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Pullback dynamical behaviors of the non-autonomous micropolar fluid flows

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Abstract. The purpose of this work is to investigate the pullback asymptotic behaviors of solutions for non-autonomous micropolar fluid flows in two-dimensional bounded domains. On the base of the known results concerning the global well-posedness of the solutions, we apply the technique of enstrophy equality, combining with the estimates on the solutions, to prove the existence and regularity of the pullback attractors for the generated evolution process for the universe of fixed bounded sets and for another universe with a tempered condition in different phase spaces. Then we use the estimates of the solutions to analyze the tempered behavior and H^2-boundedness of the pullback attractors.

Contents

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1. Introduction

The purpose of this work is to investigate the pullback asymptotic behaviors of solutions for the micropolar fluid model. The micropolar fluid model were firstly established by Eringen [**11**] in 1966, which describe fluids consisting of randomly oriented particles suspended in a viscous medium. According to [**11**], the model equations for micropolar fluid flows can be described by the following system:

$$\frac{\partial u}{\partial t} - (\nu+\nu_r)\Delta u - 2\nu_r \text{rot}\omega + (u\cdot\nabla)u + \nabla p = g, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$\frac{\partial \omega}{\partial t} - (c_a+c_d)\Delta\omega + 4\nu_r\omega + (u\cdot\nabla)\omega - (c_0+c_d-c_a)\nabla\text{div}\omega - 2\nu_r\text{rot}u = \tilde{g}, \tag{1.3}$$

where $u=(u_1,u_2,u_3)$ is the velocity, p represents the pressure, $\omega=(\omega_1,\omega_2,\omega_3)$ is the microrotation field interpreted as the angular velocity field of rotation of particles. $g=(g_1,g_2,g_3)$ and $\tilde{g}=(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3)$ are external force and moments, respectively. The positive parameters ν,ν_r,c_0,c_a,c_d, represent viscosity coefficients. In fact, ν represents the usual Newtonian viscosity and ν_r is called the microrotation viscosity. From [**11, 23**], we see that system (1.1)-(1.3) expresses the balance of momentum, mass, and moment of momentum. If $\nu_r=0$ and $\omega=0$, then equations (1.1) and (1.2) reduce to the Navier-Stokes equations. Therefore, the micropolar fluid model can be regarded as an essential generalization of the Navier-Stokes model in the sense that it takes into account the microstructure of the fluid [**24**].

Due to the wide applications in the real world, the micropolar fluid flows have drawn much attention from mathematicians and physicists and have been well studied. Here we only illustrate some known results. First, we must mention that Łukaszewicz has obtained fruitful results in his monograph [**23**], including the existence and uniqueness of solutions for the stationary problems; the existence of weak and strong solutions for the nonstationary problems, as well as the global existence of solution for the heat-conducting flows; the applications of the micropolar fluids in lubrication theory and in porous media, etc. Also, numerous papers are devoted to the existence and uniqueness of solutions for the micropolar fluids, see, e.g. [**12, 13, 14, 18, 20, 21, 22, 23, 24, 25**]. At the same time, the long time behavior of solutions for the micropolar fluids has been investigated from various aspects. For example, the estimates of Hausdorff and fractal dimension of the L^2-global attractor was studied in [**24**]; the existence of H^2-compact global attractor was proved in [**6**]; the global and uniform attractor on unbounded domain was verified in [**10**] and [**26, 32, 38**], respectively; the uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains was proved in [**7**]; the H^1-pullback attractor was obtained in [**8, 27**]. The existence of L^2-pullback attractor for the micropolar fluid flows in a Lipschitz bounded domain with non-homogeneous boundary conditions was established in [**9**]. However, the pullback asymptotic behaviors of the micropolar fluid flows as studied in this paper have not been considered so far.

In this work, we will concentrate on studying the pullback asymptotic behaviors of solutions for system (1.1)-(1.3) in two-dimensional bounded domains. More precisely, we consider a cross section $x_3=$ constant of the three-dimensional domain $\Omega\times\mathbb{R}$ when the external fields and the flow itself do not depend on the x_3 coordinate.

Then, we may assume that the velocity component u_3 in the x_3 direction is zero and the axes of rotation of particles are parallel to the x_3 axis. In this case, the fields $u, \omega, g, \tilde{g}$ are of the form $u = (u_1, u_2, 0)$, $\omega = (0, 0, \omega_3)$, $g = (g_1, g_2, 0)$, $\tilde{g} = (0, 0, \tilde{g}_3)$ and system (1.1)-(1.3) can be reduced to the following two-dimensional non-autonomous incompressible micropolar fluid flow

$$\frac{\partial u}{\partial t} - (\nu + \nu_r)\Delta u - 2\nu_r \nabla \times \omega + (u \cdot \nabla)u + \nabla p = g(x, t), \tag{1.4}$$

$$\nabla \cdot u = 0, \tag{1.5}$$

$$\frac{\partial \omega}{\partial t} - \alpha \Delta \omega + 4\nu_r \omega - 2\nu_r \nabla \times u + (u \cdot \nabla)\omega = \tilde{g}(x, t), \tag{1.6}$$

where $t > \tau$ for some $\tau \in \mathbb{R}$, $\alpha := c_a + c_d$, $x := (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, $u := (u_1, u_2)$, $g := (g_1, g_2)$, ω and $\tilde{g}$ are scalar functions;

$$\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \text{and} \quad \nabla \times \omega := (\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}).$$

In addition, we impose the following boundary and initial conditions:

$$u|_{\partial\Omega} = 0, \qquad \omega|_{\partial\Omega} = 0, \tag{1.7}$$

$$u|_{t=\tau} = u_\tau, \quad \omega|_{t=\tau} = \omega_\tau, \tag{1.8}$$

where $u_\tau(\cdot)$ and $\omega_\tau(\cdot)$ are given functions of x. Our goal is to study the existence and reveal some properties of the pullback attractors for the processes associated with equations (1.4)-(1.8). For simplicity, we assume Ω is a bounded smooth domain such that the following Poincaré inequality holds

$$\lambda_1 \|\varphi\|^2_{L^2(\Omega)} \leqslant \|\nabla \varphi\|^2_{L^2(\Omega)}, \ \ \forall \varphi \in H^1_0(\Omega), \tag{1.9}$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ in $L^2(\Omega)$ with domain $H^1_0(\Omega) \cap H^2(\Omega)$ and satisfies the Dirichlet boundary condition. Note that λ_1 is a constant depending only on Ω.

We remark that the existence of the pullback attractor of Navier-Stokes equations in space V (for definition, see Section 2) and its tempered behavior were studied in [**16**]. Motivated by this work and following its main idea, we generalize their results to micropolar fluid flows. In contrast to the Navier-Stokes equations ($\omega = 0$, $\nu_r = 0$), we emphasize that the micropolar fluid flows consist the angular velocity field ω of the micropolar particles, which leads to a different nonlinear term $B(u, w)$ and an additional term $N(u)$ in the abstract equation (see (2.8)). Due to these differences, more delicate estimates and analysis for the solutions are required in our study.

The paper is organized as follows.

In section 2, we recall the known results concerning the global existence and uniqueness of solutions for equations (1.4)-(1.8). According to the estimates on the solutions, we see that the evolution process generated by the solution maps is continuous, which possesses a family of bounded pullback absorbing sets.

In section 3, we prove the existence and regularity of the pullback attractors in L^2 and H^1 norms for the universe of fixed bounded sets and for another universe with a tempered condition, respectively. Note that Łukaszewicz and Tarasińska proved, using a method based on the concept of the Kuratowski measure of non-compactness of a bounded set as well as some new estimates of solutions, the existence of the H^1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain ([27]). Here our key steps are to verify the existence of the pullback absorbing set and the asymptotical compactness of the generated evolution process. To establish the existence of the pullback absorbing set, we use the Galerkin approximate solutions and combine the embedding between functional spaces (see Lemma 3.1) to prove the higher regular estimates of the solutions. Then we employ these estimates and the method of enstrophy equality to verify the asymptotical compactness of the generated evolution process.

In sections 4 and 5, using the regular estimates of the Galerkin approximate solutions and the embedding between the relevant functional spaces, we prove the tempered behaviors of the pullback attractors as the initial time tends to $-\infty$ in $\widehat{H}$, $\widehat{V}$ and $(H^2(\Omega))^3$ norms, respectively, as well as the boundedness of the pullback attractors in $\widehat{V}$ and $(H^2(\Omega))^3$ norms, respectively. Note that the above spaces $\widehat{H}$, $\widehat{V}$ and $(H^2(\Omega))^3$ will be introduced in Section 2. We want to point out that the earlier research on the H^2 global attractor (see [6]) was from the viewpoint of measuring noncompactness, and the semidistance in the attracting property of the H^2 compact global attractor are still in $\widehat{H}$ space. Here the regularity of the obtained pullback attractor, as well as it's tempered behaviors and boundedness in $(H^2(\Omega))^3$ norm, illustrates the pullback asymptotic smoothing effect of the addressed miropolar fluid flows in the sense that the solutions become eventually more regular (lying in $(H^2(\Omega))^3$) than the initial data (lying in $\widehat{H}$).

2. Global existence and uniqueness of solutions

In this section, we will establish the global existence and uniqueness of solutions for system (1.4)-(1.8). To state our investigations in a clear way, we first introduce the following notations.

Let us denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ the usual Lebesgue space and Sobolev space (see [**1, 3**]) endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively,

$$\|\varphi\|_p := (\int_\Omega |\varphi|^p \mathrm{d}x)^{1/p} \quad \text{and} \quad \|\varphi\|_{m,p} := (\sum_{|\beta|\leqslant m} \int_\Omega |D^\beta \varphi|^p \mathrm{d}x)^{1/p}.$$

Especially, we denote $H^m(\Omega) := W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ the closure of $\{\varphi \in \mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$ norm. Then we introduce the following functions spaces:

$$\mathcal{V} := \{\varphi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) |\, \varphi = (\varphi_1, \varphi_2), \nabla \cdot \varphi = 0\},$$

$$H := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega) \times L^2(\Omega), \text{ with norm } \|\cdot\|_H \text{ and dual space } H^*,$$

$$V := \text{closure of } \mathcal{V} \text{ in } H^1(\Omega) \times H^1(\Omega), \text{ with norm } \|\cdot\|_V \text{ and dual space } V^*,$$

$$\widehat{H} := H \times L^2(\Omega) \text{ with norm } \|\cdot\|_{\widehat{H}} \text{ and dual space } \widehat{H}^*,$$

$$\widehat{V} := V \times H_0^1(\Omega) \text{ with norm } \|\cdot\|_{\widehat{V}} \text{ and dual space } \widehat{V}^*.$$

Note that $\|\cdot\|_H$, $\|\cdot\|_V$, $\|\cdot\|_{\widehat{H}}$ and $\|\cdot\|_{\widehat{V}}$ are defined by

$$\|(u,v)\|_H := (\|u\|_2^2 + \|v\|_2^2)^{1/2}, \qquad \|(u,v)\|_V := (\|u\|_{H^1}^2 + \|v\|_{H^1}^2)^{1/2},$$

$$\|(u,v,w)\|_{\widehat{H}} := (\|(u,v)\|_H^2 + \|w\|_2^2)^{1/2}, \quad \|(u,v,w)\|_{\widehat{V}} := (\|(u,v)\|_V^2 + \|w\|_{H^1}^2)^{1/2}.$$

Throughout this article, we simplify the notations $\|\cdot\|_2$, $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$, if there is no confusion. According to the above notations, we further denote

$L^p(I;X) :=$ space of strongly measurable functions on the closed interval I, with values in a Banach space X, endowed with norm

$$\|\varphi\|_{L^p(I;X)} := (\int_I \|\varphi\|_X^p \mathrm{d}t)^{1/p}, \quad \text{for } 1 \leqslant p < \infty;$$

$\mathcal{C}(I;X) :=$ space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm;

$L^2_{loc}(I;\widehat{H}) :=$ space of locally integrable functions from the interval I to $\widehat{H}$;

$W^{1,2}_{loc}(I;\widehat{H}) := \{G|\ G \in L^2_{loc}(I;\widehat{H}) \text{ and } G' \in L^2_{loc}(I;\widehat{H})\}$, here " $'$ " means the derivative with respect to time variable.

In addition, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, H or $\widehat{H}$, and by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$.

To write equations (1.4)-(1.6) into the abstract form, we further introduce the following three operators:

$$\langle Aw,\varphi\rangle := (\nu+\nu_r)(\nabla u,\nabla\Phi) + \alpha(\nabla\omega,\nabla\phi),\ \forall w=(u,\omega)\in\widehat{V},\ \forall\varphi=(\Phi,\phi)\in\widehat{V},$$
$$\langle B(u,w),\varphi\rangle := ((u\cdot\nabla)w,\varphi),\ \forall u\in V,\ w\in\widehat{V},\ \forall\varphi\in\widehat{V},$$
$$N(w) := (-2\nu_r\nabla\times\omega, -2\nu_r\nabla\times u + 4\nu_r\omega),\ \forall w=(u,\omega)\in\widehat{V}.$$

From the above definitions, one can check that A is a linear continuous operator both from $\widehat{V}$ to $\widehat{V}^*$ and from $D(A) := \widehat{V}\cap\left(H^2(\Omega)\right)^3$ to $\widehat{H}$; $B(\cdot,\cdot)$ is continuous from $V\times\widehat{V}$ to $\widehat{V}^*$ and $N(\cdot)$ is continuous from $\widehat{V}$ to $\widehat{H}$. Some useful estimations for the operators $A, B(\cdot,\cdot)$ and $N(\cdot)$ have been established in the works [**24, 26**]. For completeness, we recall them as following.

LEMMA 2.1.

(1) *There are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw,w\rangle \leqslant \|w\|^2_{\widehat{V}} \leqslant c_2\langle Aw,w\rangle,\ \ \forall w\in\widehat{V}. \tag{2.1}$$

Furthermore, for any $w\in D(A)$, there holds

$$\min\{\nu+\nu_r,\alpha\}\|\nabla w\|^2 \leqslant \langle Aw,w\rangle \leqslant \|w\|\|Aw\| \leqslant \lambda_1^{-\frac{1}{2}}\|\nabla w\|\|Aw\|. \tag{2.2}$$

(2) *There exists a positive constant λ which depends only on Ω, such that for any $(u,w,\varphi)\in V\times\widehat{V}\times\widehat{V}$ there holds*

$$|\langle B(u,w),\varphi\rangle| \leqslant \lambda\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}\|\nabla w\|^{\frac{1}{2}}\|\nabla\varphi\|. \tag{2.3}$$

Moreover, if $(u,w,\varphi)\in V\times D(A)\times D(A)$, then

$$|\langle B(u,w),A\varphi\rangle| \leqslant \lambda\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\nabla w\|^{\frac{1}{2}}\|Aw\|^{\frac{1}{2}}\|A\varphi\|. \tag{2.4}$$

(3) *There exists a positive constant $c(\nu_r)$ such that*

$$\|N(w)\| \leqslant c(\nu_r)\|w\|_{\widehat{V}},\ \ \forall w\in\widehat{V}. \tag{2.5}$$

In addition, there holds

$$-\langle N(w), Aw\rangle \leqslant \begin{cases} \frac{1}{2}\|Aw\|^2 + 2c^2(\nu_r)\|w\|^2_{\widehat{V}}; \\ \frac{1}{4}\|Aw\|^2 + c^2(\nu_r)\|w\|^2_{\widehat{V}}, \end{cases} \quad \forall w \in D(A), \tag{2.6}$$

$$\langle Aw, w\rangle + \langle N(w), w\rangle \geqslant \delta\|w\|^2_{\widehat{V}}, \ \forall\, w \in \widehat{V}, \tag{2.7}$$

hereinafter $\delta := \min\{\nu, \alpha\}$.

According to the above notations, we can formulate the weak version of equations (1.4)-(1.8) as following (see [**38**]):

$$\frac{\partial w}{\partial t} + Aw + B(u, w) + N(w) = G(x, t), \ \ w = (u, \omega) \in \widehat{V}, \ \ t > \tau, \tag{2.8}$$

$$w|_{t=\tau} = w_\tau = (u_\tau, \omega_\tau), \ \ \tau \in \mathbb{R}, \tag{2.9}$$

hereinafter $G(x, t) := (g(x, t), \tilde{g}(x, t))$ and (2.8) is understood in the $\mathcal{D}'([\tau, t]; \widehat{V}^*)$ distribution sense. Therefore, given $\tau \in \mathbb{R}$, we say that a function $w = (u, \omega) \in \mathcal{C}([\tau, T]; \widehat{H}) \cap L^2([\tau, T]; \widehat{V})$ for $T > \tau$ is a *weak solution* of system (1.4)-(1.8) if $w|_{t=\tau} = w_\tau = (u_\tau, \omega_\tau)$ and (2.8) holds in the $\mathcal{D}'([\tau, T]; \widehat{V}^*)$ distributions sense.

The following global existence and uniqueness result of weak solutions can be found in [**24**].

Proposition 2.1. *Assume* $G(x, t) \in L^2_{loc}(\mathbb{R}; \widehat{H})$.

(1) *If* $w_\tau \in \widehat{H}$, *then system* (2.8)-(2.9) *has a unique solution* w *satisfying*

$$w \in L^\infty([\tau, \infty); \widehat{H}) \cap \mathcal{C}([\tau, \infty); \widehat{H}) \cap L^2_{loc}([\tau, \infty); \widehat{V}), \ w' \in L^2_{loc}([\tau, \infty); \widehat{V}^*).$$

Moreover, the solution w *depends continuously on the initial value* w_τ *with respect to the* $\widehat{H}$ *norm.*

(2) *If* $w_\tau \in \widehat{V}$, *then system* (2.8)-(2.9) *has a unique solution* w *satisfying*

$$w \in L^\infty([\tau, \infty); \widehat{V}) \cap \mathcal{C}([\tau, +\infty); \widehat{V}) \cap L^2_{loc}([\tau, \infty); D(A)), \ w' \in L^2_{loc}([\tau, \infty); \widehat{H}).$$

Furthermore, the solution w *depends continuously on the initial value* w_τ *with respect to the* $\widehat{V}$ *norm.*

Remark 2.1. Let w be the solution of system (2.8)-(2.9) with initial value $w_\tau \in \widehat{V}$, then w satisfies the following "enstrophy equality":

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw, w\rangle + \|Aw\|^2 + \langle B(u, w), Aw\rangle + \langle N(w), Aw\rangle = (G(t), Aw). \tag{2.10}$$

This enstrophy equality will play an important role in establishing the pullback asymptotic compactness of the process in space $\widehat{V}$. In fact, since the uniqueness of the strong solution to problem (2.8)-(2.9), equation (2.9) can be obtained by passing limit $n \to \infty$ in equation (3.18). Here we omit the detailed proof.

According to Proposition 2.1, we see that the maps defined by

$$U(t, \tau): \ w_\tau \longmapsto U(t, \tau; w_\tau) = w(t), \ t \geqslant \tau, \tag{2.11}$$

where $w(t)$ is the weak solution of system (2.8)-(2.9), generate a continuous process $\{U(t, \tau)\}_{t \geqslant \tau}$ in $\widehat{H}$ and $\widehat{V}$, respectively.

3. Existence and regularity of pullback attractors

In this section, we will prove that the process $\{U(t,\tau)\}_{t\geqslant\tau}$ defined by (2.11) possesses pullback attractors for universe of fixed bounded sets and for another universe given by a tempered condition in spaces $\widehat{H}$ and $\widehat{V}$, respectively. Also, we reveal the regularity result of the pullback attractors by showing that these two attractors coincide with each other.

For convenience, in the sequel, we denote by X the space $\widehat{H}$ or $\widehat{V}$, and by $\mathcal{P}(X)$ the family of all nonempty subsets of X. Let $\mathcal{D}$ be a given nonempty class of families parameterized in time $\widehat{D}=\{D(t)|\, t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$. We will denote by $\mathcal{D}$ a universe in $\mathcal{P}(X)$, if $\hat{D}_1\in\mathcal{D}$ and $D_2(t)\subset D_1(t)$ for all $t\in\mathbb{R}$, then $\hat{D}_2\in\mathcal{D}$. Furthermore, we introduce some definitions related to the pullback attractors. One can refer to [**5, 15, 16, 29, 30, 34**] for general definitions and theory, as well as the applications of the theory to [**2, 17, 28, 27, 31, 37**].

Definition 3.1.

(1) *A family of sets* $\widehat{D}_0=\{D_0(t)|\, t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is called pullback* $\mathcal{D}$-*absorbing for the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in* X *if for any* $t\in\mathbb{R}$ *and any* $\widehat{D}=\{D(t)|\, t\in\mathbb{R}\}\in\mathcal{D}$, *there exists a* $\tau_0(t,\widehat{D})\leqslant t$ *such that* $U(t,\tau)D(\tau)\subseteq D_0(t)$ *for all* $\tau\leqslant\tau_0(t,\widehat{D})$.

(2) *The process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is said to be pullback* $\widehat{D}_0$-*asymptotically compact in* X *if for any* $t\in\mathbb{R}$, *any sequences* $\{\tau_n\}\subseteq(-\infty,t]$ *and* $\{x_n\}\subseteq X$ *satisfying* $\tau_n\to-\infty$ *as* $n\to\infty$ *and* $x_n\in D_0(\tau_n)$ *for all* n, *the sequence* $\{U(t,\tau_n;x_n)\}$ *is relatively compact in* X. $\{U(t,\tau)\}_{t\geqslant\tau}$ *is called pullback* $\mathcal{D}$-*asymptotically compact in* X *if it is pullback* $\widehat{D}$-*asymptotically compact for any* $\widehat{D}\in\mathcal{D}$.

(3) *A family of sets* $\widehat{\mathcal{A}}_{\mathcal{D}}=\{\mathcal{A}_{\mathcal{D}}(t)|\, t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is called a pullback* $\mathcal{D}$-*attractor for the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *on* X *if it has the following properties:*

- Compactness*: for any* $t\in\mathbb{R},\mathcal{A}_{\mathcal{D}}(t)$ *is a nonempty compact subset of* X*;*
- Invariance*:* $U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau)=\mathcal{A}_{\mathcal{D}}(t),\ \forall\, t\geqslant\tau$*;*
- Pullback attracting*:* $\widehat{\mathcal{A}}_{\mathcal{D}}$ *is pullback* $\mathcal{D}$-*attracting in the following sense:*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X\left(U(t,\tau)D(\tau),\mathcal{A}_{\mathcal{D}}(t)\right)=0,\ \forall\widehat{D}=\{D(s)|\, s\in\mathbb{R}\}\in\mathcal{D},t\in\mathbb{R},$$

where $\mathrm{dist}_X(Y,Z):=\sup_{y\in Y}\inf_{z\in Z}\mathrm{dist}_X(y,z)$ *means the Hausdorff semi-distance*

from $Y\subseteq X$ *to* $Z\subseteq X$ *in the metric space* X.

- Minimality*: the family of sets* $\widehat{\mathcal{A}}_{\mathcal{D}}$ *is minimal in the sense that if* $\widehat{O}=\{O(t)|\, t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ *is another family of closed sets such that*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),O(t))=0,\quad \textit{for any } \widehat{D}=\{D(t)|\, t\in\mathbb{R}\}\in\mathcal{D},$$

then $\mathcal{A}_{\mathcal{D}}(t)\subseteq O(t)$ *for* $t\in\mathbb{R}$.

To guarantee the existence of pullback attractors, we need the function $G(x,t)$ satisfies the following assumption:

$$\text{(H1)}\quad G(x,t)\in L^2_{\rm loc}(\mathbb{R};\widehat{H})\ \text{and}\ \int_{-\infty}^{0}\mathrm{e}^{c_3 s}\|G(s)\|^2\mathrm{d}s<+\infty.$$

It is not difficult to check that the second condition of (H1) is equivalent to

$$\int_{-\infty}^{t}\mathrm{e}^{-c_3(t-s)}\|G(s)\|^2\mathrm{d}s<+\infty,\ \ \forall\, t\in\mathbb{R}. \tag{3.1}$$

3.1. Preliminary results.

Before to prove the existence of pullback attractors, we first recall and establish some useful lemmas in the subsection, which play important roles in proving the existence, regularity, tempered behavior and H^2 boundedness of the pullback attractors.

Lemma 3.1. (See [**33**].) *Let X, Y be two Banach spaces such that X is reflexive, and the inclusion $X\subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty([t_0,T];X)$ such that $u_n\rightharpoonup u$ weakly in $L^q([t_0,T];X)$ for some $q\in[1,+\infty)$ and $u\in\mathcal{C}([t_0,T];Y)$. Then $u(t)\in X$ and $\|u(t)\|_X\leqslant\liminf_{n\to\infty}\|u_n\|_{L^\infty([t_0,T];X)}$ for all $t\in[t_0,T]$.*

Lemma 3.2. *Let $G\in L^2_{loc}(\mathbb{R};\widehat{H})$ and w be the solution of equations* (2.8)-(2.9) *with initial value $w_\tau\in\widehat{H}$, then*

$$\|w(t)\|^2\leqslant\|w_\tau\|^2\mathrm{e}^{-c_3(t-\tau)}+c_4\mathrm{e}^{-c_3 t}\int_\tau^t\mathrm{e}^{c_3 s}\|G(s)\|^2\mathrm{d}s,\ \ \textit{for}\ t\geqslant\tau, \tag{3.2}$$

$$\|w(t)\|^2+\delta\int_\tau^t\|w(s)\|^2_{\widehat{V}}\mathrm{d}s\leqslant\|w_\tau\|^2+c_4\int_\tau^t\|G(s)\|^2\mathrm{d}s,\ \ \textit{for}\ t\geqslant\tau, \tag{3.3}$$

where c_3 and c_4 are positive constants depending only on ν,α and Ω.

Proof. Let $w(t)$ be the solution of equations (2.8)-(2.9) with initial value $w_\tau\in\widehat{H}$. Following the same derivation of (2.22) in [**24**], we can also obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+c_3\|w(t)\|^2\leqslant c_4\|G(t)\|^2, \tag{3.4}$$

where

$$c_3:=\min\{\nu\tilde{\lambda}_1,\alpha\lambda_1\}\quad\text{and}\quad c_4:=\max\{\nu^{-1}\tilde{\lambda}_1^{-1},\alpha^{-1}\lambda_1^{-1}\},$$

and $\tilde{\lambda}_1>0$ is the first eigenvalue of the Stokes operator $-P\Delta$ in H (see [**35**]). Note that P is the orthogonal projection from $L^2(\Omega)\times L^2(\Omega)$ to H with domain $V\cap(H^2(\Omega))^2$, and $\tilde{\lambda}_1$ a constant depending only on Ω. Changing the variable t of (3.4) by s, multiplying it by $\mathrm{e}^{-c_3(t-s)}$, and integrating it from $s=\tau$ to $s=t$, we have

$$\|w(t)\|^2\leqslant\|w_\tau\|^2\mathrm{e}^{-c_3(t-\tau)}+c_4\mathrm{e}^{-c_3 t}\int_\tau^t\mathrm{e}^{c_3 s}\|G(s)\|^2\mathrm{d}s. \tag{3.5}$$

Hence, the inequality (3.2) holds. The inequality (3.3) can be proved similarly as that of (2.24) in [**24**]. Here we omit the details. □

From now on, we use $\mathcal{D}^{\widehat{H}}$ to denote the class of all families of nonempty subsets $\widehat{D}=\{D(t)|\,t\in\mathbb{R}\}\subseteq\mathcal{P}(\widehat{H})$ such that

$$\lim_{\tau\to-\infty}\big(\mathrm{e}^{c_3\tau}\sup_{w\in D(\tau)}\|w\|^2\big)=0. \tag{3.6}$$

Also, we denote by $\mathcal{D}_F^{\widehat{H}}$ the class of families $\widehat{D}=\{D(t)=D|\, t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of $\widehat{H}$. Evidently, we have $\mathcal{D}_F^{\widehat{H}}\subseteq\mathcal{D}^{\widehat{H}}$.

LEMMA 3.3. *Assume* (H1) *holds. Then for any* $t\in\mathbb{R}$ *and* $\widehat{D}=\{D(t)|\, t\in\mathbb{R}\}\in\mathcal{D}^{\widehat{H}}$*, there exists some* $\tau_0(\widehat{D},t)<t-3$*, such that for any* $\tau\leqslant\tau_0(\widehat{D},t)$ *and* $w_\tau\in D(\tau)$*, there hold*

$$\|w(r;\tau,w_\tau)\|^2\leqslant\rho_1(t),\quad \forall r\in[t-3,t], \tag{3.7}$$

$$\|w(r;\tau,w_\tau)\|^2_{\widehat{V}}\leqslant\rho_2(t),\quad \forall r\in[t-2,t], \tag{3.8}$$

$$\int_{r-1}^{r}\|Aw(\theta;\tau,w_\tau)\|^2\mathrm{d}\theta\leqslant\rho_3(t),\quad \forall r\in[t-1,t], \tag{3.9}$$

$$\int_{r-1}^{r}\|w'(\theta;\tau,w_\tau)\|^2\mathrm{d}\theta\leqslant\rho_4(t),\quad \forall r\in[t-1,t], \tag{3.10}$$

where

$$\rho_1(t):=1+c_4\mathrm{e}^{c_3(3-t)}\int_{-\infty}^{t}\mathrm{e}^{c_3\theta}\|G(\theta)\|^2 d\theta, \tag{3.11}$$

$$\begin{aligned}\rho_2(t):=\max_{r\in[t-2,t]}&\Big\{c_2c_5\big(\rho_1(r)+\int_{r-1}^{r}\|G(\theta)\|^2\mathrm{d}\theta\big)\\ &\times\exp\big\{c_6\big[\big(\rho_1(r)+\int_{r-1}^{r}\|G(\theta)\|^2\mathrm{d}\theta\big)^2+1\big]\big\}\Big\},\end{aligned} \tag{3.12}$$

$$\rho_3(t):=c_7\big(2\rho_2(t)+\rho_1(t)\rho_2^2(t)+\int_{t-2}^{t}\|G(\theta)\|^2\mathrm{d}\theta\big), \tag{3.13}$$

$$\rho_4(t):=\max\{4,2c_1^{-1},c_8\}\big(\rho_2(t)+\rho_2(t)\rho_3(t)+\int_{t-2}^{t}\|G(\theta)\|^2\mathrm{d}\theta\big), \tag{3.14}$$

and all c_i *are positive constants.*

PROOF. Obviously, (3.7) can be deduced from (3.2). To prove (3.8)-(3.10), we need a higher regularity of the solutions. Hence, we consider the Galerkin approximate solutions. For each integer $n\geqslant 1$, let us denote by

$$w_n(t)=w_n(t;\tau,w_\tau):=\sum_{j=1}^{n}\xi_{nj}(t)e_j, \tag{3.15}$$

the Galerkin approximation of the solution $w(t)$ of equations (2.8)-(2.9), where $\xi_{nj}(t)$ is the solution of the following Cauchy problem of ODEs:

$$\frac{\mathrm{d}}{\mathrm{d}t}(w_n(t),e_j)+\langle Aw_n(t)+B(u_n(t),w_n(t))+N(w_n(t)),e_j\rangle=(G(t),e_j), \tag{3.16}$$

$$(w_n(\tau),e_j)=(w_\tau,e_j),\ \ j=1,2,\cdots,n, \tag{3.17}$$

here $\{e_j:j\geqslant 1\}\subseteq D(A)$, which forms a Hilbert basis of $\widehat{V}$ and is orthonormal in $\widehat{H}$. Multiplying equation (3.16) by $A\xi_{nj}(t)$ and summing them for $j=1$ to n, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n,w_n\rangle+\|Aw_n\|^2+\langle B(u_n,w_n),Aw_n\rangle+\langle N(w_n),Aw_n\rangle&\\ =(G(t),Aw_n).&\end{aligned} \tag{3.18}$$

By (2.4) and the facts

$$\|u_n\|^2 \leqslant \|w_n\|^2 \quad \text{and} \quad \|\nabla u_n\|^2 \leqslant \|w_n\|^2_{\widehat{V}}, \tag{3.19}$$

we can use the Young inequality to obtain

$$\begin{aligned} -\langle B(u_n,w_n),Aw_n\rangle &\leqslant |\langle B(u_n,w_n),Aw_n\rangle| \leqslant \lambda\|u_n\|^{\frac12}\|\nabla u_n\|^{\frac12}\|\nabla w_n\|^{\frac12}\|Aw_n\|^{\frac32} \\ &\leqslant \frac14\|Aw_n\|^2+\lambda^4\|w_n\|^2\|w_n\|^4_{\widehat{V}}. \end{aligned} \tag{3.20}$$

Replacing the variable t with θ, we follows from (2.6) and (3.18)-(3.20) that

$$\begin{aligned} \frac12\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n,w_n\rangle &\leqslant -\frac34\|Aw_n\|^2+\|G(\theta)\|^2-\langle B(u_n,w_n),Aw_n\rangle-\langle N(w_n),Aw_n\rangle \\ &\leqslant -\frac34\|Aw_n\|^2+\|G(\theta)\|^2+\frac14\|Aw_n\|^2+\lambda^4\|w_n\|^2\|w\|^4_{\widehat{V}} \\ &\quad +2c^2(\nu_r)\|w_n\|^2_{\widehat{V}}+\frac12\|Aw_n\|^2 \\ &= \|G(\theta)\|^2+\|w_n\|^2_{\widehat{V}}\left(\lambda^4\|w_n\|^2\|w_n\|^2_{\widehat{V}}+2c^2(\nu_r)\right). \end{aligned} \tag{3.21}$$

By (2.1) and (3.21), we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n,w_n\rangle \leqslant 2\|G(\theta)\|^2+\langle Aw_n,w_n\rangle\left(2c_2\lambda^4\|w\|^2\|w_n\|^2_{\widehat{V}}+4c_2c^2(\nu_r)\right). \tag{3.22}$$

Let us set

$$\begin{aligned} &H_n(\theta):=\langle Aw_n(\theta),w_n(\theta)\rangle, \quad I(\theta):=2\|G(\theta)\|^2 \\ &K_n(\theta):=2c_2\lambda^4\|w_n(\theta)\|^2\|w_n(\theta)\|^2_{\widehat{V}}+4c_2c^2(\nu_r). \end{aligned}$$

Then (3.22) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H_n(\theta) \leqslant K_n(\theta)H_n(\theta)+I(\theta). \tag{3.23}$$

Using Gronwall inequality to (3.23), for all $\tau \leqslant r-1 \leqslant s \leqslant r$, we have

$$H_n(r) \leqslant \big(H_n(s)+\int_{r-1}^r I(\theta)\mathrm{d}\theta\big)\exp\big\{\int_{r-1}^r K_n(\theta)\mathrm{d}\theta\big\}. \tag{3.24}$$

Integrating (3.24) from $s=r-1$ to $s=r$, we can obtain

$$H_n(r) \leqslant \big(\int_{r-1}^r H_n(s)\mathrm{d}s+\int_{r-1}^r I(\theta)\mathrm{d}\theta\big)\exp\big\{\int_{r-1}^r K_n(\theta)\mathrm{d}\theta\big\}. \tag{3.25}$$

In addition, by (2.1) and (3.3), we have

$$\begin{aligned} \int_{r-1}^r H_n(s)\mathrm{d}s+\int_{r-1}^r I(\theta)\mathrm{d}\theta &= \int_{r-1}^r \langle Aw_n(s),w_n(s)\rangle\mathrm{d}s+\int_{r-1}^r 2\|G(\theta)\|^2\mathrm{d}\theta \\ &\leqslant {c_1}^{-1}\int_{r-1}^r \|w_n(s)\|^2_{\widehat{V}}\mathrm{d}s+2\int_{r-1}^r\|G(\theta)\|^2\mathrm{d}\theta \\ &\leqslant c_5\big(\|w_n(r-1)\|^2+\int_{r-1}^r\|G(\theta)\|^2\mathrm{d}\theta\big), \end{aligned} \tag{3.26}$$

where $c_5 := \max\{c_1^{-1}\delta^{-1}, 2 + c_4 c_1^{-1}\delta^{-1}\}$. Also, by (3.3), we see that

$$\begin{aligned}
\int_{r-1}^{r} K_n(\theta)\mathrm{d}\theta &= \int_{r-1}^{r} \big(2c_2\lambda^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2 + 4c_2c^2(\nu_r)\big)\mathrm{d}\theta \\
&\leqslant 2c_2\lambda^4 \int_{r-1}^{r} \big[\big(\|w_n(r-1)\|^2 \\
&\quad + c_4\int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big)\|w_n(\theta)\|_{\widehat{V}}^2\big]\mathrm{d}\theta + 4c_2c^2(\nu_r) \\
&\leqslant 2c_2\lambda^4\big(\|w_n(r-1)\|^2 + c_4\int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big)\int_{r-1}^{r} \|w_n(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta + 4c_2c^2(\nu_r) \\
&\leqslant 2c_2\lambda^4\big(\|w_n(r-1)\|^2 + c_4\int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big) \\
&\quad \times \big(\frac{\|w_n(r-1)\|^2}{\delta} + c_4\delta^{-1}\int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big) + 4c_2c^2(\nu_r) \\
&\leqslant c_6\Big[\big(\|w_n(r-1)\|^2 + \int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big)^2 + 1\Big], \qquad (3.27)
\end{aligned}$$

where $c_6 := \max\big\{2c_2\lambda^4 \cdot \max\big\{1, \delta^{-2}, c_4^2\delta^{-2}, c_4^2\big\},\ 4c_2c^2(\nu_r)\big\}$. Then, by (2.1) and (3.25)-(3.27), we conclude that for any $r \in [t-2, t]$ and $\tau \leqslant \tau_0(\hat{D}, t)$,

$$\begin{aligned}
\|w_n(r)\|_{\widehat{V}}^2 \leqslant c_2H_n(r) &\leqslant c_2c_5\big(\|w_n(r-1)\|^2 + \int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big) \\
&\quad \times \exp\big\{c_6\big[\big(\|w_n(r-1)\|^2 + \int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta\big)^2 + 1\big]\big\}. \qquad (3.28)
\end{aligned}$$

On the other hand, by (2.6) and (3.18)-(3.20), we have, replacing the variable t with θ,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n, w_n\rangle &\leqslant \frac{1}{4}\|Aw_n\|^2 + \|G(\theta)\|^2 - \|Aw_n\|^2 - \langle B(u_n, w_n), Aw_n\rangle - \langle N(w_n), Aw_n\rangle \\
&\leqslant -\frac{3}{4}\|Aw_n\|^2 + \|G(\theta)\|^2 + \frac{1}{4}\|Aw_n\|^2 + \lambda^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4 \\
&\quad + \frac{1}{4}\|Aw_n\|^2 + c^2(\nu_r)\|w_n\|_{\widehat{V}}^2 \\
&= -\frac{1}{4}\|Aw_n\|^2 + \|G(\theta)\|^2 + \lambda^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4 + c^2(\nu_r)\|w_n\|_{\widehat{V}}^2,
\end{aligned}$$

which implies

$$2\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n, w_n\rangle + \|Aw_n\|^2 \leqslant 4\|G(\theta)\|^2 + 4\lambda^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4 + 4c^2(\nu_r)\|w_n\|_{\widehat{V}}^2. \qquad (3.29)$$

Integrating (3.29) from $r-1$ to r and using (2.1), we have

$$\begin{aligned}
\int_{r-1}^{r} \|Aw_n(\theta)\|^2\mathrm{d}\theta &\leqslant c_7\big(1 + \sup_{\theta\in[r-1,r]} \|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2\big)\int_{r-1}^{r} \|w_n(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta \\
&\quad + c_7\int_{r-1}^{r} \|G(\theta)\|^2\mathrm{d}\theta + c_7\|w_n(r-1)\|_{\widehat{V}}^2, \qquad (3.30)
\end{aligned}$$

for $\tau \leqslant r-1$, where $c_7 := \max\{4\lambda^4, 4c^2(\nu_r), 4, 2c_1^{-1}\}$.

Finally, multiplying equation (3.16) by $\xi'_{nj}(t)$, summing them from $j=1$ to n and replacing the variable t with θ, we obtain

$$\|w_n'(\theta)\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n(\theta),w_n(\theta)\rangle+\langle B(u_n(\theta),w_n(\theta)),w_n'(\theta)\rangle+\langle N(w_n(\theta)),w_n'(\theta)\rangle$$
$$=(G(\theta),w_n'(\theta))\leqslant\frac{1}{4}\|w_n'(\theta)\|^2+\|G(\theta)\|^2. \tag{3.31}$$

Moreover, by (2.2), (2.4) and the fact (3.19), we obtain

$$\begin{aligned}-\langle B(u_n(\theta),w_n(\theta)),w_n'(\theta)\rangle&\leqslant\lambda\|u_n(\theta)\|^{\frac12}\|\nabla u_n(\theta)\|^{\frac12}\|\nabla w_n(\theta)\|^{\frac12}\|Aw_n(\theta)\|^{\frac12}\|w_n'(\theta)\|\\&\leqslant\frac{\lambda}{\sqrt{\delta_1\sqrt{\lambda_1}}}\|u_n(\theta)\|^{\frac12}\|\nabla u_n(\theta)\|^{\frac12}\|Aw_n(\theta)\|\|w_n'(\theta)\|\\&\leqslant\frac{\lambda}{\sqrt{\delta_1\sqrt{\lambda_1}}}\|w_n(\theta)\|_{\widehat V}\|Aw_n(\theta)\|\|w_n'(\theta)\|\\&\leqslant\frac14\|w_n'(\theta)\|^2+\frac{\lambda}{\delta_1\sqrt{\lambda_1}}\|w_n(\theta)\|^2_{\widehat V}\|Aw_n(\theta)\|^2,\end{aligned} \tag{3.32}$$

where $\delta_1:=\min\{\nu+\nu_r,\alpha\}$. Also, by (2.6), we have

$$-\langle N(w_n(\theta)),w_n'(\theta)\rangle\leqslant\frac14\|w_n'(\theta)\|^2+c^2(\nu_r)\|w_n(\theta)\|^2_{\widehat V}. \tag{3.33}$$

Then it follows from equations (3.31)-(3.33) and the Cauchy inequality that

$$\begin{aligned}&\|w_n'(\theta)\|^2+\frac12\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n(\theta),w_n(\theta)\rangle\\&\leqslant\frac14\|w_n'(\theta)\|^2+\|G(\theta)\|^2+\frac14\|w_n'(\theta)\|^2+\frac{\lambda}{\delta_1\sqrt{\lambda_1}}\|w_n(\theta)\|^2_{\widehat V}\|Aw_n(\theta)\|^2\\&\quad+\frac14\|w_n'(\theta)\|^2+c^2(\nu_r)\|w_n(\theta)\|^2_{\widehat V}\\&=\frac34\|w_n'(\theta)\|^2+\|G(\theta)\|^2+c^2(\nu_r)\|w_n(\theta)\|^2_{\widehat V}+\frac{\lambda}{\delta_1\sqrt{\lambda_1}}\|w_n(\theta)\|^2_{\widehat V}\|Aw_n(\theta)\|^2,\end{aligned}$$

that is

$$\begin{aligned}&\|w_n'(\theta)\|^2+2\frac{\mathrm{d}}{\mathrm{d}\theta}\langle Aw_n(\theta),w_n(\theta)\rangle\\&\leqslant4\|G(\theta)\|^2+c_8\|w_n(\theta)\|^2_{\widehat V}\big(1+\|Aw_n(\theta)\|^2\big),\end{aligned} \tag{3.34}$$

where $c_8:=\max\left\{4c^2(\nu_r),4\lambda\delta_1^{-1}\lambda_1^{-\frac12}\right\}$. Integrating (3.34) from $r-1$ to r and using (2.1), we see that

$$\begin{aligned}\int_{r-1}^r\|w_n'(\theta)\|^2\mathrm{d}\theta&\leqslant\frac{2}{c_1}\|w_n(r-1)\|^2_{\widehat V}+4\int_{r-1}^r\|G(\theta)\|^2\mathrm{d}\theta\\&\quad+c_8\sup_{\theta\in[r-1,r]}\|w_n(\theta)\|^2_{\widehat V}\Big(1+\int_{r-1}^r\|Aw_n(\theta)\|^2\mathrm{d}\theta\Big),\end{aligned} \tag{3.35}$$

for any $r\in[t-1,t]$ and $\tau\leqslant\tau_0(\hat D,t)$. Note that reference [**24**] has proved the facts that $w_n(\cdot;\tau,w_\tau)\rightharpoonup w(\cdot;\tau,w_\tau)$ weakly in $L^2([t-3,t];D(A))$, $w_n'(\cdot;\tau,w_\tau)\rightharpoonup w'(\cdot;\tau,w_\tau)$ weakly in $L^2([t-3,t];\widehat H)$, and $w(\cdot;\tau,w_\tau)\in\mathcal{C}([t-3,t];V)$. By Lemma 3.1, we can pass to the limit in (3.28), (3.30) and (3.35) to obtain the inequalities (3.8)-(3.10). The proof is complete. □

Note that if (H1) hold, then we have $\lim_{t\to-\infty} \mathrm{e}^{c_3 t}\rho_1(t) = 0$.

3.2. Pullback attractors in space $\widehat{H}$.
We now prove the existence and regularity of the pullback attractors for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in space $\widehat{H}$.

According to the estimate (3.2), we can easily obtain the following lemma.

LEMMA 3.4. *Assume* (H1) *holds. Then the family of sets* $\widehat{D}_0 = \{D_0(t) | \, t \in \mathbb{R}\}$ *with* $D_0(t) = \overline{\mathcal{B}}_{\widehat{H}}\big(0, \mathcal{R}^{\frac{1}{2}}_{\widehat{H}}(t)\big)$ *is pullback* $\mathcal{D}^{\widehat{H}}$*-absorbing for the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in* $\widehat{H}$, *where*

$$\bar{\mathcal{B}}_{\widehat{H}}\big(0, \mathcal{R}^{\frac{1}{2}}_{\widehat{H}}(t)\big) := \big\{w \in \widehat{H} | \, \|w\| \leqslant \mathcal{R}^{\frac{1}{2}}_{\widehat{H}}(t)\big\} \tag{3.36}$$

is a closed ball in $\widehat{H}$ *and*

$$\mathcal{R}_{\widehat{H}}(t) := 1 + c_4 \mathrm{e}^{-c_3 t} \int_{-\infty}^{t} \mathrm{e}^{c_3 s} \|G(s)\|^2 \mathrm{d}s. \tag{3.37}$$

LEMMA 3.5. *Assume* (H1) *holds. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}^{\widehat{H}}$*-asymptotically compact in* $\widehat{H}$.

PROOF. The proof is similar to that of Lemma 3.8 in §3.3 and we omit the details here. □

By Lemmas 3.4 and 3.5, we can use the abstract result of [**16**] to obtain the existence of the pullback attractors for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in space $\widehat{H}$.

THEOREM 3.6. *Assume* (H1) *holds. Then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *defined by* (2.11) *in* $\widehat{H}$ *possesses the minimal pullback* $\mathcal{D}^{\widehat{H}}_F$*- and* $\mathcal{D}^{\widehat{H}}$*-attractors*

$$\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}_F} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H}}_F}(t) | \, t \in \mathbb{R}\} \quad \textit{and} \quad \widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t) | \, t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H}},$$

respectively. Furthermore,

$$\mathcal{A}_{\mathcal{D}^{\widehat{H}}_F}(t) \subseteq \mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t) \subseteq \overline{\mathcal{B}}_{\widehat{H}}\big(0, \mathcal{R}^{\frac{1}{2}}_{\widehat{H}}(t)\big), \ \forall t \in \mathbb{R}.$$

3.3. Pullback attractors in space $\widehat{V}$.
Hereinafter, we will use $\mathcal{D}^{\widehat{H},\widehat{V}}$ to denote the class of all families $\widehat{D}_{\widehat{V}}$ of elements of $\mathcal{P}(\widehat{V})$ of the form

$$\widehat{D}_{\widehat{V}} = \{D_{\widehat{V}}(t) = D(t) \cap \widehat{V} | \, t \in \mathbb{R}\} \text{ with } \widehat{D} = \{D(t) | \, t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H}}.$$

Also, we use $\mathcal{D}^{\widehat{V}}_F$ to denote the universe of nonempty fixed bounded subsets of $\widehat{V}$. It is clear that both classes $\mathcal{D}^{\widehat{H},\widehat{V}}$ and $\mathcal{D}^{\widehat{V}}_F$ are universes in $\mathcal{P}(\widehat{V})$ and $\mathcal{D}^{\widehat{V}}_F \subseteq \mathcal{D}^{\widehat{H},\widehat{V}}$. Moreover, let us denote by $\big\{\bar{\mathcal{B}}_{\widehat{H}}\big(0, \rho_1^{\frac{1}{2}}(t)\big) | \, t \in \mathbb{R}\big\}$ the family of closed balls in $\widehat{H}$ centered at zero and with radius $\rho_1^{\frac{1}{2}}(t)$, then

$$\big\{\bar{\mathcal{B}}_{\widehat{H}}\big(0, \rho_1^{\frac{1}{2}}(t)\big) | \, t \in \mathbb{R}\big\} \in \mathcal{D}^{\widehat{H}}.$$

According to the above notations and Lemma 3.3, we can easily derive the following existence of bounded family of pullback $\mathcal{D}^{\widehat{H},\widehat{V}}$-absorbing sets for $\{U(t,\tau)\}_{t\geqslant\tau}$ in space $\widehat{V}$.

LEMMA 3.7. *Assume* (H1) *holds, then the family of sets*

$$\widehat{D}_{0,\widehat{V}} := \big\{D_{0,\widehat{V}}(t) = \bar{\mathcal{B}}_{\widehat{H}}\big(0, \rho_1^{\frac{1}{2}}(t)\big) \cap \widehat{V} | \, t \in \mathbb{R}\big\} \in \mathcal{D}^{\widehat{H},\widehat{V}}. \tag{3.38}$$

Furthermore, for any $t \in \mathbb{R}$, $\widehat{D}_{\widehat{V}} = \{D_{\widehat{V}}(t) = D(t) \cap \widehat{V} | \, t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H},\widehat{V}}$ *with any* $\widehat{D} = \{D(t)| \; t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H}}$, *there exists a* $\tau_1(\widehat{D}, t) < t$ *such that*

$$U(t,\tau)D(\tau) \subseteq \widehat{D}_{0,\widehat{V}}(t), \;\; \forall \tau \leqslant \tau_1(\widehat{D}, t). \tag{3.39}$$

Next, we prove the pullback asymptotically compact of $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{V}$ for the universe $\mathcal{D}^{\widehat{H},\widehat{V}}$ by using the enstrophy equality.

LEMMA 3.8. *Assume* (H1) *holds, then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}^{\widehat{H},\widehat{V}}$*-asymptotically compact in space* $\widehat{V}$.

PROOF. Let us fix some $t \in \mathbb{R}$ and consider any family $\widehat{D}_{\widehat{V}} = \{D_{\widehat{V}}(t) = D(t) \cap \widehat{V} | \, t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H},\widehat{V}}$, any sequences $\{\tau_n\} \subseteq (-\infty, t]$ and $\{w_{\tau_n}\} \subseteq \widehat{V}$, satisfying $\tau_n \to -\infty$ as $n \to +\infty$, and $w_{\tau_n} \in D_{\widehat{V}}(\tau_n) = D(\tau_n) \cap \widehat{V}$ for all n. Our goal is to show that the sequence $\{w^{(n)}(t)\}$ defined by

$$w^{(n)}(\cdot) := w^{(n)}(\cdot; \tau_n, w_{\tau_n}) = U(\cdot, \tau_n; w_{\tau_n}). \tag{3.40}$$

is relatively compact in $\widehat{V}$.

By Lemma 3.3 we see that there exist a $\tau_0(\widehat{D}_{\widehat{V}}, t) < t-3$ such that the subsequence $\{w^{(n)}(\cdot)| \; \tau_n \leqslant \tau_0(\widehat{D}_{\widehat{V}}, t)\}$ is uniformly bounded in

$$L^\infty([t-2,t]; \widehat{V}) \cap L^2([t-2,t]; D(A)),$$

and $\{(w^{(n)})'(\cdot)\}$ is uniformly bounded in $L^2([t-2,t]; \widehat{H})$. Then, following the standard diagonal procedure, there exists a function $w(\cdot)$ such that (by extracting a subsequence if necessary)

$$w^{(n)}(\cdot) \rightharpoonup^* w(\cdot) \;\text{ weakly star in }\; L^\infty([t-2,t]; \widehat{V}), \tag{3.41}$$

$$w^{(n)}(\cdot) \rightharpoonup w(\cdot) \;\text{ weakly in }\; L^2([t-2,t]; D(A)), \tag{3.42}$$

$$(w^{(n)})'(\cdot) \rightharpoonup w'(\cdot) \;\text{ weakly in }\; L^2([t-2,t]; \widehat{H}). \tag{3.43}$$

By the Aubin-Lions compactness theorem (see, e.g. [**19, 4, 36**]) and the compact embedding $D(A) \hookrightarrow \widehat{V} \hookrightarrow \widehat{H}$, it follows from (3.42)-(3.43) that

$$w^{(n)}(\cdot) \longrightarrow w(\cdot) \;\text{ strongly in }\; \widehat{V}, \;\text{ a.e. on}\, [t-2,t], \tag{3.44}$$

$$w^{(n)}(\cdot) \in \mathcal{C}\big([t-2,t]; \widehat{V}\big), \;\; w(\cdot) \in \mathcal{C}\big([t-2,t]; \widehat{V}\big), \tag{3.45}$$

and $\{w^n(\cdot)\}$ is uniformly bounded in $\mathcal{C}([t-2,t]; \widehat{V})$. According to (3.41)-(3.44), we see that w satisfies equation (2.8). Since

$$w^{(n)}(s_2) - w^{(n)}(s_1) = \int_{s_1}^{s_2} (w^{(n)})'(r)\mathrm{d}r \;\text{ in }\; \widehat{H}, \;\; \forall s_1, s_2 \in [t-2,t], \tag{3.46}$$

and $\{(w^{(n)})'\}$ is uniformly bounded in $L^2([t-2,t]; \widehat{H})$, we conclude that $\{w^{(n)}(\cdot)\}$ is equicontinuous on the interval $[t-2,t]$ with values in $\widehat{H}$. Thus, by the Ascoli-Arzelá theorem, we have

$$w^{(n)}(\cdot) \longrightarrow w(\cdot) \quad \text{strongly in}\; \mathcal{C}([t-2,t]; \widehat{H}). \tag{3.47}$$

Then, by the uniform boundedness of $\{w^{(n)}(\cdot)\}$ in $\mathcal{C}([t-2,t];\widehat{V})$ and (3.47), for any sequence $\{s_n\}\subseteq[t-2,t]$ with $s_n\to s_*$ as $n\to\infty$, there holds

$$w^{(n)}(s_n)\rightharpoonup w(s_*)\ \text{ weakly in } \widehat{V}. \tag{3.48}$$

Next, we claim that

$$w^{(n)}(\cdot)\longrightarrow w(\cdot)\ \text{ strongly in } \mathcal{C}([t-1,t];\widehat{V}), \tag{3.49}$$

which implies the relative compactness of $w^{(n)}(t)$ in $\widehat{V}$. Suppose (3.49) is false, then there exists an $\epsilon_0>0$ and a sequence $\{t_n\}\subset[t-1,t]$ satisfying $t_n\to t_*$ such that

$$\|w^{(n)}(t_n)-w(t_*)\|_{\widehat{V}}\geqslant\epsilon_0,\ \forall\, n\geqslant 1. \tag{3.50}$$

Since the norm $\|w\|_{\widehat{V}}$ is equivalent to the norm induced by $\langle Aw,w\rangle$ (see (2.1)), we can assume that

$$\langle A(w^{(n)}(t_n)-w(t_*)),w^{(n)}(t_n)-w(t_*)\rangle\geqslant\epsilon_0,\ \forall\, n\geqslant 1. \tag{3.51}$$

Then, by (3.48) we have

$$\langle Aw(t_*),w(t_*)\rangle\leqslant\liminf_{n\to\infty}\langle Aw^{(n)}(t_n),w^{(n)}(t_n)\rangle. \tag{3.52}$$

On the other hand, similar to the derivation of (3.29) and using the enstrophy equality as that as (2.10) for $w^{(n)}$ and w, we have

$$\begin{aligned}
&\langle Aw^{(n)}(s_2),w^{(n)}(s_2)\rangle+\frac{1}{2}\int_{s_1}^{s_2}\|Aw^{(n)}(\theta)\|^2\mathrm{d}\theta\\
&\leqslant\langle Aw^{(n)}(s_1),w^{(n)}(s_1)\rangle+2\int_{s_1}^{s_2}\|G(\theta)\|^2\mathrm{d}\theta+2c^2(\nu_r)\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\quad+2\lambda^4\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|^2\|w^{(n)}(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta\\
&\leqslant\langle Aw^{(n)}(s_1),w^{(n)}(s_1)\rangle+c_9\int_{s_1}^{s_2}\|G(\theta)\|^2\mathrm{d}\theta+c_9\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\quad+c_9\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta,\quad\text{for } t-2\leqslant s_1\leqslant s_2\leqslant t,
\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}
&\langle Aw(s_2),w(s_2)\rangle+\frac{1}{2}\int_{s_1}^{s_2}\|Aw(\theta)\|^2\mathrm{d}\theta\\
&\leqslant\langle Aw(s_1),w(s_1)\rangle+2\int_{s_1}^{s_2}\|G(\theta)\|^2\mathrm{d}\theta+2c^2(\nu_r)\int_{s_1}^{s_2}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\quad+2\lambda^4\int_{s_1}^{s_2}\|w(\theta)\|^2\|w(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta\\
&\leqslant\langle Aw(s_1),w(s_1)\rangle+c_9\int_{s_1}^{s_2}\|G(\theta)\|^2\mathrm{d}\theta+c_9\int_{s_1}^{s_2}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta\\
&\quad+c_9\int_{s_1}^{s_2}\|w(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta,
\end{aligned} \tag{3.54}$$

where $c_9:=\max\left\{2,\ 2c^2(\nu_r),\ 2\lambda^4\rho_1(t)\right\}$. Now, for $s\in[t-2,t]$, let us define

$$\Gamma_n(s) := \left\langle Aw^{(n)}(s), w^{(n)}(s)\right\rangle - c_9\int_{t-2}^{s}\|G(\theta)\|^2\mathrm{d}\theta - c_9\int_{t-2}^{s}\|w^{(n)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta$$

$$- c_9\int_{t-2}^{s}\|w^{(n)}(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta, \tag{3.55}$$

$$\Gamma(s) := \langle Aw(s), w(s)\rangle - c_9\int_{t-2}^{s}\|G(\theta)\|^2\mathrm{d}\theta - c_9\int_{t-2}^{s}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta$$

$$- c_9\int_{t-2}^{s}\|w(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta. \tag{3.56}$$

It is obvious from the regularity of w and all $w^{(n)}$ that $\Gamma_n(\cdot)$ and $\Gamma(\cdot)$ are continuous on $[t-2,t]$. By (3.53), for any $t-2\leqslant s_1\leqslant s_2\leqslant t$, we have

$$\begin{aligned}
&\Gamma_n(s_2)-\Gamma_n(s_1)\\
&= \left\langle Aw^{(n)}(s_2), w^{(n)}(s_2)\right\rangle - \langle Aw^{(n)}(s_1), w^{(n)}(s_1)\rangle - c_9\int_{s_1}^{s_2}\|G(\theta)\|^2\mathrm{d}\theta\\
&\quad - c_9\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta - c_9\int_{s_1}^{s_2}\|w^{(n)}(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta\\
&\leqslant -\frac{1}{2}\int_{s_1}^{s_2}\|Aw^{(n)}(\theta)\|^2\mathrm{d}\theta \leqslant 0.
\end{aligned} \tag{3.57}$$

Thus, for each n, $\Gamma_n(\cdot)$ is non-increasing on $[t-2,t]$. Similarly, using the definition of $\Gamma(\cdot)$ and (3.54), we see that the function $\Gamma(s)$ is also non-increasing on $[t-2,t]$. Then, from (3.44), we obtain when $n\to\infty$ that

$$\|w^{(n)}(\cdot)\|_{\widehat{V}}^2 \longrightarrow \|w(\cdot)\|_{\widehat{V}}^2,\quad \|w^{(n)}(\cdot)\|_{\widehat{V}}^4 \longrightarrow \|w(\cdot)\|_{\widehat{V}}^4,\ \text{a.e. on } [t-2,t]. \tag{3.58}$$

Since, $w^{(n)}(\cdot)$ is a bounded sequence in $L^\infty([t-2,t];\widehat{V})$, we see that both $\|w^{(n)}(\cdot)\|_{\widehat{V}}^2$ and $\|w^{(n)}(\cdot)\|_{\widehat{V}}^4$ are bounded in $L^\infty([t-2,t])$. Then, by the Lebesgue dominated convergence theorem, it follows that

$$\int_{t-2}^{s}\|w^{(n)}(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta \longrightarrow \int_{t-2}^{s}\|w(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta,\ \text{a.e. } s\in[t-2,t],\ n\to\infty, \tag{3.59}$$

$$\int_{t-2}^{s}\|w^{(n)}(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta \longrightarrow \int_{t-2}^{s}\|w(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta,\ \text{a.e. } s\in[t-2,t],\ n\to\infty. \tag{3.60}$$

By (3.44), (3.59), (3.60) and again the equivalence between the norm $\|w\|_{\widehat{V}}$ and the norm induced by $\langle Aw, w\rangle$, we have

$$\Gamma_n(s) \longrightarrow \Gamma(s)\ \text{ a.e. } s\in[t-2,t],\ n\to\infty. \tag{3.61}$$

Therefore, there exists an increasing sequence $\{\widetilde{t}_k\}\subseteq[t-2,t_*]$ such that

$$\lim_{k\to\infty}\widetilde{t}_k = t_* \quad\text{and}\quad \lim_{n\to\infty}\Gamma_n(\widetilde{t}_k) = \Gamma(\widetilde{t}_k),\ \text{ for all } k\in\mathbb{N}. \tag{3.62}$$

By the continuity of $\Gamma(\cdot)$, for any $\epsilon>0$, there exists some k_ε such that

$$|\Gamma(\widetilde{t}_k)-\Gamma(t_*)| < \epsilon/2,\ \text{ for all } k\geqslant k_\varepsilon. \tag{3.63}$$

Also, by the non-increasing property of Γ_n and (3.62), for each $n > n(k_\varepsilon)$ we may choose $t_n > \widetilde{t}_{k_\varepsilon}$ such that

$$\begin{aligned}\Gamma_n(t_n) - \Gamma(t_*) &\leqslant \Gamma_n(\widetilde{t}_{k_\epsilon}) - \Gamma(t_*)\\ &\leqslant |\Gamma_n(\widetilde{t}_{k_\epsilon}) - \Gamma(\widetilde{t}_{k_\epsilon})| + |\Gamma(\widetilde{t}_{k_\epsilon}) - \Gamma(t_*)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{3.64}$$

Then we obtain

$$\begin{aligned}\limsup_{n\to\infty} \Gamma_n(t_n) &= \limsup_{n\to\infty} \Big(\langle Aw^{(n)}(t_n), w^{(n)}(t_n)\rangle - c_9 \int_{t-2}^{t_n} \|G(\theta)\|^2 \mathrm{d}\theta\\ &\quad - c_9 \int_{t-2}^{t_n} \|w^{(n)}(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta - c_9 \int_{t-2}^{t_n} \|w^{(n)}(\theta)\|_{\widehat{V}}^4 \mathrm{d}\theta\Big)\\ &\leqslant \Gamma(t_*) = \langle Aw(t_*), w(t_*)\rangle - c_9 \int_{t-2}^{t_*} \|G(\theta)\|^2 \mathrm{d}\theta\\ &\quad - c_9 \int_{t-2}^{t_*} \|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta - c_9 \int_{t-2}^{t_*} \|w(\theta)\|_{\widehat{V}}^4 \mathrm{d}\theta.\end{aligned} \tag{3.65}$$

By (3.44) and (3.59)-(3.60), we further obtain

$$\int_{t-2}^{t_n} \|w^{(n)}(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta \longrightarrow \int_{t-2}^{t_*} \|w(\theta)\|_{\widehat{V}}^2 \mathrm{d}\theta, \ \ n \to \infty, \tag{3.66}$$

$$\int_{t-2}^{t_n} \|w^{(n)}(\theta)\|_{\widehat{V}}^4 \mathrm{d}\theta \longrightarrow \int_{t-2}^{t_*} \|w(\theta)\|_{\widehat{V}}^4 \mathrm{d}\theta, \ \ n \to \infty. \tag{3.67}$$

Hence, it follows from (3.65)-(3.67) and the fact $\int_{t-2}^{t_n} \|G(\theta)\|^2 \mathrm{d}\theta \to \int_{t-2}^{t_*} \|G(\theta)\|^2 \mathrm{d}\theta$ that

$$\langle Aw(t_*), w(t_*)\rangle \geqslant \limsup_{n\to\infty} \langle Aw^{(n)}(t_n), w^{(n)}(t_n)\rangle. \tag{3.68}$$

Since $\widehat{V}$ is a Hilbert space, (3.52) and (3.68) give a contradiction with (3.51). Hence the claim (3.49) follows and the proof is complete. □

Combining the results of Lemmas 3.7, 3.8 and the abstract result of [**16**], we can obtain the main result of this section as follows.

THEOREM 3.9. *Assume* (H1) *holds, then the process* $\{U(t,\tau)\}_{t\geqslant\tau}$ *possesses the minimal pullback* $\mathcal{D}_F^{\widehat{V}}$*- and* $\mathcal{D}^{\widehat{H},\widehat{V}}$*-attractors*

$$\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{V}}} = \{\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t) |\, t \in \mathbb{R}\} \quad \textit{and} \quad \widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H},\widehat{V}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t) |\, t \in \mathbb{R}\},$$

respectively. Furthermore, the following statements hold.

(1) *For any* $t \in \mathbb{R}$, *we have*

$$\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t) \subseteq \mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t) \subseteq \mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t) = \mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t), \tag{3.69}$$

where $\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t) |\, t \in \mathbb{R}\}$ *and* $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}} = \{\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t) |\, t \in \mathbb{R}\}$ *are the minimal pullback* $\mathcal{D}_F^{\widehat{H}}$*- and* $\mathcal{D}^{\widehat{H}}$*-attractors of* $\{U(t,\tau)\}_{t\geqslant\tau}$ *in space* $\widehat{H}$, *which are obtained in Theorem 3.6.*

(2) *For any* $t \in \mathbb{R}$ *and* $\widehat{D} \in \mathcal{D}^{\widehat{H}}$, *there holds*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{\widehat{V}} \big(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)\big) = 0. \tag{3.70}$$

(3) *Suppose G satisfies*

$$\sup_{s\leqslant 0}\big(\mathrm{e}^{-c_3 s}\int_{-\infty}^{s}\mathrm{e}^{c_3\theta}\|G(\theta)\|^2\mathrm{d}\theta\big)<+\infty. \tag{3.71}$$

Then, for any $t\in\mathbb{R}$ and fixed bounded subset $\mathcal{B}$ of $\widehat{H}$, we have

$$\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)=\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H},\widehat{V}}}(t) \tag{3.72}$$

and

$$\lim_{\tau\to-\infty}\mathrm{dist}_{\widehat{V}}\big(U(t,\tau)\mathcal{B},\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)\big)=0. \tag{3.73}$$

Proof. By [**16**, Theorem 3.11 and Corollary 3.13] and Lemmas 3.3, 3.7-3.8, we can obtain the existence of the minimal pullback attractors $\widehat{\mathcal{A}}_{\mathcal{D}_F^{\widehat{V}}}$ and $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H},\widehat{V}}}$. Also, (3.69) follows form [**16**, Corollary 3.13 and Theorem 3.15] and Lemma 3.7. Obviously, (3.70) can be obtained immediately from (3.69). Moreover, if G satisfies (3.71), then the equality $\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)=\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)$ is a corollary of [**16**, Remark 3.14] and (3.11). Again, by [**16**, Theorem 3.15] and (3.69)-(3.70), we have $\mathcal{A}_{\mathcal{D}_F^{\widehat{V}}}(t)=\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)$ and (3.73) follows obviously. The proof is complete. □

4. Tempered behaviors of the pullback attractors

In this section, we will investigate the tempered behaviors of the pullback attractor $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ with respect to the norms of $\widehat{H}$, $\widehat{V}$ and $(H^2(\Omega))^3$ as t tends to $-\infty$.

First, we consider the tempered behavior of $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ in $\widehat{H}$ norm. Since $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}\in\mathcal{D}^{\widehat{H}}$, by Theorem 3.6, we see that the tempered behavior is given by

$$\lim_{t\to-\infty}\big(\mathrm{e}^{c_3 t}\sup_{w\in\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)}\|w\|^2\big)=0. \tag{4.1}$$

Next, for the tempered behavior of $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ in $\widehat{V}$, we have the following result.

Theorem 4.1. *Assume* (H1) *and* (3.71) *hold, then*

$$\lim_{t\to-\infty}\big(\mathrm{e}^{c_3 t}\sup_{w\in\mathcal{A}_{\mathcal{D}^{\widehat{H}}}(t)}\|w\|_{\widehat{V}}^2\big)=0. \tag{4.2}$$

Proof. First, by [**39**, Theorem 4.1], we see that the condition (3.71) is equivalent to

$$\sup_{s\leqslant 0}\int_{s-1}^{s}\|G(\theta)\|^2\mathrm{d}\theta<+\infty. \tag{4.3}$$

According to the equivalence, it is easy to verify that the tempered behavior (4.2) is a consequence of the invariance of the pullback attractor $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ and the estimates (3.8), (3.11) and (3.12). The proof is complete. □

Finally, we study the tempered behavior of $\widehat{\mathcal{A}}_{\mathcal{D}^{\widehat{H}}}$ in $(H^2(\Omega))^3$. To investigate the tempered behavior, we further assume G satisfies the condition:

(H2) $G\in W^{1,2}_{loc}(\mathbb{R};\widehat{H})$ satisfies (3.71), and

$$\lim_{t\to-\infty}\big(\mathrm{e}^{c_3 t}\|G(t)\|^2\big)=0\quad\text{and}\quad\lim_{t\to-\infty}\big(\mathrm{e}^{c_3 t}\int_{t-1}^{t}\|G'(\theta)\|^2\mathrm{d}\theta\big)=0. \tag{4.4}$$

First, we establish some estimates of a higher regularity of the solutions.

LEMMA 4.2. *Assume* (H1) *and* (H2) *hold. For any* $t \in \mathbb{R}$, $\widehat{D} = \{D(t) | t \in \mathbb{R}\} \in \mathcal{D}^{\widehat{H}}$, *there exists a* $\tau_2(\widehat{D}, t) < t - 3$ *such that for all* $r \in [t-1, t]$, *there hold*

$$\|w'(r;\tau,w_\tau)\|^2 \leqslant \rho_5(t), \ \ \forall \tau \leqslant \tau_2(\widehat{D},t), \ \ w_\tau \in D(\tau), \tag{4.5}$$

$$\|Aw(r;\tau,w_\tau)\|^2 \leqslant \rho_6(t), \ \ \forall \tau \leqslant \tau_2(\widehat{D},t), \ \ w_\tau \in D(\tau), \tag{4.6}$$

where

$$\rho_5(t) := \big(\rho_4(t) + \delta^{-1}(\lambda_1^{-1/2} + \tilde{\lambda}_1^{-1/2}) \int_{t-2}^{t} \|G'(\theta)\|^2 \mathrm{d}\theta\big) \cdot \exp\big(4\lambda^4\delta^{-1}\rho_2^2(t)\big), \tag{4.7}$$

$$\rho_6(t) := c_{11}\big(\rho_5(t) + \max_{r\in[t-1,t]} \|G(r)\|^2 + \rho_2(t) + \rho_1(t)\rho_2^2(t)\big), \tag{4.8}$$

$$c_{11} := \max\{8, 4\lambda^4, 4c^2(\nu_r)\}.$$

PROOF. Let $w_n(t) = w_n(t;\tau,w_\tau)$ be the Galerkin approximate solution defined by (3.15). Differentiating equation (3.16) with respect to t, multiplying the resulting equalities by $\xi'_{nj}(t)$ and summing them from $j = 1$ to n, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n'(\theta)\|^2 + \delta\|w_n'(\theta)\|_{\widehat{V}}^2 + \langle B(u_n'(\theta), w_n(\theta)), w_n'(\theta)\rangle \\
&\qquad + \langle B(u_n(\theta), w_n'(\theta)), w_n'(\theta)\rangle \\
&\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n'(\theta)\|^2 + \langle Aw_n'(\theta), w_n'(\theta)\rangle + \langle N(w_n'(\theta)), w_n'(\theta)\rangle \\
&\qquad + \langle B(u_n'(\theta), w_n(\theta)), w_n'(\theta)\rangle + \langle B(u_n(\theta), w_n'(\theta)), w_n'(\theta)\rangle \\
&= (G'(\theta), w_n'(\theta)) \\
&\leqslant \frac{\delta\tilde{\lambda}_1^{1/2}}{2}\|u_n'(\theta)\|^2 + \frac{\delta\lambda_1^{1/2}}{2}\|\omega_n'(\theta)\|^2 + \frac{1}{2\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\|G'(\theta)\|^2 \\
&\leqslant \frac{\delta}{2}\|w_n'(\theta)\|_{\widehat{V}}^2 + \frac{1}{2\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\|G'(\theta)\|^2,
\end{aligned} \tag{4.9}$$

where we have also used (2.7) and the Pioncaré inequality. Also, by (2.3) and the facts $\|u_n'\| \leqslant \|w_n'\|$, $\|\nabla u_n'\| \leqslant \|w_n'\|_{\widehat{V}}$, we can obtain

$$\begin{aligned}
|\langle B(u_n'(\theta), w_n(\theta)), w_n'(\theta)\rangle| &\leqslant \lambda\|u_n'(\theta)\|^{\frac{1}{2}}\|\nabla u_n'(\theta)\|^{\frac{1}{2}}\|w_n(\theta)\|^{\frac{1}{2}}\|\nabla w_n(\theta)\|^{\frac{1}{2}}\|\nabla w_n'(\theta)\| \\
&\leqslant \lambda\|w_n'(\theta)\|^{\frac{1}{2}}\|w_n'(\theta)\|_{\widehat{V}}^{\frac{1}{2}}\|w_n(\theta)\|^{\frac{1}{2}}\|w_n(\theta)\|_{\widehat{V}}^{\frac{1}{2}}\|w_n'(\theta)\|_{\widehat{V}} \\
&\leqslant \lambda\|w_n'(\theta)\|^{\frac{1}{2}}\|w_n(\theta)\|_{\widehat{V}}\|w_n'(\theta)\|_{\widehat{V}}^{\frac{3}{2}} \\
&\leqslant \frac{\delta}{4}\|w_n'(\theta)\|_{\widehat{V}}^2 + \frac{\lambda^4}{\delta}\|w_n'(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^4.
\end{aligned} \tag{4.10}$$

Similarly, we have

$$|\langle B(u_n(\theta), w_n'(\theta)), w_n'(\theta)\rangle| \leqslant \frac{\delta}{4}\|w_n'(\theta)\|_{\widehat{V}}^2 + \frac{\lambda^4}{\delta}\|w_n'(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^4. \tag{4.11}$$

Then, for a.e. $\theta > \tau$, it follows from (4.9)-(4.11) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n'(\theta)\|^2 &\leqslant \delta\|w_n'(\theta)\|_{\widehat{V}}^2 + \frac{1}{\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\|G'(\theta)\|^2 + \delta\|w_n'(\theta)\|_{\widehat{V}}^2 \\ &\quad + \frac{4\lambda^4}{\delta}\|w_n'(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^4 - 2\delta\|w_n'(\theta)\|_{\widehat{V}}^2 \\ &= \frac{1}{\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\|G'(\theta)\|^2 + \frac{4\lambda^4}{\delta}\|w_n'(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^4. \qquad (4.12)\end{aligned}$$

Integrating (4.12), for $\tau \leqslant r-1 \leqslant s \leqslant r$, we have

$$\begin{aligned}\|w_n'(r)\|^2 &\leqslant \|w_n'(s)\|^2 + \frac{1}{\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\int_{r-1}^{r}\|G'(\theta)\|^2\mathrm{d}\theta \\ &\quad + \frac{4\lambda^4}{\delta}\int_{s}^{r}\|w_n'(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta. \qquad (4.13)\end{aligned}$$

Applying Gronwall inequality to (4.13), we see that

$$\begin{aligned}\|w_n'(r)\|^2 &\leqslant \big(\|w_n'(s)\|^2 + \frac{1}{\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\int_{r-1}^{r}\|G'(\theta)\|^2\mathrm{d}\theta\big) \\ &\quad \times \exp\big(\frac{4\lambda^4}{\delta}\int_{r-1}^{r}\|w_n(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta\big). \qquad (4.14)\end{aligned}$$

Furthermore, integrating (4.14) with respect to s for $s \in [r-1, r]$, we obtain

$$\begin{aligned}\|w_n'(r)\|^2 &\leqslant \big(\int_{r-1}^{r}\|w_n'(s)\|^2\mathrm{d}s + \frac{1}{\delta}(\tilde{\lambda}_1^{-1/2} + \lambda_1^{-1/2})\int_{r-1}^{r}\|G'(\theta)\|^2\mathrm{d}\theta\big) \\ &\quad \times \exp\big(\frac{4\lambda^4}{\delta}\int_{r-1}^{r}\|w_n(\theta)\|_{\widehat{V}}^4\mathrm{d}\theta\big). \qquad (4.15)\end{aligned}$$

Hence, by (3.8), (3.10), (4.15) and the fact that $w_n(\cdot;\tau,w_\tau) \rightharpoonup w(\cdot;\tau,w_\tau)$ weakly in $L^2([t-1,t];\widehat{V})$ and $w(\cdot;\tau,w_\tau) \in \mathcal{C}([t-1,t];\widehat{V})$, the inequality (4.5) follows.

Next, we prove the inequality (4.6). Similar to (3.18), we have

$$\begin{aligned}(w_n'(r), Aw_n(r)) + \|Aw_n(r)\|^2 &+ \langle B(u_n(r), w_n(r)), Aw_n(r)\rangle \\ &+ \langle N(w_n(r)), Aw_n(r)\rangle \\ = (G(r), Aw_n(r)) &\leqslant 2\|G(r)\|^2 + \frac{1}{8}\|Aw_n(r)\|^2. \qquad (4.16)\end{aligned}$$

By (2.4) and (2.6), we easily get

$$|\langle B(u_n(r), w_n(r)), Aw_n(r)\rangle| \leqslant \frac{1}{4}\|Aw_n(r)\|^2 + \lambda^4\|w_n(r)\|^2\|w_n(r)\|_{\widehat{V}}^4, \qquad (4.17)$$

$$|\langle N(w_n(r)), Aw_n(r)\rangle| \leqslant \frac{1}{4}\|Aw_n(r)\|^2 + c^2(\nu_r)\|w_n(r)\|_{\widehat{V}}^2. \qquad (4.18)$$

From (4.16)-(4.18) and the fact

$$|(w_n'(r), Aw_n(r))| \leqslant \|w_n'(r)\|\|Aw_n(r)\| \leqslant 2\|w_n'(r)\|^2 + \frac{1}{8}\|Aw_n(r)\|^2,$$

we can obtain

$$\begin{aligned}\|Aw_n(r)\|^2 &\leqslant \Big(4\lambda^4\|w_n(r)\|^2\|w_n(r)\|_{\widehat{V}}^2 + 4c^2(\nu_r)\Big)\|w_n(r)\|_{\widehat{V}}^2 \\ &\quad + 8\|G(r)\|^2 + 8\|w_n'(r)\|^2. \qquad (4.19)\end{aligned}$$

Since the embedding $W^{1,2}_{loc}(\mathbb{R};\widehat{H}) \hookrightarrow \mathcal{C}(\mathbb{R};\widehat{H})$ is continuous, we see that $G \in \mathcal{C}(\mathbb{R};\widehat{H})$. According to (3.7)-(3.8), (4.5) and (4.19), we have for all $r \in [t-1,t]$ that

$$\|Aw_n(r;\tau,w_\tau)\|^2 \leqslant \rho_6(t),\ \forall\,\tau \leqslant \tau_2(\widehat{D},t),\ \forall\,w_\tau \in D(\tau), \tag{4.20}$$

where $\rho_6(t)$ is given by (4.8). Then, combining Lemma 3.1, (4.20), (H2) and the fact that $w_n(\cdot;\tau,w_\tau) \rightharpoonup w(\cdot;\tau,w_\tau)$ weakly in $L^2([t-1,t];\widehat{V})$, and $w(\cdot;\tau,w_\tau) \in \mathcal{C}([t-1,t];\widehat{V})$, we can obtain the inequality (4.6). The proof is complete. □

THEOREM 4.3. *Assume* (H1) *and* (H2) *hold. Then we have*

$$\lim_{t\to-\infty}\big(\mathrm{e}^{c_3 t}\sup_{w\in\mathcal{A}_{\mathcal{D}\widehat{H}}(t)}\|w\|^2_{(H^2(\Omega))^3}\big) = 0. \tag{4.21}$$

PROOF. Obviously, we have for all $r \in [t-1,t]$ that

$$\|G(r)\| \leqslant \|G(t-1)\| + \big(\int_{t-1}^{t}\|G'(\theta)\|^2\mathrm{d}\theta\big)^{\frac{1}{2}}. \tag{4.22}$$

Then, combining Lemmas 3.3,4.2, Theorem 4.1, the equivalence between (3.71) and (4.3), and the invariance of $\widehat{\mathcal{A}}_{\mathcal{D}\widehat{H}}(t)$, we have (4.21). □

5. H^2-boundedness of the pullback attractors

In this section, we will use the estimates of the Galerkin approximate solutions sequence $\{w_n(t;\tau,w_\tau)\}_{n\geqslant1}$ defined by (3.15) to show the H^2-boundedness of the pullback attractors.

LEMMA 5.1. *Assume* (H2) *holds. For any $\tau\in\mathbb{R}$, $\varepsilon>0$, $t>\tau+\varepsilon$ and bounded set $\mathcal{B}\subset\widehat{H}$, we have the following properties:*

(1) *$\{w_n(r;\tau,w_\tau)|\, r\in[\tau+\varepsilon,t], w_\tau\in\mathcal{B}\}_{n\geqslant1}$ is a bounded subset of $\widehat{V}$;*
(2) *$\{w_n(\cdot;\tau,w_\tau)|\, w_\tau\in\mathcal{B}\}_{n\geqslant1}$ is a bounded subset of $L^2([\tau+\varepsilon,t];D(A))$;*
(3) *$\{w_n'(\cdot;\tau,w_\tau)|\, w_\tau\in\mathcal{B}\}_{n\geqslant1}$ is bounded subset of $L^2([\tau+\varepsilon,t];\widehat{H})$;*
(4) *$\{w_n'(r;\tau,w_\tau)|\, r\in[\tau+\varepsilon,t], w_\tau\in\mathcal{B}\}_{n\geqslant1}$ is a bounded subset of $\widehat{H}$;*
(5) *$\{w_n(r;\tau,w_\tau)|\, r\in[\tau+\varepsilon,t], w_\tau\in\mathcal{B}\}_{n\geqslant1}$ is a bounded subset of $D(A)$.*

PROOF. We can obtain, with small modifications on the interval $[\tau+\varepsilon,t]$, the similar estimates as (3.7)-(3.10) and (4.5)-(4.6) for the Galerkin approximate solution $w_n(t;\tau,w_\tau)$ defined by (3.15). Hence, the properties of parts (1)-(5) follow from theses estimates directly. The proof is complete. □

LEMMA 5.2. *Assume* (H2) *holds. Then for any $\tau\in\mathbb{R}$, $\varepsilon>0$, $t>\tau+\varepsilon$ and bounded set $\mathcal{B}\subset\widehat{H}$, the set $\bigcup_{s\in[\tau+\varepsilon,t]}U(s,\tau)\mathcal{B}$ is bounded in $\widehat{V}$.*

PROOF. In [**24**], the authors had proved that for any $w_\tau\in\mathcal{B}\subset\widehat{H}$, the Galerkin approximate solutions $\{w_n(\cdot;\tau,w_\tau)\}_{n\geqslant1}$ converge weakly to $w(\cdot;\tau,w_\tau)$ in $L^2([\tau,t];\widehat{V})$ and $w(\cdot;\tau,w_\tau)\in\mathcal{C}([\tau,t];\widehat{H})$. Then, the desired result is a straightforward consequence of Lemma 3.1 and part (1) of Lemma 5.1. The proof is complete. □

Summing up the above lemmas, we have the following results.

THEOREM 5.3. *Assume* (H1) *and* (H2) *hold.*

(1) *For any* $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$ *and bounded set* $\mathcal{B} \subset \widehat{H}$, *the set* $\bigcup_{s\in[\tau+\varepsilon,t]} U(s,\tau)\mathcal{B}$ *is a bounded subset of* $D(A) = \widehat{V} \cap \big(H^2(\Omega)\big)^3$.

(2) *For any* $T_1, T_2 \in \mathbb{R}$ *with* $T_1 < T_2$, *the set* $\bigcup_{t\in[T_1,T_2]} \mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t)$ *is a bounded subset of* $\widehat{V} \cap \big(H^2(\Omega)\big)^3$.

Proof. (1) According to [**24**], one can see that for all $w_\tau \in \mathcal{B} \subset \widehat{H}$, the Galerkin approximate solutions $\{w_n(\cdot;\tau,w_\tau)\}_{n\geqslant 1}$ converge weakly to $w(\cdot;\tau,w_\tau)$ in $L^2([\tau,t];\widehat{V})$ and $w(\cdot;\tau,w_\tau) \in \mathcal{C}([\tau+\varepsilon,t];\widehat{V})$. Hence, it follows from Lemma 3.1, part (5) of Lemma 5.1 that the assertion holds.

(2) It is easy to see that if $\tau < T_1 - 1$ is fixed, then

$$\bigcup_{t\in[T_1,T_2]} \mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(t) \subseteq \bigcup_{t\in[\tau+1,T_2]} U(t,\tau)\mathcal{A}_{\mathcal{D}_F^{\widehat{H}}}(\tau).$$

Hence, combining Lemma 5.2 and the result of part (1), we obtain the boundedness result. The proof is complete. □

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