$,

$$\frac{\alpha}{1+\alpha}u(y) + \frac{1}{1+\alpha}y\cdot\nabla_y u(y) + u\cdot\nabla_y u(y) + \nabla_y\Big((T-t)^{\frac{2\alpha}{1+\alpha}}\bar{p}(y,t)\Big) = 0. \tag{2.13}$$

For any fixed $t < T$, denoting $f(y,t) = (T-t)^{\frac{2\alpha}{1+\alpha}}\bar{p}(y,t)$, then the vector-valued function $\nabla_y f(y,t) =: g(y)$ depends only on y on the domain $D(t) := \{y : |y| \leq$

$\rho(T-t)^{-\frac{1}{1+\alpha}}\}$. Thus from the fundamental theorem of calculus, we deduce that for all $y \in D(t)$,

$$\begin{aligned}
&(T-t)^{\frac{2\alpha}{1+\alpha}}\bar{p}(y,t) - (T-t)^{\frac{2\alpha}{1+\alpha}}\bar{p}(0,t) \\
&= f(y,t) - f(0,t) = \int_0^1 \frac{d}{ds} f(sy,t)\,\mathrm{d}s = \int_0^1 y\cdot\nabla f(sy,t)\mathrm{d}s \\
&= \int_0^1 y\cdot g(sy)\,\mathrm{d}s =: q(y),
\end{aligned} \tag{2.14}$$

that is,

$$p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{1+\alpha}}} q\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right) + c(t), \quad \forall\, x\in B_\rho(x_0), \tag{2.15}$$

with $c(t) = p(x_0,t)$. Inserting (2.15) into (2.13) yields the equation (2.11) on $\mathbb{R}^N$. Now we prove that $q(y)$ is indeed a tempered distribution. The proof is quite similar to the deduction in [**3**, Lemma 2.1], and we here include it for completeness. Since we have the energy conservation of the velocity v and (1.2), we infer that $\|p(x,t)\|_{L^1_{\text{weak}}} \lesssim \|v(x,t)\|_{L^2}^2 \lesssim \|v_0\|_{L^2}^2 \lesssim 1$, which implies $|\{x : |p(x,t)| > \lambda\}| \le \frac{C}{\lambda}$ for all $t < T$. Thus there exists a small number $\eta > 0$ so that $|\{x : |p(x,t)| > \frac{1}{\eta}\frac{1}{(T-t)^{N/(1+\alpha)}}\}| \le \frac{|B_1(0)|}{2}\rho^N(T-t)^{\frac{N}{1+\alpha}}$, which yields that there is a point x_t in the ball $\{x : |x-x_0| \le \rho(T-t)^{\frac{1}{1+\alpha}}\}$ so that $|p(x_t,t)| \le \frac{1}{\eta}\frac{1}{(T-t)^{N/(1+\alpha)}}$. Hence with this x_t and the corresponding $y_t = \frac{x_t - x_0}{(T-t)^{1/(1+\alpha)}} \in B_\rho(0)$ at our disposal, we have

$$|c(t)| \le (T-t)^{-\frac{2\alpha}{1+\alpha}}|q(y_t)| + \eta^{-1}(T-t)^{-\frac{N}{1+\alpha}} \lesssim (T-t)^{-\frac{2\alpha}{1+\alpha}} + (T-t)^{-\frac{N}{1+\alpha}},$$

where we have used the fact that $|q(y_t)| \le C$ from $q(y) \in C^2_{\text{loc}}(\mathbb{R}^N)$. From (2.15), we see that

$$q(y) = (T-t)^{\frac{2\alpha}{1+\alpha}} p\big(x_0 + y(T-t)^{\frac{1}{1+\alpha}}, t\big) + (T-t)^{\frac{2\alpha}{1+\alpha}}c(t), \quad \forall |y| \le \rho(T-t)^{-\frac{1}{1+\alpha}},$$

thus if $1 \lesssim |u(y)| \lesssim |y|^\delta$, $\forall |y| \ge M$ for some $\delta \in [0,1[$, we get that for any $\tilde{p} \in]2,\infty[$,

$$\begin{aligned}
&\int_{\frac{\rho}{4(T-t)^{1/(1+\alpha)}} \le |y| \le \frac{\rho}{2(T-t)^{1/(1+\alpha)}}} |q(y)|^{\frac{\tilde{p}}{2}}\,\mathrm{d}y \\
&\lesssim (T-t)^{-\frac{N}{1+\alpha}} + (T-t)^{\frac{(2\alpha-N)\frac{\tilde{p}}{2}-N}{1+\alpha}} + (T-t)^{\frac{\tilde{p}\alpha-N}{1+\alpha}} \int_{\frac{\rho}{4}\le|x-x_0|\le\frac{\rho}{2}} |p(t)|^{\frac{\tilde{p}}{2}}\,\mathrm{d}x \\
&\lesssim (T-t)^{-\frac{N}{1+\alpha}} + (T-t)^{\frac{(2\alpha-N)\frac{\tilde{p}}{2}-N}{1+\alpha}} + \\
&\quad + (T-t)^{\frac{\tilde{p}\alpha-N}{1+\alpha}}\left(\int_{\frac{\rho}{8}\le|x-x_0|\le\rho} |v(x,t)|^{\tilde{p}}\,\mathrm{d}x + \|v(t)\|_{L^2}^{\tilde{p}}\right) \\
&\lesssim (T-t)^{-\frac{N}{1+\alpha}} + (T-t)^{\frac{(2\alpha-N)\tilde{p}/2-N}{1+\alpha}} + (T-t)^{\frac{\tilde{p}\alpha-N}{1+\alpha}} + \\
&\quad + (T-t)^{\frac{\tilde{p}\alpha}{1+\alpha}} \int_{\frac{\rho}{8(T-t)^{1/(1+\alpha)}} \le |y| \le \frac{\rho}{(T-t)^{1/(1+\alpha)}}} |u(y)|^{\tilde{p}}\,\mathrm{d}y,
\end{aligned} \tag{2.16}$$

which leads to that for some $m_1 \in \mathbb{N}$,

$$\int_{\frac{\rho}{4(T-t)^{1/(1+\alpha)}} \le |y| \le \frac{\rho}{2(T-t)^{1/(1+\alpha)}}} |q(y)|^{\frac{\tilde{p}}{2}}\,\mathrm{d}y \lesssim (T-t)^{-m_1}; \tag{2.17}$$

while if $u \in L^p(\mathbb{R}^N)$ for some $p \in]2,\infty[$, we see that (2.16) also holds true for $\tilde{p} = p$ and so does (2.17) for some number m_1. In the above deduction of (2.16) from the second line to the third line, we have used the decomposition that for all $\frac{\rho}{4} \leq |x - x_0| \leq \frac{\rho}{2}$,

$$\begin{aligned} p(x,t) = & -\frac{1}{N}|v(x,t)|^2 + \\ & + \left(\text{p.v.}\int_{|z-x_0|\leq\frac{\rho}{8}} + \int_{\frac{\rho}{8}\leq|z-x_0|\leq\rho} + \int_{|z-x_0|\geq\rho}\right)\Big(K_{ij}(x-z)v_iv_j(z,t)\,\mathrm{d}z\Big) \\ := & -\frac{1}{N}|v(x,t)|^2 + p_1(x,t) + p_2(x,t) + p_3(x,t), \end{aligned}$$

and the following estimates that

$$\begin{aligned} \|p_1(x,t)\|_{L^\infty_x(\{\frac{\rho}{4}\leq|x-x_0|\leq\frac{\rho}{2}\})} & \lesssim \|v(x,t)\|^2_{L^2_x}, \\ \int_{\frac{\rho}{4}\leq|x-x_0|\leq\frac{\rho}{2}} |p_2(x,t)|^{\frac{\tilde{p}}{2}}\,\mathrm{d}x & \lesssim \int_{\frac{\rho}{8}\leq|x-x_0|\leq\rho} |v(x,t)|^{\tilde{p}}\,\mathrm{d}x, \\ \|p_3(x,t)\|_{L^\infty_x(\{\frac{\rho}{4}\leq|x-x_0|\leq\frac{\rho}{2}\})} \lesssim \int_{|z-x_0|\geq\rho} \frac{1}{|z-x_0|^N}|v(z,t)|^2\,\mathrm{d}z & \lesssim \|v(x,t)\|^2_{L^2_x}. \end{aligned}$$

According to (2.17), we infer that $q(y)$ at most has the polynomial growth near infinity and is thus a tempered distribution of $\mathbb{R}^N$.

Now we show that q and I are equal up to a first-order harmonic polynomial. Since they both satisfy the Poisson equation $\Delta I = -\mathrm{divdiv}(u\otimes u) = \Delta q$, and are both tempered distributions of $\mathbb{R}^N$, the difference

$$q - I =: h \tag{2.18}$$

is a harmonic polynomial. In the following we prove that the order of h is at most one. For all $|y| \leq \frac{\rho}{4(T-t)^{\frac{1}{1+\alpha}}}$, inserting (1.3) into (1.2), and using (2.6) with $4L = \rho(T-t)^{-\frac{1}{1+\alpha}}$, we have

$$\begin{aligned} & (T-t)^{\frac{2\alpha}{1+\alpha}} p(x_0 + y(T-t)^{\frac{1}{1+\alpha}}, t) = \\ = & -\frac{1}{N}|u(y)|^2 + \\ & + \text{p.v.}\int_{\mathbb{R}^N} K_{ij}\left(x_0 + y(T-t)^{\frac{1}{1+\alpha}} - z\right)\phi_\rho(z-x_0)\,(u_iu_j)\left(\frac{z-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right)\mathrm{d}z \\ & + (T-t)^{\frac{2\alpha}{1+\alpha}}\int_{\mathbb{R}^N} K_{ij}\left(x_0 + y(T-t)^{\frac{1}{1+\alpha}} - z\right)(1-\phi_\rho(z-x_0))\,(v_iv_j)(z,t)\,\mathrm{d}z \\ = & -\frac{1}{N}|u(y)|^2 + \text{p.v.}\int_{\mathbb{R}^N} K_{ij}(y-z)\,\phi\left(\frac{z}{\rho(T-t)^{-1/(1+\alpha)}}\right)(u_iu_j)(z)\mathrm{d}z + \tilde{p}(y,t), \\ = & \, I(y) - \tilde{I}(y,t) + \tilde{p}(y,t), \end{aligned}$$

with $\phi_\rho(z) = \phi(\frac{z}{\rho})$ (introduced below (2.5)),

$$\tilde{p}(y,t) := (T-t)^{\frac{2\alpha}{1+\alpha}}\int_{\mathbb{R}^N} K_{ij}\big(x_0 + y(T-t)^{\frac{1}{1+\alpha}} - z\big)\big(1-\phi_\rho(z-x_0)\big)(v_iv_j)(z,t)\,\mathrm{d}z,$$

and

$$\tilde{I}(y,t) := \int_{|z|\geq M} \Big(\widetilde{K}_{ij}(y,z) - K_{ij}(y-z)\Big)\,\phi\left(\frac{z}{\rho(T-t)^{-1/(1+\alpha)}}\right) u_i u_j(z)\,\mathrm{d}z + \int_{\mathbb{R}^N} \widetilde{K}_{ij}(y,z)\left(1-\phi\left(\frac{z}{\rho(T-t)^{-1/(1+\alpha)}}\right)\right) u_i u_j(z)\,\mathrm{d}z.$$

On the other hand, thanks to (2.15), we see that

$$(T-t)^{\frac{2\alpha}{1+\alpha}} p(x_0 + y(T-t)^{\frac{1}{1+\alpha}}, t) = q(y) + d(t) \tag{2.19}$$

with $d(t) := (T-t)^{\frac{2\alpha}{1+\alpha}} c(t)$. Hence, by virtue of (2.18)-(2.19), we see that

$$|h(y) - d(t)| \leq |\tilde{p}(y,t)| + |\tilde{I}(y,t)|, \qquad \forall |y| \leq \frac{\rho}{4(T-t)^{\frac{1}{1+\alpha}}}. \tag{2.20}$$

For $\tilde{p}$, due to that $z \in B^c_{\frac{\rho}{2}}(x_0)$ and $(T-t)^{\frac{1}{1+\alpha}} y \in B_{\frac{\rho}{4}}(0)$, we see that $\big|K_{ij}(x_0 + y(T-t)^{\frac{1}{1+\alpha}} - z)\big| \lesssim \frac{1}{\rho^N}$, and thus

$$|\tilde{p}(y,t)| \lesssim (T-t)^{\frac{2\alpha}{1+\alpha}} \|v\|^2_{L^2} \lesssim (T-t)^{\frac{2\alpha}{1+\alpha}}. \tag{2.21}$$

For $\tilde{I}$, by arguing as (2.7)-(2.10) (with $L = \frac{1}{4}\rho(T-t)^{-\frac{1}{1+\alpha}}$), we get

$$|\tilde{I}(y,t)| \lesssim \begin{cases} (T-t)^{-\frac{2\delta}{1+\alpha}}, & \text{if } 1 \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [0,1[, \\ (T-t)^{\frac{2N}{p(1+\alpha)}}, & \text{if } u \in L^p(\mathbb{R}^N),\, p \in]2,\infty[. \end{cases} \tag{2.22}$$

Since $\alpha > -1$, $\delta \in [0,1[$ and the above estimates hold for all $y \leq \frac{\rho}{4(T-t)^{1/(1+\alpha)}}$, we infer that the order of harmonic polynomial $h(y)$ is at most one (if not, $|h(y)| \gtrsim (T-t)^{-\frac{2}{1+\alpha}}$ for some $|y| \leq \frac{\rho}{4(T-t)^{1/(1+\alpha)}}$, which is not compatible with (2.20)-(2.22)) and

$$|d(t)| \lesssim \begin{cases} 1 + (T-t)^{\frac{2\alpha}{1+\alpha}} + (T-t)^{-\frac{2\delta}{1+\alpha}}, & \text{if } 1 \lesssim |u(z)| \lesssim |z|^\delta, \delta \in [0,1[, \\ 1 + (T-t)^{\frac{2\alpha}{1+\alpha}} + (T-t)^{\frac{2N}{p(1+\alpha)}}, & \text{if } u \in L^p(\mathbb{R}^N),\, p \in]2,\infty[, \end{cases} \tag{2.23}$$

which proves (2.2). In particular, if $\alpha > -1/2$, $\delta \in [0, 1/2[$ or $\alpha > -\frac{1}{2}$, $u \in L^p(\mathbb{R}^N)$ ($p \in]2,\infty[$), $h(y)$ moreover should be a uniform constant. □

LEMMA 2.2. *Assume that $u \in C^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ is a locally regular vector field. Suppose u additionally satisfies that*

$$\begin{aligned} &|u(y)| \lesssim |y|^\delta, \ \ \forall |y| \geq M, \quad \textit{with} \quad 0 \leq \delta < 1 \quad \textit{and} \\ &\int_{|y|\leq L} |u(y)|^2\,\mathrm{d}y \lesssim L^b, \ \ \forall L \geq M, \quad \textit{with} \quad 0 \leq b \leq N + 2\delta, \end{aligned} \tag{2.24}$$

and $M > 0$ a fixed number. Let q be a scalar field defined from u by that for every $|y| \leq L$,

$$\begin{aligned} q(y) =\ & c_0|u(y)|^2 + A\cdot y + \mathrm{p.v.}\int_{|y|\leq M} K_{ij}(y-z) u_i(z) u_j(z)\,\mathrm{d}z + \\ &+ \begin{cases} \int_{|z|\geq M} \big(K_{ij}(y-z) - K_{ij}(z)\big) u_i u_j(z)\,\mathrm{d}z, & \textit{if } \delta \in [0,\frac{1}{2}[, \\ \int_{|z|\geq M} \big(K_{ij}(y-z) - K_{ij}(z) - y\cdot\nabla K_{ij}(z)\big) u_i u_j(z)\,\mathrm{d}z, & \textit{if } \delta \in [\frac{1}{2},1[, \end{cases} \end{aligned} \tag{2.25}$$

with $c_0 \in \mathbb{R}$, $A \in \mathbb{R}^N$ and $K_{ij}(z)$ $(i,j = 1,\cdots,N)$ some Calderón-Zygmund kernel, then we have

$$\int_{|y|\leq L} |q(y)||u(y)|\,\mathrm{d}y \lesssim \begin{cases} L^{b+\delta} + L^{\frac{N+b}{2}+1}, & \text{if } (b,\delta) \neq (N+1,\frac{1}{2}), \\ L^{\frac{N+3}{2}}[\log_2 L], & \text{if } (b,\delta) = (N+1,\frac{1}{2}). \end{cases} \tag{2.26}$$

In particular, if $\delta \in [0,\frac{1}{2}[$ and $A = 0$, we also have

$$\int_{|y|\leq L} |q(y)||u(y)|\,\mathrm{d}y \lesssim \begin{cases} L^{b+\delta}, & \text{if } b \geq N-2\delta, (b,\delta) \neq (N,0), \\ L^N[\log_2 L], & \text{if } (b,\delta) = (N,0), \\ L^{\frac{N+b}{2}}, & \text{if } b \leq N-2\delta, (b,\delta) \neq (N,0). \end{cases} \tag{2.27}$$

Proof of Lemma 2.2. We decompose $q(y)$ as

$$q(y) = c_0|u(y)|^2 + q_1(y,L) + q_2(y,L) + q_3(y,L) + q_4(y,L) \tag{2.28}$$

with

$$q_1(y,L) = A\cdot y, \qquad q_2(y,L) = \text{p.v.}\int_{|y|\leq 2L} K_{ij}(y-z)u_i(z)u_j(z)\,\mathrm{d}z,$$

$$q_3(y,L) = \begin{cases} \int_{|z|\geq 2L}\big(K_{ij}(y-z) - K_{ij}(z)\big)u_i(z)u_j(z)\,\mathrm{d}z, & \text{if } \delta \in [0,\frac{1}{2}[\\ \int_{|z|\geq 2L}\big(K_{ij}(y-z) - K_{ij}(z) - y\cdot\nabla K_{ij}(z)\big)u_i(z)u_j(z)\,\mathrm{d}z, & \text{if } \delta \in [\frac{1}{2},1[, \end{cases}$$

$$q_4(y,L) = \begin{cases} -\int_{M\leq|z|\leq 2L} K_{ij}(z)u_i(z)u_j(z)\,\mathrm{d}z, & \text{if } \delta \in [0,\frac{1}{2}[, \\ -\int_{M\leq|z|\leq 2L}\big(K_{ij}(z) + y\cdot\nabla K_{ij}(z)\big)u_i(z)u_j(z)\,\mathrm{d}z, & \text{if } \delta \in [\frac{1}{2},1[. \end{cases}$$

We first directly have $\int_{|y|\leq L}|u(y)|^3\,\mathrm{d}y \lesssim L^{b+\delta}$, and

$$\int_{|y|\leq L}|q_1(y,L)||u(y)|\,\mathrm{d}y \leq |A|L^{N/2+1}\Big(\int_{|y|\leq L}|u(y)|^2\,\mathrm{d}y\Big)^{1/2} \lesssim |A|L^{\frac{N+b}{2}+1}.$$

For the term involving $q_2(y,L)$, by the Hölder inequality and Calderón-Zygmund theorem, we get

$$\begin{aligned}\int_{|y|\leq L}|q_2(y,L)||u(y)|\,\mathrm{d}y &\leq \Big(\int_{|y|\leq L}|q_2(y,L)|^{\frac{3}{2}}\,\mathrm{d}y\Big)^{\frac{2}{3}}\Big(\int_{|y|\leq L}|u(y)|^3\,\mathrm{d}y\Big)^{\frac{1}{3}} \\ &\lesssim \int_{|y|\leq 2L}|u(y)|^3\,\mathrm{d}y \lesssim L^{b+\delta}.\end{aligned}$$

For the term containing $q_3(y,L)$, using the support property and the dyadic decomposition again, we infer that if $\delta \in [0,1/2[$,

$$\begin{aligned}\int_{|y|\leq L}|q_3(y,L)||u(y)|\,\mathrm{d}y &\lesssim L^{N+\delta}\sup_{|y|\leq L}|q_3(y,L)| \\ &\lesssim L^{N+\delta}\sup_{|y|\leq L}\Big(\sum_{k=1}^{\infty}\int_{2^kL\leq|z|\leq 2^{k+1}L}\frac{|y|}{|z|^{N+1}}|u(z)|^2\,\mathrm{d}z\Big) \\ &\lesssim L^{N+\delta+1}\sum_{k=1}^{\infty}\frac{1}{(2^kL)^{N+1}}\int_{|z|\approx 2^kL}|u(z)|^2\,\mathrm{d}z \\ &\lesssim L^{N+\delta+1}\sum_{k=1}^{\infty}(2^kL)^{b-N-1} \lesssim L^{b+\delta},\end{aligned}$$

and if $\delta \in [1/2, 1[$,

$$\begin{aligned}\int_{|y|\leq L} |q_3(y,L)||u(y)|\,\mathrm{d}y &\lesssim L^{N+\delta} \sup_{|y|\leq L} |q_3(y,L)| \\ &\lesssim L^{N+\delta} \sup_{|y|\leq L} \Big(\sum_{k=1}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{|y|^2}{|z|^{N+2}} |u(z)|^2 \,\mathrm{d}z \Big) \\ &\lesssim L^{N+\delta+2} \sum_{k=1}^{\infty} \frac{1}{(2^k L)^{N+2}} \int_{|z|\approx 2^k L} |u(z)|^2 \,\mathrm{d}z \\ &\lesssim L^{N+\delta+2} \sum_{k=1}^{\infty} (2^k L)^{b-N-2} \lesssim L^{b+\delta}.\end{aligned}$$

For the last term, from Hölder's inequality and the dyadic decomposition we deduce that if $\delta \in [0, \frac{1}{2}[$,

$$\begin{aligned}\int_{|y|\leq L} |q_4(y,L)||u(y)|\,\mathrm{d}y &\lesssim L^{N/2} \Big(\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \Big)^{1/2} \Big(\sup_{|y|\leq L} |q_4(y,L)| \Big) \\ &\lesssim L^{\frac{N+b}{2}} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} \int_{\frac{L}{2^{k+1}} \leq |z| \leq \frac{L}{2^k}} \frac{1}{|z|^N} |u(z)|^2 \,\mathrm{d}z \\ &\lesssim L^{\frac{N+b}{2}} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} \Big(\frac{L}{2^k} \Big)^{-N+b} \lesssim \begin{cases} L^{\frac{3b-N}{2}}, & \text{if } b > N, \\ L^N [\log_2 L], & \text{if } b = N, \\ L^{\frac{N+b}{2}}, & \text{if } b < N, \end{cases} \\ &\lesssim \begin{cases} L^{b+\delta}, & \text{if } b \geq N, (b,\delta) \neq (N,0), \\ L^N [\log_2 L], & \text{if } (b,\delta) = (N,0), \\ L^{\frac{N+b}{2}}, & \text{if } b < N, \end{cases}\end{aligned}$$

and if $\delta \in [\frac{1}{2}, 1[$,

$$\begin{aligned}&\int_{|y|\leq L} |q_4(y,L)||u(y)|\,\mathrm{d}y \lesssim \\ &\lesssim L^{N/2} \Big(\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \Big)^{1/2} \Big(\sup_{|y|\leq L} |q_4(y,L)| \Big) \\ &\lesssim L^{\frac{N+b}{2}} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} \int_{\frac{L}{2^{k+1}} \leq |z| \leq \frac{L}{2^k}} \Big(\frac{1}{|z|^N} + \frac{L}{|z|^{N+1}} \Big) |u(z)|^2 \,\mathrm{d}z \\ &\lesssim L^{\frac{N+b}{2}+1} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} \Big(\frac{L}{2^k} \Big)^{-N-1+b} \lesssim \begin{cases} L^{\frac{3b-N}{2}}, & \text{if } b > N+1, \\ L^{N+\frac{3}{2}} [\log_2 L], & \text{if } b = N+1, \\ L^{\frac{N+b}{2}+1}, & \text{if } b < N+1, \end{cases} \\ &\lesssim \begin{cases} L^{b+\delta}, & \text{if } b \geq N+1,\ (b,\delta) \neq (N+1, \frac{1}{2}), \\ L^{N+\frac{3}{2}} [\log_2 L], & \text{if } (b,\delta) = (N+1, \frac{1}{2}), \\ L^{\frac{N+b}{2}+1}, & \text{if } b < N+1. \end{cases}\end{aligned}$$

Collecting the above estimates leads to (2.26). □

3. Proof of the main result

As mentioned in the introduction section, the starting point of the main proof is the following local energy inequality of the profiles (u,q):

$$\begin{aligned}&\left|\frac{1}{l_2^{N-2\alpha}}\int_{|y|\le l_2}|u(y)|^2\phi\Big(\frac{y}{l_2}\Big)\mathrm{d}y-\frac{1}{l_1^{N-2\alpha}}\int_{|y|\le l_1}|u(y)|^2\phi\Big(\frac{y}{l_1}\Big)\mathrm{d}y\right|\le\\ &\le C\int_{\frac12 l_1\le|y|\le l_2}\frac{|u(y)|^3+|q(y)||u(y)|}{|y|^{N-2\alpha+1}}\mathrm{d}y,\end{aligned}\tag{3.1}$$

where $0<l_1<l_2$, and $\phi\in C_c^\infty(\mathbb{R}^N)$ is a cutoff function such that $0\le\phi\le 1$, $\phi\equiv 1$ on $B_{1/2}(0)$ and $\phi\equiv 0$ on $B_1^c(0)$. This inequality (3.1) is derived from the energy equality of the original velocity,

$$\begin{aligned}&\int_{\mathbb{R}^N}|v(t_2,x)|^2\chi(x)\,\mathrm{d}x-\int_{\mathbb{R}^N}|v(t_1,x)|^2\chi(x)\,\mathrm{d}x=\\ &=\int_{t_1}^{t_2}\int_{\mathbb{R}^N}\big(|v|^2v+2(p-d(t))v\big)\cdot\nabla\chi(x)\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.2}$$

where $0<t_1<t_2<T$ and $\chi\in C_c^\infty(\mathbb{R}^N)$ is a test function. By considering (3.2) on the region of self-similarity, and inserting the self-similar scenario (1.3)-(1.4) into (3.2), and through changing of variables, we can show (3.1) (e.g. see [**5**]). Notice that $l_i=(T-t_i)^{-\frac{1}{1+\alpha}}$, $i=1,2$ in (3.1).

3.1. Proof of Theorem 1.1-(1). First we consider the case $-1<\alpha<-\delta$. Note that from (1.6), we have that in this case

$$\frac{1}{l_2^{N-2\alpha}}\int_{|y|\le l_2}|u(y)|^2\mathrm{d}y\lesssim l_2^{2\delta+2\alpha}\longrightarrow 0,\quad\text{as}\quad l_2\to\infty.$$

Thus by letting $l_1=2L\gg 1$ and $l_2\to\infty$ in (3.1), we get

$$\int_{|y|\le L}|u(y)|^2\mathrm{d}y\le CL^{N-2\alpha}\int_{|y|\ge L}\frac{|u(y)|^3+|q(y)||u(y)|}{|y|^{N-2\alpha+1}}\mathrm{d}y,\tag{3.3}$$

where q takes the formula as (2.3). By the dyadic decomposition, we infer that

$$\begin{aligned}&CL^{N-2\alpha}\sum_{k=0}^\infty\int_{2^kL\le|y|\le 2^{k+1}L}\frac{|u(y)|^3+|q(y)||u(y)|}{|y|^{N-2\alpha+1}}\,\mathrm{d}y\\ &\le\frac{C}{L}\sum_{k=0}^\infty 2^{-k(N-2\alpha+1)}\int_{2^kL\le|y|\le 2^{k+1}L}\big(|u(y)|^3+|q(y)||u(y)|\big)\,\mathrm{d}y\end{aligned}\tag{3.4}$$

By using the following estimate

$$\int_{|y|\le L}|u(y)|^2\,\mathrm{d}y\lesssim L^{N+2\delta},\quad\forall L\gg 1,\tag{3.5}$$

the estimate (2.26) in Lemma 2.2 ensures that

$$\begin{aligned}\int_{|y|\le 2^{k+1}L}|u(y)||q(y)|\,\mathrm{d}y&\lesssim\begin{cases}(2^kL)^{N+3\delta}+(2^kL)^{N+\delta+1}, & \text{if }\ \delta\neq\frac12,\\ (2^kL)^{N+\frac32}[\log_2(2^kL)], & \text{if }\ \delta=\frac12,\end{cases}\\ &\lesssim\begin{cases}(2^kL)^{N+3\delta}, & \text{if }\delta>\frac12,\\ (2^kL)^{N+\frac32+\epsilon}, & \text{if }\delta=\frac12,\\ (2^kL)^{N+\delta+1}, & \text{if }\delta<\frac12,\end{cases}\end{aligned}$$

with $0 < \epsilon \ll 1/2$ a small number. Thus for all $-1 < \alpha < -\delta$, we first obtain a rough bound

$$\begin{aligned}\int_{|y|\le L} |u|^2 \mathrm{d}y &\le \begin{cases} \frac{C}{L}\sum_{k=0}^{\infty} 2^{-k(N-2\alpha+1)}(2^k L)^{\max\{N+3\delta, N+1+\delta\}}, & \text{if } \delta \neq \frac{1}{2}, \\ \frac{C}{L}\sum_{k=0}^{\infty} 2^{-k(N-2\alpha+1)}(2^k L)^{N+\frac{3}{2}+\epsilon}, & \text{if } \delta = \frac{1}{2}, \end{cases} \\ &\le \begin{cases} CL^{N+3\delta-1}, & \text{if } \delta \in]\frac{1}{2}, 1[, \\ CL^{N+\delta+\epsilon}, & \text{if } \delta \in]\epsilon_0, \frac{1}{2}]. \end{cases}\end{aligned} \tag{3.6}$$

Next we will use (3.6) to show a more refined bound. By using Lemma 2.2 again, and noting that

$$\max\left\{b+\delta, \frac{N+b}{2}+1\right\} = \begin{cases} b+\delta, & \text{if } b \ge N+2(1-\delta), \\ \frac{N+b}{2}+1, & \text{if } b < N+2(1-\delta), \end{cases} \tag{3.7}$$

we get

$$\int_{|y|\le 2^{k+1}L} |u(y)||q(y)|\,\mathrm{d}y \lesssim \begin{cases} (2^k L)^{N+4\delta-1}, & \text{if } \delta \in [\frac{3}{5}, 1[, \\ (2^k L)^{N+\frac{3\delta+1}{2}}, & \text{if } \delta \in]\frac{1}{2}, \frac{3}{5}], \\ (2^k L)^{N+\frac{\delta+\epsilon}{2}+1}, & \text{if } \delta \in]\epsilon_0, \frac{1}{2}], \end{cases} \tag{3.8}$$

Plugging it into (3.4), we have

$$\begin{aligned}\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y &\le \begin{cases} \frac{C}{L}\sum_{k=0}^{\infty} 2^{-k(N-2\alpha+1)}(2^k L)^{N+4\delta-1}, & \text{if } \delta \in [\frac{3}{5}, 1[, \\ \frac{C}{L}\sum_{k=0}^{\infty} 2^{-k(N-2\alpha+1)}(2^k L)^{N+\frac{3\delta+1}{2}}, & \text{if } \delta \in]\frac{1}{2}, \frac{3}{5}], \\ \frac{C}{L}\sum_{k=0}^{\infty} 2^{-k(N-2\alpha+1)}(2^k L)^{N+\frac{\delta+\epsilon}{2}+1}, & \text{if } \delta \in]\epsilon_0, \frac{1}{2}], \end{cases} \\ &\le \begin{cases} CL^{N+4\delta-2}, & \text{if } \delta \in [\frac{3}{5}, 1[, \\ CL^{N+\frac{3\delta-1}{2}}, & \text{if } \delta \in]\frac{1}{2}, \frac{3}{5}], \\ CL^{N+\frac{\delta+\epsilon}{2}}, & \text{if } \delta \in]\epsilon_0, \frac{1}{2}]. \end{cases}\end{aligned} \tag{3.9}$$

We can repeat the above process for $n+1$ times to show that

$$\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y \le \begin{cases} CL^{N+2\delta+(n+1)(\delta-1)}, & \text{if } \delta \in [\frac{n+2}{n+4}, 1[, \\ CL^{N+\frac{2\delta+n(\delta-1)}{2}}, & \text{if } \delta \in [\frac{n+1}{n+3}, \frac{n+2}{n+4}], \\ CL^{N+\frac{2\delta+(n-1)(\delta-1)}{2^2}}, & \text{if } \delta \in [\frac{n}{n+2}, \frac{n+1}{n+3}], \\ \cdots \quad \cdots & \\ CL^{N+\frac{2\delta+(\delta-1)}{2^n}}, & \text{if } \delta \in]\frac{1}{2}, \frac{3}{5}], \\ CL^{N+\frac{\delta+\epsilon}{2^n}}, & \text{if } \delta \in]\epsilon_0, \frac{1}{2}]. \end{cases} \tag{3.10}$$

For each $\delta \in]\epsilon_0, \frac{1}{2}]$, and for n sufficiently large, we get that

$$\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y \le L^{N+\epsilon_0}, \tag{3.11}$$

where ϵ_0 is just the number appearing in (1.11); while for each $\delta \in]\frac{1}{2}, 1[$, there is some $m \in \mathbb{N}^+$ so that $\delta \in]\frac{m+1}{m+3}, \frac{m+2}{m+4}]$, thus after repeating the above process for $m+n+1$ times, we get

$$\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y \le CL^{N+\frac{2\delta+m(\delta-1)}{2^{n+1}}}, \quad \text{for } \delta \in \left]\frac{m+1}{m+3}, \frac{m+2}{m+4}\right],$$

and for n large enough, we infer that (3.11) also holds true. But this obviously contradicts with the following estimate from the condition (1.11)

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \gtrsim \int_{M\leq|y|\leq L} |y|^{2\epsilon_0} \,\mathrm{d}y \gtrsim L^{N+2\epsilon_0}, \tag{3.12}$$

which means that there is no possibility to admit nontrivial velocity profiles in the case $-1<\alpha<-\delta$.

Next we consider the case $\alpha \geq -\delta$, and for all $\alpha\geq -\delta$ and $\delta \in]\epsilon_0, 1[$, we shall prove

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq C L^{N-2\alpha}, \quad \forall L \gg 1, \tag{3.13}$$

which, combined with (3.12), ensures that the range of α admitting possible nontrivial profiles belongs to $\{\alpha : -\delta \leq \alpha \leq -\epsilon_0\}$. By letting $l_1 = 2M$ and $l_2 = 2L \gg 1$, we begin with (3.1) to get

$$\int_{|y|\leq L} |u(y)|^2 \mathrm{d}y \leq C L^{N-2\alpha} + C L^{N-2\alpha} \int_{M\leq |y|\leq 2L} \frac{|u(y)|^3 + |q(y)||u(y)|}{|y|^{N-2\alpha+1}} \mathrm{d}y. \tag{3.14}$$

In view of the dyadic decomposition, we infer that

$$\begin{aligned} & C L^{N-2\alpha} \int_{M\leq |y|\leq 2L} \frac{|u(y)|^3 + |q(y)||u(y)|}{|y|^{N-2\alpha+1}} \mathrm{d}y \\ \leq\ & C L^{N-2\alpha} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} \int_{\frac{L}{2^{k+1}}\leq |y|\leq \frac{L}{2^k}} \frac{|u(y)|^3 + |q(y)||u(y)|}{|y|^{N-2\alpha+1}} \mathrm{d}y \\ \leq\ & \frac{C}{L} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)} \int_{\frac{L}{2^{k+1}}\leq |y|\leq \frac{L}{2^k}} \big(|u(y)|^3 + |q(y)||u(y)|\big) \,\mathrm{d}y. \end{aligned} \tag{3.15}$$

By using (1.10), (3.5) and (2.26) in Lemma 2.2, we have a rough bound:

$$\begin{aligned} & \int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq \\ & \leq \begin{cases} C L^{N-2\alpha} + \frac{C}{L} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)} \Big(\big(\frac{L}{2^k}\big)^{N+3\delta} + \big(\frac{L}{2^k}\big)^{N+\delta+1} \Big), & \text{if } \delta \neq \frac{1}{2}, \\ C L^{N-2\alpha} + \frac{C}{L} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)} \big(\frac{L}{2^k}\big)^{N+3/2} [\log \frac{L}{2^k}], & \text{if } \delta = \frac{1}{2}, \end{cases} \\ & \leq \begin{cases} C L^{N-2\alpha} [\log_2 L], & \text{if } \alpha \leq -\frac{3\delta-1}{2},\ \delta \in]\frac{1}{2}, 1[, \\ C L^{N+3\delta-1}, & \text{if } \alpha > -\frac{3\delta-1}{2},\ \delta \in]\frac{1}{2}, 1[, \\ C L^{N-2\alpha} [\log_2 L]^2, & \text{if } \alpha \leq -\frac{\delta}{2},\ \delta \in]\epsilon_0, \frac{1}{2}], \\ C L^{N+\delta+\epsilon}, & \text{if } \alpha > -\frac{\delta}{2},\ \delta \in]\epsilon_0, \frac{1}{2}], \end{cases} \end{aligned} \tag{3.16}$$

with $0<\epsilon \ll \delta$ a small number. If $\{\alpha \leq -\frac{3\delta-1}{2}, \delta \in]\frac{1}{2}, 1[\}$ or $\{\alpha \leq -\frac{\delta}{2}, \delta \in]\epsilon_0, \frac{1}{2}]\}$, we can improve the bound to drop the additional logarithmic term: indeed, let $0<\epsilon<1$ be a small number chosen later (cf. (3.18)), then we get

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq C_\epsilon L^{N-2\alpha+\epsilon}, \quad \text{if } \begin{cases} \alpha \leq -\frac{3\delta-1}{2}, \delta \in]\frac{1}{2}, 1[, & \text{or} \\ \alpha \leq -\frac{\delta}{2}, \delta \in]\epsilon_0, \frac{1}{2}], \end{cases}$$

and inserting this estimate into (3.15) yields that for all such (α,δ),

$$
\begin{aligned}
&\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y \lesssim \\
&\lesssim L^{N-2\alpha} + \frac{1}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\left(\left(\frac{L}{2^k}\right)^{N-2\alpha+\epsilon+\delta} + \left(\frac{L}{2^k}\right)^{N-\alpha+\frac{\epsilon}{2}+1}\right)\\
&\lesssim L^{N-2\alpha} + L^{N-2\alpha+\epsilon+\delta-1}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(1-\epsilon-\delta)} + L^{N-\alpha+\frac{\epsilon}{2}}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(-\alpha-\frac{\epsilon}{2})}\\
&\lesssim L^{N-2\alpha} + L^{N-2\alpha+\epsilon+\delta-1}\left(\frac{L}{M}\right)^{1-\epsilon-\delta} + L^{N-2\alpha+\frac{\epsilon}{2}}\left(\frac{L}{M}\right)^{-\alpha-\frac{\epsilon}{2}}\\
&\lesssim L^{N-2\alpha},
\end{aligned}
\tag{3.17}
$$

as long as $0<\epsilon<\min\{1-\delta,-2\alpha\}$ (the scope of ϵ guarantees the third line of (3.17)), which can be satisfied by choosing

$$
\epsilon = \begin{cases} \frac{1-\delta}{2}, & \text{for } \delta \in]\frac{1}{2},1[,\\ \frac{\delta}{2}, & \text{for } \delta \in]\epsilon_0,\frac{1}{2}]. \end{cases}
\tag{3.18}
$$

If $\alpha > -\frac{3\delta-1}{2}$ for $\delta \in]\frac{1}{2},1[$ or $\alpha > -\frac{\delta}{2}$ for $\delta \in]\epsilon_0,\frac{1}{2}]$, we shall use the iterative method to reduce the power of L in the right-hand-side of (3.16). Thanks to (2.26) in Lemma 2.2, we get an estimate similar to (3.8) only with $\frac{L}{2^k}$ in place of $2^{k+1}L$ and 2^kL, and by inserting it into (3.15), we find

$$
\begin{aligned}
&\int_{|y|\le L} |u(y)|^2\,\mathrm{d}y \lesssim \\
&\lesssim \begin{cases}
L^{N-2\alpha} + \frac{1}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\left(\frac{L}{2^k}\right)^{N+4\delta-1}, & \text{if } \alpha > -\frac{3\delta-1}{2}, \delta\in[\frac{3}{5},1[,\\
L^{N-2\alpha} + \frac{1}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\left(\frac{L}{2^k}\right)^{N+\frac{3\delta-1}{2}+1}, & \text{if } \alpha > -\frac{3\delta-1}{2}, \delta\in]\frac{1}{2},\frac{3}{5}],\\
L^{N-2\alpha} + \frac{1}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\left(\frac{L}{2^k}\right)^{N+\frac{\delta+\epsilon}{2}+1}, & \text{if } \alpha > -\frac{\delta}{2}, \delta\in]\epsilon_0,\frac{1}{2}],
\end{cases}\\
&\lesssim \begin{cases}
L^{N-2\alpha}[\log_2 L], & \text{if } \alpha \in]-\frac{3\delta-1}{2}, -(2\delta-1)], \delta\in[\frac{3}{5},1[,\\
L^{N+4\delta-2}, & \text{if } \alpha > -(2\delta-1), \delta\in[\frac{3}{5},1[,\\
L^{N-2\alpha}[\log_2 L], & \text{if } \alpha \in]-\frac{3\delta-1}{2}, -\frac{3\delta-1}{2^2}], \delta\in]\frac{1}{2},\frac{3}{5}],\\
L^{N+\frac{3\delta-1}{2}}, & \text{if } \alpha > -\frac{3\delta-1}{2^2}, \delta\in]\frac{1}{2},\frac{3}{5}],\\
L^{N-2\alpha}[\log_2 L], & \text{if } \alpha \in]-\frac{\delta}{2}, -\frac{\delta+\epsilon}{2^2}], \delta\in]0,\frac{1}{2}],\\
L^{N+\frac{\delta+\epsilon}{2}}, & \text{if } \alpha > -\frac{\delta+\epsilon}{2^2}, \delta\in]\epsilon_0,\frac{1}{2}].
\end{cases}
\end{aligned}
\tag{3.19}
$$

If $\{\alpha \in]-\frac{3\delta-1}{2}, -(2\delta-1)], \delta\in[\frac{3}{5},1[\}$, or $\{\alpha \in]-\frac{3\delta-1}{2}, -\frac{3\delta-1}{2^2}], \delta\in]\frac{1}{2},\frac{3}{5}],\}$ or $\{\alpha \in]-\frac{\delta}{2}, -\frac{\delta+\epsilon}{2^2}], \delta\in]\epsilon_0,\frac{1}{2}]\}$, we can also improve the above bound by removing the logarithmic term: indeed, this is as the treating in (3.17), and we only need to choose $0<\epsilon<1$ so that $\epsilon<\min\{1-\delta,-2\alpha\}$, e.g., setting

$$
\epsilon = \begin{cases} \frac{1-\delta}{2}, & \text{for } \delta \in [\frac{3}{5},1[,\\ \frac{\delta}{2^2}, & \text{for } \delta \in]\epsilon_0,\frac{3}{5}[, \end{cases}
$$

then the bound of $\int_{|y|\leq L}|u(y)|^2\,\mathrm{d}y$ can be likewise improved from $C_\epsilon L^{N-2\alpha+\epsilon}$ to the expected $C_\epsilon L^{N-2\alpha}$. If $\{\alpha>-(2\delta-1),\delta\in[\frac{3}{5},1[\}$, or $\{\alpha>-\frac{3\delta-1}{2^2},\delta\in]\frac12,\frac35]\}$, or $\{\alpha>-\frac{\delta+\epsilon}{2^2},\delta\in]\epsilon_0,\frac12]\}$, we can further improve the bound in an identical fashion as obtaining (3.19) from (3.16). In conclusion, after repeating the above process for $n+1$ times, we infer

(3.20)
$$\int_{|y|\leq L}|u|^2\,\mathrm{d}y\leq\begin{cases}CL^{N+2\delta+(n+1)(\delta-1)}, & \text{if } \alpha>-\frac{(n+3)\delta-(n+1)}{2},\delta\in[\frac{n+2}{n+4},1[,\\ CL^{N+\frac{2\delta+n(\delta-1)}{2}}, & \text{if } \alpha>-\frac{(n+2)\delta-n}{2^2},\delta\in[\frac{n+1}{n+3},\frac{n+2}{n+4}],\\ CL^{N+\frac{2\delta+(n-1)(\delta-1)}{2^2}}, & \text{if } \alpha>-\frac{(n+1)\delta-(n-1)}{2^3},\delta\in[\frac{n}{n+2},\frac{n+1}{n+3}],\\ \cdots\quad\cdots & \\ CL^{N+\frac{2\delta+(\delta-1)}{2^n}}, & \text{if } \alpha>-\frac{3\delta-1}{2^{n+1}},\delta\in]\frac12,\frac35],\\ CL^{N+\frac{\delta+\epsilon}{2^n}}, & \text{if } \alpha>-\frac{\delta+\epsilon}{2^{n+1}},\delta\in]\epsilon_0,\frac12],\\ CL^{N-2\alpha}, & \text{if for other scopes of } (\alpha,\delta).\end{cases}$$

From (3.20), by arguing as the deduction after (3.10), we deduce that if (α,δ) is not in the scope so that $\int_{|y|\leq L}|u(y)|^2\,\mathrm{d}y$ is bounded by $CL^{N-2\alpha}$, then the scope of such α will eventually exceed the range $\{-\delta\leq\alpha\leq-\epsilon_0\}$ and the quantity $\int_{|y|\leq L}|u(y)|^2\mathrm{d}y$ will be bounded by $CL^{N+\epsilon_0}$, which thus is not compatible. Hence for all $-\delta\leq\alpha\leq-\epsilon_0$ and $\delta\in]0,1[$, we obtain (3.13).

At last we prove (1.12), and in order to do that, it suffices to prove the following inequality for all $-\delta\leq\alpha\leq-\epsilon_0$,

$$\int_{|y|\leq L}|u(y)|^2\mathrm{d}y\gtrsim L^{N-2\alpha},\quad\forall L\gg1.\tag{3.21}$$

Suppose it is not true, then there exists a sequence of numbers $L_k\gg1$ such that

$$\frac{1}{L_k^{N-2\alpha}}\int_{|y|\leq L_k}|u(y)|^2\,\mathrm{d}y\to0,\quad\text{as }\ L_k\to\infty.$$

Thus by setting $l_2=L_k\to\infty$ and $l_1=2L>0$ in (3.1), we get

$$\int_{|y|\leq L}|u(y)|^2\mathrm{d}y\leq CL^{N-2\alpha}\int_{|y|\geq L}\frac{|u(y)|^3+|q(y)||u(y)|}{|y|^{N-2\alpha+1}}\mathrm{d}y.\tag{3.22}$$

From (3.13), and by using the decomposition (3.4) and (2.26) in Lemma 2.2 with $b=N-2\alpha$, we have

$$\begin{aligned}&\int_{|y|\leq L}|u(y)|^2\,\mathrm{d}y\leq\\ &\leq\begin{cases}\frac{C}{L}\sum_{k=0}^\infty\frac{1}{2^{k(N-2\alpha+1)}}(2^kL)^{\max\{N-2\alpha+\delta,N-\alpha+1\}}, & \text{if }\ \alpha\neq-\frac12,\delta\neq\frac12,\\ \frac{C}{L}\sum_{k=0}^\infty\frac{1}{2^{k(N-2\alpha+1)}}(2^kL)^{\frac32}[\log_2(2^kL)], & \text{if }\ \alpha=-\frac12,\delta=\frac12,\end{cases}\\ &\leq\begin{cases}CL^{N-2\alpha+\delta-1}, & \text{if } \alpha\in[-\delta,\delta-1],\delta\in]\frac12,1[,\\ CL^{N+\frac12+\epsilon}, & \text{if } \alpha=-\frac12,\delta=\frac12,\\ CL^{N-\alpha}, & \text{if } \alpha\in[\delta-1,-\epsilon_0],\delta\in]\epsilon_0,1-\epsilon_0],(\alpha,\delta)\neq(-\frac12,\frac12)\end{cases}\end{aligned}$$

with $0 < \epsilon \ll 1/2$ a small number. Using this improved estimate and Lemma 2.2 again, similarly as above we find

$$\int_{|y|\le L} |u(y)|^2 \, \mathrm{d}y \le$$
$$\le \begin{cases} \frac{C}{L}\sum_{k=0}^{\infty} \frac{1}{2^{k(N-2\alpha+1)}}(2^k L)^{N-2\alpha+2\delta-1}, & \text{if } \alpha \in [-\delta, \frac{3}{2}(\delta-1)], \delta \in [\frac{3}{5}, 1[, \\ \frac{C}{L}\sum_{k=0}^{\infty} \frac{1}{2^{k(N-2\alpha+1)}}(2^k L)^{N-\alpha+\frac{\delta-1}{2}+1}, & \text{if } \alpha \in [\frac{3}{2}(\delta-1), \delta-1], \delta \in]\frac{1}{2}, 1[, \\ \frac{C}{L}\sum_{k=0}^{\infty} \frac{1}{2^{k(N-2\alpha+1)}}(2^k L)^{N+\frac{-\alpha+\epsilon}{2}+1}, & \text{if } \alpha \in [\delta-1, -\epsilon_0], \delta \in]\epsilon_0, 1-\epsilon_0], \end{cases}$$
$$\le \begin{cases} L^{N-2\alpha+2\delta-2}, & \text{if } \alpha \in [-\delta, \frac{3}{2}(\delta-1)], \delta \in [\frac{3}{5}, 1[, \\ L^{N-\alpha+\frac{\delta-1}{2}}, & \text{if } \alpha \in [\frac{3}{2}(\delta-1), \delta-1], \delta \in]\frac{1}{2}, 1[, \\ L^{N+\frac{-\alpha+\epsilon}{2}}, & \text{if } \alpha \in [\delta-1, -\epsilon_0], \delta \in]\epsilon_0, 1-\epsilon_0], \end{cases}$$

By repeating the above process for $n+1$ times leads to

$$\int_{|y|\le L} |u(y)|^2 \, \mathrm{d}y \le$$
$$\le \begin{cases} L^{N-2\alpha+(n+1)(\delta-1)}, & \text{if } \alpha \in [-\delta, \frac{n+2}{2}(\delta-1)], \delta \in [\frac{n+2}{n+4}, 1[, \\ L^{N-\alpha+\frac{n}{2}(\delta-1)}, & \text{if } \alpha \in [\frac{n+2}{2}(\delta-1), \frac{n+1}{2}(\delta-1)], \delta \in [\frac{n+1}{n+3}, 1[, \\ \cdots \quad \cdots \\ L^{N-\frac{\alpha}{2^{n-1}}+\frac{1}{2^n}(\delta-1)}, & \text{if } \alpha \in [\frac{3}{2}(\delta-1), (\delta-1)], \delta \in]\frac{1}{2}, 1[, \\ L^{N+\frac{-\alpha+\epsilon}{2^n}}, & \text{if } \alpha \in [\delta-1, -\epsilon_0], \delta \in]\epsilon_0, 1-\epsilon_0]. \end{cases} \tag{3.23}$$

From (3.23), we claim that for all $-\delta \le \alpha \le -\epsilon_0$ and $\epsilon_0 < \delta < 1$,

$$\int_{|y|\le L} |u(y)|^2 \, \mathrm{d}y \lesssim L^{N+\epsilon_0}, \quad \forall L \gg 1, \tag{3.24}$$

Indeed, we divide into three cases: if $\delta \in]\epsilon_0, \frac{1}{2}[$, then the scope $[-\delta, -\epsilon_0] \subset [\delta - 1, -\epsilon_0]$, and thus for n large enough, we get (3.24) for all $-\delta \le \alpha \le -\epsilon_0$; if $\delta \in [\frac{n+1}{n+3}, \frac{n+2}{n+4}[$ for some $n \in \mathbb{N}^+$ and $\delta \le 1-\epsilon_0$, then $-\delta > \frac{n+2}{2}(\delta-1)$, and $\alpha \in [-\delta, -\epsilon_0] \subset [\frac{n+2}{2}(\delta-1), \frac{n+1}{2}(\delta-1)] \cup \cdots \cup [\frac{3}{2}(\delta-1), \delta-1] \cup [\delta-1, \epsilon_0]$, thus after repeating the above process for $m+n+1$ times, we get for all $-\delta \le \alpha \le -\epsilon_0$,

$$\int_{|y|\le L} |u(y)|^2 \, \mathrm{d}y \lesssim \begin{cases} L^{N+\frac{-\alpha}{2^m}+\frac{n(\delta-1)}{2^{m+1}}}, & \text{if } \alpha \in [\frac{n+2}{2}(\delta-1), \frac{n+1}{2}(\delta-1)], \\ \cdots \quad \cdots \\ L^{N+\frac{-\alpha}{2^{m+n-1}}+\frac{(\delta-1)}{2^{m+n}}}, & \text{if } \alpha \in [\frac{3}{2}(\delta-1), (\delta-1)], \\ L^{N+\frac{-\alpha+\epsilon}{2^{m+n}}}, & \text{if } \alpha \in [\delta-1, -\epsilon_0], \end{cases}$$
$$\lesssim L^{N+\epsilon_0}, \qquad \forall L \gg 1, \tag{3.25}$$

where in the second line we have chosen m large enough; finally, if $\delta \in [\frac{n+1}{n+3}, \frac{n+2}{n+4}[$ for some $n \in \mathbb{N}^+$ and $\delta > 1-\epsilon_0$, then $-\delta > \frac{n+2}{2}(\delta-1)$, $\delta - 1 > -\epsilon_0$, and $\alpha \in [-\delta, -\epsilon_0] \subset [\frac{n+2}{2}(\delta-1), \frac{n+1}{2}(\delta-1)] \cup \cdots \cup [\frac{3}{2}(\delta-1), \delta-1]$, we can obtain (3.24) similarly as deriving (3.25) for all $-\delta \le \alpha \le -\epsilon_0$. However, the estimate (3.24) clearly contradicts with (3.12) obtained from the assumption (1.11), and thus (3.21) is not compatible and the desired estimate (1.12) is followed.

3.2. Proof of Theorem 1.1-(2). Since $\delta < \frac{1}{2}$ and $\alpha > -\frac{1}{2}$, we have $A = 0$ in the formula of q (2.3), and we can use the better estimate (2.27) instead of (2.26) in the above proof. We first consider the case $-1 < \alpha < -\delta$. Similarly as above, we also begin with (3.3), and by virtue of (3.5) and (2.27), we get

$$\int_{|y|\leq 2^{k+1}L} |u(y)||q(y)| \,\mathrm{d}y \lesssim (2^k L)^{N+3\delta},$$

and

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq \frac{C}{L}\sum_{k=0}^{\infty} \frac{1}{2^{k(N-2\alpha+1)}}(2^k L)^{N+3\delta} \leq CL^{N+3\delta-1}.$$

We can repeatedly use this process to show that

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq CL^{N+2\delta-(n+1)(1-\delta)},$$

as long as $N + 2\delta - n(1-\delta) \geq N - 2\delta$, that is, $n \leq \frac{4\delta}{1-\delta}$. Set $n_0 = [\frac{4\delta}{1-\delta}]$, then we obtain

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \leq CL^{N+2\delta-(n_0+1)(1-\delta)} \leq CL^{N-2\delta},$$

which clearly contradicts with the estimation from the condition (1.13)

$$\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y \gtrsim \int_{M\leq|y|\leq L} 1 \,\mathrm{d}y \gtrsim L^N, \tag{3.26}$$

and guarantees that the case $-1 < \alpha < -\delta$ is not compatible.

Next we consider the case $\alpha \geq -\delta$ to prove (3.13). We similarly begin with (3.14), and from (3.15) and (2.27), we see that

$$\int_{|y|\leq \frac{L}{2^k}} |u(y)||q(y)| \,\mathrm{d}y \lesssim \Big(\frac{L}{2^k}\Big)^{N+3\delta},$$

and

$$\begin{aligned}\int_{|y|\leq L} |u(y)|^2 \,\mathrm{d}y &\leq CL^{N-2\alpha} + \frac{C}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\Big(\frac{L}{2^k}\Big)^{N+3\delta} \\ &\leq \begin{cases} CL^{N+3\delta-1}, & \text{if } \alpha > -\frac{3\delta-1}{2}, \\ CL^{N-2\alpha}[\log_2 L], & \text{if } \alpha \leq -\frac{3\delta-1}{2}. \end{cases}\end{aligned} \tag{3.27}$$

For $\alpha \leq -\frac{3\delta-1}{2}$, we can replace the bound by $C_\epsilon L^{N-2\alpha+\epsilon}$ with $0 < \epsilon < 1-\delta$, then by repeating this process once more, we obtain

$$\int_{|y|\leq L} |u|^2 \,\mathrm{d}y \lesssim L^{N-2\alpha} + \frac{1}{L}\sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)}\Big(\frac{L}{2^k}\Big)^{N-2\alpha+\delta+\epsilon} \lesssim L^{N-2\alpha}. \tag{3.28}$$

For $\alpha > -\frac{3\delta-1}{2}$, and if $\delta < \frac{1}{3}$, we see that (3.27) contradicts with (3.26), and such a case is incompatible; otherwise, if $\delta \geq \frac{1}{3}$, we can further improve the bound by iteration:

$$\int_{|y|\leq L} |u(y)|^2 \mathrm{d}y \lesssim L^{N-2\alpha} + \frac{1}{L} \sum_{k=-1}^{[\log_2 \frac{L}{M}]} 2^{k(N-2\alpha+1)} \Big(\frac{L}{2^k}\Big)^{N+4\delta-1}$$
$$\lesssim \begin{cases} L^{N+4\delta-2}, & \text{if } \alpha > -(2\delta-1), \\ L^{N-2\alpha}[\log_2 L], & \text{if } \alpha \in]-\frac{3\delta-1}{2}, -(2\delta-1)]. \end{cases}$$

For $\alpha \in]-\frac{3\delta-1}{2}, -(2\delta-1)]$, we can similarly obtain (3.28) in this case; while for $\alpha > -(2\delta-1)$, the bound $CL^{N+4\delta-2}$ contradicts with (3.26) due to $\delta < \frac{1}{2}$, which means such a case is incompatible. Therefore, for all $\alpha \geq -\delta$ and $\delta < \frac{1}{2}$, we have

$$\int_{|y|\leq L} |u(y)|^2 \, \mathrm{d}y \leq CL^{N-2\alpha}, \quad \forall L \gg 1. \tag{3.29}$$

Combined with (3.26), we furthermore infer that the scope of α admitting possible nontrivial velocity profiles is $\{\alpha : -\delta \leq \alpha \leq 0\}$.

In the end we prove (1.12) for all $-\delta \leq \alpha \leq 0$, and it suffices to prove (3.21) for α in this range. We prove by contradiction, and similarly as above, we begin with (3.22) to get

$$\int_{|y|\leq L} |u(y)|^2 \, \mathrm{d}y \leq \frac{C}{L} \sum_{k=0}^{\infty} \frac{1}{2^{k(N-2\alpha+1)}} (2^k L)^{N-2\alpha+\delta} \leq CL^{N-2\alpha+\delta-1}.$$

By iteration, we can show that, as long as $N + 2\alpha - n(1-\delta) \geq N - 2\delta$, we have

$$\int_{|y|\leq L} |u(y)|^2 \, \mathrm{d}y \leq CL^{N-2\alpha-(n+1)(1-\delta)}.$$

Set $n_0' = [\frac{2\alpha+2\delta}{1-\delta}]$, thus we find

$$\int_{|y|\leq L} |u(y)|^2 \, \mathrm{d}y \leq CL^{N-2\alpha-(n_0'+1)(1-\delta)} \leq CL^{N-2\delta},$$

which contradicts with the estimate (3.26), and thus concludes (3.21) and (1.12) for $-\delta \leq \alpha \leq 0$.

Acknowledgments. Part of the work was done when the author was a post-doctor of Université Paris-Est from 2012-09 to 2013-08, and he would like to express his gratitude for the guidance of Prof. Régis Monneau and Prof. Marco Cannone. He also thanked the anonymous referees for the helpful suggestion to improve the expression of this paper. The author is partially supported by NSFC grant 11401027 and the Youth Scholars Program of Beijing Normal University.

References

[1] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.* **94** (1984), 61-66.

[2] O. Boratav, R. Pelz, Direct numerical simulation of transition to turbulence from a high-symmentry initial condition. *Physics of Fluids* **6, no. 8** (1994), 2757-2784.

[3] A. Bronzi, R. Shvydkoy, On the energy behavior of locally self-similar blow-up for the Euler equation. *Indiana Univ. Math. J.* **64, no. 5** (2015), 1291-1302.

[4] D. Chae, Nonexistence of self-similar singularities for the 3D incompressible Euler equations. *Comm. Math. Phys.* **273, no. 1** (2007), 203-215.

[5] D. Chae, R. Shvydkoy, On formation of a locally self-similar collapse in the incompressible Euler equations. *Arch. Rational Mech. Anal.* **209, no. 3** (2013), 999-1017.

[6] K. Choi, A. Kiselev, Y. Yao, Finite time blow up for a 1D model of 2D Boussinesq system. accepted by *Commun. Math. Phys.* **334, no. 3** (2015), 1667-1679.

[7] A. Córdoba, D. Córdoba, M. A. Fontelos, Formation of singularities for a transport equation with nonlocal velocity. *Ann. Math.* **162** (2005), 1375-1387.

[8] F. de la Hoz, M. Fontelos, The structure of singularities in nonlocal transport equations. *J. Phys. A: Math. Theor.* **41, no. 18** (2008), 185204.

[9] J. Eggers, M. A. Fontelos, The role of self-similary in singularities of PDE's. *Nonlinearity* **22, no. 1** (2009), 1-44.

[10] X. He, Self-similar singularities of the 3D Euler equations. *Appl. Math. Lett.* **13, no. 5** (2000), 41-46.

[11] X. He, An example of finite-time singularities in the 3d Euler equations. *J. Math. Fluid Mech.* **9, no. 3** (2007), 398-410.

[12] T. Y. Hou, P. Liu, Self-similar singularity of a 1D model for the 3D axisymmetric Euler equations. *Res. Math. Sci.* **2, no. 5** (2015), 1-26.

[13] T. Y. Hou, G. Luo, Potentially Singular Solutions of the 3D Axisymmetric Euler Equations, *PNAS* **111, no. 36** (2014), 12968-12973.

[14] T. Y. Hou, G. Luo, Toward the finite-time blowup of the 3D incompressible Euler equations: a numerical investigation, *SIAM Multiscale Model. Simul.* **12, no. 4** (2014), 1722-176.

[15] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Functional Analysis* **9** (1972), 296-305.

[16] R. Kerr, Evidence for a singularity of the three-dimensional, incompressible Euler equations. *Physics of Fluids: Fluid Dynamics* **7, no. 5** (1993), 1725-1746.

[17] Y. Kimura, Self-similar collapse of 2d and 3d vortex filament models. *Theo. Comp. Fluid Dyna.* **24** (2010), 389-394.

[18] P. G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC, 2002.

[19] R. B. Pelz, Locally self-similar, finite-time collapse in a high-symmetry vortex filament model. *Phys. Rev. E* **55** (1997), 1617-1626.

[20] M. Schonbek, Nonexistence of pseudo-similar solutions to incompressible Euler equations. *Acta Math. Sci.* **31B, no. 6** (2011), 1-8.

[21] R. Shvydkoy, A study of energy concentration and drain in incompressible fluids. *Nonlinearity* **26, no. 2** (2013), 425-436.

[22] R. Takada, Nonexistence of backward self-similar weak solutions to the Euler equations. *Kyoto University Research Information Repository* **1690** (2010), 147-155.

School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China

Université Paris-Est, CERMICS, Ecole des Ponts ParisTech, Cité Descartes, Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France

E-mail address: xuelt@bnu.edu.cn