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On a non-homogeneous and non-linear heat equation

Luca Bisconti and Matteo Franca

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ABSTRACT. We consider the Cauchy-problem for a parabolic equation of the following type:

$$\frac{\partial u}{\partial t} = \Delta u + f(u, |x|),$$

where $x \in \mathbb{R}^n$, $n > 2$, $f = f(u, |x|)$ is supercritical. We supplement this equation by the initial condition $u(x, 0) = \phi$, and we allow ϕ to be either bounded or unbounded in the origin but smaller than stationary singular solutions. We discuss local existence and long time behaviour for the solutions $u(t, x; \phi)$ for a wide class of non-homogeneous non-linearities f. We show that in the supercritical case, ground states with slow decay lie on the threshold between initial data corresponding to blow-up solutions, and the basin of attraction of the null solution. Our results extend previous ones in that we allow f to be a Matukuma-type potential and in that we allow it to depend on u in a more general way.

We explore such a threshold in the subcritical case too, and we obtain a result which is new even for the model case $f(u) = u|u|^{q-2}$. We find a family of initial data $\psi(x)$ which have fast decay (i.e. $\sim |x|^{2-n}$), are arbitrarily small in L^∞- norm, but which correspond to blow-up solutions.

CONTENTS

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1. Introduction

The purpose of this paper is to study the asymptotic behavior of positive solutions of the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u + f(u,|x|), \tag{1.1}$$

$$u(x,0) = \phi(x), \tag{1.2}$$

where $x \in \mathbb{R}^n$, $n > 2$, and $f = f(u,|x|)$ is a function which is null for $u = 0$.

In the last 20 years this problem has been of great interest, starting from the models $f(u,|x|) = u^{q-1}$ and $f(u,|x|) = |x|^\delta u^{q-1}$, $\delta > -2$. Due to symmetry considerations, we use from now on a notation which is standard for the stationary problem, so we refer to $f(u,|x|) = u^{q-1}$ as the model case. Thereby we clarify the relationship between the critical values for (1.7) appearing below, and their meaning in other contexts.

We assume that f is *supercritical* with respect to the Serrin exponent, $2_* := \frac{2(n-1)}{n-2}$, and for some specific results we require f to be *supercritical* also to the Sobolev critical exponent, i.e. $2^* := \frac{2n}{n-2}$. The exponents 2_* and 2^* are related respectively to the continuity of the trace operator in L^q and to the possibility of embedding H^1 in L^q.

Our attention is on the structure of the border of the basin of attraction of the null solution, and the set of initial data ϕ of solutions of (1.1)–(1.2) which blow up in finite time. Our main aim is to extend the discussion to a wide class of functions f. For the remainder of the paper we will always assume the following:

- ***F*0**: The function $f(u,r)$ is locally Lipschitz in u and r, for any $u \geq 0$, and $r > 0$. Moreover $f(0,r) \equiv 0$, $f(u,r) > 0$ and $f(u,r)$ is increasing in u, for any $u > 0$ and any $r > 0$, and there is a constant $C(u) > 0$ such that $f(u,r)r^2 \leq C(u)$ for $0 < r \leq 1$.

Further hypotheses on f will be given in the sequel (see conditions ***G*0**, ***Gu***, and ***Gs*** in Section 2). We list possible examples:

$$f(u,|x|) = k_1(|x|)|u|^{q_1-1}, \tag{1.3}$$

$$f(u,|x|) = k_1(|x|)|u|^{q_1-1} + k_2(|x|)|u|^{q_2-1}, \tag{1.4}$$

$$f(u,|x|) = k_1(|x|)\min\{u^{q_1-1}, u^{q_2-1}\}, \tag{1.5}$$

where $q_1 < q_2$ and $k_i = k_i(|x|)$, $i = 1,2$, are supposed non-negative and Lipschitz continuous, and such that

$$k_i(r) \sim A_i r^{\delta_i} \quad \text{as } r \to 0, \qquad k_i(r) \sim B_i r^{\eta_i} \quad \text{as } r \to +\infty, \tag{1.6}$$

where $A_i, B_i \geq 0$, $\sum_i A_i > 0$, $\sum_i B_i > 0$, $q_i > 2$ and $\delta_i, \eta_i > (n-2)q - (n-1)$, for $i = 1,2$ (so for $\delta_i = \eta_i = 0$ we require $q_i > 2_*$).

Due to the nature of the considered nonlinearities, in general we cannot expect the solutions of (1.1)–(1.2) to be differentiable, or even continuous, everywhere. In fact, we deal also with solutions that may be not defined at $x = 0$ since they become unbounded.

In Section 3 we prove the existence of a suitable class of weak solutions to the considered problem (see Lemma 3.7 and Theorem 3.8) and we analyze their regularity properties. We consider the classes of C_B-mild and C_S-mild solutions to

(1.1)–(1.2) (see the definitions 3.2 and 3.3 below, see also [**27**]) proving local and global existence as well as uniqueness.

Let $u(x,t;\phi)$ be the solution of (1.1)–(1.2). The analysis of the long time behavior of $u(x,t;\phi)$ is strongly based on the separation properties of the stationary solutions of (1.1), i.e. functions $u(x)$ solving

$$\Delta u + f(u,|x|) = 0\,, \tag{1.7}$$

and in particular on the properties of radial solutions. If $u(x)$ is a radial solutions of (1.7), setting $U(r)=u(x)$, for $r=|x|$, then $U=U(r)$ solves

$$U'' + \frac{n-1}{r}U' + f(U,r) = 0\,, \tag{1.8}$$

where "$'$" denotes derivative with respect to r.

In the whole paper we use the following notation: $U(r)$ is *regular* if $U(0)=\alpha>0$, so we set $U(r)=U(r,\alpha)$, and we say that $U(r)$ has a *non-removable singularity* (or shortly that it is *singular*) if $\lim_{r\to 0}U(r)=+\infty$. Similarly, we affirm that a positive solution $V(r)$ of (1.8) has *fast decay* if $\lim_{r\to+\infty}V(r)r^{n-2}=\beta>0$ and in this case we set $V(r)=V(r,\beta)$, and that $V(r)$ has *slow decay* if $\lim_{r\to+\infty}V(r)r^{n-2}=+\infty$.

Further, $U(r)$ is a ground state (GS) if it is a regular solution of (1.8) which is positive for any $r>0$. In addition, we say that $U(r)$ is a singular ground state (SGS) if it is a singular solution of (1.8) which is positive for any $r>0$. The asymptotic behavior of singular and slow decay solutions is well understood and it will be discussed in more details in Section 2.

Roughly speaking, the ω-limit set of (1.1) is (usually) made of the union of solutions of Equation (1.8), see e.g. [**19, 20, 21**], and these solutions are one of the ingredients necessary to construct sub and super-solutions to (1.1), (see, e.g., [**27, 12**]).

We briefly review some known results concerning (1.7) and (1.1)–(1.2). Since $2_*:=2\frac{n-1}{n-2}$ and $2^*:=\frac{2n}{n-2}$, we have

$$\begin{gathered} P_F<2_*<2^*<\sigma^* \quad \text{where} \quad P_F:=2\frac{n+1}{n}, \\ \sigma^*:=\begin{cases} \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n>10 \\ +\infty & \text{if } n\le 10 \end{cases} \end{gathered} \tag{1.9}$$

The parameters $P_F<2_*<2^*$, with P_F-1 the so called Fujita exponent, are critical for this problem.

We assume first that f is of type (1.3), with $k_1(r):=K_0r^{\delta_0}$, where K_0 is a positive constant and $\delta_0>-2$. Define

$$l_0:=2\frac{q+\delta_0}{2+\delta_0} \quad \text{and} \quad m_0:=\frac{2}{l_0-2}=\frac{2+\delta_0}{q-2}, \tag{1.10}$$

by considering $m_0=m_0(l_0)$.

In this case, whenever $l_0>2_*$, we have at least one SGS with slow decay $\phi_s(x)=P_1|x|^{-m_0}$ (unique for $l_0\neq 2^*$), where $P_1>0$ is a computable constant. Moreover, all the regular solutions of (1.8) have a non-degenerate zero for $l_0\in(2,2^*)$, they are GS with fast decay for $l_0=2^*$, and they are GS with slow decay for $l_0>2^*$ (see, e.g., [**27**]). Note that if $\delta_0=0$ then $l_0=q$, and $m_0=2/(q-2)$.

Again, if $2^* \le l_0 < \sigma^*$, then all the regular solutions cross each other, while if $l_0 \ge \sigma^*$ and $\alpha_2 > \alpha_1$, then $U(r,\alpha_2) > U(r,\alpha_1)$ for any $r \ge 0$, see [**27**]. In fact, when the structure of positive solutions of (1.8) changes, the asymptotic behavior of solutions of (1.1)–(1.2) changes too.

Let us recall that all the solutions of (1.1)–(1.2) blow up in finite time if $l_0 \le P_F$, so the null solution is unstable in any reasonable sense (see [**10, 16**]). If $l_0 > P_F$ the null solution is stable with the suitable weighted L^∞-norm, but still "large" solutions blow up in finite time.

There are several papers devoted to exploring the threshold between the basin of attraction of the null solution and the set of initial data which blow up in finite time (see, e.g. [**27, 12, 13**] and also [**19, 20, 21**]). It seems that radial GS of (1.7) play a key role in determining such a border. In particular Gui et al. in [**12**] (see also [**27**]) proved the following:

1.1. THEOREM. [**12**] *Assume* $f(u,r) = K_0 r^{\delta_0} u^{q-1}$ *and* $2^* \le l_0 < \sigma^*$. *Then*

(1) *If* $\phi(x) \lneqq U(|x|,\alpha)$ *for some* $\alpha > 0$, *then* $\|u(x,t;\phi)\|_\infty \to 0$ *as* $t \to +\infty$.

(2) *If* $\phi(x) \gneqq U(|x|,\alpha)$ *for some* $\alpha > 0$, *then* $u(x,t;\phi)$ *must blow up in finite time, i.e. there is* $T_\phi \in (0,\infty)$ *such that* $\lim_{t\to T_\phi} \|u(x,t;\phi)\|_\infty = +\infty$.

This result was extended in [**1**] to functions f of the form (1.3) where

$$k_1(r) = k_0(r) := r^{\delta_0} K(r) \tag{1.11}$$

with $K(r)$ varying monotonically between two positive constants. Then, in [**28**] it was extended to f of the form (1.4) where $k_i(r) = r^{\delta_i} k_i$ and $k_i > 0$ is a constant. An interesting related topic is the rate of decay of fading solutions and of blow up (see, e.g. [**27, 2**]).

It is worth mentioning that in [**27, 12**] the authors proved that for $f(u) = u^{q-1}$ the situation is very different if $q \ge \sigma^*$, i.e. GS are stable and weakly asymptotically stable with respect to a pertinent weighted L^∞-norm, see (3.6) below. This result still holds true also for f as in (1.3) if $k_0(r) = r^\delta K(r)$, when K is decreasing, uniformly positive and bounded, and $l_0 \ge \sigma^*$, where l_0 is defined in (1.10). The same result holds for (1.4) when $k_i = 1$, for $i = 1,2$ and $q_2 > q_1 \ge \sigma^*$ (see [**28**]). The extension of this stability result to the functions considered in this paper will be matter of future investigations.

A first contribution of the present paper is the extension of Theorem 1.1 to a family of non-linearities including (1.3) and (1.4). In fact, we propose a unifying approach that allows us to consider a broad class of functions that also includes (1.5), among others.

As we have already seen, the sub- and supercriticality of (1.1) in the non-homogenous case, e.g. if the function f is as in (1.3), depends on the interplay between the exponent $q = q_1$ and the asymptotic behaviour of $k = k_1$. The same circumstance happens for the asymptotic behavior of positive solutions of (1.8). The following parameters are useful to combine the two effects:

$$l = l(q,\delta) := 2\frac{q+\delta}{2+\delta} \quad \text{and} \quad m(l) := \frac{2}{l-2}. \tag{1.12}$$

If $f(u,r) = r^\delta u^{q-1}$, as in the Wang case, we have a subcritical behavior for $l < 2^*$ and supercritical behavior for $l > 2^*$; the same happens with the other critical parameters defined in (1.9).

We stress that singular and slow decay solutions $U(r)$ of Equation (1.8) behave as $P_1 r^{-m(l)}$ as $r \to 0$ and as $r \to +\infty$ respectively (where P_1 is a computable constant).

Using different values of l, we can allow two different behaviors for the solutions considered, namely: Denote by l_u and $m(l_u)$ the parameters ruling the asymptotic behavior of singular solutions $U(r)$, i.e. $U(r) \sim r^{-m(l_u)}$ as $r \to 0$; similarly, l_s and $m(l_s)$ are those associated with the asymptotic behavior of slow decay solutions $V(r)$, i.e. $V(r) \sim r^{-m(l_s)}$ as $r \to +\infty$.

This will allow us also to consider Matukuma potentials (see, e.g., [**29**]): Thus, for instance in the case (1.3) with $k_1 = k$ as in (1.6), we have that

$$f \text{ as in (1.3), then } \begin{cases} l_u = 2\frac{q+\delta}{2+\delta}\,, \\ m(l_u) = \frac{2+\delta}{q-2}\,, \end{cases} \quad \text{and} \quad \begin{cases} l_s = 2\frac{q+\eta}{2+\eta}\,, \\ m(l_s) = \frac{2+\eta}{q-2}\,, \end{cases} \tag{1.13}$$

Analogously, in the cases (1.4) and (1.5) we have

$$\begin{aligned} &f \text{ as in (1.4), then } l_u = \max\left\{2\tfrac{q_i+\delta_i}{2+\delta_i} \mid i=1,2\right\},\ l_s = \min\left\{2\tfrac{q_i+\eta_i}{2+\eta_i} \mid i=1,2\right\} \\ &f \text{ as in (1.5), then } l_u = 2\tfrac{q_2+\delta}{2+\delta}, \qquad\qquad l_s = 2\tfrac{q_1+\eta}{2+\eta} \end{aligned} \tag{1.14}$$

and, according to (1.12), we also obtain $m(l_u) = \frac{2}{l_u-2}$ and $m(l_s) = \frac{2}{l_s-2}$.

Let us state the following sub and super-criticality conditions related to $k_i(r)$, $i = 1, 2$, that replace the fact that, for $k_0(r) = r^{\delta_0} K(r)$ in (1.11), $K(r)$ is monotone:

- $\boldsymbol{H}+$: $\int_0^r s^{\frac{n-2}{2}q_i} \frac{d}{ds}[k_i(s) s^{\frac{n-2}{2}(2^*-q_i)}]ds \geq 0$ for any $r > 0$ and any i, strictly for some i and $r > 0$.
- $\boldsymbol{H}-$: $\int_0^r s^{\frac{n-2}{2}q_i} \frac{d}{ds}[k_i(s) s^{\frac{n-2}{2}(2^*-q_i)}]ds \leq 0$ for any $r > 0$ and any i, strictly for some i and $r > 0$.

Assume that f is either of type (1.3), (1.4) or (1.5). If $\boldsymbol{H}+$ holds true, then regular solutions of (1.8) are crossing, while, if $\boldsymbol{H}-$ is verified, then they are G.S with slow decay (see [**5**]).

Now we can state the following:

1.2. Proposition. Assume that f is either of the form (1.3), (1.4), or (1.5) and satisfies $\boldsymbol{H}-$. Further, assume that $l_u \geq 2^*$, and $2^* \leq l_s < \sigma^*$. Then all the regular solutions $U(r, \alpha)$ of (1.8) are GS with slow decay, and there is at least one SGS, $U(r, \infty)$, with slow decay. Moreover, if $0 < \alpha_1 < \alpha_2 \leq \infty$, for any $M > 0$ there is $R = R(\alpha_2, \alpha_1) > M$ such that $U(R, \alpha_2) = U(R, \alpha_1)$ and $\frac{\partial}{\partial r}U(R, \alpha_2) - \frac{\partial}{\partial r}U(R, \alpha_1) < 0$.

This result is a direct consequence of Proposition 2.12 and Proposition 2.10 below. In fact, this intersection property of the GS is a second contribution of this paper.

In this setting we can extend Theorem 1.1 as follows

1.3. Theorem. *Assume that the assumptions of Proposition 1.2 are verified, then the same conclusion as in Theorem 1.1 still holds true.*

The above result is obtained as a special case of Theorem 4.1 below, which is somewhat more general.

When f is of type (1.3), Theorem 1.3 generalizes the result of [**1**] to the case where $k_1(r)$ is not monotone decreasing and may even be increasing in some cases.

For instance, let f be of type (1.3) with $k(r) = k_1(r) = 1 + r^a$; assume $q \geq 2^*$ and $a \geq 2^*(q-2)-2$, so that from (1.12) we have $l_u = q$ and $l_s = 2(q+a)/(2+a) \geq 2^*$: In this case Theorem 1.3 applies directly.

Notice that Theorem 1.3 actually requires a weaker condition on l_u rather than on l_s. Hence, Theorem 1.3 applies also to the case (1.3) even for $q \geq \sigma^*$, with the condition that $a \in \big(\frac{2}{\sigma^*-2}(q-\sigma^*), \frac{n-2}{2}(q-2^*) \big]$, while from [**27**] and [**1**] we know that in this case, GS are stable, if $k(r)$ is a constant or a decreasing function varying between two positive values, so we are in the opposite situation.

We emphasize that Theorem 1.3 extends [**1**, Theorem 1] also to Matukuma type functions (see, e.g. [**29**] for more details), which are of interest in astrophysics, i.e. to f of the form (1.3) where $q \in [2^*, \sigma^*)$, $k(r) = 1/(1+r^a)$, and $a \in (0, 2-\sigma^*(q-2))$.

When f is of type (1.4) we extend the result in [**28**] to the case where $k_i(r)$, $i = 1, 2$ are r-dependent functions, and we can deal with a generic family of nonlinearities including (1.4).

Let us go back to the case of $f(u,r) = f(u) = u^{q-1}$: The singular solution $\phi_s(x) := P_1|x|^{-2/(q-2)}$ seems to play a key role in determining the threshold between solutions converging to zero and solutions blowing up in finite time.

In [**17**] Ni shows that if $2_* < q < \sigma^*$ and $\phi(x) < \phi_s(x)$, then $u(x,t;\phi)$ converges to the null solution as $t \to +\infty$. Let λ_1 denote the first eigenvalue of the Laplace operator in the ball of radius $r = 1$; if $\liminf_{|x|\to+\infty} \phi(x)|x|^{2/(q-2)} > (\lambda_1)^{1/(q-2)}$ then $u(x,t;\phi)$ blows up in finite time.

Wang in [**27**] shows that if $q \geq \sigma^*$ and $\liminf_{|x|\to+\infty} \phi(x)|x|^{2/(q-2)} > P_1$ then $u(x,t;\phi)$ blows up in finite time. This result is optimal since, for $q \geq \sigma^*$, there are uncountably many GS with slow decay asymptotic to $P_1|x|^{-2/(q-2)}$ as $|x| \to +\infty$. On the other hand, in [**27**, Theorem 0.2, point (ii)], Wang proved the following result.

1.4. Theorem. [**27**] *Consider $f(u,r) = f(u) = u^{q-1}$, where $2_* < q < \sigma^*$; then for any $\beta > 0$ there is a radial decreasing upper solution of (1.7) $\chi_\beta(x)$ such that $\chi_\beta(0) = \beta$ and $\chi_\beta(x)|x|^{m(q)} \to P_1 := [m(q)(n-2-m(q))]^{1/(q-2)}$ as $|x| \to +\infty$. Moreover $\lim_{t\to+\infty} \|u(x,t;\chi_\beta)(1+|x|^\nu)\|_\infty = 0$ for any $0 < \nu < m(q)$.*

Actually this theorem is proved for a slightly more general function $f(u,r) = r^\delta u^{q-1}$, for a suitable δ. These results seem to indicate $r^{-2/(q-2)}$ (or, more in general, the decay rate of slow decay solutions of (1.7)) as the optimal decay rate for having solutions which are continuable for all $t \in \mathbb{R}$ (see the introduction in [**13**] for a detailed discussion on this topic).

From now till the end of this section we consider f as follows:

- **(i)** f as in (1.3) and k_1 satisfies (1.6) with $l_u, l_s > 2_*$;
- **(ii)** f as in (1.4) and k_i, $i = 1, 2$, satisfy (1.6) with $l_u, l_s > 2_*$;
- **(iii)** f as in (1.5) and k satisfies (1.6) with $l_u, l_s > 2_*$;

As a consequence of Theorem 4.2 we generalize this result to the present setting, i.e.

1.5. Theorem. *Assume f either of the form* (**i**), (**ii**), *or* (**iii**). *Assume either* $\boldsymbol{H}+$ *with $l_u, l_s \in (2_*, 2^*]$ or* $\boldsymbol{H}-$ *with $l_u \geq 2^*$, $l_s \in [2^*, \sigma^*)$. Then we have the same conclusion as in Theorem 1.4 but $m(q)$ is replaced by $m(l_s)$ and P_1 is replaced by* *the computable constant* $P_1^{+\infty}$ *(e.g.* $P_1^{+\infty} = [m(l_s)(n-2-m(l_s))]^{1/(q-2)}$*, if* f *is of type* (**i**)*).*

To the best of our knowledge, this result is new whenever we consider $f(u,r)$ as in (1.3) but $k(r) \not\equiv r^\delta$, and for (1.4) even for $k_1 = k_2 = 1$.

The main contribution of this paper is the following result (a special case of the slightly more general Theorem 4.3) which goes in the opposite direction with respect to Theorem 1.4 (and hence to Theorem 1.5), and shows how delicate the situation is.

1.6. Theorem. *Assume* f *either of the form* (**i**), (**ii**), *or* (**iii**). *Further assume that either* l_u, l_s *are in* $(2_*, 2^*]$*, and* $\boldsymbol{H}+$ *holds, or that* l_u, l_s *are in* $[2^*, +\infty)$*, and* $\boldsymbol{H}-$ *holds. Then there are one parameter families of upper and lower radial solutions with fast decay of (1.7), denoted by* $\zeta_\tau(x)$ *and* $\psi_\tau(x)$ *respectively; hence*

$$\psi_\tau(0) = D(\tau) = \zeta_\tau(0) > 0,$$

$$\lim_{|x|\to+\infty} |x|^{n-2}\zeta_\tau(x) = L_\zeta(\tau)$$

and

$$\lim_{|x|\to+\infty} |x|^{n-2}\psi_\tau(x) = L_\psi(\tau) > 0$$

where $L_\zeta(\tau) < L_\psi(\tau)$. *The solution* $u(x,t;\zeta)$ *blow up in finite time, while the limit* $\lim_{t\to+\infty} \|u(x,t;\psi)(1+|x|)^\nu\|_\infty = 0$ *for any* $0 < \nu < n-2$.

Moreover $\|\zeta_\tau(x)\|_\infty = \|\psi_\tau(x)\|_\infty = D(\tau) \to +\infty$*, while* $L_\zeta(\tau) < L_\psi(\tau) \to 0$ *as* $\tau \to -\infty$*, while* $D(\tau) \to 0$ *and* $L_\psi(\tau) > L_\zeta(\tau) \to +\infty$ *as* $\tau \to +\infty$.

1.7. Remark. For any fixed $\tau \in \mathbb{R}$, $\psi_\tau(x) \geq \zeta_\tau(x)$ when $x \in \mathbb{R}^n$. From the argument used in the proof it follows also that both $\|\zeta_\tau(x)(1+|x|^\nu)\|_\infty$ and $\|\psi_\tau(x)(1+|x|^\nu)\|_\infty$ tend to 0, as $\tau \to +\infty$, for any $0 \leq \nu < m(l_s)$, while they are uniformly positive for $\nu = m(l_s)$.

A novelty of Theorem 1.6, beyond the more general potentials we can deal with, is the fact that we can find fast decay initial data, with L^∞-norm arbitrarily small, which blow up in finite time, while the critical decay indicated in the literature (also by results such as Theorem 1.5) for such a phenomenon seems to be slow decay, i.e. $|x|^{-m(l_s)}$ (see [**13**]).

We highlight that this result is new even when $f(u,|x|) = u^{q-1}$. Notice that, the dichotomy depicted in Theorem 1.6 and in Corollary 1.8 below, takes place even for solutions slightly above or below a GS if we are in the hypotheses of Theorem 1.3. The novelty here is that we can look at a much larger range of parameters and that these families of sub- and super-solution have fast decay: It is possible to find solutions with fast decay and L^∞-norm small which blow up in finite time, see the appendix in [**11**].

The relevance of Theorem 1.6 follows from the next corollary. This latter result is an immediate consequence of the comparison principle.

1.8. Corollary. Assume that we are under the hypotheses of Theorem 1.6. Then for any $\varepsilon > 0$ we can find a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$, such that $\|\phi\|_\infty < \varepsilon$ and there is $T_\phi > 0$ such that the classic solution $u = u(x,t;\phi)$ of (1.1)–(1.2) satisfies $\lim_{t\to T_\phi} \|u(x,t;\phi)\|_\infty = +\infty$. On the other hand we can find a smooth function

$\phi : \mathbb{R}^n \to \mathbb{R}$, such that $\|\phi\|_\infty > 1/\varepsilon$ and the classic solution $u = u(x, t; \phi)$ of (1.1)–(1.2) is defined for any $t \geq 0$ and satisfies $\lim_{t\to+\infty} \|u(x, t; \phi)(1 + |x|^\nu)\|_\infty = 0$ for any $0 \leq \nu < n - 2$.

From this corollary we see how sensitive is, with respect to the initial data, system (1.1)–(1.2): We can find "large" initial data ϕ which converge to the null solution and "small" initial data which blow up in finite time. Indeed, we can also construct initial data ϕ_1 and ϕ_2 such that for any $\varepsilon > 0$ small we have $\|(\phi_1 - \phi_2)(x)[1 + |x|^\nu]\|_\infty < \varepsilon$, whenever $0 < \nu < m(l_s)$, and $u(x, t; \phi_1)$ blows up in finite time, while $u(x, t; \phi_2)$ is defined for any t and has the null solution as ω-limit set. However we need to choose $\|(\phi_1 - \phi_2)(x)[1 + |x|^{m(l_s)}]\|_\infty$ uniformly positive and bounded.

Plan of the paper. In Section 2 we collect all the preliminary results we need concerning regular and singular solutions of (1.8) and, in particular, we prove new ordering properties. Section 3 is devoted to prove local existence of the solutions, in the classical, and in the mild case, giving new results concerning singular solutions (which are slightly smaller than SGS of (1.8)), by a suitable weighted L^∞-norm. Finally, in Section 4, we state and prove our main results on stability and long time behavior of the considered solutions.

2. Ordering results and asymptotic estimates for the elliptic problem.

The results of this sections, which are a key point for the whole argument, are obtained applying Fowler's transformation to (1.8). Thus, we set

$$\begin{aligned} &r = \mathrm{e}^s\,, \quad y_1(s,l) = U(\mathrm{e}^s)\mathrm{e}^{m(l)s}\,, \quad y_2(s,l) = U'(\mathrm{e}^s)\mathrm{e}^{[m(l)+1]s} \\ &m(l) = \frac{2}{l-2}\,, \quad g(y_1, s; l) = f(y_1\mathrm{e}^{-m(l)s}, \mathrm{e}^s)\mathrm{e}^{(m(l)+2)s} \end{aligned} \tag{2.1}$$

Here and in the sequel l denotes a parameter which is always assumed to be larger than 2, so that $m(l) > 0$ (see the exemplifying case in (1.12) and also the parameters related to problem (2.3) below). Using this change of variables, we pass from (1.8) to the following system to which dynamical tools apply:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} m(l) & 1 \\ 0 & -[n-2-m(l)] \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -g(y_1, s; l) \end{pmatrix}. \tag{2.2}$$

In the whole section the "$\cdot$" indicates differentiation with respect to s, and we introduce the following further notation.

We write $\boldsymbol{y}(s, \tau; \boldsymbol{Q}; \bar{l}) = (y_1(s, \tau; \boldsymbol{Q}; \bar{l}), y_2(s, \tau; \boldsymbol{Q}); \bar{l})$ for a trajectory of (2.2) where $l = \bar{l}$, evaluated at s and departing from $\boldsymbol{Q} \in \mathbb{R}^2$ at $s = \tau$.

For illustrative purpose we assume first $f(u, r) = r^\delta u^{q-1}$, so that we can set $l = 2\frac{q+\delta}{2+\delta}$ and system (2.2) reduces to the following autonomous system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} m(l) & 1 \\ 0 & -[n-2-m(l)] \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -(y_1)^{q-1} \end{pmatrix} \tag{2.3}$$

We stress that in this case we passed from a singular non-autonomous O.D.E. to an autonomous system from which the singularity has been removed. Also note that when $\delta = 0$ we can simply take $l = q$.

System (2.3) admits three critical points for $l > 2_* = 2\frac{n-1}{n-2}$: The origin $O = (0, 0)$, $\boldsymbol{P} = (P_1, P_2)$ and $-\boldsymbol{P}$, where $P_2 = -m(l)P_1$ and $P_1 > 0$. The origin is a saddle point and admits a one-dimensional C^1 stable manifold $\overline{M}^s$ and a one-dimensional C^1 unstable manifold $\overline{M}^u$, see Figure 1. The origin splits $\overline{M}^s$ (respectively $\overline{M}^u$) in two relatively open components: We denote by M^s (resp. by M^u) the component which leaves the origin and enters the semi-plane $y_1 \geq 0$. Since we are just interested on positive solutions we will call, with a little abuse of notation, M^s and M^u unstable and stable manifold.

To complete the description of the phase portrait in Figure 1, we recall the following result (see, e.g., [**7**]).

2.1. Remark. The critical point $\boldsymbol{P}$ of (2.3) is such that

$$\boldsymbol{P} \text{ is} \begin{cases} \text{an unstable node} & \Leftrightarrow \quad 2_* < l \leq \sigma_* \\ \text{an unstable focus} & \Leftrightarrow \quad \sigma_* < l < 2^* \\ \text{a center} & \Leftrightarrow \quad l = 2^* \\ \text{a stable focus} & \Leftrightarrow \quad 2^* < l < \sigma^* \\ \text{stable node} & \Leftrightarrow \quad l \geq \sigma^* \end{cases}$$

where $2_*, 2^*, \sigma^*$ are as in (1.9) and

$$\sigma_* := 2\frac{n-2+2\sqrt{n-1}}{n+2\sqrt{n-1}-4}. \tag{2.4}$$

From some asymptotic estimates we deduce the following useful result (see, e.g., [**4, 5**] for the proof in the p-Laplace context).

2.2. Remark. Regular solutions $u(r)$ of Equation (1.8) correspond to trajectories $\boldsymbol{Y}(s;l)$ of system (2.2) departing from points in M^u and viceversa. Positive solutions with fast decay $u(r)$ of (1.8), correspond to trajectories $\boldsymbol{Y}(s;l)$ of system (2.2) departing from points in M^s and viceversa.

Using the Pohozaev identity, introduced in [**18**] and adapted to this context in [**4**], we can draw a picture of the phase portrait of (2.3), see Figure 1, and deduce information on positive solutions of (1.8). We postpone a sketch of the proof to the next subsection, where the general non-autonomous case is considered (anyway see [**4**] or [**5**] for a detailed proof in the more general p-Laplace context). Then it is easy to classify positive solutions: In the supercritical case ($l > 2^*$) all the regular solutions are GS with slow decay, there is a unique SGS with slow decay; in the critical case ($l = 2^*$) all the regular solutions are GS with fast decay and there are uncountably many SGS with slow decay; in the subcritical case ($2 < l < 2^*$) all the regular solutions are crossing, there are uncountably many SGS with fast decay and a unique SGS with slow decay.

Since (2.3) is autonomous we also get the following useful consequence.

2.3. Lemma. *Fix $\boldsymbol{U} \in M^u$ and $\boldsymbol{S} \in M^s$. Consider the trajectories $\boldsymbol{y}(s,\tau;\boldsymbol{U})$, $\boldsymbol{y}(s,\tau;\boldsymbol{S})$ of (2.2) and the corresponding regular solution $U(r,D)$ and fast decay solution $V(r,L)$ of (1.8). Then*

$$D = D(\tau) = D(0)e^{-m\tau} \qquad L = L(\tau) = L(0)e^{(n-m)\tau}$$
$$U(r,D) = DU(rD^{1/m},1) \qquad V(r,L) = LV(rL^{1/m},1)$$

Proof. Since $y_1(s+\tau,\tau,\boldsymbol{Q}) = y_1(s,0,\boldsymbol{Q})$ we get $U(r\mathrm{e}^{\tau},D(\tau))\mathrm{e}^{m(l)\tau} = U(r,D(0))$; hence letting $r \to 0$ we find $D(\tau) = D(0)\mathrm{e}^{-m(l)\tau}$, and this concludes the proof concerning U. Similarly we find $V(\mathrm{e}^{s+\tau},L(\tau))\mathrm{e}^{n(s+\tau)}\mathrm{e}^{(m-n)\tau} = V(\mathrm{e}^s,L(0))\mathrm{e}^{ns}$, hence, letting $s \to +\infty$ we get $L(\tau) = L(0)\mathrm{e}^{(n-m)\tau}$. Then, again from $y_1(s+\tau,\tau,\boldsymbol{Q}) = y_1(s,0,\boldsymbol{Q})$, we get $V(r,L) = LV(rL^{1/m},1)$: this concludes the proof. □

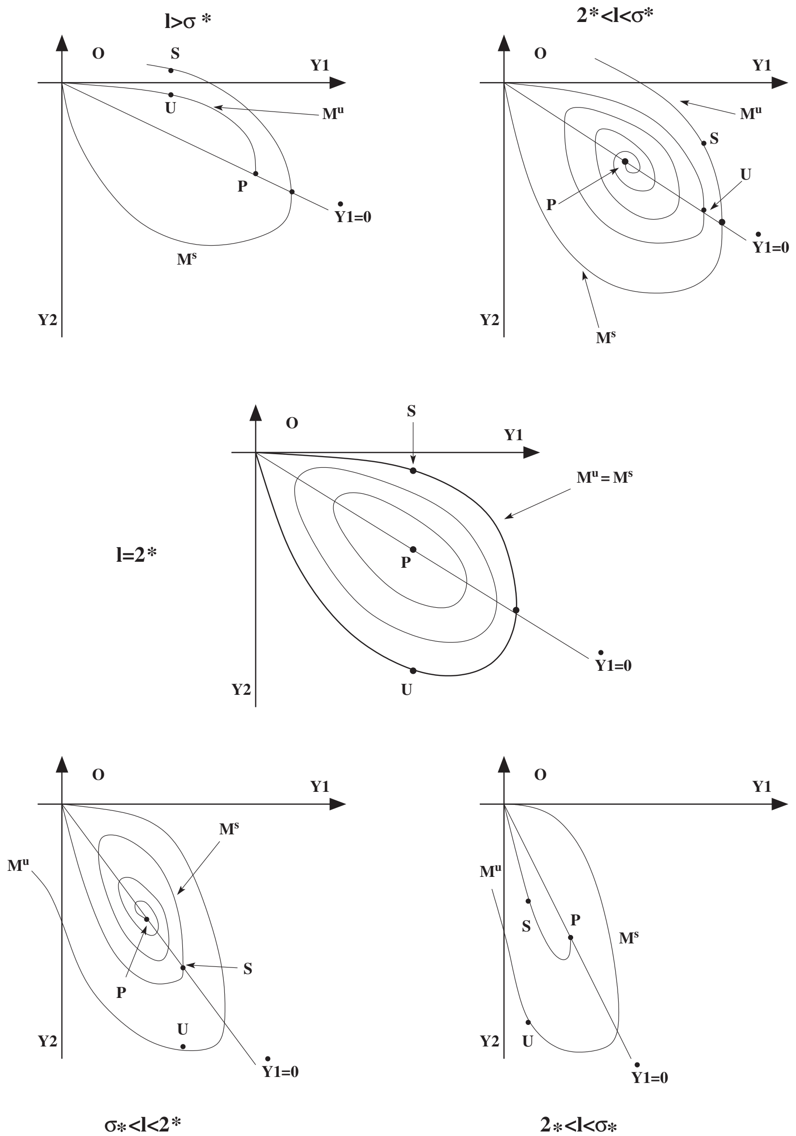

Figure 1. Sketches of the phase portrait of (2.2), for $q > 2$ fixed.

We stress that all the previous arguments concerning the autonomous Equation (2.2) still hold true for any autonomous super-linear system (2.2), i.e. whenever $g(y_1, s; l) \equiv g(y_1; l)$ and $g(y_1; l)$ enjoys the following property, denoted by $\boldsymbol{G0}$

$\boldsymbol{G0}$: $\frac{\partial g}{\partial y_1}(y_1; l)$ is a strictly increasing function for $y_1 > 0$ and

$$\lim_{y_1 \to 0} \frac{g(y_1; l)}{y_1} = 0\,, \qquad \lim_{y_1 \to +\infty} \frac{g(y_1; l)}{y_1} = +\infty\,.$$

In particular Remarks 2.1, 2.2, and Lemma 2.3 continue to be valid (see [**5**] for a proof in the general p-Laplace context).

We emphasize that $\boldsymbol{G0}$ implies that $\frac{g(y_1;l)}{y_1}$ is strictly increasing for $y_1 > 0$; in this case we easily find that $g(y_1;l)$ is strictly increasing too.

To draw correctly the analogous of Figure 1 for the present case, we need to use the Pohozaev identity (see [**18**], and[**5, 3**] for more details). Let us introduce the Pohozaev function

$$P(u,u';r) := \frac{n-2}{2} r^{n-1} u u' + \frac{r^n |u'|^2}{2} + F(u,r) r^n,$$

where $F(u,r) = \int_0^u f(a,r)da$. Now, consider the non-autonomous system (2.2) and denote by $G(y_1,s;l) = \int_0^{y_1} g(a,s;l)da$. In this dynamical setting the transposition of $P(u,u';r)$ is given by

$$H(y_1,y_2,s;l) = \frac{n-2}{2} y_1 y_2 + \frac{|y_2|^2}{2} + G(y_1,s;2^*);$$

then, if $\boldsymbol{y}(s;2^*) = (y_1(s,2^*), y_2(s,2^*))$ solves (2.2) with $l = 2^*$ we have the following

$$\frac{dH}{ds}(y_1(s,2^*), y_2(s,2^*), s; 2^*) = \frac{\partial G}{\partial s}(y_1(s,2^*), s; 2^*). \tag{2.5}$$

Moreover, if $\boldsymbol{y}(s;2^*)$ and $\boldsymbol{y}(s;l)$ are trajectories of (2.2) corresponding to the same solution $U(r)$ of (1.8), we get

$$H(\boldsymbol{y}(s,2^*), s; 2^*) = \mathrm{e}^{A(l)s} H(\boldsymbol{y}(s,l), s; l). \tag{2.6}$$

where $A(l) = n-2-2m(l)$. We stress that (2.5) and (2.6) hold for the general non-autonomous system (2.2).

Let us fix $\tau \in \mathbb{R}$ and $l > 2_*$ and denote by

$$\begin{aligned} K(b) &:= \{(y_1,y_2) \mid H(y_1,y_2,\tau;l) = b\} \\ K_+(b) &:= \{(y_1,y_2) \mid H(y_1,y_2,\tau;l) = b\,,\ y_1 > 0\} \end{aligned} \tag{2.7}$$

Then, there is $b^*(\tau,l) < 0$ such that $K(b)$ is empty for $b < b^*(\tau,l)$, $K(b)$ is made up by two closed bounded curves contained in $y_2 < 0 < y_1$ and in $y_1 < 0 < y_2$ for $b^* < b < 0$ (the graph of the former gives $K_+(b)$), $K(b)$ is a 8-shaped curve having the origin as center for $b = 0$, and it is a closed bounded curve surrounding the origin for $b > 0$.

From (2.5) we see that $H(\boldsymbol{y}(s,2^*), s; 2^*)$ is increasing in s (respectively decreasing) along the trajectories $\boldsymbol{y}(s,2^*)$ of (2.2) whenever $G(y_1,s;2^*)$ is increasing in s (resp. decreasing in s). Moreover from (2.6) we see that $H(\boldsymbol{y}(s,2^*), s; 2^*)$ and $H(\boldsymbol{y}(s,l), s; l)$ have the same sign. Thus, if we consider system (2.3), for any $\boldsymbol{Q} \in M^u$ and $\boldsymbol{R} \in M^s$ we get $H(\boldsymbol{Q},s;l) < 0 < H(\boldsymbol{R},s;l)$ when $l > 2^*$, $H(\boldsymbol{R},s;l) < 0 < H(\boldsymbol{Q},s;l)$ when $2 < l < 2^*$, and $H(\boldsymbol{Q},s;l) = 0 = H(\boldsymbol{R},s;l)$ when $l = 2^*$. Using (2.5) and (2.6), it can be proved that the phase portrait of the autonomous system (2.3) is again depicted as in Fig. 1, see, e.g., [**5, 7**].

We collect here the values of several constants and parameters which will be used through the whole paper. Recalling that $m(l) = \frac{2}{l-2}$, we set

$$A(l) = n-2-2m(l)\,, \quad C(l) = m(l)[n-2-m(l)]. \tag{2.8}$$

Notice that $(P_1, -m(l)P_1)$ is a critical point of (2.2) if it is s-independent, so P_1 is the unique positive solution in y of $g_l(y;l) = C(l)y$. When $g(y,l) = y^{q-1}$ then

$P_1 = (C(l))^{1/(q-2)}$. Let $n > 2$ we denote by $\sigma_* < \sigma^*$ the real solutions of the equation in l given by

$$A(l)^2 - 4[C(l) + \frac{\partial g}{\partial y}(P_1, l)] = 0\,. \tag{2.9}$$

which reduces to $A(l)^2 - 4(q-2)C(l) = 0$ for $g(y) = y^{q-1}$. In this case the value of σ^* coincide with the one given in (1.9).

2.1. The stationary problem: the spatial dependent case. Now we turn to consider (2.2) in the s-dependent case. The first step is to extend invariant manifold theory to the non-autonomous setting; there are several ways to achieve the result: using skew-product semi-flow (see, e.g. [**15**]), or through Wazewski's principle, see e.g. [**5**]. Here, we follow a simpler construction which is less general but preserve more properties (in particular the ordering results Propositions 2.8, 2.9), used e.g. in [**7, 8**]. So we introduce an extra variable, either $z(s) = e^{\varpi s}$ or $\zeta(s) = e^{-\varpi s}$, in order to deal with a 3-dimensional autonomous system. We use z and ζ in order to investigate the behavior respectively as $s \to -\infty$ (i.e. $r \to 0$), and as $s \to +\infty$ (i.e. $r \to +\infty$).

We collect here below the assumptions used in the main results:

- ***Gu***: There is $l_u > 2_*$ such that for any $y_1 > 0$ the function $g(y_1, s; l_u)$ converges to a s-independent locally Lipschitz function $g(y_1, -\infty; l_u) \not\equiv 0$ as $s \to -\infty$, uniformly on compact intervals. The function $g(y_1, -\infty; l_u)$ satisfies ***G0***. Moreover there is $\varpi > 0$ such that $\lim_{s\to-\infty} e^{-\varpi s}\frac{\partial}{\partial s} g(y_1, s; l_u) = 0$. Furthermore if $l_u = 2^*$, we also assume that there is $M > 0$ such that $g(y_1, s; 2^*)$ is monotone in s for for any $y_1 > 0$ and any $s < -M$.
- ***Gs***: There is $l_s > 2_*$ such that for any $y_1 > 0$ the function $g(y_1, s; l_s)$ converges to a s-independent locally Lipschitz function $g(y_1; l_s) \not\equiv 0$ as $s \to +\infty$, uniformly on compact intervals. The function $g^{+\infty}(y_1; l_s)$ satisfies ***G0***. Moreover there is $\varpi > 0$ such that $\lim_{s\to+\infty} e^{+\varpi s}\frac{\partial}{\partial s} g(y_1, s; l_s) = 0$. Furthermore if $l_s = 2^*$, we also assume that there is $M > 0$ such that $g(y_1, s; 2^*)$ is monotone in s for for any $y_1 > 0$ and any $s > M$.
- $\boldsymbol{A^-}$: The function $G(y_1, s; 2^*) := \int_0^{y_1} g(a, s; 2^*)da$ is decreasing in s for any $y_1 > 0$, strictly for some s.
- $\boldsymbol{A^+}$: $G(y_1, s; 2^*)$ is increasing in s for any $y_1 > 0$, strictly for some s.

Hypotheses ***Gu***, ***Gs*** are used to construct unstable and stable manifolds for Equation (2.2) when it depends on s, while $\boldsymbol{A^-}$ and $\boldsymbol{A^+}$ mean that the system is respectively supercritical and subcritical with respect to 2^*, and are used to understand the position of these manifolds.

2.4. Remark. Observe that if f is as in (1.3), (1.4), (1.5) and (1.6) hold then ***Gu*** and ***Gs*** hold with l_u and l_s defined as in (1.13) and (1.14).

Assume ***Gu***. We introduce the following 3-dimensional autonomous system, obtained from (2.2) by adding the extra variable $z = e^{\varpi t}$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{pmatrix} = \begin{pmatrix} m(l_u) & 1 & 0 \\ 0 & -[n-2-m(l_u)] & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ -g(y_1, \frac{\ln(z)}{\varpi}; l_u) \\ 0 \end{pmatrix} \tag{2.10}$$

Similarly if $\boldsymbol{Gs}$ is satisfied we set $l = l_s$ and $\zeta(t) = \mathrm{e}^{-\varpi t}$ and we consider

$$(2.11)\quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} m(l_s) & 1 & 0 \\ 0 & -[n-2-m(l_s)] & 0 \\ 0 & 0 & -\varpi \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \zeta \end{pmatrix} + \begin{pmatrix} 0 \\ -g(y_1, -\frac{\ln(\zeta)}{\varpi}; l_s) \\ 0 \end{pmatrix}$$

The technical assumption at the end of $\boldsymbol{Gu}$ (and $\boldsymbol{Gs}$) is needed in order to ensure that the system is smooth for $z = 0$ and $\zeta = 0$ too. Consider (2.10) (respectively (2.11)): each trajectory that may be continued for any $s \leq 0$ (resp. for any $s \geq 0$) is such that its α-limit set is contained in the $z = 0$ plane (resp. its ω-limit set is contained in the $\zeta = 0$ plane); moreover such a plane is invariant and the dynamic reduced to $z = 0$ (resp. $\zeta = 0$) coincides with the one of the autonomous system (2.2) where $g(y_1, s; l_u) \equiv g(y_1, -\infty; l_u)$ (resp. $g(y_1, s; l_s) \equiv g(y_1, +\infty; l_s)$).

Observe that the origin of (2.10) admits a 2-dimensional unstable manifold $\mathbf{W^u}(l_u)$ which is transversal to $z = 0$ (and a 1-dimensional stable manifold M^s contained in $z = 0$), while the origin of (2.11) admits a 2-dimensional stable manifold $\mathbf{W^s}(l_s)$ which is transversal to the plane $\zeta = 0$ (and a 1-dimensional unstable manifold M^u contained in $\zeta = 0$). Following [**8**], see also [**15, 5**] we see that, for any $\tau \in \mathbb{R}$,

$$\begin{aligned} W^u(\tau; l_u) &= \boldsymbol{W^u}(l_u) \cap \{z = \mathrm{e}^{\varpi\tau}\}, \qquad & W^s(\tau; l_s) &= \boldsymbol{W^s}(l_s) \cap \{\zeta = \mathrm{e}^{-\varpi\tau}\} \\ W^u(-\infty; l_u) &= \boldsymbol{W^u}(l_u) \cap \{z = 0\}, \qquad & W^s(+\infty; l_s) &= \boldsymbol{W^s}(l_s) \cap \{\zeta = 0\} \end{aligned}$$

are 1-dimensional manifolds. Moreover they inherit the same smoothness properties as (2.10) and (2.11). Indeed, let K be a segment which intersects $W^u(\tau_0; l_u)$ (respectively $W^s(\tau_0; l_s)$) transversally in a point $\boldsymbol{Q}(\tau_0)$ for $\tau_0 \in [-\infty, +\infty)$ (resp. for $\tau_0 \in (-\infty, +\infty]$), then there is a neighborhood I of τ_0 such that $W^u(\tau; l_u)$ (resp. $W^s(\tau; l_s)$) intersects K in a point $\boldsymbol{Q}(\tau)$ for any $\tau \in I$, and $\boldsymbol{Q}(\tau)$ is as smooth as (2.10) (resp. as (2.11)). Since we need to compare $W^u(\tau; l_u)$ and $W^s(\tau; l_s)$ we introduce the manifolds:

$$(2.12)\quad \begin{aligned} W^u(\tau; l_s) &:= \big\{\boldsymbol{R} = \boldsymbol{Q}\exp\big\{-\big(m(l_u) - m(l_s)\big)\tau\big\} \in \mathbb{R}^2 \mid \boldsymbol{Q} \in W^u(\tau; l_u)\big\} \\ W^s(\tau; l_u) &:= \big\{\boldsymbol{Q} = \boldsymbol{R}\exp\big\{\big(m(l_u) - m(l_s)\big)\tau\big\} \in \mathbb{R}^2 \mid \boldsymbol{R} \in W^s(\tau; l_s)\big\} \end{aligned}$$

As in the s-independent case, we see that *regular solutions correspond to trajectories in W^u, while fast decay solutions correspond to trajectories in W^s*, see [**8, 5**]. More precisely, from Lemma 3.5 in [**5**] we get the following.

2.5. Lemma. *Consider the trajectory $\boldsymbol{y}(s, \tau, \boldsymbol{Q}; l_u)$ of (2.2) with $l = l_u$, the corresponding trajectory $\boldsymbol{y}(t, \tau, \boldsymbol{R}; l_s)$ of (2.2) with $l = l_s$ and let $u(r)$ be the corresponding solution of (1.8). Then $\boldsymbol{R} = \boldsymbol{Q}exp[(m(l_s) - m(l_u))\tau]$.*
Assume $\boldsymbol{Gu}$; then $u(r)$ is a regular solution if and only if $\boldsymbol{Q} \in W^u(\tau; l_u)$ or equivalently $\boldsymbol{R} \in W^u(\tau; l_s)$.
Assume $\boldsymbol{Gs}$; then $u(r)$ is a fast decay solution if and only if $\boldsymbol{R} \in W^s(\tau; l_s)$ or equivalently $\boldsymbol{Q} \in W^s(\tau; l_u)$.

Moreover, if $\boldsymbol{Q} \in W^s(\tau; l_u)$ and $l_u > 2_*$, then $\lim_{s\to+\infty} \boldsymbol{y}(s, \tau, \boldsymbol{Q}; l_u) = (0, 0)$, and if $\boldsymbol{R} \in W^u(\tau; l_s)$ and $l_s > 2$ then $\lim_{s\to-\infty} \boldsymbol{y}(s, \tau, \boldsymbol{R}; l_s) = (0, 0)$.

For the reader's convenience we now describe a result proved in [**5**] which explains further the relationship between (1.8) and (2.2): We recall that, close to the origin, $W^u(\tau; l_u)$ is locally a graph on the y_1 axis, while $W^s(\tau; l_s)$ is locally a graph on its tangent space, i.e. the line $y_2 = -(n-2)y_1$. So let us consider a ball $B(\delta)$ of radius $\delta > 0$ centered in the origin. Follow $W^u(\tau; l_u)$ (respectively $W^s(\tau; l_s)$)

from the origin towards $y_1 > 0$. If $\delta > 0$ is small enough, we can choose a segment $L \subset B(\delta)$, parallel to the y_2 axis such that $W^u(\tau; l_u)$ (respectively $W^s(\tau; l_s)$) intersects L transversally a first time exactly in a point, say $\boldsymbol{Q^u}(\tau)$ (resp. $\boldsymbol{Q^s}(\tau)$). We know that this point depends on τ as smoothly as (2.2), so it is at least C^1.

Moreover, we have the following result analogous to Lemma 2.3, see [**5, 8**].

2.6. Remark. Assume $\boldsymbol{Gu}$. Consider $\boldsymbol{y}(s, \tau, \boldsymbol{Q^u}(\tau); l_u)$ and the corresponding regular solution $U(r, \alpha(\tau))$ of (1.8). Then $\alpha(\tau) \to 0$ as $\tau \to -\infty$ and $\alpha(\tau) \to +\infty$ as $\tau \to +\infty$.

Similarly, assume $\boldsymbol{Gs}$. Consider $\boldsymbol{y}(s, \tau, \boldsymbol{Q^s}(\tau); l_s)$ and the corresponding fast decay solution $V(r, \beta(\tau))$ of (1.8). Then $\beta(\tau) \to 0$ as $\tau \to -\infty$ and $\beta(\tau) \to +\infty$ as $\tau \to +\infty$.

Now we turn to consider singular and slow decay solutions of (1.8).
We observe that if $l_u > 2_*$ then (2.10) has a critical point in $y_1 > 0$, say $(\boldsymbol{P^{-\infty}}, 0)$, where $\boldsymbol{P^{-\infty}} = (P_1^{-\infty}, -m(l_u)P_1^{-\infty})$ is the critical point of the autonomous system (2.2) where $g(y_1, s; l_u) \equiv g(y_1, -\infty; l_u)$, and $P_1^{-\infty} > 0$. It is easy to check that $(\boldsymbol{P^{-\infty}}, 0)$ admits an exponentially unstable manifold transversal to $z = 0$ which is 1-dimensional (the graph of a trajectory which will be denoted by $\boldsymbol{y^u}(s; l_u)$) if $l_u \geq 2^*$, and 3-dimensional if $2_* < l_u < 2^*$.

Analogously, if $l_s > 2_*$ then (2.11) has a critical point in $y_1 > 0$, say $(\boldsymbol{P^{+\infty}}, 0)$, where $\boldsymbol{P^{+\infty}} = (P_1^{+\infty}, -m(l_s)P_1^{+\infty})$ is the critical point of the autonomous system (2.2) where $g(y_1, s; l_s) \equiv g(y_1, +\infty; l_s)$, and $P_1^{+\infty} > 0$. $(\boldsymbol{P^{+\infty}}, 0)$ admits an exponentially stable manifold transversal to $\zeta = 0$ which is 1-dimensional (the graph of a trajectory which will be denoted by $\boldsymbol{y^s}(s; l_s)$) if $2_* < l_s \leq 2^*$ and 3-dimensional if $l_s > 2^*$.

In the whole paper we denote by $U(r, \infty)$ the solution of (1.8) corresponding to $\boldsymbol{y^u}(s; l_u)$, and by $\boldsymbol{y^u}(s; l_s)$ the corresponding trajectory of (2.2) with $l = l_s$; similarly we denote by $V(r, \infty)$ the slow decay solution corresponding to $\boldsymbol{y^s}(s; l_s)$, and by $\boldsymbol{y^s}(s; l_u)$ the corresponding trajectory of (2.2) with $l = l_u$.

Note that, if $2_* < l_s < 2^* < l_u$,the manifolds $W^u(-\infty; l_u)$ and $W^s(+\infty; l_s)$ are paths connecting the origin respectively with $\boldsymbol{P^{-\infty}}$ and $\boldsymbol{P^{+\infty}}$, and contained in $y_2 < 0 < y_1$, see Figure 1 (we emphasize that this is not the case when $2_* < l_u \leq 2^* \leq l_s$). Using a connection argument we get the following.

2.7. Remark. Assume $\boldsymbol{Gu}$, $\boldsymbol{Gs}$ with $2_* < l_s < 2^* < l_u$, then $W^u(\tau; l_u)$ and $W^u(\tau; l_s)$ are paths connecting the origin respectively with $\boldsymbol{y^u}(\tau; l_u)$ and $\boldsymbol{y^u}(\tau; l_s)$ for any $\tau \in [-\infty, +\infty)$; similarly $W^s(\tau; l_u)$ and $W^s(\tau; l_s)$ are paths connecting the origin respectively with $\boldsymbol{y^s}(\tau; l_u)$ and $\boldsymbol{y^s}(\tau; l_s)$ for any $\tau \in (-\infty, +\infty]$.

2.8. Remark. Assume $\boldsymbol{Gu}$ with $l_u > 2_*$; then there is at least one singular solution $U(r, \infty)$ of (1.8). Moreover $U(r, \infty)r^{m(l_u)}$ converges to $P_1^{-\infty}$ as $r \to 0$. Furthermore $U(r, \infty)$ is the unique singular solution if $l_u > 2^*$.

A specular argument gives us a similar condition for slow decay solutions.

2.9. Remark. Assume $\boldsymbol{Gs}$ with $l_s > 2_*$; then there is at least one slow decay solution $V(r, \infty)$ of (1.8): moreover $V(r, \infty)r^{m(l_s)}$ converges to $P_1^{+\infty}$ as $r \to +\infty$. Such a solution is unique if $2_* < l_s < 2^*$.

Now we give a further result concerning separation properties, which will be useful to construct sub and super-solutions for (1.8).

2.10. Proposition. Assume $\boldsymbol{Gs}$ with $l_s \in [2^*, \sigma^*)$, and consider two slow decay solutions $\bar{U}(r)$ and $\tilde{U}(r)$ of (1.8). Then $\bar{U}(r) - \tilde{U}(r)$ changes sign infinitely many times as $r \to +\infty$. Analogously assume $\boldsymbol{Gu}$ with $l_u \in (\sigma_*, 2^*]$, and consider two singular solutions $\bar{V}(r)$ and $\tilde{V}(r)$ of (1.8). Then $\bar{V}(r) - \tilde{V}(r)$ changes sign indefinitely as $r \to 0$. See (1.9), (2.4) for a definition of σ^* and σ_*.

Proof. Denote by $\bar{\boldsymbol{y}}(s) = \bar{\boldsymbol{y}}(s; l_s)$, $\tilde{\boldsymbol{y}}(s) = \tilde{\boldsymbol{y}}(s; l_s)$ the solutions of (2.2) corresponding to $\bar{U}(r)$ and $\tilde{U}(r)$, respectively. Now assume $l_s = 2^*$ (and $g(y_1, s; 2^*)$ monotone in s for s large). Then $H(\bar{\boldsymbol{y}}(s), s; 2^*) \to \bar{b}$, and $H(\tilde{\boldsymbol{y}}(s), s; 2^*) \to \tilde{b}$ as $s \to +\infty$, and $\bar{b}, \tilde{b}$ are both negative. If $\bar{b} \geq \tilde{b} > b^*$ then $\bar{\boldsymbol{y}}(s)$ converges to $K_+(\bar{b})$, and $\tilde{\boldsymbol{y}}(s)$ to $K_+(\tilde{b})$, see (2.7): by construction $K_+(\tilde{b})$ lies in the interior of the bounded set enclosed by $K_+(\bar{b})$. Denote by $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ the point of $K_+(\bar{b})$ respectively with largest and smallest component y_1. When $\bar{\boldsymbol{y}}(s)$ passes close to $\boldsymbol{A}^+$ we have $\bar{y}_1(s) - \tilde{y}_1(s) > 0$, while when $\bar{\boldsymbol{y}}(s)$ passes close to $\boldsymbol{A}^-$ we have $\bar{y}_1(s) - \tilde{y}_1(s) < 0$, so the result is proved.

The argument works also if $\bar{b} > \tilde{b} = b^*$, so assume now $\bar{b} = \tilde{b} = b^*$, i.e. both $\bar{\boldsymbol{y}}(s)$ and $\tilde{\boldsymbol{y}}(s)$ converge to $\boldsymbol{P}^{+\infty}$.
We denote by $h(s) = \bar{y}_1(s) - \tilde{y}_1(s)$: note that $h(s) \to 0$ as $s \to +\infty$ and that it satisfies

$$\ddot{h}(s) + Bh(s) + N(h(s), s) = 0 \tag{2.13}$$

where $B = B(l_s) = \frac{\partial g^{+\infty}}{\partial y_1}(P_1^{+\infty}) - m(l_s)[n-2-m(l_s)] = \frac{\partial g^{+\infty}}{\partial y_1}(P_1^{+\infty}) - \frac{(n-2)^2}{4}$ and

$$\begin{aligned} N(h(s), s) :=& g(\tilde{y}_1(s) + h(s), s) - g(\tilde{y}_1(s), s) - \frac{\partial g^{+\infty}}{\partial y_1}(P_1^{+\infty})h(s) = \\ =& h(s) \int_0^1 \Big[\frac{\partial g}{\partial y_1}(\tilde{y}_1(s) + \sigma h(s), s) - \frac{\partial g^{+\infty}}{\partial y_1}(P_1^{+\infty})\Big] d\sigma \end{aligned} \tag{2.14}$$

So from (2.14), $\boldsymbol{Gs}$, and the fact that $|\tilde{y}_1(s)| + |h(s)| \to P_1^{+\infty}$ as $s \to +\infty$ we see that $N(h(s), s) = o(h(s))$. Therefore for any $\varepsilon > 0$ we find $S = S(\varepsilon)$ such that $|N(h(s), s)| \leq \varepsilon |h(s)|$ for any $s > S$. Note also that from $\boldsymbol{Gs}$ we get $B > 0$. Setting

$$h(s) = \rho(s)\frac{\cos(\theta(s))}{\sqrt{B}}, \qquad \dot{h}(s) = \rho(s)\sin(\theta(s)) \tag{2.15}$$

from (2.13) we get

$$\dot{\theta}(s) = -\sqrt{B} - \cos(\theta(s))\frac{N(\rho(s)\frac{\cos(\theta(s))}{\sqrt{B}}, s)}{\rho(s)} < -\sqrt{B}(1-\varepsilon) < -\frac{\sqrt{B}}{2} \tag{2.16}$$

for any $s > S$ and S large enough. Since $\theta(s) \to -\infty$, and $\rho(s) \to 0$ as $s \to +\infty$, but $\rho(s) > 0$ for any $s \in \mathbb{R}$, then $h(s)$ changes sign indefinitely, and the Proposition follows.

Assume now $l_s \in (2^*, \sigma^*)$: then both $\bar{\boldsymbol{y}}(s)$, $\tilde{\boldsymbol{y}}(s)$ converge exponentially to $\boldsymbol{P}^{+\infty}$, therefore $h(s) = \bar{y}(s) - \tilde{y}(s) \to 0$ as $s \to +\infty$. In this case (2.13) is replaced by

$$\ddot{h}(s) - A\dot{h}(s) + Bh(s) + N(h(s), s) = 0 \tag{2.17}$$

where $A = A(l_s) = n - 2 - 2m(l_s) < 0$ and $B = B(l_s) = \frac{\partial g^{+\infty}}{\partial y_1}(P_1^{+\infty}) - m(l_s)[n - 2 - m(l_s)] > 0$. Note that $\sqrt{B} - \frac{A}{2} = 2\varepsilon > 0$ for $l_s \in (\sigma_*, 2^*]$ and it equals 0 for

$l_s = \sigma_*$. Thus, using again (2.15), and passing to polar coordinates we get

$$\begin{aligned}\dot\theta(s) &= -\sqrt{B} - \frac{A\sin(2\theta(s))}{2} - \cos(\theta(s))\frac{N(\rho(s)\frac{\cos(\theta(s))}{\sqrt{B}}, s)}{\rho(s)} \\ &< -\sqrt{B} + \frac{A}{2} + \varepsilon < -\varepsilon\end{aligned} \tag{2.18}$$

So we find again that $\theta(s) \to -\infty$, and $\rho(s) \to 0$ as $s \to +\infty$, thus $h(s)$ changes sign indefinitely, and the Proposition follows.

The case of singular solutions $\tilde V(r)$ and $\bar V(r)$ can be obtained from the previous, repeating the argument but reversing the direction of s. □

Following [**5**] we can show that if $\boldsymbol{A}^-$ holds then (1.8) is supercritical, while if $\boldsymbol{A}^+$ holds then (1.8) is subcritical. To be more precise we have the following (see [**5**, Theorems 4.2 and 4.3]).

2.11. Proposition. [**5**] Assume $\boldsymbol{Gu}$, $\boldsymbol{Gs}$, with $l_s, l_u \in (2_*, 2^*]$, and $\boldsymbol{A}^+$, then all the regular solutions $U(r,\alpha)$ are crossing, i.e. there is $R(\alpha)$ such that $U(r,\alpha) > 0$ for $0 \le r < R(\alpha)$ and $U(R(\alpha),\alpha) = 0$. Furthermore $R(\alpha)$ is continuous and $R(\alpha) \to +\infty$ as $\alpha \to 0$, and if $l_u < 2^*$ then $R(\alpha) \to 0$ as $\alpha \to +\infty$.

Moreover, all the fast and slow decay solutions are SGS. So for any $\beta > 0$ the fast decay solution $V(r,\beta)$ is a SGS with fast decay; if $l_s < 2^*$ there is a unique SGS with slow decay, say $V(r,\infty)$, while if $l_s = 2^*$ there are uncountably many SGS with slow decay.

Proof. This result is borrowed from [**5**, Theorem 4.2] , where it is proved in the p-Laplace context in a more general framework, so here we just sketch the proof. The main idea is to use the Pohozaev identity as done in the previous subsection: From (2.5) we know that the function $H(\boldsymbol{y}, s; 2^*)$ is decreasing along the trajectories, and it is null for $\boldsymbol{y} = 0$. Using also (2.6) we see that if $\boldsymbol{Q} \in W^u(\tau; l_u)$ and $\boldsymbol{R} \in W^s(\tau; l_s)$ we get $H(\boldsymbol{Q}, \tau; l_u) > 0 > H(\boldsymbol{R}, \tau; l_s)$. Recalling which is the form of the level set $K(b)$ of H (see (2.7) and the discussion just after it) we deduce which is the position of $W^u(\tau; l_u)$ and $W^s(\tau; l_s)$ and using Lemma 2.5, Remark 2.7 we conclude the proof. □

With a specular argument we get the following.

2.12. Proposition. Assume $\boldsymbol{Gu}$, $\boldsymbol{Gs}$ with $l_u, l_s \ge 2^*$, and $\boldsymbol{A}^-$, then all the regular solutions $U(r,\alpha)$ are GS with slow decay. Moreover all the fast decay solutions $V(r,\beta)$ have a positive non-degenerate zero $r = R(\beta)$, i.e. $V(r,\beta)$ is positive for any $r > R(\beta)$ and it is null for $r = R(\beta)$. Furthermore $R(\beta)$ is continuous and $R(\beta) \to 0$ as $\beta \to +\infty$, and if $l_u > 2^*$ then $R(\beta) \to +\infty$ as $\beta \to 0$. Further if $l_u = 2^*$ there are uncountably many SGS with slow decay, while if $l_u > 2^*$ there is a unique SGS with slow decay say $U(r,\infty)$.

Assume $\boldsymbol{A}^+, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $2 < l_u < 2^*$ and $2_* < l_s \le 2^*$. Follow $W^u(\tau; l_u)$ from the origin towards $\mathbb{R}^2_+ := \{(y_1, y_2) \mid y_1 > 0\}$: it intersects the y_2 positive semi-axis in a point, say $\boldsymbol{Q^u}(\tau)$. We denote by $\bar W^u(\tau; l_u)$ the branch of $W^u(\tau; l_u)$ between the origin and $\boldsymbol{Q^u}(\tau)$, and by $\bar E^u(\tau)$ the bounded set enclosed by $\bar W^u(\tau; l_u)$ and the y_2 axis. Similarly assume $\boldsymbol{A}^-, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $l_u \ge 2^*$ and $l_s > 2^*$. Follow $W^s(\tau; l_s)$ from the origin towards $y_1 \ge 0$: it intersects the y_2 negative semi-axis in a point, say $\boldsymbol{Q^s}(\tau)$. We denote by $\bar W^s(\tau; l_s)$ the branch of $W^s(\tau; l_s)$ between the origin and $\boldsymbol{Q^s}(\tau)$, and by $\bar E^s(\tau)$ the bounded set enclosed by $\bar W^u(\tau; l_u)$ and the y_2 axis.

Now we give a Lemma, consequence of Propositions 2.11 and 2.12, which allows to extend picture 1 to the non-autonomous setting.

2.13. LEMMA. *Assume* $\boldsymbol{A^+}, \boldsymbol{Gu}, \boldsymbol{Gs}$ *with* $2_* < l_u < 2^*$ *and* $2_* < l_s \le 2^*$. *Then for any* $\tau \in \mathbb{R}$, $W^s(\tau; l_u) \subset \bar{E}^u(\tau)$; *assume further* $l_s < 2^*$, *then* $W^s(\tau; l_u)$ *is a path joining the origin and* $\boldsymbol{y^s}(\tau; l_u)$.
Assume $\boldsymbol{A^-}, \boldsymbol{Gu}, \boldsymbol{Gs}$ *with* $l_u \ge 2^*$ *and* $l_s > 2^*$. *Then for any* $\tau \in \mathbb{R}$, $W^u(\tau; l_s) \subset \bar{E}^s(\tau)$; *assume further* $l_u > 2^*$, *then* $W^u(\tau; l_s)$ *is a path joining the origin and* $\boldsymbol{y^u}(\tau; l_s)$.

PROOF. Assume $\boldsymbol{A^+}, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $2_* < l_u < 2^*$ and $2_* < l_s \le 2^*$. In this case $H(\boldsymbol{Q}, \tau; l_s) < 0 < H(\boldsymbol{R}, \tau; l_s)$ for any $\boldsymbol{Q} \in W^u(\tau; l_s)$, $\boldsymbol{R} \in W^s(\tau; l_s)$, so we conclude the first part of the proof using again Remark 2.7. The second part is analogous. □

Lemma 2.13 is useful to construct a new family of sub and super-solutions.

We emphasize that the sets $\bar{E}^u(\tau)$, $\bar{E}^s(\tau)$ have the following property: let $\boldsymbol{Q} \in \bar{E}^u(\tau)$, $\boldsymbol{R} \in \bar{E}^s(\tau)$, then $\boldsymbol{y}(s, \tau, \boldsymbol{Q}; l_u) \in \bar{E}^u(t)$ for any $s \le \tau$, and $\boldsymbol{y}(s, \tau, \boldsymbol{R}; l_s) \in \bar{E}^s(t)$ for any $s \ge \tau$.

When $l_u = l_s = 2^*$ we have a slightly different situation. Denote by $\boldsymbol{P^*}(\tau) = (P_1^*(\tau), P_2^*(\tau))$ the critical point of the autonomous system (2.2) where $l = 2^*$ and $g(y_1, s; 2^*) \equiv g(y_1, \tau; 2^*)$. Let $P_1^* := \inf\{P_1^*(\tau) \mid \tau \in \mathbb{R}\}$; in this framework we have $P_1^* > 0$ and we set $\bar{P}^* = P_1^*/2$. We denote by $\boldsymbol{\bar{P}^*} = (\bar{P}^*, -m(2^*)\bar{P}^*)$.

2.14. LEMMA. *Assume* $\boldsymbol{Gu}, \boldsymbol{Gs}$ *with* $l_u = l_s = 2^*$. *Assume further* $\boldsymbol{A^+}$, *then for any* $\tau \in \mathbb{R}$ *the line* $y_1 = \bar{P}^*$ *intersects the manifold* $W^u(\tau)$ *in two points, say* $\boldsymbol{Q^{u,+}}(\tau) = (\bar{P}^*, Q_2^{u,+}(\tau))$ *and* $\boldsymbol{Q^{u,-}}(\tau) = (\bar{P}^*, Q_2^{u,-}(\tau))$, *and it intersects* $W^s(\tau)$ *in one point, denoted by* $\boldsymbol{Q^{s,-}}(\tau) = (\bar{P}^*, Q_2^{s,-}(\tau))$, *where* $Q_2^{u,-}(\tau) < Q_2^{s,-}(\tau) < -m(2^*)\bar{P}^* < Q_2^{u,+}(\tau)$. *Moreover, if* $\boldsymbol{y}(s)$ *corresponds to a SGS with slow decay, there is* $\boldsymbol{Q} = (Q_1, Q_2) \in W^u(\tau)$ *such that* $Q_1 = y_1(\tau)$ *and* $Q_2 > y_2(\tau)$ *for any* $\tau \in \mathbb{R}$.

Now, assume $\boldsymbol{A^-}$, *then for any* $\tau \in \mathbb{R}$ *the line* $y_1 = \bar{P}^*$ *intersect the manifold* $W^s(\tau)$ *in* $\boldsymbol{Q^{s,\pm}}(\tau) = (\bar{P}^*, Q_2^{s,\pm}(\tau))$, *and* $W^u(\tau)$ *in* $\boldsymbol{Q^{u,+}}(\tau) = (\bar{P}^*, Q_2^{u,+}(\tau))$ *and* $Q_2^{s,-}(\tau) < Q_2^{u,-}(\tau) < -m(2^*)\bar{P}^* < Q_2^{s,+}(\tau)$. *Moreover if* $\boldsymbol{y}(s)$ *corresponds to a SGS with slow decay, there is* $\boldsymbol{Q} = (Q_1, Q_2) \in W^s(\tau)$ *such that* $Q_1 = y_1(\tau)$ *and* $Q_2 < y_2(\tau)$ *for any* $\tau \in \mathbb{R}$.

PROOF. We recall that $W^u(\tau; 2^*)$ and $W^s(\tau; 2^*)$ depend smoothly on τ, and that they become the graph of a homoclinic trajectory as $\tau \to -\infty$ and as $\tau \to +\infty$ respectively. Denote by

$$S(\tau) := \{(y_1, y_2) \mid H(y_1, y_2, \tau; 2^*) = 0\,,\ y_1 > 0\}$$

and by $\boldsymbol{H}^+(\tau) = (\bar{P}^*, H^+(\tau))$, and $\boldsymbol{H}^-(\tau) = (\bar{P}^*, H^-(\tau))$ the intersection of $S(\tau)$ with the line $y_1 = \bar{P}^*$, where $H^-(\tau) < H^+(\tau)$.

From an analysis of the phase portrait relying on Wazewski's principle it follows that $W^u(\tau; 2^*)$ (respectively $W^s(\tau; 2^*)$) intersects the line $y_1 = \bar{P}^*$ for any $\tau \in \mathbb{R}$, see [**6**] for a proof in the p-Laplace context. Follow $W^u(\tau; 2^*)$ and $W^s(\tau; 2^*)$ from the origin towards $y_1 > 0$: we denote by $\boldsymbol{Q^{u,+}}(\tau)$ the first intersection of $W^u(\tau; 2^*)$ (resp. of $W^s(\tau; 2^*)$) with the line $y_1 = \bar{P}^*$, and by $\boldsymbol{Q^{s,-}}(\tau)$ the first intersection of $W^s(\tau; 2^*)$ with $y_1 = \bar{P}^*$. Using transversal smoothness of the manifold $W^u(\tau; 2^*)$ and $W^s(\tau; 2^*)$, see subsection 2.1, we see that we have at least a further intersection

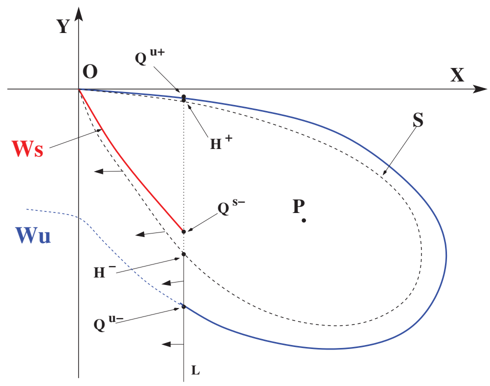

FIGURE 2. Sketch of the proof of Lemma 2.13, when $\boldsymbol{A}^{+}$ holds.

with such a line, respectively for $\tau \ll 0$ and for $\tau \gg 0$. We denote by $\boldsymbol{Q^{u,-}}(\tau_u)$ the second intersection of $W^u(\tau_u; 2^*)$ with the line $y_1 = \bar{P}^*$ and by $\boldsymbol{Q^{s,+}}(\tau_s)$ the second intersection of $W^s(\tau_s; 2^*)$ with the line $y_1 = \bar{P}^*$, for any $\tau_u \leq -N$ and $\tau_s \geq N$ and $N > 0$ large enough. Set $\boldsymbol{Q^{u,\pm}}(\tau_u) = (\bar{P}^*, Q_2^{u,\pm}(\tau_u))$ and $\boldsymbol{Q^{s,\pm}}(\tau_s) = (\bar{P}^*, Q_2^{s,\pm}(\tau_s))$: possibly choosing a larger N we can assume w.l.o.g. that $Q_2^{u,+}(\tau_u) > -m(2^*)\bar{P}^* > Q_2^{u,-}(\tau_u)$ and $Q_2^{s,+}(\tau_u) > -m(2^*)\bar{P}^* > Q_2^{s,-}(\tau_s)$. We denote by $\bar{W}^u(\tau_u)$ the branch of $W^u(\tau_u; 2^*)$ between the origin and $\boldsymbol{Q^{u,-}}(\tau_u)$ and by $\bar{W}^s(\tau_s)$ the branch of $W^s(\tau_s; 2^*)$ between the origin and $\boldsymbol{Q^{s,+}}(\tau_s)$.

Assume $\boldsymbol{A}^{+}$; then $W^u(\tau; 2^*)$ lies in the exterior of the bounded set enclosed by $S(\tau)$ for any τ: we claim that $\boldsymbol{Q^{u,-}}(\tau)$ exists for any $\tau \in \mathbb{R}$. In fact consider the semi-line $L(\tau) = \{(P^*, y_2) \mid y_2 < H^-(\tau)\}$; the flow of (2.2) on $L(\tau)$ points towards $y_1 < \bar{P}^*$ for any $\tau \in \mathbb{R}$. Hence the trajectory $\boldsymbol{y}(s, -N, \boldsymbol{Q^{u,-}}(-N); 2^*)$ crosses the line $y_1 = P^*$ for $s = -N$ and then the $y_1 < 0$ semi-plane, and similarly for any $\boldsymbol{Q} \in \bar{W}^u(\tau)$ the trajectory $\boldsymbol{y}(s, -N, \boldsymbol{Q}; 2^*)$ will cross the line $y_1 = P^*$ for a certain $s > -N$ and then the $y_1 < 0$ semi-plane. Hence, for any $\tau \geq -N$, the branch of the manifold $W^u(\tau; 2^*)$ between the origin and $\boldsymbol{y}(\tau, -N, \boldsymbol{Q^{u,-}}(-N); 2^*)$ will surround $S(\tau)$ untill it crosses a second time the line $y_1 = P^*$ and the claim is proved, so we get picture 2.

Now denote by $D^u(\tau)$ the bounded set enclosed by $\bar{W}^u(\tau)$, the segment between $\boldsymbol{Q^{u,-}}(\tau)$ and $\boldsymbol{H}^{-}(\tau)$, the branch of $S(\tau)$ between $\boldsymbol{H}^{-}(\tau)$ and the origin: observe that by construction if $\boldsymbol{Q} \in D^u(\tau)$, then $\boldsymbol{y}(t, \tau, \boldsymbol{Q}; 2^*) \in D^u(t)$ for any $t \leq \tau$. Since $S(\tau) \subset D^u(\tau)$ we see that if $\boldsymbol{y}(t)$ corresponds to a SGS with slow decay, then $\boldsymbol{y}(t) \in S(t)$ for any $t \in \mathbb{R}$.
Reasoning in the same way but reversing the direction of t we see that if $\boldsymbol{A}^{-}$ holds, we can construct $\bar{W}^s(\tau)$ for any $\tau \in \mathbb{R}$. Denote by $D^s(\tau)$ the bounded set enclosed by $\bar{W}^s(\tau)$, the segment between $\boldsymbol{Q^{s,+}}(\tau)$ and $\boldsymbol{H}^{+}(\tau)$, the branch of $S(\tau)$ between $\boldsymbol{H}^{+}(\tau)$ and the origin. If $\boldsymbol{y}(t)$ corresponds to a singular solution, then $\boldsymbol{y}(t) \in S(t)$ for any $t \in \mathbb{R}$. So Lemma 2.13 follows. □

Now we give a Lemma which is useful to detect the ω-limit set of solutions of (1.1)–(1.2) when ϕ is a radial upper or lower solution of (1.8).

2.15. LEMMA. *Let $U(r)$ and $V(r)$ be positive solutions of (1.8), either regular or singular, and assume that there is $Z > 0$ such that $U(Z) = V(Z)$ and $U'(Z) <$*

$V'(Z)$. Denote by

$$\zeta(r)=\begin{cases} V(r) & \text{if } r\le Z \\ U(r) & \text{if } r\ge Z \end{cases}, \qquad \psi(r)=\begin{cases} U(r) & \text{if } r\le Z \\ V(r) & \text{if } r\ge Z \end{cases} \tag{2.19}$$

Assume $\boldsymbol{A^-}, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $l_u \ge 2^*$ and $l_s \in [2^*, \sigma^*)$.

Then (1.8) admits no solutions $\phi(r)$ either regular or singular such that $0 < \phi(r) \le \zeta(r)$, and no solutions $\phi(r)$ such that $\phi(r) \ge \psi(r)$ for any $r > 0$.

Proof. From Proposition 2.12 we know that all the positive solutions have slow decay. Assume first $l_s \in [2^*, \sigma^*)$. Then from Proposition 2.10 all the slow decay solutions of (1.8) cross each other indefinitely as $r \to +\infty$, so the Lemma easily follows. □

Reasoning in the same way we get the following:

2.16. Lemma. *Let $U(r)$, $V(r)$, $\zeta(r)$ and $\psi(r)$ be as in Lemma 2.15. Assume $\boldsymbol{A^+}, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $l_u \in (\sigma_*, 2^*]$ and $l_s \in (2_*; 2^*]$.*

Then (1.8) admits no solutions $\phi(r)$ either regular or singular such that $0 < \phi(r) \le \zeta(r)$, and no solutions $\phi(r)$ such that $\phi(r) \ge \psi(r)$ for any $r > 0$.

3. Local existence

In this section we introduce some basic facts and definitions related to the problem (1.1)–(1.2), and exploiting techniques similar to those used in [**27**, §1, §2] (see also [**22**, Ch. II]), we prove local existence for the solutions of problem (1.1)–(1.2). For the remainder of this section we will make the following assumptions on the function f in (1.1), in addition to **F0**:

- ***Fu***: ***Gu*** holds and there is $D > 0$ such that $\frac{\partial}{\partial y_1}[g(y_1, s; l_u)] \le D|y_1|^\delta$ for any $0 \le y_1 \le 1$, and $0 < s \le 1$.
- ***Fs***: There are $\ell \ge 0$, $\bar{C} > 0$, $\delta > 0$, $\varepsilon > 0$ such that $|f(u,r) - f(u+h,r)| \le \bar{C} h u^\delta r^\ell$ whenever $r \ge 1$, $0 \le u \le 1$, and $0 \le h \le \varepsilon$.

Assumption ***Fu*** is very close to ***Gu*** (and it is actually satisfied in all the motivating examples given in the introduction), while ***Fs*** is more standard and it is adapted from [**27**]. Let us introduce the following map

$$w(x)=\begin{cases} |x|^\nu & \text{if } |x|\le 1 \\ |x|^{\ell/\delta} & \text{if } |x|\ge 1 \end{cases} \quad \text{where } 0\le \nu < m(l_u). \tag{3.1}$$

We emphasize that if $\ell = 0$ then $w(x) \equiv 1$ for $|x| \ge 1$. Moreover if we set $\nu = 0$ then $w(x) \equiv 1$ for $|x| \le 1$, so we are dealing with bounded solutions, while if we set $\nu > 0$ we can deal with solutions which are unbounded for $|x|$ small, and are not defined for $x = 0$.

Let us recall the definitions of *continuous weak solution* and C_B-*mild solution* to the problem (1.1)–(1.2).

3.1. Definition. We say that a function u is a continuous weak (c.w.) solution of (1.1)–(1.2) if u is continuous, and it is a distributional solution: i.e. if $u(x,0) = \phi(x)$ and, for any $\eta \in C^{2,1}(\mathbb{R}^n \times [0,T])$ with $\eta \ge 0$ and $\operatorname{supp}\eta(\cdot,t) \Subset \mathbb{R}^n$ for all

$t \in [0,T]$, it holds true that

$$\int_{\mathbb{R}^n} u(x,s)\eta(x,s)dx|_0^{T_1} = \int_0^{T_1}\int_{\mathbb{R}^n}\Big[u(x,s)\big(\eta_t+\Delta\eta\big)(x,s)+f(u,|x|)\eta(x,s)\Big]dxds \tag{3.2}$$

if $T_1 \in [0,T]$. Further, u is a c.w. lower (respectively upper) solution of (1.1)–(1.2) if $u(x,0) \geq \phi(x)$ (resp. $u(x,0) \leq \phi(x)$) and we replace " $=$ " in (3.2) by " $\geq$ " (resp. by " $\leq$ "). We call a function u a classical solution if it satisfies (1.1)–(1.2) and $u \in C^{2,1}(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$.

Let $\phi \in C_B(\mathbb{R}^n) := C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We introduce the following operators

$$\begin{aligned} e^{t\Delta}\phi &:= (4\pi t)^{-n/2}\int_{\mathbb{R}^n}\exp\Big(-\frac{|x-y|^2}{4t}\Big)\phi(y)dy. \\ F_\phi(u) &= \Big(e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}f(u(\cdot,s),|\cdot|)ds\Big)(x) \end{aligned} \tag{3.3}$$

3.2. Definition. We say that u is a C_B-mild solution of (1.1)–(1.2) on $\mathbb{R}^n \times [0,T)$ if, for any $T' \in [0,T)$, we have

$$u \in C_B(\mathbb{R}^n \times [0,T']) := C(\mathbb{R}^n \times [0,T']) \cap L^\infty(\mathbb{R}^n \times [0,T']), \tag{3.4}$$

$$u(x,t) = F_\phi(u(x,t)), \text{ for } (x,t) \in \mathbb{R}^n \times [0,T). \tag{3.5}$$

Also, we say that u is a C_B-mild lower solution (or upper) solution if " $=$ " in (3.5) is replaced by " $\geq$ " (" $\leq$ " respectively).

Consider the norm $\|\phi\|_X := \|\phi w\|_{L^\infty(\mathbb{R}^n)}$, where w is defined in (3.1). We introduce the weighted space

$$X = L^\infty_w(\mathbb{R}^n) = \{\phi \mid \|\phi\|_X < +\infty\} \tag{3.6}$$

We denote by $C_S(\mathbb{R}^n \times [0,T']) := C((\mathbb{R}^n\setminus \{0\}) \times [0,T']) \cap L^\infty([0,T'],X)$ and we give the following definition.

3.3. Definition. We say that u is a C_S-mild solution to (1.1)–(1.2) on $(\mathbb{R}^n\setminus\{0\})\times[0,T)$ if $u \in C_S(\mathbb{R}^n \times [0,T'])$ for any $0 < T' < T$ and satisfies $u(x,t) = F_\phi(u(x,t))$ for $(x,t) \in (\mathbb{R}^n\setminus \{0\}) \times [0,T)$.

Note that if $\nu > 0$ and u is a C_S-mild solution then it may be unbounded as $x \to 0$. Therefore we can deal with initial data and solutions having a singularity in the origin, and we will prove local existence and uniqueness for such initial data. It is worth recalling that, with our assumptions, we have singular stationary solutions ϕ_S which behave like $|x|^{-m(l_u)}$ as $x \to 0$.

An interesting question, still open even for the starting case $f(u) = u^{q-1}$ (and not addressed in the present paper), is whether stationary SGS are stable or not. One of the difficulties is in fact to prove local existence and uniqueness for nearby initial data (which is in general violated, but maybe recovered in some special space and for some parameters, see [**25**]). In fact we cannot even hope for a general local uniqueness result, due to the presence of self-similar solutions converging to singular data (see, e.g. [**26, 22**]), and to a new class of solutions with moving singularity recently described in [**23, 24**]: We need to prescribe a class of functions in which local uniqueness may hold.

We stress that here we are forced to stay below ϕ_S since we need to require $\nu < m(l_u)$, so we cannot start a stability analysis for stationary singular solutions.

3.4. Lemma. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$. *Let* u *be a continuous weak upper (lower) solution of (1.1)–(1.2) with* $n \geq 3$. *Assume that there exist* $k, \beta > 0$, *and* $0 < \alpha < 2$ *such that* $f(u,x) < k\exp(\beta|x|^{\alpha})$ *on* $\mathbb{R}^n \times [0,T]$. *Then* $u(x,t) \geq (\leq)\ F_\phi(u(x,t))$ *on* $\mathbb{R}^n \times [0,T]$.

3.5. Remark. By this lemma it follows that a c.w. solution of (1.1)–(1.2) satisfying either (3.4) in Definition 3.2, or the analogous weighted condition in Definition 3.3, is also respectively either a C_B-mild solution or a C_S solution. The converse is also true (see the proof of [**27**, Lemma 1.5]).

To prove Lemma 3.4 it is sufficient to adapt the proof of [**27**, Lemma 1.5] to the present case. By Lemma 3.4 we also have the next result.

3.6. Proposition. Assume $\boldsymbol{Fu}$, $\boldsymbol{Fs}$. Suppose that u is a continuous weak upper (resp. lower) solution of (1.7) in $\mathbb{R}^n \backslash \{0\}$ such that $\|u\|_X$ is bounded, then u is a C_S-mild upper (resp. lower) solution of (1.1)–(1.2). The converse is also true, provided $\phi(x) \geq u(x,0)$ (resp. $\phi(x) \leq u(x,0)$). In particular if we set $\nu = 0$ we see that a continuous bounded weak upper (resp. lower) solution of (1.7) is a C_B-mild solution of (1.1)–(1.2) and viceversa.

Take $\rho > 0$ and $\phi \in X$, and denote by $B_\rho := B_\rho^T(0) \subseteq L^\infty([0,T];X)$ the ball of center 0 and radius ρ, in X; the radius ρ will be chosen properly later. We now prove local existence and uniqueness for C_B and C_S-mild solutions.

3.7. Lemma. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$. *If the initial datum* $\phi \in X$, *there is* $T_\phi > 0$ *such that the operator* $F_\phi(u)$ *defined by (3.3) has a unique fixed point in* $B_\rho^T(\phi)$ *for any* $0 < T < T_\phi$, *and if* $T_\phi < +\infty$, *then* $\lim_{t \to T_\phi^-} \|u(\cdot,t)\|_X = +\infty$.

Proof. We claim that the operator F_ϕ maps B_ρ in itself and it is a contraction: Then the Banach fixed point theorem provides existence and uniqueness of a fixed point u for F_ϕ.

Observe first that if $u \in B_\rho$, then $|u(x,t;\phi)| \leq \rho w(x)^{-1}$ a.e. in $\mathbb{R}^n \times [0,T]$. Here, we take $\rho = 2(2D_1 + 2^{\nu+1} + 2^{\ell/\delta})\|\phi\|_X$, where $D_1 := \mathrm{e}^{-\nu/2}(16\nu)^{\nu/2}$.

From $\boldsymbol{Fu}$, for $|x| \leq 1$, we get

$$
\begin{aligned}
f(u,|x|) \leq f(\rho\omega^{-1}(x),|x|) &\leq g(\rho|x|^{m(l_u)-\nu}, \ln(|x|); l_u)|x|^{-(2+m(l_u))} \\
&\leq \bar{k}^-|x|^{\delta(m(l_u)-\nu)-2-\nu},
\end{aligned}
\tag{3.7}
$$

where $\bar{k}^- = D\rho^{1+\delta}$. Then, for any $v \in B_\rho$, we also have

$$
\begin{aligned}
|f(u,|x|) - f(v,|x|)| &= \left|\int_0^1 \frac{\partial f}{\partial u}(su + (1-s)v), |x|)[u-v]ds\right| \\
&\leq \frac{\partial g}{\partial y_1}\big(\rho|x|^{m(l_u)-\nu}, \ln(|x|); l_u\big)|x|^{-(2+m(l_u))}|(u-v)(x)|\,|x|^{m(l_u)} \\
&\leq \frac{\partial g}{\partial y_1}\big(\rho|x|^{m(l_u)-\nu}, \ln(|x|); l_u\big)|x|^{-(2+m(l_u))}\|(u-v)(x)|\,|x|^{\nu}\|_\infty \\
&\leq k^-|x|^{\delta(m(l_u)-\nu)-2-\nu}\|u-v\|_X,
\end{aligned}
\tag{3.8}
$$

where $k^- = D\max\{\rho^{1+\delta},\rho^{\delta}\}$. From $\boldsymbol{Fs}$, for $|x|\geq 1$, we also get

$$\begin{aligned} f(u,|x|) &= f(u,|x|) - f(0,|x|) \leq \bar{C}|u|^{1+\delta}|x|^{\ell} \\ &\leq \bar{C}\rho^{1+\delta}|x|^{-\ell/\delta} \leq k^+ w(x)^{-1}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} |f(u,|x|) - f(v,|x|)| &\leq |u|^{\delta}|v-u||x|^{\ell} \leq \bar{C}\frac{\rho^{\delta}}{|x|^{\ell}}\|u-v\|_{\infty}|x|^{\ell} \\ &\leq k^+ w(x)^{-1}\|u-v\|_X, \end{aligned} \tag{3.10}$$

with $k^+ = \bar{C}\max\{\rho^{1+\delta},\rho^{\delta}\}$. Till the end of the proof we need the following straightforward estimate: Let $A\in(0,n)$ and denote by Γ the Euler Gamma function, then:

$$\int_{\mathbb{R}^n} \frac{\mathrm{e}^{-\frac{|\eta|^2}{4}}}{(4\pi)^{n/2}|\eta|^A} d\eta \leq \frac{1}{2^{n-1}(n-A)\Gamma(n/2)} + \int_{|\eta|\geq 1} \mathrm{e}^{-\frac{|\eta|^2}{4}} d\eta \leq C(A)\,, \tag{3.11}$$

and we can set $C(A)=2$ if $A\in(0,n-1)$. We now proceed to prove that $F_\phi\colon B_\rho \to B_\rho$. From (3.3) we have that

$$\begin{aligned} |F_\phi(u)|(x,t) &\leq |e^{t\Delta}\phi|(x) + \int_0^t \int_{\mathbb{R}^n} \frac{\exp\Big(-\frac{|x-y|^2}{4(t-s)}\Big)}{\big(4\pi(t-s)\big)^{n/2}} f(u(y,s),|y|)dyds \\ &\leq w^{-1}(x)\|e^{t\Delta}\phi\|_X(t) + w^{-1}(x)I, \end{aligned} \tag{3.12}$$

where

$$I = w(x)\int_0^t \Big(\int_{|y|\leq 1} + \int_{|y|\geq 1}\Big) \frac{\exp\Big(-\frac{|x-y|^2}{4s}\Big)}{(4\pi s)^{n/2}} f\big(u,|y|\big)dyds =: I_a + I_b.$$

Using (3.7)-(3.8) we get the following

$$\begin{aligned} I_a &\leq k^- \int_0^t \Big(\int_{\frac{|x|}{2}\leq|y|\leq 1} + \int_{|y|\leq\frac{|x|}{2}}\Big) \frac{w(x)\exp\Big(-\frac{|x-y|^2}{4s}\Big)}{(4\pi s)^{n/2}|y|^{2-\delta(m(l_u)-\nu)+\nu}} dyds \\ &=: k^-(I_a^- + I_a^+) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I_a^- &\leq 2^{\nu}\int_0^t \int_{\frac{|x|}{2}\leq|y|\leq 1} \frac{\exp\Big(-\frac{|x-y|^2}{4s}\Big)}{(4\pi s)^{n/2}|y|^{2-\delta(m(l_u)-\nu)}} dyds \\ &\leq 2^{\nu}\int_0^t \int_{\mathbb{R}^n} \frac{\exp\Big(-\frac{|y|^2}{4s}\Big)}{(4\pi s)^{n/2}|y|^{2-\delta(m(l_u)-\nu)}} dyds \\ &\leq 2^{\nu}\int_0^t \int_{\mathbb{R}^n} \frac{\exp(-|\eta|^2)|\eta|^{\delta(m(l_u)-\nu)-2}}{(\pi)^{n/2}s^{1-\delta(m(l_u)-\nu)/2}} d\eta ds \leq K^- t^{\delta(m(l_u)-\nu)/2}, \end{aligned}$$

where $K^->0$ is a positive constant, and we used the fact that the convolution of radial decreasing function is radial decreasing too (see [**27**, Lemma 1.4]), and that $n-3+\delta(m(l_u)-\nu) > -1$.

Since $|x-y| \geq \big||x|-|y|\big|$ we get

$$
\begin{aligned}
I_a^+ &\leq \int_0^t \int_{|y|\leq \frac{|x|}{2}} \frac{w(x)\exp\Big(-\frac{|x|^2}{16s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{2+\nu-\delta(m(l_u)-\nu)}}dyds \\
&\leq \int_0^t \int_{|y|\leq \frac{|x|}{2}} \frac{w(x)\exp\Big(-\frac{|x|^2}{32s}\Big)\exp\Big(-\frac{|y|^2}{8s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{2+\nu-\delta(m(l_u)-\nu)}}dyds \\
&\leq \int_0^t \frac{w(x)\exp\Big(-\frac{|x|^2}{32s}\Big)}{s^{1-[\delta(m(l_u)-\nu)/2]+\nu/2}}ds \int_{\mathbb{R}^n} \frac{\exp\Big(-\frac{|\eta|^2}{8}\Big)}{\big(4\pi\big)^{n/2}|\eta|^{2+\nu-\delta(m(l_u)-\nu)}}d\eta \\
&\leq 2^{n/2}C\int_0^t \frac{w(x)}{s^{1-\delta m(l_u)/2+\nu/2}}\exp\Big(-\frac{|x|^2}{32s}\Big)ds
\end{aligned}
$$

where we used that $2+\nu-\delta(m(l_u)-\nu) < 2+m(l_u) < n$, and $C = C(2+m(l_u))$ is the constant defined in (3.11). Now we need to distinguish between the $|x| \leq 1$ and the $|x| \geq 1$ case. Assume the former so that $w(x) = |x|^\nu$, and observe that $h(a) := \mathrm{e}^{-a/32}a^{\nu/2} \leq D_1 = h(16\nu)$; for any $a \geq 0$ we get

$$
I_a^+ \leq D_1 C\int_0^t \frac{1}{s^{1-[\delta(m(l_u)-\nu)/2]}}ds \leq K^+ t^{[\delta(m(l_u)-\nu)/2]}. \tag{3.14}
$$

where $K^+ = \frac{D_1 C}{\delta(m(l_u)-\nu)/2} > 0$ is a constant. When $|x| \geq 1$, so that $w(x) = |x|^{\ell/\delta}$, setting $\bar{h}(a) := \mathrm{e}^{-a/64}a^{\ell/2\delta}$ we find $\bar{h}(a) \leq D_2 := \bar{h}(32\ell/\delta)$ for any $a \geq 0$; similarly $\tilde{h}(a) := \mathrm{e}^{-a/64}a^{1-[\delta(m(l_u)-\nu-1/\ell)/2]+\nu/2}$ is bounded, say $\tilde{h}(a) \leq D_3$. Thus

$$
\begin{aligned}
I_a^+ &\leq C\int_0^t \left[\Big(\frac{|x|^2}{s}\Big)^{\frac{\ell}{2\delta}}\exp\Big(-\frac{|x|^2}{64s}\Big)\right]\frac{\exp\Big(-\frac{1}{64s}\Big)}{s^{1-[\delta(m(l_u)-\nu-1/\ell)/2]+\nu/2}}ds \\
&\leq CD_2\int_0^t \bar{h}(\frac{|x|^2}{s})\tilde{h}(\frac{1}{s})ds \leq CD_2D_3 t \leq K^+ t
\end{aligned} \tag{3.15}
$$

with a possibly larger constant K^+. Now we estimate I_b. From (3.9)–(3.10) we get:

$$
\begin{aligned}
I_b &\leq k^+ w(x)\int_0^t \Big(\int_{|y|\geq \max\{\frac{|x|}{2},1\}} + \int_{1\leq|y|\leq\frac{|x|}{2}}\Big)\frac{\exp\Big(-\frac{|x-y|^2}{4s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{\ell/\delta}}dyds \\
&=: k^+(I_b^- + I_b^+)\,.
\end{aligned} \tag{3.16}
$$

Observing that $\hat{h}(a) = a^{\ell/(2\delta)}\mathrm{e}^{-a/32} \leq \hat{h}(8\ell/\delta)$ and $C_b = 2^{3n/2}\hat{h}(8\ell/\delta)$, we find

$$
I_b^- \leq w(x)\int_0^t \int_{|y|\geq\max\{\frac{|x|}{2},1\}} \frac{\exp\Big(-\frac{|x-y|^2}{4s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{\ell/\delta}}dyds \leq \int_0^t \frac{w(x)}{w(\max\{\frac{|x|}{2},1\})} \leq 2^{\ell/\delta}t,
$$

$$
I_b^+ \leq \int_0^t w(x)\mathrm{e}^{-\frac{|x|^2}{32s}}ds\int_{1\leq|y|\leq\frac{|x|}{2}} \frac{\mathrm{e}^{-\frac{|x|^2}{32s}}}{\big(4\pi s\big)^{n/2}}dy \leq 2^{3n/2}|x|^{\ell/\delta}t\mathrm{e}^{-\frac{|x|^2}{32t}} \leq C_b t^{1+\frac{\ell}{2\delta}},
$$

Therefore, there is $K > 0$ such that $I \leq K\max\{t^{1+\ell/(2\delta)}, t, t^{\delta(m(l_u)-\nu)/2}\}$, with $K = K(n,\ell,\nu,\|\phi\|_X)$ and (3.12) reduces to

$$
|F_\phi(u)|(x,t) \leq w^{-1}(x)\|e^{t\Delta}\phi\|_X + w^{-1}(x)K\max\{T^{1+\ell/(2\delta)}, T, T^{\delta(m(l_u)-\nu)/2}\}. \tag{3.17}
$$

To estimate the term $w^{-1}(x)\|e^{\Delta t}\phi\|_X$ we follow an approach similar to the one used above. Indeed, we rewrite $e^{t\Delta}\phi(x)$ as follows

$$e^{t\Delta}\phi(x) = \left(\int_{|y|\le 1} + \int_{|y|\ge 1}\right)\frac{\exp\left(-\frac{|x-y|^2}{4t}\right)}{(4\pi t)^{n/2}}\phi(y)dy =: I_\alpha + I_\beta. \tag{3.18}$$

Hence we get

$$I_\alpha \le \|\phi\|_X\left(\int_{\frac{|x|}{2}\le|y|\le 1} + \int_{|y|\le \min\{\frac{|x|}{2},1\}}\right)\frac{\exp\left(-\frac{|x-y|^2}{4t}\right)}{(4\pi t)^{n/2}w(y)}dy =: I_\alpha^- + I_\alpha^+.$$

For $\frac{|x|}{2} \le |y| \le 1$ we reach

$$\begin{aligned} I_\alpha^- &\le \frac{\|\phi\|_X 2^\nu}{w(x)}\int_{\frac{|x|}{2}\le|y|\le 1}\frac{\exp\left(-\frac{|x-y|^2}{4t}\right)}{(4\pi t)^{n/2}}dy \\ &\le \frac{\|\phi\|_X}{w(x)}2^\nu\int_{\mathbb{R}^n}\frac{\exp\left(-\frac{|y|^2}{4t}\right)}{(4\pi t)^{n/2}}dy = \frac{\|\phi\|_X 2^\nu}{w(x)}. \end{aligned} \tag{3.19}$$

For the term I_α^+, using (3.11), we obtain

$$\begin{aligned} I_\alpha^+ &\le \|\phi\|_X \exp\left(\frac{-|x|^2}{32t}\right)\int_{|y|\le\min\{\frac{|x|}{2},1\}}\frac{\exp\left(-\frac{|y|^2}{8t}\right)}{(4\pi t)^{n/2}|y|^\nu}dy \\ &\le \|\phi\|_X \exp\left(\frac{-|x|^2}{32t}\right)\int_{\mathbb{R}^n}\frac{\exp\left(-\frac{|\eta|^2}{4}\right)}{(4\pi)^{n/2}|\eta|^\nu t^{\nu/2}}d\eta \le 2\|\phi\|_X\frac{\exp\left(\frac{-|x|^2}{32t}\right)}{t^{\nu/2}}. \end{aligned}$$

Now, arguing as in (3.14), (3.15), for $|x|\le 1$ we find

$$I_\alpha^+ \le 2\|\phi\|_X\frac{\left(\frac{|x|^2}{t}\right)^{\nu/2}\exp\left(\frac{-|x|^2}{32t}\right)}{|x|^\nu} \le \frac{2D_1\|\phi\|_X}{w(x)} \tag{3.20}$$

while for $|x|\ge 1$, since $\bar{h}(a) = a^{\ell/(2\delta)}\mathrm{e}^{-a/64} \le D_2$, we get

$$I_\alpha^+ \le 2\|\phi\|_X\frac{\left(\frac{|x|^2}{t}\right)^{\ell/(2\delta)}\exp\left(\frac{-|x|^2}{64t}\right)}{w(x)}\,\frac{\exp\left(\frac{-|x|^2}{64t}\right)}{t^{\nu/2-\ell/(2\delta)}} \le \varepsilon(t)\,\frac{2D_2\|\phi\|_X}{w(x)}$$

where $\varepsilon(t) := \exp\left(\frac{-1}{64t}\right)t^{\ell/(2\delta)-nu/2} \to 0$ as $t\to 0$. So choosing t small enough we can assume that (3.20) holds for $|x|\ge 1$, too. Take into account I_β, to get

$$\begin{aligned} I_\beta &\le \frac{\|\phi\|_X}{w(x/2)}\int_{|y|\ge\frac{|x|}{2}}\frac{\exp\left(-\frac{|x-y|^2}{4t}\right)}{(4\pi t)^{n/2}} \\ &\le \frac{\|\phi\|_X}{w(x/2)}\|\phi\|_X\int_{\mathbb{R}^n}\frac{\exp\left(-\frac{|\eta|^2}{4}\right)}{(4\pi)^{n/2}}d\eta \\ &\le \frac{\|\phi\|_X}{w(x/2)} \le \frac{\|\phi\|_X}{w(x)}\max\{2^\nu, 2^{\ell/\delta}\}\,. \end{aligned} \tag{3.21}$$

Collecting the estimates (3.18)-(3.19)-(3.20)-(3.21), we have that relation (3.17); hence (3.12), for $T \le 1$ gives

$$\begin{aligned}\|F_\phi(u)\|_X(T) &\le \big(2D_1 + 2^{\nu+1} + 2^{\ell/\delta}\big)\|\phi\|_X + K\max\{T, T^{\delta(m(l_u)-\nu)/2}\}\\ &\le \rho/2 + CT\|\phi\|_X + K\max\{T, T^{\delta(m(l_u)-\nu)/2}\}.\end{aligned}$$

Let $T_0 = T_0(n, \ell, \|\phi\|_X, \nu, \rho) > 0$ be such that, for any $T \le T_0$,

$$K\max\{T, T^{\delta(m(l_u)-\nu)/2}\} < \rho/2.$$

Then, we have that

$$\|F_\phi(u)\|_{L^\infty(0,T;X)} \le \rho, \text{ for any } T \le T_0\,,$$

hence F_ϕ maps B_ρ into B_ρ, for $T \le T_0$.

Analogously, let $u, v \in B_\rho$, we get

$$\begin{aligned}|F_\phi(u) - F_\phi(v)|(x,t) \le \int_0^t \Big(\int_{|y|\le 1} \frac{k^- \exp\Big(-\frac{|x-y|^2}{4s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{2+\nu-\delta(m(l_u)-\nu)}}dy +\\ + \int_{|y|\ge 1}\frac{k^+ \exp\Big(-\frac{|x-y|^2}{4s}\Big)}{\big(4\pi s\big)^{n/2}|y|^{\ell/\sigma}}dy\Big)\|u-v\|_X ds.\end{aligned} \tag{3.22}$$

Repeating the argument of (3.13) and (3.14) we get

$$\|[F_\phi(u) - F_\phi(v)]\|_X(t) \le k^-[K^- t^{\delta(m(l_u)-\nu)/2} + K^+ t] + k^+[2^{\ell/\delta} + C_b t^{\ell/(2\delta)}]t. \tag{3.23}$$

Therefore, taking $T < T_0$ sufficiently small, it follows that F_ϕ maps B_ρ into B_ρ and it is actually a contraction. From the contraction principle, we obtain existence and uniqueness of a fixed point u in B_ρ, which in turn implies the existence and local uniqueness of a C_B-mild solution to (1.1)–(1.2). Then we can restart the reasoning by setting $\phi(x) = u(x,T)$, and go up to T_ϕ by a ladder argument. Note that if $T_\phi < \infty$ and $\lim_{t\to T_\phi}\|u(x,t)\|_X$ is bounded, we can restart the ladder argument and obtain a continuation interval $[0,T'] \supset [0,T_\phi)$ and this is a contradiction. Hence if $T_\phi < \infty$ we get $\lim_{t\to T_\phi}\|u(x,t)\|_X = +\infty$. □

As a direct consequence of Lemma 3.7 we have the following existence result.

3.8. Theorem. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$. *Let* $\phi \in X$ *be the initial datum for Equation* (1.1). *Then problem* (1.1)–(1.2) *has a unique weak solution* u *on* $\mathbb{R}^n \times [0, T_\phi)$, $T_\phi > 0$. *If* $\phi \in C_B(\mathbb{R}^n) = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *then* $u \in C\big(\mathbb{R}^n \times [0,T_\phi)\big) \cap L^\infty\big(\mathbb{R}^n \times [0,T_\phi)\big)$, *and if* $T_\phi < +\infty$ *then* $\lim_{t\to T_\phi^-}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = +\infty$. *Similarly, if* $\phi \in C_S(\mathbb{R}^n) = C(\mathbb{R}^n\backslash\{0\}) \cap L^\infty_w(\mathbb{R}^n)$, *then* $u \in C\big((\mathbb{R}^n\backslash\{0\}) \times [0,T_\phi)\big) \cap L^\infty([0,T_\phi), L^\infty_w(\mathbb{R}^n))$, *and if* $T_\phi < +\infty$, *then* $\lim_{t\to T_\phi^-}\|u(\cdot,t)w(\cdot)\|_{L^\infty(\mathbb{R}^n)} = +\infty$.

Furthermore, if $\phi \ge 0$, *then* $u \ge 0$; *if* ϕ *is radial, then* u *is radial in* x; *if* ϕ *is radial and radially non-increasing, then* u *is non-increasing in* $|x|$.

3.9. Remark. Assume that f is locally Hölder continuous in x, and that there exists $l \ge 0$ such that $f(u,r)$ is locally Lipschitz in u, uniformly with respect to r in any bounded subset of $r > 0$ (as in the examples (1.3)–(1.5)). Then, using [**27**, Lemma 1.2] and arguing as in [**27**], one can verify that the C_B-solutions are actually classical.

As a consequence of Proposition 3.6 and Lemma 3.7 we have the following.

3.10. Theorem. *Assume that* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$ *are verified. Then*

- **(i)** *Suppose that* $\bar{u}$ *and* $\underline{u}$ *are* C_S*-mild upper and lower solutions of* (1.1)-(1.2) *on* $\mathbb{R}^n \times [0,T)$. *Then* $\bar{u} > \underline{u}$ *on* $\mathbb{R}^n \times [0,T)$, *and the unique* C_S*-mild solution of* (1.1)-(1.2) *on* $\mathbb{R}^n \times [0,T_\phi)$ *satisfies* $\underline{u} \le u \le \bar{u}$ *on* $\mathbb{R}^n \times [0,T)$ *and* $T_\phi > T$.
- **(ii)** *If the initial value* ϕ *in* (1.2) *is a c.w. upper (lower) solution of (1.7), then the* C_S*-mild solution* u *of* (1.1)-(1.2) *is non-increasing (non-decreasing) in* $t \in [0,T_\phi)$.
- **(iii)** *If* ϕ *is radial, then* $\bar{u}$, $\underline{u}$ *and* u *are radial for any* t *in their dominion of definition.*
- **(iv)** *If* ϕ *is a c.w. upper (lower) solution but not a solution of (1.7), then* $u_t(x,t) < (>) 0$, $t > 0$.

The proof of this theorem is omitted because it can be easily derived by adapting the one of [**27**, Theorem 2.4] to the current case. We point out that claim **(iv)** follows directly by exploiting a comparison principle, arguing as in [**27**, Lemma 2.6] (see, e.g., [**11**] for a full-fledged proof of this well-established comparison argument. See also [**14**]). Further, a result analogous to Theorem 3.10 holds true also in the even simpler case of the C_B-mild solutions of (1.1)–(1.2) on $\mathbb{R}^n \times [0,T)$ (see [**27**, Theorem 2.4]).

To conclude this section we give a result about the global solution $u(x,t;\phi)$ of the problem (1.1)–(1.2), for $\phi \in C_B(\mathbb{R}^n)$ or $\phi \in C_S(\mathbb{R}^n)$.

3.11. Remark. Assume that ϕ is a singular upper (respectively lower) solution. From Theorem 3.10 point **(iv)**, which translates [**27**, Lemma 2.6] to the current case, it follows that $\lim_{t\to T_\phi} u(x,t;\phi) = u(x,T_\phi;\phi)$ exists for any $x \neq 0$. Following the proof of Claim 2 of [**27**, Theorem 3.6], using Lebesgue dominated convergence theorem and regularity theory for elliptic equation, we see that $u(x,T_\phi;\phi)$ is a distributional solution of (1.7). Moreover if ϕ is radial then $u(x,T_\phi;\phi)$ is radial too.

4. Long time behavior: main results.

Now we are ready to state and prove our results in their general form, from which Theorems 1.5, 1.6, and Corollary 1.8 follow.

Let $w(x)$ be defined as in (3.1).

4.1. Theorem. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$, $\boldsymbol{A^-}$, $\boldsymbol{Gu}$ *and* $\boldsymbol{Gs}$, *with* $2^* \le l_s < \sigma^*$ *and* $l_u \ge 2^*$.

- **(i)** *If* $\phi(x) \lneqq U(|x|,\alpha)$ *for some* $\alpha > 0$, *then it holds that* $\|u(x,t;\phi)\|_\infty \to 0$ *and* $\|u(x,t;\phi)(1+|x|^\nu)\|_\infty \to 0$ *as* $t \to +\infty$ *for any* $0 \le \nu < m(l_s)$.
- **(ii)** *Let* ϕ *be a continuous initial datum or a singular one such that* $\|\phi(x)w(x)\|_\infty$ *is bounded, for some* $\nu \in [0, m(l_s))$. *If* $\phi(x) \gneqq U(|x|,\alpha)$ *for some* $\alpha > 0$, *then* $\|u(x,t;\phi)w(x)\|_\infty$ *must blow up in finite time.*

This result establishes that GS with slow decay are on the border between the basin of attraction of the null solutions, and initial data which blow up in finite time, in the range of the considered parameters. We emphasize that we need $l_s < \sigma^*$, but l_u has no upper bound: this allows e.g. $f(u,r) = (1+r^a)u^q$ where $q > \sigma^*$ and $a \in \big(\frac{2}{\sigma^*-2}(q-\sigma^*), \frac{n-2}{2}(q-2^*)\big)$. It is easy to check that $\boldsymbol{A^-}$ might be replaced by $\boldsymbol{H-}$, and that Theorem 4.1 implies Theorem 1.3.

Next, we state Theorems 4.2 and 4.3 from which Theorems 1.5, 1.6, and Corollary 1.8 follow. These results concern a wider range of parameters, and enable us to understand something about the border of the basin of attraction of the null solution and of infinity, i.e. blowing up solutions. We start from a generalization of a result by Wang in [**27**] concerning slow decay solutions.

4.2. THEOREM. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$, $\boldsymbol{Gu}$, $\boldsymbol{Gs}$. *Assume either* $\boldsymbol{A}^+$ *with* $l_u, l_s \in (2_*, 2^*]$ *or* $\boldsymbol{A}^-$ *with* $l_u \geq 2^*$ *and* $2^* \leq l_s < \sigma^*$. *Then there exists a one-parameter family of upper radial solutions of (1.7) denoted by* $\chi_\tau(x)$, *such that* $u(x,t;\chi)$ *converge to the null solution as* $t \to +\infty$, *and with the properties described in Theorem 1.5; in particular they have slow decay.*

Such a result is proved in [**27**] for $f(u,r) = |x|^\delta u^{q-1}$, but as far as we are aware is new even for the nonlinearities considered in [**1**] and [**28**], i.e. even for f as in (1.3) with $k(r) \neq r^\delta$, and f as in (1.4) also when $k_1 = k_2 = 1$. Theorem 4.2 seems to suggest that slow decay is the optimal decay rate to have solutions of (1.1) defined for any t. In fact we have the following result which is in contradiction with this idea (as far as we are aware these results are new even for the case $f(u) = u^{q-1}$).

4.3. THEOREM. *Assume* $\boldsymbol{Fu}$, $\boldsymbol{Fs}$, $\boldsymbol{Gu}$, $\boldsymbol{Gs}$. *Assume either* $\boldsymbol{A}^+$ *with* $l_u, l_s \in (2_*, 2^*]$ *or* $\boldsymbol{A}^-$ *with* $l_u, l_s \geq 2^*$. *Then there are one-parameter families of upper and lower radial solution of (1.7), denoted by* $\zeta_\tau(x)$ *and* $\psi_\tau(x)$ *having the properties described in Theorem 1.6.*

We stress that Remark 1.7 holds also in this case.
The relevance of Theorems 4.2 and 4.3 is more clear if we recall the comparison principle: if we choose any $\phi(x)$ even non radial, such that $\phi \leq\not\equiv \zeta_\tau$ for some τ then $u(x,t;\phi)$ converge to the null solution, while if $\phi \geq\not\equiv \psi_\tau(x)$ then $u(x,t;\phi)$ blows up in finite time. In fact Corollary 1.8 holds in this more general context too.

4.1. Construction of upper and lower-solutions for the stationary problem. From now to the end we always assume $\boldsymbol{F0}, \boldsymbol{Fu}, \boldsymbol{Fs}$ (in order to guarantee local existence of solutions) without further mentioning. We introduce the following notation; we denote by $\boldsymbol{y^u}(s,\alpha;l_u)$ the trajectory of (2.2) corresponding to the regular solution $U(r,\alpha)$ of (1.8), by $\boldsymbol{y^u}(s,\infty;l_u)$ the trajectory corresponding to the singular solution $U(r,\infty)$, by $\boldsymbol{y^s}(s,\beta;l_s)$ the trajectory corresponding to the fast decay solution $V(r,\beta)$, and by $\boldsymbol{y^s}(s,\infty;l_s)$ the trajectory corresponding to the slow decay solution $V(r,\infty)$.

4.4. LEMMA. *Fix* $S \in \mathbb{R}$. *Assume* $l_u > 2^*$, *then* $\boldsymbol{y^u}(s,\alpha;l_u)$ *converges to* $\boldsymbol{y^u}(s,\infty;l_u)$ *as* $\alpha \to +\infty$, *uniformly for* $s \leq S$.
Assume $2_* < l_s < 2^*$, *then* $\boldsymbol{y^s}(s,\beta;l_s)$ *converges to* $\boldsymbol{y^s}(s,\infty;l_s)$, *uniformly for* $s \geq S$.

We think it is worth recalling that, in the cases considered, $U(r,\infty)$ and $V(r,\infty)$ are the unique singular and slow decay solutions of (1.8).

PROOF OF LEMMA 4.4. Assume $l_u > 2^*$, fix $\tau \in \mathbb{R}$ and set $\boldsymbol{Q}(\alpha) = \boldsymbol{y^u}(\tau,\alpha;l_u)$. We recall that $W^u(\tau;l_u)$ is a path joining the origin and $\boldsymbol{R} = \boldsymbol{y^u}(\tau,+\infty;l_u)$; moreover $\boldsymbol{Q}(\alpha) \to \boldsymbol{R}$ as $\alpha \to \infty$, see Remark 2.6 and [**5**]. If $I \subset \mathbb{R}$ is a compact interval, then $\boldsymbol{y^u}(t,\alpha;l_u) \to \boldsymbol{y^u}(\tau,+\infty;l_u)$ for any $t \in I$. But $\boldsymbol{y^u}(t,\alpha;l_u)$ and $\boldsymbol{y^u}(\tau,+\infty;l_u)$ are solutions of (2.2), hence $\boldsymbol{y^u}(t,\alpha;l_u)$ is equibounded and equicontinuous for α

large, so we conclude using Ascoli theorem. The case $2_* < l_s < 2^*$ is completely analogous. □

Analogously, using the fact that

$$\boldsymbol{y^u}(s, \alpha; l_u) \to (0,0)$$

as $s \to -\infty$, and

$$\boldsymbol{y^s}(s, \beta; l_s) \to (0,0)$$

as $s \to +\infty$ for any $\alpha > 0$ $\beta > 0$, we get the following.

4.5. Lemma. *Assume* $\boldsymbol{Gu}$ *with* $l_u > 2_*$*, and fix* $S \in \mathbb{R}$*; then the trajectory* $\boldsymbol{y^u}(s, \alpha_2; l_u)$ *converges to* $\boldsymbol{y^u}(s, \alpha_1; l_u)$ *as* $\alpha_2 \to \alpha_1$*, uniformly for* $s \le S$.
Assume $\boldsymbol{Gs}$ *with* $l_s > 2_*$*, and fix* $S \in \mathbb{R}$*; then* $\boldsymbol{y^s}(s, \beta_2; l_s)$ *converges to* $\boldsymbol{y^s}(s, \beta_1; l_s)$ *as* $\beta_2 \to \beta_1$*, uniformly for* $s \ge S$.

From Lemmas 4.4 and 4.5 we easily get the following.

4.6. Lemma. *Let* $\rho > 0$ *be arbitrarily small. Assume* $\boldsymbol{Gu}$*,* $\boldsymbol{Gs}$ *with* $l_u, l_s \ge 2^*$*, and* $\boldsymbol{A^-}$*. Then for any* $\alpha_1 \ge 0$*,* $U(r, \alpha_2)$ *converges to* $U(r, \alpha_1)$ *as* $\alpha_2 \to \alpha_1$*, uniformly for* $r \ge 0$*. Further, assume* $l_u > 2^*$*, then* $U(r, \alpha_2)$ *converges to* $U(r, \infty)$ *as* $\alpha_2 \to +\infty$*, uniformly for* $r \ge \rho$.
Similarly assume $\boldsymbol{Gu}$*,* $\boldsymbol{Gs}$ *with* $l_u, l_s \in (2_*, 2^*]$*, and* $\boldsymbol{A^+}$*. Then for any* $\beta_1 \ge 0$*,* $V(r, \beta_2)$ *converges to* $V(r, \beta_1)$ *as* $\beta_2 \to \beta_1$*, uniformly for* $r \ge \rho$*. Moreover, if* $2_* < l_s < 2^*$ *then we can also take* $\beta_1 = +\infty$.

Proof. Assume $\boldsymbol{Gu}$, $\boldsymbol{Gs}$ with $l_u, l_s \ge 2^*$, and $\boldsymbol{A^-}$ and choose $\alpha_1 \in (0, +\infty)$. Then from Lemma 4.5 we easily see that $U(r, \alpha_2)$ converges to $U(r, \alpha_1)$ uniformly for r in compact subsets of $(0, \infty)$. However using Ascoli theorem and working directly on (1.8) we easily get uniform convergence for $r \in [0, \rho)$ too, see e.g. the Appendix of [**9**] for more details (in a much more general context); so we have uniform convergence in $[0, R]$. From Proposition 2.12 we know that $U(r, \alpha)$ is a GS with slow decay for any $\alpha > 0$, hence $y_1^u(s, \alpha; l_s)$ is positive and bounded for $s \ge 0$. Thus setting $r = e^s \ge 1$ we find

$$|U(r, \alpha_2) - U(r, \alpha_1)| = |y_1^u(s, \alpha_2; l_s) - y_1^u(s, \alpha_1; l_s)| e^{-m(l_s)s} < K r^{-m(l_s)} .$$

Hence for any $\varepsilon > 0$ we can choose $R_0 = (K/\varepsilon)^{1/m(l_s)}$ so that $|U(r, \alpha_2) - U(r, \alpha_1)| < \varepsilon$ for $r > R_0$. Then we can choose $R > R_0$ and we have that $U(r, \alpha_2)$ converges to $U(r, \alpha_1)$ uniformly for $r \ge 0$.

When $l_u > 2_*$ we simply repeat the argument for $U(r, \infty)$ using Lemma 4.4 instead of Lemma 4.5.
Now, assume $l_u, l_s \in (2_*, 2^*]$ and that $\boldsymbol{A^+}$ holds, so that $V(r, \beta_1)$ is a SGS with fast decay. Then we get uniform convergence for $r \ge \rho$ directly from Lemma 4.5 and if $l_s < 2^*$, we conclude by using Lemma 4.4. □

4.7. Lemma. *Assume* $\boldsymbol{A^-}, \boldsymbol{Gu}, \boldsymbol{Gs}$ *with* $l_u \ge 2^*$ *and* $l_s \in [2^*, \sigma^*)$*, then for any* $0 < \alpha_1 < \alpha_2 \le \infty$ *there is* $Z(\alpha_2, \alpha_1) > 0$ *such that* $U(r, \alpha_2) > U(r, \alpha_1)$ *for* $0 < r < Z(\alpha_2, \alpha_1)$ *and* $U(r, \alpha_2) = U(r, \alpha_1)$*,* $U'(r, \alpha_2) < U'(r, \alpha_1)$ *for* $r = Z(\alpha_2, \alpha_1)$.
Assume $\boldsymbol{A^+}, \boldsymbol{Gu}, \boldsymbol{Gs}$ *with* $\sigma_* < l_u \le 2^*$ *and* $l_s \in (2_*, 2^*]$*, then for any* $0 < \beta_1 < \beta_2 \le \infty$ *there is* $W(\beta_2, \beta_1) > 0$ *such that* $V(r, \beta_2) > V(r, \beta_1)$ *for* $r > W(\beta_2, \beta_1)$ *and* $V(r, \beta_2) = V(r, \beta_1)$*,* $V'(r, \beta_2) > V'(r, \beta_1)$ *for* $r = W(\beta_2, \beta_1)$.

PROOF. Assume $\boldsymbol{A^-}, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $l_u \geq 2^*$ and $l_s \in [2^*, \sigma^*)$. Then from Proposition 2.12 we know that $U(r, \alpha)$ is a GS with slow decay and that it is a SGS with slow decay for $\alpha = \infty$. Observe that $U(r, \alpha_2) > U(r, \alpha_1)$ for r in a right neighborhood of 0, since $U(0, \alpha_2) = \alpha_2 > \alpha_1 = U(0, \alpha_1)$, and they are continuous functions in r. Since they are slow decay solutions, from Proposition 2.10 we see that there is $R > 0$ (depending on α_1, α_2) such that $U(R, \alpha_2) = U(R, \alpha_1)$. Then we denote by

$$Z(\alpha_2, \alpha_1) := \min\{R > 0 \mid U(R, \alpha_2) = U(R, \alpha_1)\} \tag{4.1}$$

Thus by construction we get $\frac{\partial}{\partial r} U(R, \alpha_2) \leq \frac{\partial}{\partial r} U(R, \alpha_1)$ for $R = Z(\alpha_2, \alpha_1)$, but from the uniqueness of the solution of Cauchy problem for ODEs we see that the inequality is actually strict.

The case $\boldsymbol{A^+}, \boldsymbol{Gu}, \boldsymbol{Gs}$ with $\sigma_* < l_u \leq 2^*$ and $l_s \in (2_*, 2^*]$ is completely analogous, and its proof can be obtained repeating the argument for fast decay solutions $V(r, \beta)$ and reversing the direction of s. □

4.2. Proof of Theorem 4.1. In this subsection we assume the hypotheses of Theorem 4.1 without further mentioning. Let us set

$$\begin{aligned} \psi(x) &= \begin{cases} U(|x|, \alpha_2) & \text{if } |x| \leq Z(\alpha_2, \alpha_1) \\ U(|x|, \alpha_1) & \text{if } |x| \geq Z(\alpha_2, \alpha_1) \end{cases} \\ \zeta(x) &= \begin{cases} U(|x|, \alpha_2) & \text{if } |x| \geq Z(\alpha_2, \alpha_1) \\ U(|x|, \alpha_1) & \text{if } |x| \leq Z(\alpha_2, \alpha_1) \end{cases} \end{aligned} \tag{4.2}$$

where $Z(\alpha_2, \alpha_1)$ is defined in (4.1). Then by construction $\zeta(x)$ and $\psi(x)$ are radial C_B-mild upper and lower solutions for (1.7). From Theorem 3.10 and Remark 3.11 it follows that $u(x, t; \zeta)$ is radial, decreasing in t, and converges uniformly to a radial non-negative solution $u(x, T_\zeta; \zeta)$ of (1.7) for $t < T_\zeta$. Thus $\|u(x, t; \zeta)(1 + |x|^{m(l_s)})\|_\infty$ is bounded for $t < T_\zeta$, hence $T_\zeta = +\infty$. Since the null solution is the unique radial solution of (1.7) staying below $\zeta(x)$ for any $x \in \mathbb{R}^n$, see Lemma 2.15, then $\lim_{t\to\infty} \|u(x, t; \zeta)(1 + |x|^\nu)\|_\infty = 0$ for any $0 \leq \nu < m(l_s)$.

Now let $\phi \in C_B$ such that there is $\alpha_2 > 0$ and $\phi(x) \lneqq U(|x|, \alpha_2)$. From strong maximum principle for parabolic equations (see, e.g. the appendix in [**11**]), we get $u(x, t; \phi) < U(|x|, \alpha_2)$ for any x and any $t > 0$. So, up to a time translation, we can assume $\phi(x) < U(|x|, \alpha_2)$ for any $x \in \mathbb{R}^n$. From Lemma 4.6 we see that for any $\varepsilon > 0$ we can find $\alpha_1 < \alpha_2$ such that $|U(|x|, \alpha_1) - U(|x|, \alpha_2)| < \varepsilon$ for any $|x| \in \mathbb{R}^n$. Let $\zeta(x)$ be the upper solution defined by (4.2), then if $\varepsilon > 0$ is small enough we can assume $\phi(x) < \zeta(x)$ for any $x \in \mathbb{R}^n$. Hence $0 < u(x, t; \phi) < u(x, t; \zeta)$ for any $x \in \mathbb{R}^n$ and any $t > 0$; so $T_\phi = +\infty$ and $\|u(x, t; \phi)\|_\infty \to 0$, $\|u(x, t; \phi)(1 + |x|^\nu)\|_\infty \to 0$ as $t \to \infty$, for any $0 \leq \nu < m(l_s)$.

Similarly consider $\phi \in C_B$ and assume that there is $\alpha_1 > 0$ such that $\phi(x) \geq U(|x|, \alpha_1)$, and $\phi(x) \not\equiv U(|x|, \alpha_1)$. Reasoning as above we can assume $\phi(x) > U(|x|, \alpha_1)$ for any $x \in \mathbb{R}^n$, and we can find $\alpha_2 > \alpha_1$, with $\alpha_2 - \alpha_1$ small enough so that the lower solution $\psi(x)$ defined in (4.2) satisfies $\psi(x) < \phi(x)$ for any $x \in \mathbb{R}^n$. From Theorem 3.10 and Remark 3.11 it follows that $u(x, t; \psi)$ is radial and increasing in t, and converges uniformly to a radial non-negative solution $U(x)$ of (1.7) for $t < T_\psi$ and $U(x) \geq \psi(x)$ for any x. But from Lemma 2.15 we see that such a solution $U(x)$ does not exist, hence $T_\psi < \infty$ and $\lim_{t\to T_\psi} \|u(x, t; \psi)\|_\infty = +\infty$. □

4.3. Proof of Theorems 4.2 and 4.3. The construction of the family of upper and lower solutions of Theorems 4.2 and 4.3 is based on Remark 2.7, Lemmas 2.13, 2.14.

Proof of Theorem 4.3. Assume first $\boldsymbol{A^-}, \boldsymbol{Gu}, \boldsymbol{Gs}$, with $l_u \geq 2^*$ and $l_s > 2^*$. We recall that the critical point $(\boldsymbol{P^{-\infty}}, 0)$ of (2.10) admits a 1-dimensional unstable manifold and that we denote by $(\boldsymbol{y^u}(s; l_u), z(s))$ the unique trajectory belonging to this manifold, and by $U(r, \infty)$ the corresponding solution of (1.8) which is a SGS with slow decay, and by $\boldsymbol{y^u}(s; l_s)$ the corresponding trajectory of (2.2) with $l = l_s$. From Lemma 2.13 we know that for any $\tau \in \mathbb{R}$, $W^u(\tau; l_s)$ and $\boldsymbol{y^u}(\tau; l_s)$ are contained in $\bar{E}^s(\tau)$, i.e. the bounded set enclosed by $W^s(\tau; l_s)$ and the y_2 coordinate axis (see the construction just before Lemma 2.13).
Observe that $\boldsymbol{y^u}(s; l_s) \to \boldsymbol{P^{+\infty}}$, hence the values

$$\bar{R}(l_u) := \min\{y_1^u(s; l_u) \mid s \leq 0\} \qquad \bar{R}(l_s) := \min\{y_1^u(s; l_s) \mid s \geq 0\}$$

are strictly positive and bounded. Assume first that $l_u \leq l_s$, so that $m(l_u) \geq m(l_s)$, and set

$$\tilde{R} := 1/2 \min\left(\{\bar{R}(l_u)\exp[(m(l_s) - m(l_u))\tau] \mid \tau \leq 0\} \cup \{\bar{R}(l_s)\}\right) \tag{4.3}$$

Note that $\tilde{R} > 0$ and $y_1^u(\tau; l_s) > \tilde{R}$ for any $s \in \mathbb{R}$.
If $l_u > 2^*$ then $W^u(\tau; l_s)$ is a path connecting the origin and $\boldsymbol{y^u}(\tau; l_s)$ so we can find (at least) a point $\boldsymbol{Q^{u,*}}(\tau) = (Q_1^{u,*}(\tau), Q_2^{u,*}(\tau)) \in W^u(\tau; l_s)$ such that $Q_1^{u,*}(\tau) = \tilde{R}$. Moreover there are two points, say $\boldsymbol{Q^{s,+}}(\tau)$, $\boldsymbol{Q^{s,-}}(\tau)$, belonging to $W^s(\tau; l_s)$ and such that $\boldsymbol{Q^{s,\pm}}(\tau) = (\tilde{R}, Q_2^{s,\pm}(\tau))$, and $Q_2^-(\tau) < \bar{P}_2^*(\tau) < Q_2^+(\tau)$. Let us consider now the trajectories $\boldsymbol{y}(s, \tau; \boldsymbol{Q^{s,+}}(\tau); l_s)$ and $\boldsymbol{y}(s, \tau; \boldsymbol{Q^{s,-}}(\tau); l_s)$: by construction they correspond to fast decay solutions of (1.8), say $V(r, \beta_2)$ and $V(r, \beta_1)$ respectively. We can assume w.l.o.g. that $\beta_2 > \beta_1$, see Lemma 2.5 and Remark 2.6. We denote by $U(r, \alpha^*)$ the regular solution of (1.8) corresponding to $\boldsymbol{y}(s, \tau; \boldsymbol{Q^{u,*}}(\tau); l_s)$. Since $y_2(s, \tau; \boldsymbol{Q^{s,-}}(\tau); l_s) < y_2(s, \tau; \boldsymbol{Q^{u,*}}(\tau); l_s) < y_2(s, \tau; \boldsymbol{Q^{s,+}}(\tau); l_s)$ we have $V'(R, \beta_1) < U'(R, \alpha^*) < V'(R, \beta_2)$ for $R = \mathrm{e}^\tau$. Therefore we can construct upper and lower radial solution of (1.7) , say $\zeta(x)$ and $\psi(x)$ as follows:

$$\begin{aligned} \zeta(x,\tau) &= \begin{cases} U(|x|, \alpha^*) & \text{if } |x| \leq \mathrm{e}^\tau \\ V(|x|, \beta_1) & \text{if } |x| \geq \mathrm{e}^\tau \end{cases} \\ \psi(x,\tau) &= \begin{cases} U(|x|, \alpha^*) & \text{if } |x| \geq \mathrm{e}^\tau \\ V(|x|, \beta_2) & \text{if } |x| \leq \mathrm{e}^\tau \end{cases} \end{aligned} \tag{4.4}$$

Moreover observe that $\beta_i \to +\infty$ and $\alpha^* \to 0$ as $\tau \to +\infty$, and $\beta_i \to 0$ and $\alpha^* \to +\infty$ as $\tau \to -\infty$, for $i = 1, 2$, see Remark 2.6. Hence $\zeta(0, \tau) = \psi(0, \tau) := D(\tau) \to 0$ as $\tau \to +\infty$ and $D(\tau) \to +\infty$ as $\tau \to -\infty$. Similarly $\lim_{|x| \to +\infty} \zeta(x, \tau)|x|^{n-2} = \beta_1$ and $\lim_{|x| \to +\infty} \psi(x, \tau)|x|^{n-2} = \beta_2$ go to 0 as $\tau \to +\infty$ and they go to $+\infty$ as $\tau \to 0$.

So, from Theorem 3.10, Remark 3.11, Lemma 2.15 we see that for any $\tau \in \mathbb{R}$ $\|u(x, t; \zeta(x))\|_\infty \to 0$ and $\|u(x, t; \zeta(x))(1 + |x|^\nu)\|_\infty \to 0$ as $t \to +\infty$ for any $\nu \in [0, n-2)$, and that there is T_ψ such that $\|u(x, t; \psi(x))\|_\infty \to +\infty$ as $t \to T_\psi$.
Now we go back to the case where $2^* < l_s < l_u$ so that $m(l_s) > m(l_u)$, and $\boldsymbol{A^-}$ holds. In this case we need to consider

$$\hat{R} := 1/2 \min\left(\{\bar{R}(l_s)\exp[(m(l_u) - m(l_s))\tau] \mid \tau \geq 0\} \cup \{\bar{R}(l_u)\}\right) \tag{4.5}$$

and to redefine a set $\hat{E}^s(\tau) := \{\boldsymbol{Q} = \boldsymbol{R}\mathrm{e}^{(m(l_u)-m(l_s))\tau} \mid \boldsymbol{R} \in \bar{E}^s(\tau)\}$ (note that we could redefine $\hat{E}^s(\tau)$ simply considering the bounded set enclosed by $W^s(\tau; l_u)$ and the y_2 axis), and observe that $W^u(l_u; \tau) \subset \hat{E}^s(\tau)$ for any $\tau \in \mathbb{R}$. Then we repeat the previous argument working with (2.2) with $l = l_u$ and replacing $\tilde{R}$ by $\hat{R}$.

Now assume $\boldsymbol{A^-}$, with $l_u = 2^*$ and $l_s > 2^*$. In this case $W^u(\tau; l_s)$ is a C^1 manifold departing from the origin and contained in $\bar{E}^s(\tau)$ for any $\tau \in \mathbb{R}$, but it does not have $\boldsymbol{y^u}(\tau; l_s)$ in its border. However for any $\tau \in \mathbb{R}$ we can still find at least an intersection between the line $y_1 = \tilde{R}$ and $W^u(\tau; l_s)$ and at least two intersections between $y_1 = \tilde{R}$ and $W^s(\tau; l_s)$. So it is easy to check that the whole argument can be repeated with no changes.

Now assume $\boldsymbol{A^-}$, with $l_u = l_s = 2^*$. In this case we rely on Lemma 2.14: let $U(r, \alpha^*)$ be the regular solution of (1.8) corresponding to $\boldsymbol{y}(s, \tau, \boldsymbol{Q^{u,+}}(\tau); 2^*)$, and $V(r, \beta_2)$, $U(r, \beta_1)$ be the fast decay solutions of (1.8) corresponding respectively to $\boldsymbol{y}(s, \tau, \boldsymbol{Q^{s,+}}(\tau); 2^*)$ and $\boldsymbol{y}(s, \tau, \boldsymbol{Q^{s,-}}(\tau); 2^*)$. Then we define $\zeta(x)$ and $\psi(x)$ as in (4.4): they are respectively radial upper and lower solutions for (1.7). Moreover $\zeta(0, \tau) = \psi(0, \tau) := D(\tau) \to 0$ as $\tau \to +\infty$ and $D(\tau) \to +\infty$ as $\tau \to -\infty$. Similarly $\lim_{|x| \to +\infty} \zeta(x, \tau)|x|^{n-2} = \beta_1$ and $\lim_{|x| \to +\infty} \psi(x, \tau)|x|^{n-2} = \beta_2$ go to 0 as $\tau \to +\infty$ and they go to $+\infty$ as $\tau \to 0$.

The case where $\boldsymbol{A^+}$, holds with $2_* < l_s \le 2^*$ and $2_* < l_s < 2^*$ or $l_u = l_s = 2^*$ are completely analogous and are left to the reader, so the proofs of Theorem 4.3 and Proposition 1.7 is concluded. □

Proof of Theorem 4.2. Assume $\boldsymbol{A^-}$, then Theorem 4.2 follows from Theorem 4.1.

So assume $\boldsymbol{A^+}$ and suppose first $l_u, l_s \in (2_*, 2^*)$. Then from Proposition 2.12 there is a unique SGS with slow decay say $U(r, \infty)$ of (1.8): let $\boldsymbol{y^s}(s; l_u)$ be the corresponding trajectory of (2.2). Fix $\tau \in \mathbb{R}$: from Lemma 2.13 we know that there is $\boldsymbol{Q}(\tau) = (Q_1(\tau), Q_2(\tau)) \in W^u_{l_u}(\tau)$ such that $Q_1(\tau) = y^s_1(\tau; l_u)$ and $Q_2(\tau) < y^s_2(\tau; l_u)$. Consider the trajectory $\boldsymbol{y}(s, \tau, \boldsymbol{Q}(\tau); l_u)$ and the corresponding regular solution $U(r, d(\tau))$ of (1.8): from Remark 2.6 we see that $d(\tau) \to 0$ as $\tau \to +\infty$ and $d(\tau) \to +\infty$ as $\tau \to -\infty$. Moreover the function $\chi(x, \tau)$ defined as follows

$$\chi(x, \tau) = \begin{cases} U(|x|, d(\tau)) & \text{if } |x| \le \mathrm{e}^\tau \\ U(|x|, \infty) & \text{if } |x| \ge \mathrm{e}^\tau \end{cases} \tag{4.6}$$

is a super-solution of (1.7), and it is regular and has slow decay. Hence, reasoning as above we see that $u(x, t, \chi)$ is radial, radially decreasing and well defined for any t and converges monotonically to the null solution as $t \to +\infty$ and $\lim_{t \to +\infty} \|u(x, t, \chi)[1 + |x|^\nu]\|_\infty = 0$ for any $0 \le \nu < m(l_s)$. If $2_* < l_u < l_s = 2^*$, we just lose uniqueness of the SGS with slow decay but the argument can be repeated for any such solution, so it still works. If $l_u = l_s = 2^*$ we simply need to repeat the same argument but using Lemma 2.14 instead of Lemma 2.13. □

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Dipartimento di Matematica e Informatica, Università di Firenze, Via S. Marta 3, I-50139 Firenze, Italy
E-mail address: luca.bisconti@unifi.it

Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, I-60131 Ancona, Italy
E-mail address: franca@dipmat.univpm.it