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Sobolev spaces on time scales and applications to semilinear Dirichlet problems

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ABSTRACT. In this paper, we present some theoretical results of Sobolev spaces of functions defined on an open subset of an arbitrary time scale $\mathbb{T}^n$, where $n \geq 1$ is a positive integer. As an application, we consider a class of semilinear Dirichlet problems on time scales $\mathbb{T}^n$ of the form

$$\begin{cases} -\triangle u + \lambda u^\sigma = |u^\sigma|^{p-2} u^\sigma, \\ u \geq 0, \quad u \in H^1_{0,\Delta}(\Omega_\mathbb{T}), \end{cases}$$

where $\Omega_\mathbb{T}$ is a domain of $(\mathbb{T}^\kappa)^n$ and $\triangle u = \sum_{i=1}^n D^2_{i,\Delta} u$ is the Laplace operator. Under certain conditions, the sufficient and necessary condition of the existence of a nontrivial solution is established by using the mountain pass theorem.

Contents

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers and has the topology inherited from the real numbers with the standard topology. Since time scale calculus can be used to model dynamic processes whose time domains are

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more complicated than the set of integers or real numbers, it plays a crucial role in various equations and systems arising in economy, biology, ecology and astronomy [**7, 14, 22**], etc. During the last decade, there has been an explosion of interest in the study of dynamic equations on time scales and the research in this area is rapidly growing, see [**15, 17, 19, 18, 20, 16, 21, 27**] and the references therein. On the other hand, in real and functional analysis Sobolevs spaces are regarded as one of most fundamental tools, especially in the use of variational methods to solve boundary value problems in ordinary and partial differential equations and difference equations [**11, 9, 28, 26**]. In spite of this, the theory for functions defined on an arbitrary bounded interval of the real numbers has been well established [**12, 23**], but for functions defined on an arbitrary time scale, it appears that the study just started.

In this paper, we study theoretical properties of Sobolev spaces of functions defined on an open subset of an arbitrary time scale $\mathbb{T}^n$ endowed with the Lebesgue Δ-measure. As an application, we consider the following semilinear Dirichlet problem on time scales $\mathbb{T}^n$:

$$\begin{cases} -\triangle u(t) + \lambda u^\sigma(t) = |u^\sigma(t)|^{p-2}\, u^\sigma(t), \\ u(t) \geq 0, \quad u(t) \in H^1_{0,\Delta}(\Omega_{\mathbb{T}}), \end{cases} \tag{1.1}$$

where $\Omega_{\mathbb{T}}$ is a domain of $(\mathbb{T}^\kappa)^n$ and where $\triangle u = \sum_{i=1}^n D^2_{i,\Delta} u$ is the Laplace operator. The sufficient and necessary condition of the existence of nontrivial solution is obtained by using the mountain pass theorem. Such result on the existence of nontrivial solution of the semilinear Dirichlet problem is also sharp for the corresponding difference equation ($\mathbb{T} = \mathbb{Z}$) in the general time scale setting.

The paper is organized as follows. In Section 2, we introduce some basic notions and briefly present the Mean Value Theorem on time scales. In Section 3, we discuss the Divergence Theorem on time scales. In Section 4, we prove the Rellich's Compactness Theorem and the generalized Poincaré inequalities on time scales. Section 5 presents some properties of differentiable functions on time scales. Section 6 is dedicated to the existence of nontrivial solutions of the problem (1.1) on time scales.

2. Preliminaries

In order to discuss the theory of Sobolev spaces on time scales $\mathbb{T}^n$, we start with some basic notions and the related propositions [**3, 10**] that help to better understand our main results and proofs described in next several sections. Some other relevant terminologies and concepts can be found in the references [**2, 1, 4, 5, 6, 8, 13**] etc.

Let n be a positive integer. For each $i \in \{1, 2, \cdots\}$, let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Set

$$\begin{aligned} \mathbb{T}^n &= \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \\ &= \{t = (t_1, \cdots, t_n) : t_i \in \mathbb{T}_i,\ i = 1, 2, ..., n\}. \end{aligned}$$

We call $\mathbb{T}^n$ an n-dimensional time scale. The set $\mathbb{T}^n$ is a complete metric space with the metric d defined by

$$d(t, s) = \left(\sum_{i=1}^n |t_i - s_i|^2 \right)^{\frac{1}{2}} \quad \text{for } t, s \in \mathbb{T}^n.$$

Let σ_i and ρ_i denote the forward and backward jump operators in $\mathbb{T}_i$, respectively. Specifically, for $u \in \mathbb{T}_i$, the forward jump operator $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\sigma_i(u) = \inf\{v \in \mathbb{T}_i : v > u\};$$

and the back jump operator $\rho_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\rho_i(u) = \inf\{v \in \mathbb{T}_i : v < u\}.$$

In this definition we put $\sigma_i(\max \mathbb{T}_i) = \max \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite maximum, and $\rho(\min \mathbb{T}_i) = \min \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite minimum. If $\sigma_i(u) > u$, then we say that u is *right-scattered* (in $\mathbb{T}_i$), while any u with $\rho_i(u) < u$ is called *left-scattered* (in $\mathbb{T}_i$). Moreover, if $u < \max \mathbb{T}_i$ and $\sigma(u) = u$, then u is called *right-dense* (in $\mathbb{T}_i$), and if $u > \min \mathbb{T}_i$ and $\rho_i(u) = u$, then u is called *left-dense* (in $\mathbb{T}_i$). If $\mathbb{T}_i$ has a left-scattered maximum M, then we define $\mathbb{T}_i^\kappa = \mathbb{T}_i \setminus \{M\}$, otherwise $\mathbb{T}_i^\kappa = \mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $(\mathbb{T}_i)_\kappa = (\mathbb{T}_i) \setminus \{m\}$, otherwise $(\mathbb{T}_i)_\kappa = \mathbb{T}_i$.

Let $f : \mathbb{T}^n \to \mathbb{R}$ be a function. The *partial delta derivative* of f with respect to $t_i \in (\mathbb{T}^n)^\kappa$ is defined as

$$\lim_{s_i \to t_i,\, s_i \neq \sigma_i(t_i)} \frac{f(t_1, \cdots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \cdots, t_n) - f(t_1, \cdots, t_{i-1}, s_i, t_{i+1}, \cdots, t_n)}{\sigma_i(t_i) - s_i},$$

provided that this limit exists, and denoted by $\frac{\partial f(t)}{\Delta_i t_i}$. The second order partial delta derivatives of f is denoted as $\frac{\partial^2 f(t)}{\Delta_i t_i^2}$, or $\frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i}$. Higher order partial delta derivatives are similarly defined. The *partial nabla derivative* of f with respect to $t_i \in (\mathbb{T}^n)_\kappa$ is defined as

$$\lim_{s_i \to t_i,\, s_i \neq \rho_i(t_i)} \frac{f(t_1, \cdots, t_{i-1}, \rho_i(t_i), t_{i+1}, \cdots, t_n) - f(t_1, \cdots, t_{i-1}, s_i, t_{i+1}, \cdots, t_n)}{\rho_i(t_i) - s_i},$$

provided that the limit exists, and denoted by $\frac{\partial f(t)}{\nabla_i t_i}$. In a similar way, we can define higher order partial nabla derivatives, and the mixed derivatives can be obtained by combining both delta and nabla differentiations. For instance, a second order mixed derivative is defined as $\frac{\partial^2 f(t)}{\Delta_i t_i \nabla_j t_j}$ or $\frac{\partial^2 f(t)}{\nabla_j t_j \Delta_i t_i}$.

Definition 2.1. For each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers, we denote the D_Δ^α partial derivative by

$$D_{1,\Delta}^{\alpha_1} \cdots D_{n,\Delta}^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\Delta_1 t_1^{\alpha_1} \cdots \Delta_n t_n^{\alpha_n}}$$

of the order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $D_{i,\Delta} f = \frac{\partial f}{\Delta_i t_i}$ and $\frac{\partial f}{\Delta_i t_i}$ is denoted as the first Δ-derivative of the function f with respect to t_i. If $|\alpha| = 0$, then $D_\Delta^0 = I$ (identity). Moreover, $D_{i,\Delta}^2 f$ and $\frac{\partial^2 f}{\Delta_i t_i^2}$ are denoted as the second Δ-derivative of the function f with respect to t_i. The partial nabla derivatives are similarly defined.

For a Δ-measurable set $E_\mathbb{T} \subset \mathbb{T}^n$ and a Δ-measurable function $f : E_\mathbb{T} \to \mathbb{R}$, the corresponding *Lebesgue Δ-integral* of f over $E_\mathbb{T}$ will be denoted by

$$\int_{E_\mathbb{T}} f(t_1, t_2, \cdots, t_n) \Delta t_1 \Delta t_2 \cdots \Delta t_n \text{ or } \int_{E_\mathbb{T}} f(t) \Delta t.$$

All theorems of the general Lebesgue integration theory, including the Lebesgue's dominated convergence theorem, hold for Lebesgue Δ-integrals on $\mathbb{T}^n$.

We state the mountain pass theorem for the study of the semilinear Dirichlet problem on time scales $\mathbb{T}^n$.

LEMMA 2.2. [**24**] *Let X be a Hilbert space. Suppose that $\varphi \in C^2(X, R)$, $e \in X$ and $r > 0$ such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

Then for every $\varepsilon > 0$, there exists $u \in X$ such that

(i) $c - \varepsilon \leq \varphi(u) \leq c + \varepsilon$,

(ii) $\|\varphi'(u)\| < 2\varepsilon$,

where

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \varphi(\gamma(s)),$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \ \gamma(1) = e\}.$$

If φ satisfies the $(PS)_c$ condition, Then c is a critical value of φ.

The following is the mean value theorem on time scales $\mathbb{T}^n$.

LEMMA 2.3. (Mean Value Theorem) *Let $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ be any two points in $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$. Set*

$$\alpha_i = \min\{a_i, b_i\} \text{ and } \beta_i = \max\{a_i, b_i\}, \quad i = 1, 2, \cdots, n.$$

Let f be a continuous function on $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ that has the first order partial delta derivatives $\frac{\partial f}{\Delta_i t_i}$ for each $t_i \in [\alpha_i, \beta_i)_{\mathbb{T}}$. Then there exist constants $\xi_i, \eta_i \in [\alpha_i, \beta_i)_{\mathbb{T}}$ such that

$$\begin{aligned}
& \frac{\partial f(\xi_1, a_2, \cdots, a_n)}{\Delta_1 t_1}(a_1 - b_1) + \frac{\partial f(b_1, \xi_2, a_3 \cdots, a_n)}{\Delta_2 t_2}(a_2 - b_2) \\
& + \cdots + \frac{\partial f(b_1, b_2, \cdots, b_{n-1}, \xi_n)}{\Delta_n t_n}(a_n - b_n) \\
\leq \ & f(a_1, a_2, \cdots, a_n) - f(b_1, b_2, \cdots, b_n) \\
\leq \ & \frac{\partial f(\eta_1, a_2, \cdots, a_n)}{\Delta_1 t_1}(a_1 - b_1) + \frac{\partial f(b_1, \eta_2, a_3 \cdots, a_n)}{\Delta_2 t_2}(a_2 - b_2) \\
& + \cdots + \frac{\partial f(b_1, b_2, \cdots, b_{n-1}, \eta_n)}{\Delta_n t_n}(a_n - b_n). \qquad (2.1)
\end{aligned}$$

By virtue of Lemma 2.3, we can derive that

LEMMA 2.4. *Let f be a continuous function on $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ that has the first order partial derivatives $\frac{\partial f}{\Delta_1 t_1}$, $\frac{\partial f}{\Delta_2 t_2}$, $\cdots$ and $\frac{\partial f}{\Delta_n t_n}$ for $(t_1, t_2, \cdots, t_n) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, $(t_1, t_2, \cdots, t_n) \in \mathbb{T}_1 \times \mathbb{T}_2^\kappa \times \cdots \times \mathbb{T}_n$, $\cdots$, and $(t_1, t_2, \cdots, t_n) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n^\kappa$, respectively. If these derivatives are identically zero, then f is a constant function on $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$.*

3. Divergence Theorem

Let $\Omega_{\mathbb{T}^n}$ be an open subset of $\mathbb{T}^n$ $(n \geq 1)$. The set $C(\Omega_{\mathbb{T}^n})$ of continuous real-valued functions defined on $\Omega_{\mathbb{T}^n}$ is an infinite dimensional vector space with the usual definitions of addition and scalar multiplication:

(i) $(f + g)(t) = f(t) + g(t)$ for $f, g \in C(\Omega_{\mathbb{T}^n})$, $t \in \Omega_{\mathbb{T}^n}$;

(ii) $(\alpha f)(t) = \alpha f(t)$ for $\alpha \in \mathbb{R}$ and $f \in C(\Omega_{\mathbb{T}^n})$, $t \in \Omega_{\mathbb{T}^n}$.

Let $C_0\left(\Omega_{\mathbb{T}^n}\right)$ be a subspaces of $C(\Omega_{\mathbb{T}^n})$ which consists of those functions that are continuous in $\Omega_{\mathbb{T}^n}$ and has compact support in $\Omega_{\mathbb{T}^n}$. The support of a function f is defined on $\Omega_{\mathbb{T}^n}$ as the closure of the set $\{x \in \Omega_{\mathbb{T}^n} : f(x) \neq 0\}$, denoted by $supp(f)$.

For the integers $m \geq 0$, let $C^m\left(\Omega_{\mathbb{T}^n}\right)$ be the collection of all $f \in C(\Omega_{\mathbb{T}^n})$ such that $D^\alpha f \in C(\Omega_{\mathbb{T}^n})$ for any multi-index α with the length $|\alpha| \leq m$. We say $f \in C^\infty\left(\Omega_{\mathbb{T}^n}\right)$ if and only if $f \in C^m\left(\Omega_{\mathbb{T}^n}\right)$ for any nonnegative integer m. For $m \geq 0$, we define $C_0^m\left(\Omega_{\mathbb{T}^n}\right) = C^m\left(\Omega_{\mathbb{T}^n}\right) \cap C_0\left(\Omega_{\mathbb{T}^n}\right)$. Obviously, the spaces

$$C^m\left(\Omega_{\mathbb{T}^n}\right),\ C^\infty\left(\Omega_{\mathbb{T}^n}\right),\ C_0^1\left(\Omega_{\mathbb{T}^n}\right),\ C^1\left(\Omega_{\mathbb{T}^n}\right)$$

are subspaces of the vector space $C(\Omega_{\mathbb{T}^n})$.

Let $\Omega_{\mathbb{T}}$ be a nonempty open set in $\mathbb{T}^n$. Assume $\varphi \in C^m(\Omega_{\mathbb{T}})$ and $u \in C_0^m(\Omega_{\mathbb{T}})$. It follows from integration by parts that

$$\int_{\Omega_{\mathbb{T}^n}} u D_\Delta^\alpha \varphi \Delta t = (-1)^{|\alpha|} \int_{\Omega_{\mathbb{T}^n}} v \varphi^\sigma \Delta t, \quad |\alpha| \geq m, \tag{3.1}$$

where $v = D_\Delta^\alpha u$.

For $p \in \mathbb{R}$ and $p \geq 1$, let the space

$$L_\Delta^p(\Omega_{\mathbb{T}^n}, \mathbb{R}) = \left\{ f : \Omega_{\mathbb{T}^n} \to \mathbb{R} \,\middle|\, \int_{\Omega_{\mathbb{T}^n}} |f(t)|^p \Delta t < +\infty \right\}$$

be equipped with the norm

$$\|f\|_{L_\Delta^p(\Omega_{\mathbb{T}^n}, \mathbb{R})} = \left[\int_{\Omega_{\mathbb{T}^n}} |f(s)|^p \, \Delta s \right]^{\frac{1}{p}}. \tag{3.2}$$

We know that the space $L_\Delta^p(\Omega_{\mathbb{T}^n}, \mathbb{R})$ is a Banach space with the norm defined as formula (3.2). Moreover, $L_\Delta^2(\Omega_{\mathbb{T}^n}, \mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle_{L_\Delta^2(\Omega_{\mathbb{T}^n}, \mathbb{R}^N)} = \int_{\Omega_{\mathbb{T}^n}} (f(t), g(t)) \Delta t,$$

where $(f,\, g)$ is the usual scalar product in $\mathbb{R}^n$.

For $p \in [1, \infty)$, we denote $f \in L_{loc,\Delta}^p(\Omega_{\mathbb{T}^n}, \mathbb{R})$ if and only if $f \in L_\Delta^p(K_{\mathbb{T}^n}, \mathbb{R})$ for each compact set $K_{\mathbb{T}^n}$.

Definition 3.1. Assume $u \in L_{loc,\Delta}^1(\Omega_{\mathbb{T}^n}, \mathbb{R})$. A function $v \in L_{loc,\Delta}^1(\Omega_{\mathbb{T}^n}, \mathbb{R})$ is called the α-th weak derivative of u if it satisfies

$$\int_{\Omega_{\mathbb{T}^n}} u D_\Delta^\alpha \varphi \Delta t = (-1)^{|\alpha|} \int_{\Omega_{\mathbb{T}^n}} v \varphi^\sigma \Delta t \ \text{ for all } \varphi \in C_0^{|\alpha|}(\Omega_{\mathbb{T}^n}). \tag{3.3}$$

It is easy to see that the weak derivative is well-defined. Thus we use $v = D_\Delta^\alpha u$ to indicate that v is the α-th weak derivative of u. If a function u has an ordinary α-th derivative in $L_{loc,\Delta}^1(\Omega_{\mathbb{T}^n})$, then the α-th weak derivatives coincide with the ordinary derivatives.

Suppose that $\overrightarrow{F} = \overrightarrow{F}(x_1, x_2, \cdots, x_n)$ is a differentiable vector-valued function defined on $\Omega_{\mathbb{T}}$. At each point $t = (t_1, t_2, \cdots, t_n) \in \partial\Omega_{\mathbb{T}}$ (we also assume $\partial\Omega_{\mathbb{T}}$ is C^1), let $\overrightarrow{n} = \overrightarrow{n}(t_1, t_2, \cdots, t_n)$ be the outward unit normal vector. Recall that

$$Div \overrightarrow{F} = \frac{\partial \overrightarrow{F}}{\Delta_1 t_1} + \frac{\partial \overrightarrow{F}}{\Delta_2 t_2} + \cdots + \frac{\partial \overrightarrow{F}}{\Delta_n t_n}.$$

The divergence formula is

$$\int_{\Omega_{\mathbb{T}^n}} Div\overrightarrow{F}\Delta t = \int_{\partial\Omega_{\mathbb{T}^n}} \overrightarrow{F}.\overrightarrow{n}\,ds, \tag{3.4}$$

where ds denotes the surface measure on $\partial\Omega_{\mathbb{T}^n}$. Let $\overrightarrow{F}(x_1, x_2, \cdots, x_n)$ be a vector-valued function of the form $\overrightarrow{F} = v \cdot \bigtriangledown u$, where u and v are scalar functions defined on $\Omega_{\mathbb{T}^n}$. Here, $\bigtriangledown u$ denotes the gradient of u given by

$$\bigtriangledown u = \begin{pmatrix} D_{1,\Delta}u \\ D_{2,\Delta}u \\ \vdots \\ D_{n,\Delta}u \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\Delta_1 t_1} \\ \frac{\partial u}{\Delta_2 t_2} \\ \vdots \\ \frac{\partial u}{\Delta_n t_n} \end{pmatrix}.$$

Then we have

$$Div\overrightarrow{F} = v^\sigma \triangle u + \bigtriangledown v. \bigtriangledown u,$$

where $v^\sigma = v(\sigma(t))$. According to (3.4), we have

$$\int_{\Omega_{\mathbb{T}^n}} (v^\sigma \triangle u + \bigtriangledown v. \bigtriangledown u)\,\Delta t = \int_{\partial\Omega_{\mathbb{T}^n}} v\frac{\partial u}{\partial n} ds, \tag{3.5}$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative of u on $\partial\Omega_{\mathbb{T}^n}$ given by

$$\frac{\partial u}{\partial n} = \bigtriangledown u.\overrightarrow{n}.$$

The formula (3.5) is regarded as Green's Identity-I on time scales.

Rewrite identity (3.5) as

$$\int_{\Omega_{\mathbb{T}^n}} \bigtriangledown v. \bigtriangledown u\Delta t = -\int_{\Omega_{\mathbb{T}^n}} v^\sigma \triangle u\Delta t + \int_{\partial\Omega_{\mathbb{T}^n}} v\frac{\partial u}{\partial n} ds. \tag{3.6}$$

Exchanging u and v gives

$$\int_{\Omega_{\mathbb{T}^n}} \bigtriangledown u. \bigtriangledown v\Delta t = -\int_{\Omega_{\mathbb{T}^n}} u^\sigma \triangle v\Delta t + \int_{\partial\Omega_{\mathbb{T}^n}} u\frac{\partial v}{\partial n} ds. \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$\int_{\Omega_{\mathbb{T}^n}} (u^\sigma \triangle v\Delta t - v^\sigma \triangle u)\,\Delta t = \int_{\partial\Omega_{\mathbb{T}^n}} \left(u\frac{\partial v}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

which is called Green's Identity-II on time scales.

4. Embedding Theorem

For $p \geq 1$ and a nonnegative integer k, let

$$W^{k,p}_\Delta(\Omega_{\mathbb{T}^n}) = \{u : \Omega_{\mathbb{T}^n} \to \mathbb{R}\,|u^\sigma \in L^p_\Delta(\Omega_{\mathbb{T}^n}) \text{ and } D^\alpha_\Delta u \in L^p_\Delta(\Omega_{\mathbb{T}^n}), 0 < |\alpha| \leq k\}.$$

When $k = 0$, $W^{k,p}_\Delta(\Omega_{\mathbb{T}^n})$ means $L^p_\Delta(\Omega_{\mathbb{T}^n})$. It is obvious that $W^{k,p}_\Delta(\Omega_{\mathbb{T}^n})$ is a vector space.

The corresponding norm is defined by

$$\begin{aligned} \|u\|_{k,p,\Delta} &= \|u\|_{W^{k,p}_\Delta(\Omega_{\mathbb{T}^n})} \\ &= \left(\int_{\Omega_{\mathbb{T}}} \left(|u^\sigma|^p + \sum_{0<|\alpha|\leq k} |D^\alpha_\Delta u|^p\right) \Delta t\right)^{\frac{1}{p}}. \end{aligned} \tag{4.1}$$

If $1 \le p < \infty$, the space $W^{k,p}_{\Delta}(\Omega_{\mathbb{T}^n})$ is called the Sobolev space on time sales of order k.

We define the space $W^{k,p}_{0,\Delta}(\Omega_{\mathbb{T}^n})$ as the closure of $\mathcal{D}(\Omega_{\mathbb{T}^n})$ in $W^{k,p}_{\Delta}(\Omega_{\mathbb{T}^n})$ with respect to the norm $\|u\|_{k,p,\Delta}$, and it is also a Sobolev space of order k, where

$$\mathcal{D}(\Omega_{\mathbb{T}}) = \{u : \Omega_{\mathbb{T}} \to \mathbb{R} : u \in C^{\infty} \text{ and } supp\, u \text{ is compact } \}.$$

Note that when $p = 2$, $W^{k,p}_{\Delta}(\Omega_{\mathbb{T}^n})$ and $W^{k,p}_{0,\Delta}(\Omega_{\mathbb{T}^n})$ are Hilbert spaces with the inner product defined by

$$\begin{aligned}(u,v)_{k,2} &= (u,v)_{W^{k,p}_{\Delta}(\Omega_{\mathbb{T}^n})} \\ &= \int_{\Omega_{\mathbb{T}}} \left(|u^{\sigma}.v^{\sigma}| + \sum_{0<|\alpha|\le k} D^{\alpha}_{\Delta} u D^{\alpha}_{\Delta} v \right) \Delta t.\end{aligned}$$

We will deal mostly with these spaces in the sequel, as well as the following notations

$$H^{k}_{\Delta}\left(\Omega_{\mathbb{T}^n}\right) = W^{k,2}_{\Delta}\left(\Omega_{\mathbb{T}^n}\right) \text{ and } H^{k}_{0,\Delta}\left(\Omega_{\mathbb{T}^n}\right) = W^{k,2}_{0,\Delta}\left(\Omega_{\mathbb{T}^n}\right)$$

and

$$H^1_{\Delta}(\mathbb{T}^n) = \left\{ u : \mathbb{T}^N \to \mathbb{R} \,\middle|\, u^{\sigma} \in L^2_{\Delta}(\mathbb{T}^n) \text{ and } \bigtriangledown u \in L^2_{\Delta}(\mathbb{T}^n) \right\}.$$

So $H^1_{\Delta}(\mathbb{T}^n)$ is a Hilbert space with the inner product

$$(u^{\sigma}, v^{\sigma})_{1,\Delta} := \int_{\mathbb{R}^N} \left(\bigtriangledown u. \bigtriangledown v + |u^{\sigma} v^{\sigma}| \right) \Delta t$$

and the induced norm

$$\|u^{\sigma}\|_{1,\Delta} = \left(\int_{\mathbb{T}^N} \left[|\bigtriangledown u|^2 + |u^{\sigma}|^2 \right] \Delta t \right)^{\frac{1}{2}}.$$

Let $\Omega_{\mathbb{T}}$ be an open subset of $\mathbb{T}^n$. The space $H^1_{0,\Delta}(\Omega_{\mathbb{T}})$ is the closure of $\mathcal{D}(\Omega_{\mathbb{T}})$ in $H^1_{\Delta}(\mathbb{T}^n)$. Let $N \ge 3$ and $2^* := \frac{2N}{N-2}$. The space

$$\mathcal{D}^{1,2}_{\Delta}(\mathbb{T}^n) = \left\{ u : \mathbb{T}^N \to \mathbb{R} \,\middle|\, u^{\sigma} \in L^{2^*}_{\Delta}(\mathbb{T}^n) \text{ and } \bigtriangledown u \in L^2_{\Delta}(\mathbb{T}^n) \right\}$$

is a Hilbert space with the inner product

$$(u,v) := \int_{\mathbb{R}^N} \bigtriangledown u. \bigtriangledown v \Delta t$$

and the induced norm

$$\|u\|_{\mathcal{D}^{1,2}_{\Delta}(\mathbb{T}^n)} = \left(\int_{\mathbb{T}^N} |\bigtriangledown u|^2 \Delta t \right)^{\frac{1}{2}}.$$

The space $\mathcal{D}^{1,2}_{0,\Delta}(\mathbb{T}^n)$ is the closure of $\mathcal{D}(\mathbb{T}^n)$ in $\mathcal{D}^{1,2}_{\Delta}(\mathbb{T}^n)$.

In order to prove the Rellich's compactness theorem on time scales, we introduce the following definition.

Definition 4.1. Let η be a nonnegative real-valued function in $C^{\infty}_{0,rd}(\mathbb{T}^n)$ with the properties

$$\eta(t) = 0 \text{ for } |t| \ge 1$$

and

$$\int_{\mathbb{T}^n} \eta(t) \Delta t = 1.$$

For $\varepsilon > 0$, the function $\eta_\varepsilon(t) = \varepsilon^{-n}\eta\left(\frac{t}{\varepsilon}\right)$ is nonnegative, smooth and radial such that

$$\eta_\varepsilon(t) = 0, \quad \text{if } |t| \geq \varepsilon \ \text{ and } \int_{\mathbb{T}^n} \eta_\varepsilon(t)\Delta t = 1,$$

where η_ε is called a *mollifier* and the convolution

$$\eta_\varepsilon * u(t) = u^\varepsilon(t) = \int_{\mathbb{T}^n} \eta_\varepsilon(t-y)u(y)\Delta y$$

is called the *mollification* or *regularization* of u.

There exist three other forms in which u_ε can be represented as

$$\begin{aligned} u^\varepsilon(t) &= \int_{\mathbb{T}^n} \eta_\varepsilon(t-y)u(y)\Delta y \\ &= \int_{B(t,\varepsilon)} \eta_\varepsilon(t-y)u(y)\Delta y \\ &= \int_{B(0,1)} \eta(y)u(t-\varepsilon y)\Delta y. \end{aligned}$$

Notice that u^ε vanishes outside the (open) ball $B(t,\varepsilon)$ leading to the last equality and the value of $u^\varepsilon(t)$ depends only on the value of u on the ball $B(t,\varepsilon)$. In particular, if $dist(t, supp(u)) \geq \varepsilon$, then $u^\varepsilon(t) = 0$.

A series of preliminary results will be needed to prove the Rellich's compactness theorem on time scales. We first present the Gagliardo-Nirenberg-Sobolev inequality on time scales.

Lemma 4.2. (Gagliardo-Nirenberg-Sobolev inequality on time scales) *Assume $1 \leq p < \infty$. Then there is a constant C depending only on p and n, such that*

$$\|u\|_{L^{p^*}_\Delta(\mathbb{T}^n)} \leq C\| \bigtriangledown u\|_{L^p_\Delta(\mathbb{T}^n)} \text{ for all } u \in C^1_{0,\Delta}(\mathbb{T}^n). \tag{4.2}$$

Proof. Firstly, assume that $p = 1$. Since u has compact support, we have

$$u(t) = \int_{(-\infty,t_i)_\mathbb{T}} \frac{\partial u}{\partial_\Delta t_i} u(t_1,\cdots,t_{i-1},y_i,t_{i+1},\cdots,t_n)\Delta y_i \text{ for each } i = 1,2,\cdots,n,$$

which gives

$$|u(t)| \leq \int_{(-\infty,\infty)_\mathbb{T}} |\bigtriangledown u(t_1,\cdots,t_{i-1},y_i,t_{i+1},\cdots,t_n)|\,\Delta y_i, \ i = 1,2,\cdots,n.$$

Then, we have

$$|u(t)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{(-\infty,\infty)_\mathbb{T}} |\bigtriangledown u(t_1,\cdots,t_{i-1},y_i,t_{i+1},\cdots,t_n)|\,\Delta y_i\right)^{\frac{1}{n-1}}. \tag{4.3}$$

Integrating inequality (4.3) with respect to t_1 over $(-\infty,\infty)_{\mathbb{T}}$ gives

$$\begin{aligned}
&\int_{(-\infty,\infty)_{\mathbb{T}}} |u(t)|^{\frac{n}{n-1}}\,\Delta t_1 \\
\leq\ & \int_{(-\infty,\infty)_{\mathbb{T}}} \prod_{i=1}^{n}\left(\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta y_i\right)^{\frac{1}{n-1}} \Delta t_1 \\
\leq\ & \left(\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta y_1\right)^{\frac{1}{n-1}} \int_{(-\infty,\infty)_{\mathbb{T}}} \prod_{i=2}^{n}\left(\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta y_i\right)^{\frac{1}{n-1}} \Delta t_1 \\
\leq\ & \left(\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta y_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n}\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\Delta y_i\right)^{\frac{1}{n-1}}. \qquad (4.4)
\end{aligned}$$

The last inequality results from the extended Hölder's inequality on time scales.

Integrating inequality (4.4) with respect to $t_2\in\mathbb{T}$ over $(-\infty,\infty)_{\mathbb{T}}$ yields

$$\begin{aligned}
&\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |u(t)|^{\frac{n}{n-1}}\,\Delta t_1\Delta t_2 \\
\leq\ & \left(\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\Delta y_2\right)^{\frac{1}{n-1}} \int_{(-\infty,\infty)_{\mathbb{T}}} \prod_{i=1,i\neq 2}^{n} I_i^{\frac{1}{n-1}}\,\Delta t_2,
\end{aligned}$$

where

$$I_1 = \int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta y_1$$

and

$$I_i = \int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\Delta y_i \text{ for } i=3,4,\cdots,n.$$

Applying Hölder's inequality on time scales again, we have

$$\begin{aligned}
&\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |u(t)|^{\frac{n}{n-1}}\,\Delta t_1\Delta t_2 \\
\leq\ & \left(\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\Delta y_2\right)^{\frac{1}{n-1}} \left(\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_2\Delta y_1\right)^{\frac{1}{n-1}} \\
& \prod_{i=3}^{n}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}}\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\Delta t_2\Delta y_i\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Continuing to perform integration with respect to $t_3,\cdots,t_n$ and applying Hölder's inequality on time scales, we arrive at

$$\begin{aligned}
\int_{\mathbb{T}^n} |u|^{\frac{n}{n-1}}\,\Delta t\ &\leq\ \prod_{i=1}^{n}\left(\int_{(-\infty,\infty)_{\mathbb{T}}}\cdots\int_{(-\infty,\infty)_{\mathbb{T}}} |\triangledown u|\,\Delta t_1\cdots\Delta y_i\cdots\Delta t_n\right)^{\frac{1}{n-1}} \\
&=\ \left(\int_{\mathbb{T}^n} |\triangledown u|\,\Delta t\right)^{\frac{n}{n-1}}. \qquad (4.5)
\end{aligned}$$

That is,

$$\|u\|_{L_{\Delta}^{\frac{n}{n-1}}(\mathbb{T}^n)} \leq C\|\triangledown u\|_{L_{\Delta}^{1}(\mathbb{T}^n)},$$

which implies that the estimate (4.2) holds when $p=1$.

Next, we show that the estimate (4.2) holds if $1 < p < n$.

Set $v = |u|^\gamma$, where $\gamma > 1$ is to be determined. Note that

$$\begin{aligned} (D_{i,\Delta}|u|^\gamma)^2 &= \begin{cases} \left(\gamma u^{\gamma-1} D_{i,\Delta} u\right)^2 & \text{for } u \geq 0 \\ -\left(\gamma(-u)^{\gamma-1} D_{i,\Delta} u\right)^2 & \text{for } u \leq 0 \end{cases} \\ &= \left(\gamma |u|^{\gamma-1} D_{i,\Delta} u\right)^2, \end{aligned}$$

which implies $v \in C_0^1(\mathbb{T}^n)$. From formula (4.5) and by Hölder's inequality on time scales, we have

$$\begin{aligned} \left(\int_{\mathbb{T}^n} |u|^{\frac{\gamma n}{n-1}} \Delta t\right)^{\frac{n-1}{n}} &\leq \int_{\mathbb{T}^n} |\triangledown |u|^\gamma| \, \Delta t \\ &= \gamma \int_{\mathbb{T}^n} |u|^{\gamma-1} |\triangledown u| \Delta t \\ &= \gamma \left(\int_{\mathbb{T}^n} |u|^{\frac{p(\gamma-1)}{p-1}} |\Delta t\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{T}^n} \triangledown u|^p \Delta t\right)^{\frac{1}{p}}. \end{aligned} \tag{4.6}$$

Choose γ such that

$$\frac{\gamma n}{n-1} = \frac{p(\gamma-1)}{p-1},$$

that is,

$$\gamma = \frac{p(n-1)}{n-p} > 1,$$

and

$$\frac{\gamma n}{n-1} = \frac{p(\gamma-1)}{p-1} = \frac{np}{n-p} = p^*.$$

Consequently, the estimate (4.5) becomes

$$\left(\int_{\mathbb{T}^n} |u|^{p^*} \Delta t\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{T}^n} |\triangledown u|^p \Delta t\right)^{\frac{1}{p}}.$$

□

Using Theorem 2.19 in [**25**], one can prove the following Lemma.

LEMMA 4.3. *Let $\Omega_{\mathbb{T}^n} \subset \mathbb{T}^n$ be a bounded open set with $\Omega_{\mathbb{T}^n} \subset\subset \Omega'_{\mathbb{T}^n}$ and $k \geq 1$. If $\partial\Omega_{\mathbb{T}^n} \in C^k$, then any function $u(t) \in W_\Delta^{k,p}(\Omega_{\mathbb{T}^n})$ has an extension $U(x) \in W_\Delta^{k,p}(\Omega'_{\mathbb{T}^n})$ into $\Omega'_{\mathbb{T}^n}$ with compact support. Moreover, it holds*

$$\|U\|_{W_\Delta^{k,p}(\Omega'_{\mathbb{T}^n})} \leq c_2 \|u\|_{W_\Delta^{k,p}(\Omega_{\mathbb{T}^n})},$$

where the constant $c_2 > 0$ does not depend on u.

Similarly, we have

LEMMA 4.4. *Let $\Omega_{\mathbb{T}^n} \subset \mathbb{T}^n$ be a bounded open set with $\partial\Omega_{\mathbb{T}^n} \in C^1$. Assume that $1 \leq p < n$ and $u \in W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})$. Then $u \in L_\Delta^{p^*}(\Omega_{\mathbb{T}^n})$, and there is a constant C_1 depending only on p, n and $\Omega_{\mathbb{T}^n}$, such that*

$$\|u\|_{L_\Delta^{p^*}(\Omega_{\mathbb{T}^n})} \leq C_1 \|u\|_{W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})}. \tag{4.7}$$

Proof. Since $\partial\Omega_{\mathbb{T}^n} \in C^1$, by virtue of Lemma 4.3, for any $u \in W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})$ there exists $u = \overline{u} \in W_\Delta^{1,p}(\mathbb{T}^n)$ such that

$$\begin{cases} \overline{u} = u \text{ in } \Omega_{\mathbb{T}^n}, \overline{u} \text{ has compact support }, \\ \|\overline{u}\|_{W_\Delta^{1,p}(\mathbb{T}^n)} \leq C_2 \|u\|_{W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})}, \end{cases} \tag{4.8}$$

where C_2 is a positive constant. Moreover, since $\overline{u}$ has compact support, there exists $u_m \in C_{0,rd}^\infty(\mathbb{T}^n)$ such that

$$u_m \to \overline{u} \text{ in } W_\Delta^{1,p}(\mathbb{T}^n). \tag{4.9}$$

It follows from Lemma 4.2 that

$$\begin{aligned} \left(\int_{\mathbb{T}^n} |u_m - u_l|^{p^*} \Delta t\right)^{\frac{1}{p^*}} &\leq C \left(\int_{\mathbb{T}^n} | \nabla u_m - \nabla u_l|^p \Delta t\right)^{\frac{1}{p}} \\ &\leq C\|u_m - u_l\|_{W_\Delta^{1,p}(\mathbb{T}^n)} \text{ for any } l, m \geq 1. \end{aligned} \tag{4.10}$$

Formula (4.9) implies that $\{u_m\}$ is a Cauchy sequence in $W_\Delta^{1,p}(\mathbb{T}^n)$ and inequality (4.10) indicates that $\{u_m\}$ is Cauchy in $L_\Delta^{p^*}(\mathbb{T}^n)$. So we have

$$u_m \to \overline{u} \text{ in } L_\Delta^{p^*}(\mathbb{T}^n). \tag{4.11}$$

By using the Gagliardo-Nirenberg-Sobolev inequality on time scales, we obtain

$$\|u_m\|_{L_\Delta^{p^*}(\mathbb{T}^n)} \leq C\| \nabla u_m\|_{L_\Delta^p(\mathbb{T}^n)}, \tag{4.12}$$

which, together with (4.9), (4.11) and (4.12), yields

$$\|\overline{u}\|_{L_\Delta^{p^*}(\mathbb{T}^n)} \leq C\| \nabla \overline{u}\|_{L_\Delta^p(\mathbb{T}^n)}. \tag{4.13}$$

Consequently, combining (4.8) and (4.13) leads to inequality (4.7). □

We state and prove the Rellich's compactness theorem on time scales.

Theorem 4.5. (Rellich's compactness theorem on time scales) *Let $\Omega_{\mathbb{T}^n} \subset \mathbb{T}^n$ be a bounded open set, and $\partial\Omega_{\mathbb{T}^n} \in C^1$. Assume that $1 \leq p < n$, then*

$$W_\Delta^{1,p}(\Omega_{\mathbb{T}^n}) \hookrightarrow L_\Delta^q(\Omega_{\mathbb{T}}) \textit{ for } 1 \leq q < p^*$$

is compact.

Proof. It suffices to show that for any bounded sequence $\{u_m\}$ in $W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})$, there exists a subsequence of $\{u_m\}$ which is strongly convergent in $L_\Delta^q(\Omega_{\mathbb{T}})$.

Step 1. It follows from Lemma 4.4 that

$$W_\Delta^{1,p}(\Omega_{\mathbb{T}^n}) \hookrightarrow L_\Delta^q(\Omega_{\mathbb{T}}) \text{ for } 1 \leq q \leq p^* = \frac{np}{n-p},$$

and

$$\|u\|_{L_\Delta^q(\mathbb{T}^n)} \leq C\| u\|_{W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})}.$$

Step 2. (Using extension) Without loss of generality, we assume that $\Omega_{\mathbb{T}} = \mathbb{T}^n$ and $\{u_m\}$ has compact support in some large open set $V_{\mathbb{T}} \subset \mathbb{T}^n$. Since $\{u_m\}$ is bounded in $W_\Delta^{1,p}(\Omega_{\mathbb{T}^n})$, we have

$$\sup_m \|u_m\|_{W_\Delta^{1,p}(V_{\mathbb{T}^n})} \leq M < \infty. \tag{4.14}$$

Step 3. (Using mollifier) We assume that $\{u_m^\varepsilon(t)\}$ has compact support in set $V_{\mathbb{T}} \subset \mathbb{T}^n$, which is guaranteed by Step 2.

Step 4. We claim that $u_m^\varepsilon(t) \to u_m(t)$ in $L_\Delta^q(V_{\mathbb{T}^n})$ uniformly on m as $\varepsilon \to 0$.

Since

$$\begin{aligned} |u_m^{\varepsilon}(t) - u_m(t)| &= \left| \int_{B(0,1)} \eta(y) \left(u_m(t-\varepsilon y) - u_m(t)\right) \Delta y \right| \\ &= \left| \int_{B(0,1)} \eta(y) \int_{[0,1]_{\mathbb{T}}} \frac{\partial}{\Delta x} u_m(t-\varepsilon xy) \Delta x \Delta y \right| \\ &= \left| \varepsilon \int_{B(0,1)} \eta(y) \int_{[0,1]_{\mathbb{T}}} D_{\Delta} u_m(t-\varepsilon xy) y \Delta x \Delta y \right|, \end{aligned}$$

for $u_m(t) \in W_{\Delta}^{1,p}(V_{\mathbb{T}^n})$, we have

$$\begin{aligned} \int_{V_{\mathbb{T}^n}} |u_m^{\varepsilon}(t) - u_m(t)| \Delta t &\leq \varepsilon \int_{B(0,1)} \eta(y) \int_{[0,1]_{\mathbb{T}}} \int_{V_{\mathbb{T}^n}} |D_{\Delta} u_m(t-\varepsilon xy)| \Delta t \Delta x \Delta y \\ &\leq \varepsilon \int_{V_{\mathbb{T}^n}} |D_{\Delta} u_m(z)| \Delta z. \end{aligned} \tag{4.15}$$

It follows from inequality (4.14) and Hölder's inequality on time scales that

$$\begin{aligned} \|u_m^{\varepsilon} - u_m\|_{L_{\Delta}^1(V_{\mathbb{T}^n})} &\leq \varepsilon \|D_{\Delta} u_m\|_{L_{\Delta}^1(V_{\mathbb{T}^n})} \\ &\leq C^* \varepsilon \|D_{\Delta} u_m\|_{L_{\Delta}^p(V_{\mathbb{T}^n})} \\ &\leq \varepsilon C^* M, \end{aligned} \tag{4.16}$$

where C^* is constant. Inequality (4.16) means that

$$u_m^{\varepsilon} \to u_m \text{ in } L_{\Delta}^1(V_{\mathbb{T}^n}) \text{ uniformly on } m \text{ as } \varepsilon \to 0.$$

For $1 \leq q < p^*$ and $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$, using the interpolation and inequality (4.14), we have

$$\begin{aligned} \|u_m^{\varepsilon} - u_m\|_{L_{\Delta}^q(V_{\mathbb{T}^n})} &\leq \|u_m^{\varepsilon} - u_m\|_{L_{\Delta}^1(V_{\mathbb{T}^n})}^{\theta} \|u_m^{\varepsilon} - u_m\|_{L_{\Delta}^{p^*}(V_{\mathbb{T}^n})}^{1-\theta} \\ &\leq M^{1-\theta} \|u_m^{\varepsilon} - u_m\|_{L_{\Delta}^1(V_{\mathbb{T}^n})}^{\theta}, \end{aligned}$$

which implies that

$$u_m^{\varepsilon} \to u_m \text{ in } L_{\Delta}^q(V_{\mathbb{T}^n}) \text{ uniformly on } m \text{ as } \varepsilon \to 0.$$

Step 5. We claim that for each fixed $\varepsilon > 0$, $\{u_m^{\varepsilon}\}$ is uniformly bounded and equicontinuous.

Indeed, if $t \in \mathbb{T}^n$, for any fixed $\varepsilon > 0$ we have

$$\begin{aligned} |u_m^{\varepsilon}(t)| &\leq \int_{B(x,\varepsilon)} \eta_{\varepsilon}(t-y) |u_m(y)| dy \\ &\leq \|\eta_{\varepsilon}\|_{L^{\infty}(\mathbb{T}^n)} \int_{B(x,\varepsilon)} |u_m(y)| dy \\ &\leq \|\eta_{\varepsilon}\|_{L^{\infty}(\mathbb{T}^n)} \|u_m\|_{L_{\Delta}^1(V_{\mathbb{T}^n})} \\ &\leq \varepsilon^{-n} M < \infty \text{ for } m = 1, 2, \cdots. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
|D_\Delta u_m^\varepsilon(t)| &\le \int_{B(x,\varepsilon)} |D_\Delta \eta_\varepsilon(t-y)|\,|u_m(y)|\,dy \\
&\le \|D_\Delta \eta_\varepsilon\|_{L^\infty(\mathbb{T}^n)} \int_{B(x,\varepsilon)} |u_m(y)|\,dy \\
&\le \|D_\Delta \eta_\varepsilon\|_{L^\infty(\mathbb{T}^n)} \|u_m\|_{L^1_\Delta(V_{\mathbb{T}^n})} \\
&\le \varepsilon^{-(n+1)} M < \infty \text{ for } m = 1, 2, \cdots .
\end{aligned}$$

Step 6. For a fixed $\delta > 0$, there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\lim \sup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q_\Delta(V_{\mathbb{T}^n})} \le \delta.$$

By Step 4, we can choose $\varepsilon > 0$ such that

$$\|u_m^\varepsilon - u_m\|_{L^q_\Delta(V_{\mathbb{T}^n})} \le \frac{\delta}{2}, \quad m = 1, 2, \cdots . \tag{4.17}$$

Notice that $\{u_m^\varepsilon\}$ has compact support in some fixed bounded set $V_{\mathbb{T}^n} \subset \mathbb{R}^n$, and $\{u_m^\varepsilon\}$ is smooth because it is mollification of $\{u_m\}$. It follows from Arzelà-Ascoli's theorem on time scales that there is a subsequence $\{u_{m_j}^\varepsilon\} \subset \{u_m^\varepsilon\}$ which converges uniformly on $V_{\mathbb{T}^n}$. In particular, we have

$$\lim_{j,k\to\infty} \sup \|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\|_{L^q_\Delta(V_{\mathbb{T}^n})} = 0. \tag{4.18}$$

According to (4.17) and (4.18), we have

$$\lim_{j,k\to\infty} \sup \|u_{m_j} - u_{m_k}\|_{L^q_\Delta(V_{\mathbb{T}^n})} \le \delta.$$

Step 7. Choose $\delta = 1,\ 2, \cdots$ in Step 6 and use a diagonal argument to extract a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ such that

$$\lim_{l,k\to\infty} \sup \|u_{m_l} - u_{m_k}\|_{L^q_\Delta(V_{\mathbb{T}^n})} = 0.$$

Therefore, the proof is completed. □

The following two Lemmas can be regarded as the generalized versions of Poincaré inequalities on time scales.

LEMMA 4.6. *Let $\Omega_{\mathbb{T}}$ be a bounded open subset of $\mathbb{T}^n$ with the C^1 boundary. Assume that $1 \le p \le \infty$, then there exists $C = C(p, n, \Omega_{\mathbb{T}})$ such that*

$$\|u - \overline{u}\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \le C\|D_\Delta u\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \text{ for } u \in W^{1,p}(\Omega_{\mathbb{T}^n}), \tag{4.19}$$

where $\overline{u} = \frac{1}{|\Omega_{\mathbb{T}^n}|}\int_{\Omega_{\mathbb{T}^n}} u(t)\Delta t$.

PROOF. If inequality (4.19) is not true, for each integer k there exists a function u_k such that

$$\|u_k - \overline{u}_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} > k\|D_\Delta u_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})}. \tag{4.20}$$

Define

$$v_k = \frac{u_k - \overline{u}_k}{\|u_k - \overline{u}_k\|_{L^p(\Omega_{\mathbb{T}^n})}}, \quad k = 1, 2, \cdots .$$

Then it gives

$$\begin{aligned}\overline{v}_k &= \frac{1}{|\Omega_{\mathbb{T}^n}|}\int_{\Omega_{\mathbb{T}^n}} \frac{u_k(t)-\overline{u}_k(t)}{\|u_k(t)-\overline{u}_k(t)\|_{L^p(\Omega_{\mathbb{T}^n})}}\Delta t\\ &= \frac{1}{|\Omega_{\mathbb{T}^n}|\|u_k(t)-\overline{u}_k(t)\|_{L^p(\Omega_{\mathbb{T}^n})}}\left(\int_{\Omega_{\mathbb{T}^n}} u_k(t)\Delta t-\int_{\Omega_{\mathbb{T}^n}} \overline{u}_k(t)\Delta t\right)\\ &= \frac{1}{|\Omega_{\mathbb{T}^n}|\|u_k(t)-\overline{u}_k(t)\|_{L^p(\Omega_{\mathbb{T}^n})}}\left(\int_{\Omega_{\mathbb{T}^n}} u_k(t)\Delta t-\int_{\Omega_{\mathbb{T}^n}} u_k(t)\Delta t\right)\\ &= 0.\end{aligned}$$

That is,

$$\overline{v}_k = 0 \text{ and } \|v_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} = 1. \tag{4.21}$$

It follow from (4.20) that

$$\|D_\Delta v_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} < \frac{1}{k}. \tag{4.22}$$

Hence, for any $k = 1, 2, \cdots$, we have

$$\begin{aligned}\|v_k\|_{W^{1,p}_\Delta(\Omega_{\mathbb{T}^n})} &\leq \|D_\Delta v_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} + \|v_k\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})}\\ &\leq 1+\frac{1}{k},\end{aligned} \tag{4.23}$$

which implies that $\left\{\|v_k\|_{W^{1,p}_\Delta(\Omega_{\mathbb{T}^n})}\right\}$ is bounded. According to the Rellich's compactness theorem on time scales, there exists a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ and a function $v \in L^p_\Delta(\Omega_{\mathbb{T}^n})$ such that

$$v_{k_j} \to v \text{ in } L^p_\Delta(\Omega_{\mathbb{T}^n}). \tag{4.24}$$

From (4.21), we obtain

$$\overline{v} = 0 \text{ and } \|v\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} = 1. \tag{4.25}$$

However, according to (4.23), for any $i = 1, 2, \cdots, n$ and any $\phi \in C^\infty_c(\Omega_{\mathbb{T}^n})$, we have

$$\begin{aligned}\int_{\Omega_{\mathbb{T}^n}} v\frac{\partial\phi}{\Delta_i t_i}\Delta t &= \lim_{k_j\to\infty}\int_{\Omega_{\mathbb{T}^n}} v_{k_j}\frac{\partial\phi}{\Delta_i t_i}\Delta t\\ &= -\lim_{k_j\to\infty}\int_{\Omega_{\mathbb{T}^n}} \frac{\partial v_{k_j}}{\Delta_i t_i}\phi(\sigma(t))\Delta t.\end{aligned}$$

It follows from Hölder's inequality on time scales and inequality (4.22) that

$$\begin{aligned}\left|\int_{\Omega_{\mathbb{T}^n}} v\frac{\partial\phi}{\Delta_i t_i}\Delta t\right| &\leq \lim_{k_j\to\infty}\left(\int_{\Omega_{\mathbb{T}^n}}\left|\frac{\partial v_{k_j}}{\Delta_i t_i}\right|^p\Delta t\right)^{\frac{1}{p}}\left(\int_{\Omega_{\mathbb{T}^n}}|\phi(\sigma(t))|^{\frac{p}{p-1}}\Delta t\right)^{\frac{p-1}{p}}\\ &= 0.\end{aligned}$$

So we have

$$v \in W^{1,p}_\Delta(\Omega_{\mathbb{T}^n}) \text{ with } D_\Delta v = 0 \text{ a.e. on } \Omega_{\mathbb{T}^n}.$$

Since $\Omega_{\mathbb{T}^n}$ is bounded and

$$D_\Delta v = \begin{pmatrix} \frac{\partial v}{\Delta_1 t_1} \\ \frac{\partial v}{\Delta_2 t_2} \\ \vdots \\ \frac{\partial v}{\Delta_n t_n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

it follows from Lemma 2.4 that

$$v \equiv constant.$$

In view of $\overline{v} \equiv 0$, we have $v \equiv 0$ and

$$\|v\|_{L_\Delta(\Omega_{\mathbb{T}^n})} = 0,$$

which yields a contradiction with formula (4.25). □

Similarly, we can obtain

LEMMA 4.7. *Let $\Omega_{\mathbb{T}}$ be a bounded open subset of $\mathbb{T}^n$. Assume that $1 \le p \le \infty$, then there exists a constant $C = C(p, n, \Omega_{\mathbb{T}})$ such that*

$$\|u\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \quad \le \quad C\|D_\Delta u\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \text{ for } u \in W^{1,p}_0(\Omega_{\mathbb{T}^n}). \tag{4.26}$$

From Lemmas 4.6 and 4.7, we have the following remarks.

REMARK 4.8. If $|\Omega_{\mathbb{T}}| < \infty$, then

$$\lambda_1(\Omega_{\mathbb{T}}) := \inf_{\substack{u \in H^1_{0,\Delta}(\Omega_{\mathbb{T}}) \\ \|u\|_{L^2_\Delta(\Omega_{\mathbb{T}^n})} = 1}} \|\nabla u\|^2_{L^2_\Delta(\Omega_{\mathbb{T}^n})} > 0 \tag{4.27}$$

is achieved.

REMARK 4.9. (i) It is notable that $H^1_{0,\Delta}(\Omega_{\mathbb{T}}) \subset \mathcal{D}^{1,2}_\Delta(\Omega_{\mathbb{T}})$.

(ii) If $|\Omega_{\mathbb{T}}| < +\infty$, the Poincaré's inequality implies $H^1_{0,\Delta}(\Omega_{\mathbb{T}}) = \mathcal{D}^{1,2}_\Delta(\Omega_{\mathbb{T}})$.

5. Differentiable Functions

LEMMA 5.1. *Let $\Omega_{\mathbb{T}}$ be an open subset of $\mathbb{T}^n$ and $1 \le p < \infty$. If $v_n \to u$ in $L^p_\Delta(\Omega_{\mathbb{T}})$, then there exists a subsequence $\{\omega_n\}$ of $\{v_n\}$ and $g \in L^p_\Delta(\Omega_{\mathbb{T}})$ such that*

$$\omega_n(t) \to u(t) \text{ a.e. on } \Omega_{\mathbb{T}},$$

and

$$|u(t)| \le g(t) \text{ and } |\omega_n(t)| \le g(t) \text{ a.e. on } \Omega_{\mathbb{T}}.$$

PROOF. Assume that $v_n(t) \to u(t)$ a.e. on $\Omega_{\mathbb{T}}$. There is a subsequence $\{\omega_n\}$ of $\{v_n\}$ such that

$$\|\omega_{j+1}(t) - \omega_j(t)\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \le 2^{-j} \text{ for } j \ge 1.$$

Define

$$g(t) := |\omega_1(t)| + \sum_{j=1}^{\infty} |\omega_{j+1}(t) - \omega_j(t)|.$$

So we get $g \in L^p_\Delta(\Omega_{\mathbb{T}})$ and

$$|\omega_n(t)| \le g(t) \text{ a.e. on } \Omega_{\mathbb{T}},$$

and

$$|u(t)| \le g(t) \text{ a.e. on } \Omega_{\mathbb{T}}.$$

□

LEMMA 5.2. *Let $\Omega_{\mathbb{T}}$ be an open subset of $\mathbb{T}^n$ and $|\Omega_{\mathbb{T}}| < \infty$. Suppose that $1 \le p,\ r < \infty$ and $f \in C(\overline{\Omega_{\mathbb{T}}} \times \mathbb{R})$ satisfies*

$$|f(t,u)| \le c\left(1 + |u|^{\frac{p}{r}}\right).$$

Then, for any $u \in L^p_\Delta(\Omega_{\mathbb{T}})$ and $f(.,u) \in L^r_\Delta(\Omega_{\mathbb{T}})$, the operator

$$T : u \mapsto f(t,u)$$

is continuous.

PROOF. Suppose that $u \in L^p_\Delta(\Omega_{\mathbb{T}})$. Since

$$|f(t,u)|^r \le c^r\left(1 + |u|^{\frac{p}{r}}\right)^r \in L^1_\Delta(\Omega_{\mathbb{T}}),$$

this implies that

$$f(.,u) \in L^r_\Delta(\Omega_{\mathbb{T}}).$$

Assume that $u_n \to u \in L^p_\Delta(\Omega_{\mathbb{T}})$ and $\{\omega_n\}$ is a subsequence of $\{u_n\}$. Let $\{\omega_n\}$ and g be given as in Lemma 5.1. Then we have

$$\begin{aligned} |f(t,\omega_n) - f(t,u)|^r &\le \left(2c\left(1 + |u|^{\frac{p}{r}}\right)\right)^r \\ &= 2^r c^r \left(1 + |g|^{\frac{p}{r}}\right)^r \in L^1_\Delta(\Omega_{\mathbb{T}}). \end{aligned}$$

It follows from the Lebesgue's dominated convergence theorem on time scales that

$$T\omega_n \to Tu \text{ in } L^r_\Delta(\Omega_{\mathbb{T}})$$

and

$$Tu_n \to Tu \text{ in } L^r_\Delta(\Omega_{\mathbb{T}}).$$

This implies that the operator

$$T : u \mapsto f(t,u)$$

is continuous. □

LEMMA 5.3. *Assume that $\Omega_{\mathbb{T}}$ be an open subset of $\mathbb{T}^n$ and let $2 < p \le \infty$. The functionals defined by*

$$\psi(u^\sigma) = \int_{\Omega_{\mathbb{T}}} |u^\sigma|^p \Delta t$$

and

$$\chi(u^\sigma) = \int_{\Omega_{\mathbb{T}}} |(u^\sigma)^+|^p \Delta t$$

belong to $C^2\left(L^p_\Delta\left(\Omega_{\mathbb{T}}\right), \mathbb{R}\right)$ and the following two relations hold:

$$\langle \psi'(u^\sigma), h^\sigma \rangle = p \int_{\Omega_{\mathbb{T}}} |u^\sigma|^{p-2} u^\sigma h^\sigma \Delta t$$

and

$$\langle \chi'(u^\sigma), h^\sigma \rangle = p \int_{\Omega_{\mathbb{T}}} |(u^\sigma)^+|^{p-1} h^\sigma \Delta t.$$

PROOF. We only consider the functional ψ, because the discussion for the functional χ can be done in an analogous way.

Firstly, we show the existence of the Gateaux derivative.

Let u^σ, $h^\sigma \in L^p_\Delta(\Omega_\mathbb{T})$. Given $t \in \mathbb{T}^n$ and $0 < |\lambda| < 1$, it follows from the mean value theorem on time scales that there exists $\tau_1 \in [0,1]$ such that

$$\frac{|u^\sigma + \lambda h^\sigma|^p - |u^\sigma|^p}{|\lambda|} \le p\left[|u^\sigma| + |h^\sigma|\right]^{p-1}|h^\sigma|. \tag{5.1}$$

Since u^σ, $h^\sigma \in L^p_\Delta(\Omega_\mathbb{T})$, using Hölder's inequality on time scales leads to

$$\int_{\Omega_\mathbb{T}}\left[|u^\sigma| + |h^\sigma|\right]^{p-1}|h^\sigma|\,\Delta t \le \left[\int_{\Omega_\mathbb{T}}\left[|u^\sigma| + |h^\sigma|\right]^{p}\Delta t\right]^{\frac{p-1}{p}}\left[\int_{\Omega_\mathbb{T}}|h^\sigma|^p\,\Delta t\right]^{\frac{1}{p}},$$

which implies

$$\left[|u^\sigma| + |h^\sigma|\right]^{p-1}|h^\sigma| \in L^1_\Delta(\Omega_\mathbb{T}).$$

By formula (5.1) and the Lebesgue's dominated convergence theorem on time scales, we derive that

$$\begin{aligned}\langle\psi'(u^\sigma), h^\sigma\rangle : &= \lim_{\lambda\to 0}\frac{1}{\lambda}\left[\psi\left(u^\sigma + \lambda h^\sigma\right) - \psi\left(u^\sigma\right)\right]\\ &= \lim_{\lambda\to 0} p\int_{\Omega_\mathbb{T}}|u^\sigma + \lambda\tau_1 h^\sigma|^{p-2}\left|u^\sigma + \lambda\tau_1 h^\sigma\right||h^\sigma|\,\Delta t\\ &= p\int_{\Omega_\mathbb{T}}|u^\sigma|^{p-2}u^\sigma h^\sigma\Delta t.\end{aligned}$$

Secondly, we consider the continuity of the Gateaux derivative.

Let $f(u^\sigma) = p|u^\sigma|^{p-2}u^\sigma$. Assume that $u^\sigma_n \to u^\sigma$ in $L^p_\Delta(\Omega_\mathbb{T})$. By Lemma 5.2, we get

$$f(u^\sigma_n) \to f(u^\sigma) \text{ in } L^q_\Delta(\Omega_\mathbb{T}),$$

where $q = \frac{p}{p-1}$. Using Hölder's inequality on time scales again gives

$$\begin{aligned}|\langle\psi'(u^\sigma_n) - \psi'(u^\sigma), h^\sigma\rangle| &\le \int_{\Omega_\mathbb{T}}|f(u^\sigma_n) - f(u^\sigma)|\,|h^\sigma|\Delta t\\ &\le \left(\int_{\Omega_\mathbb{T}}|f(u^\sigma_n) - f(u^\sigma)|^q\,\Delta t\right)^{\frac{1}{q}}\left(\int_{\Omega_\mathbb{T}}|h^\sigma|^p\Delta t\right)^{\frac{1}{p}}\\ &\le \|f(u^\sigma_n) - f(u^\sigma)\|_{L^q_\Delta(\Omega_\mathbb{T})}\|h^\sigma\|_{L^p_\Delta(\Omega_\mathbb{T})}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\psi'(u^\sigma_n) - \psi'(u^\sigma)\| &\le \|f(u^\sigma_n) - f(u^\sigma)\|_{L^q_\Delta(\Omega_\mathbb{T})}\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

In the following, we prove the existence of the second Gateaux derivative.

Let u^σ, h^σ, $v^\sigma \in L^p_\Delta(\Omega_\mathbb{T})$. Given $t \in \mathbb{T}^N$ and $0 < |\lambda| < 1$, it follows from the mean value theorem on time scales that there exists $\tau_2 \in [0,1]$ such that

$$\begin{aligned}&\frac{|\left[f\left(u^\sigma + \lambda h^\sigma\right) - f\left(u^\sigma\right)\right]v^\sigma|}{|\lambda|}\\ &= p(p-1)\left|u^\sigma + \lambda\tau_2 h^\sigma\right|^{p-2}|h^\sigma||v^\sigma|\\ &\le p(p-1)\left[|u^\sigma| + |h^\sigma|\right]^{p-2}|h^\sigma||v^\sigma|.\end{aligned}$$

Since u^σ, h^σ, $v^\sigma \in L^p_\Delta(\Omega_{\mathbb{T}})$, using Hölder's inequality on time scales we have

$$\begin{aligned}&\int_{\Omega_{\mathbb{T}}} [|u^\sigma| + |h^\sigma|]^{p-2} |h^\sigma v^\sigma| \Delta t \\ \leq\ & \left[\int_{\Omega_{\mathbb{T}}} [|u^\sigma| + |h^\sigma|]^p \Delta t\right]^{\frac{p-2}{p}} \left[\int_{\Omega_{\mathbb{T}}} |h^\sigma|^p \Delta t\right]^{\frac{1}{p}} \left[\int_{\Omega_{\mathbb{T}}} |v^\sigma|^p \Delta t\right]^{\frac{1}{p}},\end{aligned}$$

which implies

$$[|u^\sigma| + |h^\sigma|]^{p-2} |h^\sigma v^\sigma| \in L^1_\Delta(\Omega_{\mathbb{T}}).$$

It follows from the Lebesgue's dominated convergence theorem on time scales that

$$\begin{aligned}\langle \psi''(u^\sigma) h^\sigma, v^\sigma\rangle : &= \lim_{\lambda\to 0} \frac{1}{\lambda}\left[f(u^\sigma + \lambda h^\sigma) - f(u^\sigma)\right] \\ &= \lim_{\lambda\to 0} p(p-1) \int_{\Omega_{\mathbb{T}}} |u^\sigma + \lambda\tau_1 h^\sigma|^{p-2} h^\sigma v^\sigma \Delta t \\ &= p(p-1)\int_{\Omega_{\mathbb{T}}} |u^\sigma|^{p-2} h^\sigma v^\sigma \Delta t.\end{aligned}$$

Finally, we show the continuity of the second Gateaux derivative.
Let

$$g(u^\sigma) = p(p-1)|u^\sigma|^{p-2}.$$

Assume that $u^\sigma_n \to u^\sigma$ in $L^p_\Delta(\Omega_{\mathbb{T}})$. By Lemma 5.2, we get

$$g(u^\sigma_n) \to g(u^\sigma) \text{ in } \mathrm{L}^{\mathrm{r}}_\Delta(\Omega_{\mathbb{T}}),$$

where $r = \frac{p}{p-2}$. Using Hölder's inequality on time scales we deduce that

$$\begin{aligned}&|\langle (\psi''(u^\sigma_n) - \psi''(u^\sigma)) h^\sigma, v^\sigma\rangle| \\ \leq\ & \int_{\Omega_{\mathbb{T}}} |g(u^\sigma_n) - g(u^\sigma)|\, |h^\sigma||v^\sigma| \Delta t \\ \leq\ & \left(\int_{\Omega_{\mathbb{T}}} |f(u^\sigma_n) - f(u^\sigma)|^r \Delta t\right)^{\frac{1}{r}} \left(\int_{\Omega_{\mathbb{T}}} |h|^p \Delta t\right)^{\frac{1}{p}} \left(\int_{\Omega_{\mathbb{T}}} |v^\sigma|^p \Delta t\right)^{\frac{1}{p}} \\ \leq\ & \|f(u^\sigma_n) - f(u^\sigma)\|_{L^r_\Delta(\Omega_{\mathbb{T}})} \|h^\sigma\|_{L^p_\Delta(\Omega_{\mathbb{T}})} \|v^\sigma\|_{L^p_\Delta(\Omega_{\mathbb{T}})}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\|\psi''(u^\sigma_n) - \psi''(u^\sigma)\| &\leq \|f(u^\sigma_n) - f(u^\sigma)\|_{r,\Delta} \\ &\to 0, \text{ as } n\to\infty.\end{aligned}$$

□

In view of Lemmas 4.5 and 5.3, we have the following corollary immediately.

Corollary 5.4. **(i):** Let $2 < p \leq \infty$ if $n = 1, 2$ or let $2 < p \leq 2^*$ if $n \geq 3$. The functionals ψ and χ are of class $C^2\left(H^1_{0,\Delta}(\Omega_{\mathbb{T}}), \mathbb{R}\right)$.

(ii): Let $N \geq 3$ and $p = 2^*$, then the functionals ψ and χ are of class $C^2\left(D^{1,2}_{0,\Delta}(\Omega_{\mathbb{T}}), \mathbb{R}\right)$.

6. Existence of Solutions

In this section, we state and prove our results on the existence of nontrivial solutions of equation (1.1) on time scales.

THEOREM 6.1. *Assume that* $|\Omega_{\mathbb{T}}| < \infty$ *and* $2 < p < 2^*$ *hold. Then the problem* (1.1) *has at least one nontrivial solution if and only if* $\lambda > -\lambda_1(\Omega_{\mathbb{T}})$*, where* $\lambda_1(\Omega_{\mathbb{T}})$ *is defined by formula* (4.27)*.*

PROOF. For necessity, we assume that u is a nontrivial solution of the problem (1.1). Let e_1 be an eigenfunction of $-\triangle$ corresponding to $\lambda_1(\Omega_{\mathbb{T}})$ with $e_1 > 0$ on $\Omega_{\mathbb{T}}$. Due to $u > 0$, we have

$$\begin{aligned} \lambda \int_{\Omega_{\mathbb{T}}} u^\sigma e_1 \Delta t &= \int_{\Omega_{\mathbb{T}}} \left((u^\sigma)^{p-1} + \triangle u\right) e_1 \Delta t \\ &> \int_{\Omega_{\mathbb{T}}} \triangle u e_1 \Delta t \\ &= -\lambda_1(\Omega_{\mathbb{T}}) \int_{\Omega_{\mathbb{T}}} u^\sigma e_1 \Delta t, \end{aligned}$$

which implies that $\lambda > -\lambda_1(\Omega_{\mathbb{T}})$.

For sufficiency, we let

$$f_1(u) := \left(u^+\right)^{p-1} \quad \text{and} \quad F(u) := \frac{(u^+)^p}{p}.$$

Define the functional $A : E \to \mathbb{R}$ by

$$\begin{aligned} A(u^\sigma) : &= \int_{\Omega_{\mathbb{T}}} \left(\frac{1}{2}|\bigtriangledown u|^2 + \frac{\lambda}{2}(u^\sigma)^2\right) \Delta t - \int_{\Omega_{\mathbb{T}}} F\left(u^\sigma\right) \Delta t \\ &= \frac{1}{2}\langle L u^\sigma, u^\sigma\rangle - \psi(u^\sigma), \end{aligned}$$

where $E := H^1_{0,\Delta}(\Omega_{\mathbb{T}})$, and define the inner product by

$$(u^\sigma, v^\sigma)_{E,\Delta} := \int_{\mathbb{R}^N} \left(\bigtriangledown u. \bigtriangledown v + \lambda v^\sigma v^\sigma\right) \Delta t,$$

with the corresponding norm

$$\|u^\sigma\|_{E,\Delta} = \left(\int_{\Omega_{\mathbb{T}}} \left[|\bigtriangledown u|^2 + \lambda |u^\sigma|^2\right] \Delta t\right)^{\frac{1}{2}}.$$

Let

$$c_1 = 1 + \min\left(0, \frac{\lambda}{\lambda_1(\Omega_{\mathbb{T}})}\right).$$

By the Poincaré inequality on time scales we have

$$\int_{\Omega_{\mathbb{T}}} \left[|\bigtriangledown u|^2 + \lambda |u^\sigma|^2\right] \Delta t \geq c_1 \int_{\Omega_{\mathbb{T}}} |\bigtriangledown u|^2 \Delta t.$$

Hence, E is a Hilbert space. It follows from Corollary 5.4 that $A(u^\sigma) \in C^2\left(H^1_{0,\Delta}\left(\Omega_{\mathbb{T}}\right), \mathbb{R}\right)$.

For any $v^\sigma \in E$ and $0 < |\varepsilon| < 1$, we have

$$\frac{1}{\varepsilon}\left[A(u^\sigma+\varepsilon v^\sigma)-A(u^\sigma)\right]=\int_{\Omega_{\mathbb{T}}}\frac{1}{2}\frac{(\bigtriangledown u+\varepsilon\bigtriangledown v)^2-(\bigtriangledown u)^2}{\varepsilon}\Delta t$$
$$-\frac{\lambda}{2}\int_{\Omega_{\mathbb{T}}}\frac{(u^\sigma+\varepsilon v^\sigma)^2-(u^\sigma)^2}{\varepsilon}\Delta t+\int_{\Omega_{\mathbb{T}}}\frac{((u^\sigma)^+ + \varepsilon(v^\sigma)^+)^p-((u^\sigma)^+)^p}{\varepsilon p}\Delta t.$$

By the dominated convergence theorem on time scales and the divergence formula (3.4) we get

$$\begin{aligned}\langle A'(u^\sigma),v^\sigma\rangle &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[A(u^\sigma+\varepsilon v^\sigma)-A(u^\sigma)\right]\\ &= \int_{\Omega_{\mathbb{T}}}(\bigtriangledown u\bigtriangledown v+\lambda u^\sigma v^\sigma)\,\Delta t-\int_{\Omega_{\mathbb{T}}}f_1(u^\sigma)\,v^\sigma\Delta t\\ &= -\int_{\Omega_{\mathbb{T}}}(\triangle u v^\sigma-\lambda u^\sigma v^\sigma)\,\Delta t-\int_{\Omega_{\mathbb{T}}}f_1(u^\sigma)\,v^\sigma\Delta t. \end{aligned}\tag{6.1}$$

Equality (6.1) implies that $A'(u^\sigma)=0$ if and only if u^σ is a solution of the equation

$$-\triangle u+\lambda u^\sigma = f_1(u^\sigma).$$

By the definition of the functional A and its properties, it suffices to show that all the conditions of Lemma 2.2 hold with respect to A.

Firstly, we verify that the $(PS)_c$ condition holds. That is, if any sequence $\{u_n^\sigma\}\subset H^1_{0,\Delta}(\Omega_{\mathbb{T}})$ satisfies

$$\sup_n A(u_n^\sigma)<\infty, \qquad A'(u_n^\sigma)\to 0,$$

then $\{u_n^\sigma\}$ contains a convergent subsequence.

In the case of sufficiently large n, we have

$$\begin{aligned}\sup_n A(u_n^\sigma)+1+\|u_n^\sigma\|_{E,\Delta} &\geq A(u_n^\sigma)-\frac{1}{p}\langle A'(u_n^\sigma),u_n^\sigma\rangle\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\left(\int_{\Omega_{\mathbb{T}}}\left[|\bigtriangledown u_n|^2+\lambda|u_n^\sigma|^2\right]\Delta t\right)\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n^\sigma\|^2_{E,\Delta}, \end{aligned}\tag{6.2}$$

which indicates that $\|u_n^\sigma\|$ is bounded. So there exists a subsequence (still denoted by $\{u_n^\sigma\}$) and we assume that there is a point $u_0^\sigma\in H^1_{0,\Delta}(\Omega_{\mathbb{T}})$ such that $u_n^\sigma\rightharpoonup u_0^\sigma$ in $H^1_{0,\Delta}(\Omega_{\mathbb{T}})$. By Lemma 4.5, we have $u_n^\sigma\to u_0^\sigma$ in $L^p_\Delta(\Omega_{\mathbb{T}})$. It follows from Lemma 5.2 that $f_1(u_n^\sigma)\to f_1(u_0^\sigma)$ in $L^q_\Delta(\Omega_{\mathbb{T}})$, where $q=\frac{p}{p-1}$. This gives

$$\begin{aligned}&\left|\int_{\Omega_{\mathbb{T}}}\left[f_1(u_n^\sigma)-f_1(u_0^\sigma)\right](u_n^\sigma-u_0^\sigma)\Delta t\right|\\ \leq\ &\left(\int_{\Omega_{\mathbb{T}}}|f_1(u_n^\sigma)-f_1(u_0^\sigma)|^q\,\Delta t\right)^{\frac{1}{q}}\left(\int_{\Omega_{\mathbb{T}}}|u_n^\sigma-u_0^\sigma|^p\,\Delta t\right)^{\frac{1}{p}}\\ \to\ &0 \text{ as } n\to\infty.\end{aligned}$$

Moreover, we have

$$\langle A'(u_n^\sigma)-A'(u_0^\sigma),u_n^\sigma-u_0^\sigma\rangle\to 0 \text{ as } n\to\infty,$$

and

$$\begin{aligned}
&\langle A'(u_n^\sigma) - A'(u_0^\sigma), u_n^\sigma - u_0^\sigma\rangle \\
= &\int_{\Omega_{\mathbb{T}}} \left[| \bigtriangledown u_n - \bigtriangledown u_0|^2 + \lambda |u_n^\sigma - u_0^\sigma|^2\right] \Delta t - \int_{\Omega_{\mathbb{T}}} \left[f_1(u_n^\sigma) - f_1(u_0^\sigma)\right](u_n^\sigma - u_0^\sigma)\Delta t \\
= &\|u_n - u_0\|_{E,\Delta} - \int_{\Omega_{\mathbb{T}}} \left[f_1(u_n^\sigma) - f_1(u_0^\sigma)\right](u_n^\sigma - u_0^\sigma)\Delta t,
\end{aligned}$$

which implies that

$$\|u_n^\sigma - u_0^\sigma\|_{E,\Delta} \to 0 \text{ as } n \to \infty.$$

Consequently, $\{u_n^\sigma\}$ has a convergent sequence in $H^1_{0,\Delta}(\Omega_{\mathbb{T}})$.

Secondly, we show that the remaining conditions of Lemma 2.2 hold with respect to the functional A.

It follows from Lemma 4.5 that there exists a constant $C_1 > 0$ such that

$$\|u\|_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \le C_1 \|u\|_{E,\Delta} \text{ for } u \in H^1_{0,\Delta}(\Omega_{\mathbb{T}}).$$

It leads to

$$\begin{aligned}
A(u^\sigma) &\ge \frac{1}{2}\|u^\sigma\|^2_{E,\Delta} - \frac{1}{p}\|u^\sigma\|^p_{L^p_\Delta(\Omega_{\mathbb{T}^n})} \\
&\ge \frac{1}{2}\|u^\sigma\|^2_{E,\Delta} - \frac{C_1^2}{p}\|u^\sigma\|^p_{E,\Delta}.
\end{aligned}$$

Thus, we obtain that there exists a $r > 0$ such that

$$\begin{aligned}
r &= \inf_{\|u\|_{E,\Delta}} A(u^\sigma) \\
&> A(0) \\
&= 0.
\end{aligned}$$

Let $u^\sigma \in H^1_{0,\Delta}(\Omega_{\mathbb{T}})$ with $u^\sigma > 0$ in $\Omega_{\mathbb{T}}$. For any constant $\mu \ge 0$, we obtain that

$$\begin{aligned}
A(\mu u^\sigma) &= \frac{1}{2}\mu^2 \int_{\Omega_{\mathbb{T}}} \left(| \bigtriangledown u|^2 + \lambda |u^\sigma|^2\right) \Delta t - \frac{\mu^p}{p}\int_{\Omega_{\mathbb{T}}} |u^\sigma|^p \,\Delta t \\
&= \frac{1}{2}\mu^2 \|u^\sigma\|^2_{E,\Delta} - \frac{\mu^p}{p}\|u^\sigma\|^p_{L^p_\Delta(\Omega_{\mathbb{T}^n})}.
\end{aligned}$$

Since $p > 2$, there exists $e := \mu u^\sigma$ such that

$$\|e\|_{E,\Delta} > r \text{ and } A(e) \le 0.$$

Hence, all conditions of the mountain pass theorem are satisfied. It follows from Lemma 2.2 that the problem

$$\begin{cases} -\triangle u + \lambda u^\sigma = f(u), \\ u \in H^1_{0,\Delta}(\Omega_{\mathbb{T}}), \end{cases} \tag{6.3}$$

has a nontrivial solution u^σ. Considering the problem (6.3) on $(u^\sigma)^-$, multiplying equation (6.3) by $(u^\sigma)^-$ and integrating it over $\Omega_{\mathbb{T}}$, we obtain

$$\begin{aligned}
0 &= -\int_{\Omega_{\mathbb{T}}} \left(\triangle u^-(u^\sigma)^- - \lambda |(u^\sigma)^-|^2\right) \Delta t \\
&= \int_{\Omega_{\mathbb{T}}} \left(| \bigtriangledown u^-|^2 + \lambda |(u^\sigma)^-|^2\right) \Delta t \\
&= \|(u^\sigma)^-\|_{E,\Delta},
\end{aligned}$$

which implies that

$$(u^\sigma)^- = 0.$$

Consequently, u^σ is a solution of the problem (1.1). □

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