Trace formulas for fourth order operators on unit interval, II

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ABSTRACT. We consider self-adjoint fourth order operators on the unit interval with the Dirichlet type boundary conditions. For such operators we determine few trace formulas, similar to the case of Gelfand–Levitan formulas for second order operators.

CONTENTS

1. Introduction and main results

1.1. Introduction. We consider a self-adjoint operator H on $L^2(0,1)$ given by

$$(1.1) \qquad Hy = (\partial^4 + 2\partial p\partial + q)y, \qquad \text{where} \quad \partial = \frac{d}{dx},$$

under the Dirichlet type boundary conditions

$$(1.2) \qquad y(0) = y''(0) = y(1) = y''(1) = 0.$$

We assume that the functions p, q are real and satisfy $p, q \in L^1(0,1)$. Note that any self-adjoint fourth order operator with real coefficients may be transformed to the form (1.1). It is well known (see, e.g., [**N**, Ch. I.2, I.4]) that the spectrum of the

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operator H consists of the real eigenvalues $\mu_n, n \in \mathbb{N}$, of multiplicity $\leqslant 2$ labeled by

$$\mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant ...,$$

counted with multiplicities, and they satisfy

$$\mu_n = (\pi n)^4 + O(n^2) \qquad \text{as} \quad n \to \infty.$$

We recall the famous Gelfand–Levitan trace formula for a second order operator h with the Dirichlet boundary conditions on the interval $[0,1]$ given by

(1.3) $$hy = -y'' - py, \qquad y(0) = y(1) = 0.$$

All eigenvalues $\alpha_n, n \in \mathbb{N}$, of this operator are simple and labeled by

$$\alpha_1 < \alpha_2 < \alpha_3 < ...$$

It is well known (see e.g., [**FP**] or (2.2)), that in the case $p, p'' \in L^1(0,1)$ the eigenvalues α_n satisfy $\alpha_n = (\pi n)^2 - p_0 + O(n^{-2})$ as $n \to \infty$, where $p_0 = \int_0^1 p(t)dt$. In this case Gelfand and Levitan [**GL**] determined the famous trace formula given by

(1.4) $$\sum_{n=1}^{\infty} \left(\alpha_n - (\pi n)^2 + p_0 \right) = \frac{p(0) + p(1)}{4} - \frac{p_0}{2},$$

where the series converges absolutely.

There are a lot of other results about trace formulas for second order operators, see the book of Levitan – Sargsyan [**LS**] and references therein. Due to application to the KdV equation on the circle there are a lot of papers about an operator $-\partial^2 - p$ on the circle: Dubrovin [**Du**] and Its – Matveev [**IM**] determined trace formulas for so called finite band potentials, McKean – van Moerbeke [**MvM**] and Trubowitz [**T**] considered the case of sufficiently smooth potentials; Korotyaev [**K**] determined the trace formula for the case $p \in L^2(0,1)$. Note that the corresponding trace formulas for the Boussinesq equation were determined by McKean [**M**] and for the Camassa – Holm equation by Badanin, Klein and Korotyaev [**BKK**].

Describe briefly results about trace formulas for fourth and higher order operators. The trace formulas for operators $(\partial^2 + p)^m, p \in C_{\mathbb{R}}^{\infty}[0,1]$ with integer $m \geqslant 2$ were determined by Gelfand [**G**] and Dikii [**D1**], [**D2**]. Sadovnichii [**S1**], [**S2**] obtained trace formulas for even order operators, using Dikii's approach [**D1**], [**D2**]. Among recent papers, nearest to our subject, we mention results of Badanin – Korotyaev [**BK5**], Gül [**Gu**], Nazarov-Stolyarov-Zatitskiy [**NSZ**], see also the review of Sadovnichii–Podol'skii [**SP**] and references therein.

Eigenvalue asymptotics are important for trace formulas: their proof and the convergence in trace formulas, see e.g., (2.2) and (1.4). Eigenvalue asymptotics for fourth and higher order operators on the finite interval are much less investigated than for second order operators. An operator $\partial^4 + \partial p \partial + q$ under the 2-periodic boundary conditions was considered by Badanin and Korotyaev [**BK2**], [**BK4**] (for the simpler case $\partial^4 + q$ see [**BK1**]). The sharp eigenvalue asymptotics for the operator H in the class of complex coefficients was determined in [**BK6**]. Moreover, there was determined the eigenvalue asymptotics for the Euler-Bernoulli operator $b^{-1}(af'')''$, $a, b > 0$, on the unit interval. This operator is related with the description of the bending vibrations of thin beams and plates. Eigenvalue asymptotics for an operator $\partial^{2n} + q$ was determined by Akhmerova [**Ah**], Mikhailets–Molyboga

[**MM**]. Badanin and Korotyaev [**BK3**] determined the eigenvalue asymptotics for a general case of $2n$ order operators under the 2-periodic boundary conditions.

We discuss some other results for fourth and higher order operators. Numerous results about higher order operators with different types of boundary conditions are expounded in the books of Atkinson [**At**] and Naimark [**N**]. Many papers are devoted to the inverse spectral problems for these operators, see Barcilon [**B**], Caudill, Perry and Schueller [**CPS**], Hoppe, Laptev and Östensson [**HLO**], McLaughlin [**Mc**], Papanicolaou [**P**], Yurko [**Yu**] and so on.

The main goal of the present paper is to determine trace formulas for fourth order operators on the unit interval. In fact, our paper is a second part of our previous results from [**BK5**], where we determined trace formulas for fourth order operators on the circle.

There is a significant difference between trace formulas for fourth order operators and for second order operators. Indeed, we have two coefficients p, q in the fourth order operator, which corresponds to perturbation by second order operators. It is possible to determine the following trace formulas:

1) for fix p in terms of q,
2) for fix q in terms of p,
3) in terms of p, q.

The case 1) is simpler, since a perturbation is a function q. The perturbation in the cases 2) and 3) is a second order operator and it is stronger, than in the case 1). Therefore, the cases 2) and 3) are more difficult, than the case 1).

1.2. Perturbations by second order operators. We determine trace formulas for the cases 2) and 3), where perturbations of the fourth order operators are second order operators.

Introduce the Sobolev spaces $\mathscr{H}_m$ and $\mathscr{H}_m^0$, $m \geqslant 0$, by

$$\mathscr{H}_m = \left\{ f \in L^1(0,1) : f^{(m)} \in L^1(0,1) \right\}, \qquad \mathscr{H}_m^0 = \left\{ f \in \mathscr{H}_m : \int_0^1 f(t)dt = 0 \right\}.$$

Recall the following results from [**BK6**]. Let $(p, q) \in \mathscr{H}_3 \times \mathscr{H}_1$. Then the eigenvalues μ_n of the operator H satisfy the asymptotics

$$(1.5) \qquad \mu_n = ((\pi n)^2 - p_0)^2 - \frac{P + p_0^2}{2} + q_0 - \widehat{V}_{cn} + O(n^{-2}),$$

uniformly on any bounded subset of $(p, q) \in \mathscr{H}_3 \times \mathscr{H}_1$, where

$$(1.6) \qquad f_0 = \int_0^1 f(t)dt, \qquad P = \int_0^1 (p''(t) + p^2(t))dt,$$

$$(1.7) \qquad \widehat{V}_{cn} = \int_0^1 V(t)\cos(2\pi nt)dt \qquad \forall \ \ n \in \mathbb{Z}, \qquad V = q - \frac{p''}{2}.$$

Now we formulate our main results on the trace formulas for the operator $H = \partial^4 + 2\partial p \partial + q$. The perturbation in these formulas is the second order operator $2\partial p \partial + q$, depending on two functions p and q.

Theorem 1.1. *Let $(p, q) \in \mathscr{H}_4 \times \mathscr{H}_2^0$ and let $\mu_n, n \in \mathbb{N}$, be the eigenvalues of the operator H. Then the following trace formula holds true:*

$$(1.8) \qquad \sum_{n \geqslant 1} \left(\mu_n - ((\pi n)^2 - p_0)^2 + \tfrac{1}{2}(P + p_0^2) \right) = -\tfrac{1}{4}(P - p_0^2 + V(0) + V(1)),$$

where the series converge absolutely and uniformly on any bounded subset of $\mathscr{H}_4 \times \mathscr{H}_2^0$.

In particular, if $p = p_0 = \text{const}$, then

$$(1.9) \qquad \sum_{n=1}^{\infty} \left(\mu_n - (\pi n)^4 + 2(\pi n)^2 p_0 \right) = -\tfrac{1}{4}(q(0) + q(1)),$$

if $q = 0$, then

$$(1.10) \quad \sum_{n \geqslant 1} \left(\mu_n - ((\pi n)^2 - p_0)^2 + \tfrac{1}{2}(P + p_0^2) \right) = -\tfrac{1}{4}(P - p_0^2) + \tfrac{1}{8}(p''(0) + p''(1)).$$

Remark. In the proof of Theorem 1.1 we use the presentation $H = h^2 + q - p'' - p^2$, where h^2 is the unperturbed operator given by (1.3). The proof follows our approach from [**BK5**] and is based on the asymptotic analysis of the difference of the resolvents of the perturbed operator H and the unperturbed operator h^2.

Theorem 1.1 implies a trace formula for the operator h^2. This trace formula is known due to Dikii–Gelfand [**D1**], [**D2**], [**G**]. In the following corollary we extend the Dikii–Gelfand trace formula onto a larger class of coefficients p.

Corollary 1.2. *Let $p \in \mathscr{H}_4$ and let $\alpha_n, n \in \mathbb{N}$, be the eigenvalues of the operator h labeled by $\alpha_1 < \alpha_2 < \alpha_3 < ....$ Then the following trace formula holds true:*

$$(1.11)$$
$$\sum_{n \geqslant 1} \left(\alpha_n^2 - \left((\pi n)^2 - p_0 \right)^2 - \frac{P - p_0^2}{2} \right)$$
$$= \frac{P + p_0^2}{4} - \frac{p^2(0) + p^2(1)}{4} - \frac{p''(0) + p''(1)}{8},$$

where the series converges absolutely and uniformly on any bounded subset of $\mathscr{H}_4$.

Remark. 1) For the case $p \in C^{\infty}[0, 1]$ the trace formula (1.11) was determined by Dikii [**D1**], [**D2**] and Gelfand [**G**]. Dikii [**D1**] determined this formula without any additional restrictions. Unfortunately, the results from [**D1**] contain some mistakes (see [**FP**, Remark 5 in Sect 4] and our discussion in Section 3). In the second paper [**D2**] Dikii determined trace formula (1.11) under additional conditions $p^{(2j-1)}(0) = p^{(2j-1)}(1) = 0$ for all $j \in \mathbb{N}$. The proof of Dikii [**D1**], [**D2**] uses the analysis of the zeta function of the operator.

Gelfand [**G**] determined the trace formula (1.11) under stronger conditions: $p = 0$ in some neighborhoods of the points 0 and 1. His proof is based on the analysis of an expansion of a trace of the resolvent in powers of the spectral parameter.

2) There is an open problem to give a transparent proof of the Dikii–Gelfand trace formulas for the operators $h^m, m \geqslant 2$, and, more widely, for the polynomials of h. Corollary 1.2 makes only the first step in this direction.

Using the trace formula (1.8) we can recover the coefficient of the operator H by the other coefficients and the spectrum. Recall that there are trace formulas for second order operators, see [**Du**] for finite band potentials, [**T**], [**L**] for smooth potentials, [**K**] for potentials from $L^2(0, 1)$. Let $p \in L^2(0, 1)$ be an 1-periodic function and $p_0 = 0$. For any $\tau \in \mathbb{T}, \mathbb{T} = \mathbb{R}/\mathbb{Z}$, consider the shifted operator $h_\tau = -\partial^2 - p(\cdot + \tau)$ on the interval $(0, 1)$ with the Dirichlet boundary conditions $y(0) = y(1) = 0$. Let $\alpha_n(\tau), n \in \mathbb{N}$, be the eigenvalues of the operator h_τ labeled by $\alpha_1(\tau) < \alpha_2(\tau) < ....$ Each function $\alpha_n(\tau), n \in \mathbb{N}$, is 1-periodic and $\alpha_n'' \in L^2(0, 1)$.

Consider an operator $\widetilde{h} = -\partial^2 - p$ on the circle $2\mathbb{T}$. Its spectrum is eigenvalues $\alpha_0^+, \alpha_n^{\pm}, n \in \mathbb{N}$ labeled by

$$\alpha_0^+ < \alpha_1^- \leqslant \alpha_1^+ < \alpha_2^- \leqslant \alpha_2^+ < \ldots.$$

It is well-known that $\alpha_n(\tau) \in [\alpha_n^-, \alpha_n^+]$ for each $(n, \tau) \in \mathbb{N} \times \mathbb{T}$. Moreover, $\alpha_n(\tau)$ runs through the interval $[\alpha_n^-, \alpha_n^+]$, making n complete revolutions in unit time, and the following trace formula holds true:

$$(1.12) \qquad \alpha_0^+ + \lim_{N \to \infty} \sum_{n=1}^{N} \left(\alpha_n^- + \alpha_n^+ - 2\alpha_n(\tau) \right) = -p(\tau), \qquad \tau \in \mathbb{T},$$

where the sum converges in L^2 sense. If we know $\alpha_n(\tau)$ for all $(n, \tau) \in \mathbb{N} \times \mathbb{T}$, then we can recover p.

Recall that the functions $\alpha_n(\tau), n \in \mathbb{N}$, satisfy so called Dubrovin system of differential equations [**Du**]:

$$\alpha_n'(\tau) = \frac{\sqrt{f(\alpha_n(\tau))}}{g_n(\tau)}, \qquad n \geqslant 1,$$

with the initial conditions $S = \{\alpha_n(0), \text{ sign } \alpha_n'(0), \ n \geqslant 1\}$, where f, g_n are given by

$$f(\lambda) = (\lambda - \alpha_0^+) \prod_{j \geqslant 1} \frac{(\lambda - \alpha_j^-)(\lambda - \alpha_j^+)}{(\pi j)^4}, \quad g_n(\tau) = \frac{1}{2(\pi n)^2} \prod_{j \geqslant 1, j \neq n} \frac{\alpha_j(\tau) - \alpha_n(\tau)}{(\pi j)^2}$$

(see p. 324 in [**T**]). Here $\text{sign } \sqrt{f(\alpha_n(0))} = (-1)^{n+1} \text{sign } \alpha_n'(0)$ for all $n \geqslant 1$. For given the 2-periodic spectrum $\alpha_0^+, \alpha_n^{\pm}, n \in \mathbb{N}$ and the initial conditions, the Dubrovin system has the unique 1-periodic solution $\alpha_n(\tau)$ for all $(\tau, n) \in \mathbb{T} \times \mathbb{N}$ and then, using the trace formula (1.12) we recover $p \in L^2(0, 1)$. The open problem is to extend these results for fourth order operators even with smooth coefficients.

Consider the fourth order operators. Introduce the Sobolev spaces $\mathscr{H}_{m,per}$ and $\mathscr{H}_{m,per}^0$, $m \geqslant 0$, of periodic functions, by

$$\mathscr{H}_{m,per} = \left\{ f \in L^1(\mathbb{T}) : f^{(m)} \in L^1(\mathbb{T}) \right\}, \quad \mathscr{H}_{m,per}^0 = \left\{ f \in \mathscr{H}_{m,per} : \int_0^1 f(t) dt = 0 \right\}.$$

Let $(p, q) \in \mathscr{H}_{4,per} \times \mathscr{H}_{2,per}^0$. For any $\tau \in \mathbb{T}$ we define the shifted operator H_τ on $L^2(0, 1)$ by

$$H_\tau = \partial^4 + \partial p(x + \tau)\partial + q(x + \tau)$$

with the boundary conditions (1.2). Let $\mu_n(\tau), n \in \mathbb{N}$, be the eigenvalues of the shifted operator H_τ labeled by $\mu_1(\tau) \leqslant \mu_2(\tau) \leqslant \ldots$, counted with multiplicities. We formulate our trace formula (1.13) for the operator H_τ. This formula determines the function $V(\tau), \tau \in \mathbb{T}$, by $\mu_n(\tau)$. Unfortunately, analysis of the corresponding Dubrovin equations for fourth order operators is still not carried out. The problem is that the eigenvalues can have multiplicity 2, and we have no sufficient information how they move when τ runs through the interval $[0, 1]$.

Corollary 1.3. *Let $(p, q) \in \mathscr{H}_{4,per} \times \mathscr{H}_{2,per}^0$ and let $\mu_n(\tau), n \in \mathbb{N}$, be the eigenvalues of the operator H_τ. Then there exists $N = N(p, q) \in \mathbb{N}$ such that the*

functions $\sum_{n=1}^{N} \mu_n(\tau)$ *and each* $\mu_n(\tau), n > N$, *belong to the space* $C^1(\mathbb{T})$. *Moreover, they satisfy*

$$(1.13) \quad \sum_{n \geqslant 1} \left(\mu_n(\tau) - \left((\pi n)^2 - p_0\right)^2 + \tfrac{1}{2}(P + p_0^2) \right) = -\tfrac{1}{4}\left(P - p_0^2 + 2V(\tau)\right) \quad \forall \ \tau \in \mathbb{T},$$

where the series converges absolutely and uniformly on $\tau \in \mathbb{T}$ *and* $P = \int_0^1 p^2(t)dt$.

In particular, assume that we know $\mu_n(\tau)$ *for all* $(n, \tau) \in \mathbb{N} \times \mathbb{T}$. *Then*

a) If we know p, *then we can recover* q.

b) If we know q, p_0 *and* $\int_0^1 p^2 dt$, *then we can recover* p.

Remark. Asymptotics (1.5) shows that each eigenvalue $\mu_n(\tau)$ with n large enough is simple, and then it is a smooth function of $\tau \in \mathbb{T}$. The situation with other eigenvalues is complicated: we don't know how they depend on τ. However, due to Rouché's theorem, we can control their sum, then it is smooth.

1.3. Perturbations by functions. Now we determine trace formulas for the simplest case: the perturbation by q for fixed p. In fact, we consider a more general situation, perturbations of the operator H, given by (1.1), (1.2), by functions $Q \in \mathscr{H}_2$. Let $\lambda_n, n \in \mathbb{N}$, be the eigenvalues of $H + Q$ labeled by $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant ...$, counted with multiplicity.

Theorem 1.4. *Let* $(p, q, Q) \in \mathscr{H}_4 \times \mathscr{H}_2 \times \mathscr{H}_2$. *Let* μ_n *and* $\lambda_n, n \in \mathbb{N}$, *be the eigenvalues of the operator* H *and* $H + Q$, *respectively. Then the following trace formula holds true:*

$$(1.14) \qquad\qquad \sum_{n \geqslant 1} (\lambda_n - \mu_n) = -\tfrac{1}{4}(Q(0) + Q(1) - 2Q_0),$$

where the series converges absolutely and uniformly on any bounded subset of $\mathscr{H}_4 \times \mathscr{H}_2 \times \mathscr{H}_2$.

Remark. 1) Sadovnichii [**S1**] determined the trace formula (1.14) for the simplest case $\partial^4 + Q$, where $Q \in C^\infty[0, 1]$ and $Q^{(2j-1)}(0) = Q^{(2j-1)}(1) = 0 \ \forall \ j \in \mathbb{N}$ (see also Remark in Section 4). Nazarov, Stolyarov and Zatitskiy [**NSZ**] extended the results of Sadovnichii onto the larger class of higher order operators (see Remark in Section 2).

2) Note that perturbations by second order operators in Theorem 1.1 are stronger, than perturbations by functions in Theorem 1.4. Therefore, we need to analyze more terms of the perturbation series in the proof of Theorem 1.1, than in the proof of Theorem 1.4.

Now we apply Theorem 1.4 to the case $H = h^2$, where the operator h is given by (1.3). Let $\nu_n, n \in \mathbb{N}$, be the eigenvalues of the operator $h^2 + Q$ labeled by $\nu_1 \leqslant \nu_2 \leqslant ...$ counting with multiplicities.

Corollary 1.5. *i) Let* $(p, Q) \in \mathscr{H}_4 \times \mathscr{H}_2$. *Let* ν_n *and* $\alpha_n, n \in \mathbb{N}$, *be the eigenvalues of the operators* $h^2 + Q$ *and* h *respectively. Then the following trace formula holds true:*

$$(1.15) \qquad\qquad \sum_{n \geqslant 1} (\nu_n - Q_0 - \alpha_n^2) = -\tfrac{1}{4}\left(Q(0) + Q(1) - 2Q_0\right),$$

where the series converges absolutely and uniformly on any bounded subset of $\mathscr{H}_4 \times \mathscr{H}_2$.

ii) Let $(p, Q) \in \mathscr{H}_4 \times \mathscr{H}_{2,per}^0$. Let $\nu_n(\tau), n \in \mathbb{N}, \tau \in \mathbb{T}$, be the eigenvalues of the operators $h^2 + Q(\cdot + \tau)$, labeled by $\nu_1(\tau) \leqslant \nu_2(\tau) \leqslant ...$, counted with multiplicities. Then there exists $N = N(Q) \in \mathbb{N}$ such that the functions $\sum_{n=1}^{N} \nu_n(\tau)$ and each $\nu_n(\tau), n > N$, belong to the space $C^1(\mathbb{T})$. Moreover, they satisfy

$$(1.16) \qquad \sum_{n \geqslant 1} \left(\nu_n(\tau) - \alpha_n^2 \right) = -\tfrac{1}{2} Q(\tau) \qquad \forall \quad \tau \in \mathbb{T},$$

where the series converges absolutely and uniformly on $\tau \in \mathbb{T}$.

In particular, if we know $\alpha_n, \nu_n(\tau)$ for all $(n, \tau) \in \mathbb{N} \times [0, 1]$, then we can recover Q.

Remark. Assume that we know $\alpha_n, \nu_n(\tau)$ for all $(n, \tau) \in \mathbb{N} \times [0, 1]$. Then we recover the coefficient Q. Remark that, in the second order case all eigenvalues $\alpha_n, n \in \mathbb{N}$ do not determine p, since we need so-called norming constants, e.g. [**LS**], [**T**]. Thus, in order to recover Q we don't need to know p, it is sufficiently to know all $\alpha_n, n \geqslant 1$. Of course, the coefficient p determines all eigenvalues $\alpha_n, n \in \mathbb{N}$, and then we recover Q.

The plan of the paper is as follows. In Section 2 we consider the perturbation by the function and prove Theorem 1.4. In Section 3 we determine the trace formula for the operator h^2. In fact, this is a simplified version of the trace formula for the operator H. Some technical proofs are replaced from Section 3 into Section 5. In Section 4 we determine the trace formula for the operator H and prove Theorem 1.1.

2. The proof of Theorem 1.4

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be the sets of all trace class and Hilbert-Schmidt class operators on $L^2(0, 1)$ equipped with the norms $\| \cdot \|_1$, $\| \cdot \|_2$, respectively. Let $(p, q, Q) \in \mathscr{H}_2 \times \mathscr{H}_0 \times \mathscr{H}_0$. In order to prove Theorem 1.4 we need to study the resolvents defined by

$$(2.1) \qquad \begin{aligned} R_1(\lambda) &= \left(H + Q - \lambda \right)^{-1}, \qquad R(\lambda) = \left(H - \lambda \right)^{-1}, \\ \mathcal{R}(\lambda) &= (h^2 - \lambda)^{-1}, \qquad \mathcal{R}_0(\lambda) = \left(h_0^2 - \lambda \right)^{-1}, \end{aligned}$$

where $h_0 = -\partial^2$ is equal to h at $p = 0$. It is well known (see [**FP**, (4.21)]), that in the case $p, p'' \in L^1(0, 1)$ the eigenvalues α_n of the operator h satisfy

$$(2.2) \qquad \alpha_n = (\pi n)^2 - p_0 + \frac{P - p_0^2}{(2\pi n)^2} + \frac{O(1)}{n^4},$$

then

$$(2.3) \qquad \alpha_n^2 = ((\pi n)^2 - p_0)^2 + \frac{P - p_0^2}{2} + \frac{O(1)}{n^2}$$

as $n \to \infty$. Due to asymptotics (2.2), (2.3), all resolvents satisfy

$$R_1(\lambda), R(\lambda), \mathcal{R}(\lambda), h_0 \mathcal{R}_0(\lambda) \in \mathcal{B}_1$$

on the corresponding resolvent sets. A proof of the following results repeats the arguments for the periodic case, see [**BK5**, Lemma 2.1].

Lemma 2.1. *Let $(p, q, Q) \in \mathscr{H}_4 \times \mathscr{H}_2 \times \mathscr{H}_2$. Then the following asymptotics hold true:*

$$(2.4) \qquad \|\mathcal{R}_0(\lambda)\|_2 + \|\mathcal{R}(\lambda)\|_2 + \|R(\lambda)\|_2 + \|R_1(\lambda)\|_2 = O(k^{-3}),$$

$$(2.5) \qquad\qquad \|h_0 \mathcal{R}_0(\lambda)\|_2 = O(k^{-1}),$$

$$(2.6) \qquad \oint_{\Gamma_k} \mathrm{Tr}\Big(\mathcal{R}_0(\lambda)(h_0 p + p h_0)\mathcal{R}_0(\lambda)q\Big)d\lambda = o(1),$$

as integer $k \to \infty$, uniformly on the contours Γ_k given by

$$\Gamma_k = \big\{\lambda \in \mathbb{C} : |\lambda|^{\frac{1}{4}} = \pi\big(k + \tfrac{1}{2}\big)\big\}.$$

In the space $L^2(0,1)$ we introduce the scalar product $(f,g) = \int_0^1 f(x)g(x)dx$, the norm $\|f\|^2 = (f,f)$ and the Fourier coefficients

$$f_0 = \int_0^1 f(x)dx, \qquad \widehat{f}_{cn} = (f, \cos 2\pi n x), \qquad \widehat{f}_{sn} = (f, \sin 2\pi n x), \qquad n \in \mathbb{N}.$$

Below we need the following simple result.

Lemma 2.2. *Let* $Q \in \mathscr{H}_2^0, k \in \mathbb{N}$. *Then the following identity holds true:*

$$(2.7) \qquad \lim_{k\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_k} \mathrm{Tr}\, Q\mathcal{R}_0(\lambda)d\lambda = \frac{Q(0) + Q(1)}{4}.$$

Proof. We have the Fourier series

$$(2.8) \qquad Q(x) = 2\sum_{n=1}^{\infty} \Big(\widehat{Q}_{cn} \cos 2\pi n x + \widehat{Q}_{sn} \sin 2\pi n x\Big).$$

Let $s_n = \sqrt{2}\sin \pi n x, n \in \mathbb{N}$. Then

$$\mathrm{Tr}\, Q\mathcal{R}_0(\lambda) = \sum_{n=1}^{\infty} \frac{(Qs_n, s_n)}{(\pi n)^4 - \lambda} = \sum_{n=1}^{\infty} \frac{\widehat{Q}_{cn}}{\lambda - (\pi n)^4},$$

since $(Qs_n, s_n) = -\widehat{Q}_{cn}, n \geqslant 1$. This implies

$$(2.9) \qquad \frac{1}{2\pi i} \oint_{\Gamma_k} \mathrm{Tr}\, Q\mathcal{R}_0(\lambda)d\lambda = \sum_{n=1}^{k} \widehat{Q}_{cn}$$

for all $k \geqslant 1$. Then (2.8) and $Q \in \mathscr{H}_2^0$ yield (2.7). ∎

Introduce the norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Proof of Theorem 1.4. The series $\sum_{n\geqslant 1}(\lambda_n - \mu_n)$ converges absolutely and uniformly in (p, q, Q) due to asymptotics (1.5). We have for integer k:

$$(2.10) \qquad \sum_{n\geqslant 1}(\lambda_n - \mu_n) = -\frac{1}{2\pi i} \lim_{k\to\infty} \oint_{\Gamma_k} \lambda\, \mathrm{Tr}\big(R_1(\lambda) - R(\lambda)\big)d\lambda.$$

The resolvent identity gives

$$(2.11) \qquad\qquad R_1 - R = -RQR + R_1 QRQR.$$

Estimates (2.4) imply

$$\big|\mathrm{Tr}\big(R_1(\lambda)QR(\lambda)QR(\lambda)\big)\big| \leqslant \|R_1(\lambda)\|_2 \|R(\lambda)\|_2^2 \|Q\|_\infty^2 = O(k^{-9})$$

uniformly on Γ_k, which yields

$$(2.12) \qquad \oint_{\Gamma_k} \lambda\, \mathrm{Tr}\big(R_1(\lambda)QR(\lambda)QR(\lambda)\big)d\lambda = O(k^{-1}) \qquad \text{as} \quad k \to \infty.$$

Substituting (2.11) into (2.10) and using (2.12) we obtain

$$\sum_{n \geqslant 1} (\lambda_n - \mu_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \oint_{\Gamma_k} \lambda \operatorname{Tr} QR^2(\lambda) d\lambda.$$

Since $R^2(\lambda) = R'(\lambda)$, the integration by parts gives

$$(2.13) \quad \sum_{n \geqslant 1} (\lambda_n - \mu_n) = \frac{1}{2\pi i} \lim_{k \to \infty} \oint_{\Gamma_k} \lambda \operatorname{Tr} QR'(\lambda) d\lambda = -\frac{1}{2\pi i} \lim_{k \to \infty} \oint_{\Gamma_k} \operatorname{Tr} QR(\lambda) d\lambda.$$

The resolvent identity together with the relations $H = h^2 - v, v = p'' + p^2 - q$ implies

$$(2.14) \quad R(\lambda) = \mathcal{R}(\lambda) + R(\lambda) v \mathcal{R}(\lambda).$$

Estimates (2.4) give

$$\left| \operatorname{Tr} \big(R(\lambda) v \mathcal{R}(\lambda) Q \big) \right| \leqslant \|R(\lambda)\|_2 \|\mathcal{R}(\lambda)\|_2 \|v\|_\infty \|Q\|_\infty = O(k^{-6})$$

uniformly on Γ_k, which yields

$$(2.15) \quad \oint_{\Gamma_k} \operatorname{Tr} \big(R(\lambda) v \mathcal{R}(\lambda) Q \big) d\lambda = O(k^{-2}) \qquad \text{as} \quad k \to \infty.$$

Substituting (2.14) into (2.13) and using (2.15) we obtain

$$(2.16) \quad \sum_{n \geqslant 1} (\lambda_n - \mu_n) = -\frac{1}{2\pi i} \lim_{k \to \infty} \oint_{\Gamma_k} \operatorname{Tr} Q\mathcal{R}(\lambda) d\lambda.$$

The resolvent identity together with $h^2 = (h_0 - p)^2 = h_0^2 - h_0 p - p h_0 + p^2$ implies

$$(2.17) \quad \mathcal{R} = \mathcal{R}_0 - \mathcal{R}_0 A \mathcal{R}_0 + \mathcal{R} A \mathcal{R}_0 A \mathcal{R}_0,$$

where $A = -h_0 p - p h_0 + p^2$. Let $k \to \infty$. Estimates (2.4) give

$$\left| \operatorname{Tr} \big(\mathcal{R}_0(\lambda) p^2 \mathcal{R}_0(\lambda) Q \big) \right| \leqslant \|\mathcal{R}_0(\lambda)\|_2^2 \|p\|_\infty^2 \|Q\|_\infty = O(k^{-6})$$

uniformly on Γ_k. This asymptotics and asymptotics (2.6) yield

$$(2.18)$$
$$\oint_{\Gamma_k} \operatorname{Tr} \big(\mathcal{R}_0(\lambda) A \mathcal{R}_0(\lambda) Q \big) d\lambda$$
$$= \oint_{\Gamma_k} \operatorname{Tr} \Big(\mathcal{R}_0(\lambda)(-h_0 p - p h_0 + p^2) \mathcal{R}_0(\lambda) Q \Big) d\lambda = o(1).$$

Estimates (2.4), (2.5) give

$$\|A \mathcal{R}_0(\lambda)\|_2 \leqslant 2\|p\|_\infty \|h_0 \mathcal{R}_0(\lambda)\|_2 + \|p^2\|_\infty \|\mathcal{R}_0(\lambda)\|_2 = O(k^{-1}),$$

then

$$\left| \operatorname{Tr} \big(\mathcal{R}(\lambda) A \mathcal{R}_0(\lambda) A \mathcal{R}_0(\lambda) Q \big) \right| \leqslant \|\mathcal{R}(\lambda)\|_2 \|A \mathcal{R}_0(\lambda)\|_2^2 \|Q\|_\infty = O(k^{-5})$$

uniformly on Γ_k, which yields

$$(2.19) \quad \oint_{\Gamma_k} \operatorname{Tr} \Big(\mathcal{R}(\lambda) A \mathcal{R}_0(\lambda) A \mathcal{R}_0(\lambda) Q \Big) d\lambda = O(k^{-1}).$$

Substituting (2.17) into (2.16) and using (2.18), (2.19) we obtain

$$\sum_{n \geqslant 1} (\lambda_n - \mu_n) = -\frac{1}{2\pi i} \lim_{k \to \infty} \oint_{\Gamma_k} \operatorname{Tr} Q\mathcal{R}_0(\lambda) d\lambda.$$

Identity (2.7) implies the trace formula (1.14) for $Q \in \mathcal{H}_2^0$. The trace formula for $Q \in \mathcal{H}_2$ follows. $\blacksquare$

Remark. Nazarov, Stolyarov and Zatitskiy [**NSZ**] determined some trace formulas for the operator $\widetilde{H} + Q$, where $\widetilde{H}$ is a higher order operator with complex coefficients on the unit interval and $Q \in L^1(0,1)$ is a complex function. Authors of [**NSZ**] determined $\lim_{N \to \infty} \sum_{n=1}^N (\lambda_n - \lambda_n^0 - q_0) = X$, where $\lambda_n^0, \lambda_n, n \in \mathbb{N}$, are the eigenvalues of the operators $\widetilde{H}, \widetilde{H} + Q$ respectively, and X is expressed in terms of q and boundary conditions, the explicit expression is rather complicated. Note that there are no any control on a type of convergence of the series s.

Proof of Corollary 1.5 i). Let $H = h^2$. Then $\lambda_n = \nu_n, \mu_n = \alpha_n^2$ and the trace formula (1.14) gives (1.15). $\blacksquare$

3. Dikiĭ–Gelfand trace formula

Our proof of Theorem 1.1 is based on the identity $H = h^2 + q - p'' - p^2$. In Theorem 3.3 we will determine the trace formula for the unperturbed operator $h^2 = (\partial^2 + p)^2$. In fact, the result of Theorem 3.3 extends the result of Corollary 1.2 onto the larger class $p \in \mathcal{H}_3$. Our proof is based on the identity

$$h^2 = (\partial^2 + p)^2 = \partial^4 + 2\partial p \partial + p'' + p^2.$$

We analyze asymptotics of the difference of the resolvents of the perturbed operator h^2 and the unperturbed operator ∂^4. Note that we don't use the results and methods from [**D1**], [**D2**], [**G**] in our proof.

Let $p \in \mathcal{H}_2$. Introduce the resolvents

$$r(z) = (h - z)^{-1}, \quad r_0(z) = (h_0 - z)^{-1}.$$

Due to asymptotics (2.2) the resolvents (on the corresponding resolvent sets) satisfy $r(z), r_0(z) \in \mathcal{B}_1$. Moreover,

$$\|r_0(z)\|_2 + \|r(z)\|_2 = O(k^{-1}) \tag{3.1}$$

as $k \to \infty$ uniformly on the contours $\gamma_k \subset \mathbb{C}, k \in \mathbb{N}$, given by

$$\gamma_k = \{z \in \mathbb{C} : |z|^{\frac{1}{2}} = \pi(k + \tfrac{1}{2})\}.$$

Identity

$$\mathrm{Tr}\,(r(z) - r_0(z)) = \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n - z} - \frac{1}{(\pi n)^2 - z}\right)$$

yields

$$-\frac{1}{2\pi i} \int_{\gamma_k} z^2 \,\mathrm{Tr}\,(r(z) - r_0(z)) dz = \sum_{n=1}^{k} \left(\alpha_n^2 - (\pi n)^4\right) \tag{3.2}$$

for all $k \in \mathbb{N}$ large enough. The following two lemmas give asymptotics of the integral in (3.2).

Lemma 3.1. *Let* $p \in \mathcal{H}_2, k \in \mathbb{N}$. *Then the following asymptotics holds true:*

$$-\frac{1}{2\pi i} \int_{\gamma_k} z^2 \,\mathrm{Tr}\,(r(z) - r_0(z)) dz = \sum_{j=1}^{5} J_j(k) + O(k^{-1}), \tag{3.3}$$

as $k \to \infty$ uniformly on any bounded subset of $\mathcal{H}_2$, where

$$(3.4) \qquad J_j(k) = \frac{1}{\pi i j} \int_{\gamma_k} z \operatorname{Tr}(r_0(z)p)^j dz \qquad \forall \quad j \in \mathbb{N}.$$

Proof. Estimates (3.1) give

$$|\operatorname{Tr}(r_0(z)p)^6 r(z)| \leqslant \|p\|_\infty^6 \|r_0(z)\|_2^6 \|r(z)\|_2 = O(k^{-7})$$

as $k \to \infty$ on all contours. Using the resolvent identity

$$r(z) - r_0(z) = \sum_{j=1}^{5} \big(r_0(z)p\big)^j r_0(z) + \big(r_0(z)p\big)^6 r(z)$$

we obtain (3.3), where

$$J_j(k) = -\frac{1}{2\pi i} \int_{\gamma_k} z^2 \operatorname{Tr}(r_0(z)p)^j r_0(z) dz.$$

Identities

$$\operatorname{Tr}(r_0(z)p)^j r_0(z) = \operatorname{Tr} p \big(r_0(z)p\big)^{j-1} r_0^2(z)$$

$$= \operatorname{Tr} p \big(r_0(z)p\big)^{j-1} r_0'(z) = \frac{1}{j} \operatorname{Tr} \left(\big(r_0(z)p\big)^j \right)'$$

imply

$$J_j(k) = -\frac{1}{2\pi i j} \int_{\gamma_k} z^2 \operatorname{Tr} \left(\big(r_0(z)p\big)^j \right)' dz.$$

The integration by parts gives (3.4). ∎

The technical proof of the following lemma see in Section 4.

Lemma 3.2. Let $p \in \mathcal{H}_2, k \in \mathbb{N}$. Then the sequences $J_j(k), j = 1, ..., 5$ satisfy

$$(3.5) \qquad J_1(k) = -\sum_{n=1}^{k} \left(2p_0(\pi n)^2 - \frac{p'(1) - p'(0)}{2} \right) - \frac{1}{2} \sum_{n=1}^{k} \widehat{(p'')}_{cn},$$

$$(3.6) \qquad J_2(k) = k\frac{\|p\|^2 + p_0^2}{2} - \frac{\|p\|^2 - p_0^2}{4} - \sum_{n=1}^{k} \widehat{(p^2)}_{cn} + O(k^{-1}),$$

$$(3.7) \qquad J_3(k) = O(k^{-\frac{2}{3}}), \qquad J_4(k) = O(k^{-\frac{4}{5}}), \qquad J_5(k) = O(k^{-1})$$

as $k \to \infty$ uniformly on any bounded subset of $\mathcal{H}_2$.

We prove the Dikii–Gelfand trace formula in our class of coefficients.

Theorem 3.3. Let $p \in \mathcal{H}_3$ and let $\alpha_n, n \in \mathbb{N}$, be the eigenvalues of the operator h. Then the trace formula (1.11) holds true, where the series converges absolutely and uniformly on any bounded subset of $\mathcal{H}_3$.

Proof. Asymptotics (2.3) shows that the series in (1.11) converges absolutely and uniformly on any bounded subset of $\mathscr{H}_3$. Substituting (3.5), (3.6), (3.7) into (3.3) we obtain

$$-\frac{1}{2\pi i}\int_{\gamma_k} z^2 \operatorname{Tr}(r(z) - r_0(z))dz$$

$$= -\sum_{n=1}^{k}\left(2p_0(\pi n)^2 - \frac{P+p_0^2}{2}\right) - \frac{\|p\|^2 - p_0^2}{4} - \frac{1}{2}\sum_{n=1}^{k}(\widehat{p''})_{cn} - \sum_{n=1}^{k}(\widehat{p^2})_{cn} + O(k^{-\frac{4}{5}})$$

as $k \to \infty$. This asymptotics and (3.2) imply

$$(3.8)\qquad\begin{aligned}&\sum_{n=1}^{k}\left(\alpha_n^2 - (\pi n)^4 + 2p_0(\pi n)^2 - \frac{P+p_0^2}{2}\right)\\ &= -\frac{\|p\|^2 - p_0^2}{4} - \frac{1}{2}\sum_{n=1}^{k}(\widehat{p''})_{cn} - \sum_{n=1}^{k}(\widehat{p^2})_{cn} + O(k^{-\frac{4}{5}}).\end{aligned}$$

Using the Fourier series for $p \in \mathscr{H}_3$

$$p^2(x) = \|p\|^2 + 2\sum_{n=1}^{\infty}\left((\widehat{p^2})_{cn}\cos 2\pi nx + (\widehat{p^2})_{sn}\sin 2\pi nx\right),$$

$$p''(x) = p'(1) - p'(0) + 2\sum_{n=1}^{\infty}\left((\widehat{p''})_{cn}\cos 2\pi nx + (\widehat{p''})_{sn}\sin 2\pi nx\right),$$

we obtain

$$(3.9)\qquad\begin{aligned}&\sum_{n=1}^{\infty}(\widehat{p^2})_{cn} = \frac{p^2(0) + p^2(1)}{4} - \frac{\|p\|^2}{2},\\ &\sum_{n=1}^{\infty}(\widehat{p''})_{cn} = \frac{p''(0) + p''(1)}{4} - \frac{p(1) - p'(0)}{2}.\end{aligned}$$

Substituting identities (3.9) into (3.8) we obtain (1.11). $\blacksquare$

Remark. Let $p \in C^{\infty}[0,1]$. Dikii [**D1**, p. 189-190] determined the following asymptotics

$$(3.10)\qquad \alpha_n = (\pi n)^2 - p_0 + \frac{\widetilde{P}}{(2\pi n)^2} + ...,$$

and trace formula

$$(3.11)\qquad\begin{aligned}&\sum_{n=1}^{\infty}\left(\alpha_n^2 - (\pi n)^4 + 2p_0(\pi n)^2 - \frac{\widetilde{P} + 2p_0^2}{2}\right)\\ &= \frac{\widetilde{P} + 2p_0^2}{4} + \frac{p''(0) + p''(1)}{8} - \frac{p^2(0) + p^2(1)}{4},\end{aligned}$$

where

$$(3.12)\qquad \widetilde{P} = \|p\|^2 - 4p_0^2 + \frac{1}{3}\left(p'(1) - p'(0)\right).$$

Asymptotics (3.10) is in a disagreement with (2.2). The coefficients 4 and $\frac{1}{3}$ in (3.12) are mistaken, see also [**FP**, Remark 4.5].

Assume, in addition, that $p_0 = 0, p^{(2j-1)}(0) = p^{(2j-1)}(1) = 0$ for all $j \in \mathbb{N}$. Then $\widetilde{P} = \|p\|^2$ and (3.11) gives

$$(3.13) \qquad \sum_{n=1}^{\infty} \left(\alpha_n^2 - (\pi n)^4 - \frac{\|p\|^2}{2} \right) = \frac{\|p\|^2}{4} + \frac{p''(0) + p''(1)}{8} - \frac{p^2(0) + p^2(1)}{4}.$$

On the other hand, for this case Dikii [**D2**, id. (6.4))] determined the following trace formula

$$(3.14) \qquad \sum_{n=1}^{\infty} \left(\alpha_n^2 - (\pi n)^4 - \frac{\|p\|^2}{2} \right) = \frac{\|p\|^2}{4} - \frac{p''(0) + p''(1)}{8} - \frac{p^2(0) + p^2(1)}{4}.$$

The last two identities are in a disagreement: the signs before $\frac{1}{8}(p''(0) + p''(1))$ are not coincide. Theorem 3.3 shows that the signs in (3.11), (3.13) are incorrect. Thus, the trace formula (3.11) contains two mistakes: the coefficients in $\widetilde{P}$ and the sign before $\frac{1}{8}(p''(0) + p''(1))$ are incorrect, whereas the trace formula (3.14) is valid.

4. The proof of Theorem 1.1

We prove the main result of our paper.

Proof of Theorem 1.1. We have the identity

$$(4.1) \qquad\qquad H = h^2 + Q, \qquad Q = q - p'' - p^2.$$

The trace formula (1.15) and the identity $Q_0 = -P$ give

$$(4.2) \qquad\qquad \sum_{n \geqslant 1} \left(\mu_n + P - \alpha_n^2 \right) = -\frac{Q(0) + Q(1) + 2P}{4}.$$

Introduce the sums $S = S(p, q), S_0 = S_0(p)$ given by

$$(4.3) \qquad \begin{aligned} S &= \sum_{n \geqslant 1} \left(\mu_n - ((\pi n)^2 - p_0)^2 + \frac{P + p_0^2}{2} \right), \\ S_0 &= \sum_{n \geqslant 1} \left(\alpha_n^2 - ((\pi n)^2 - p_0)^2 - \frac{P - p_0^2}{2} \right), \end{aligned}$$

where the series converges absolutely and uniformly on p, q due to asymptotics (1.5), (2.3). The trace formula (1.11) gives

$$(4.4) \qquad S_0 = \frac{P + p_0^2}{4} - \frac{p^2(0) + p^2(1)}{4} - \frac{p''(0) + p''(1)}{8}.$$

The trace formula (4.2) and the definitions (4.3) imply

$$(4.5) \qquad S - S_0 = \sum_{n \geqslant 1} \left(\mu_n + P - \alpha_n^2 \right) = -\frac{Q(0) + Q(1) + 2P}{4}.$$

Substituting identities (4.1), (4.4) into (4.5) we get

$$S = -\frac{P - p_0^2}{4} + \frac{p''(0) + p''(1)}{8} - \frac{q(0) + q(1)}{4},$$

which yields (1.8). ∎

Remark. Sadovnichii [**S1**, p. 308-309] considered the operator $H = \partial^4 + 2\partial p\partial + q$, where
$$(4.6)$$
$$p, q \in C^\infty[0, 1], \qquad p^{(j)}(0) = p^{(j)}(1) = q^{(2j-1)}(0) = q^{(2j-1)}(1) = 0 \qquad \forall \quad j \in \mathbb{N},$$

and wrote (without a proof) the following asymptotics

$$(4.7) \qquad \mu_n = (\pi n)^4 - 2p_0(\pi n)^2 + q_0 + \frac{c_1}{n^2} + \frac{c_2}{n^4} + ...$$

as $n \to \infty$, and trace formula

$$(4.8) \qquad \sum_{n=1}^\infty \left(\mu_n - (\pi n)^4 + 2p_0(\pi n)^2 - q_0 \right) = \frac{q_0}{2} - \frac{q(0) + q(1)}{4},$$

where c_1, c_2 are some undetermined constants. Assume that the operator H has the form $H = (-\partial^2 - p)^2$, where p satisfies (4.6). Then $q = p'' + p^2$ and q satisfies (4.6). In this case asymptotics (4.7) gives

$$(4.9) \qquad \mu_n = (\pi n)^4 - 2p_0(\pi n)^2 + \|p\|^2 + \frac{c_1}{n^2} + \frac{c_2}{n^4} + ...$$

On the other hand, asymptotics (2.3) yields

$$(4.10) \qquad \mu_n = \alpha_n^2 = (\pi n)^4 - 2p_0(\pi n)^2 + \frac{\|p\|^2 + p_0^2}{2} + \frac{O(1)}{n^2}.$$

The third term in asymptotics (4.9) is in a disagreement with the corresponding term in (4.10). Therefore, the term q_0 in (4.7) is incorrect. Then the trace formula (4.8) is not correct also. Theorem 1.1 shows that if p, q satisfy (4.6) and $q_0 = 0$, then the correct trace formula has the form

$$\sum_{n=1}^\infty \left(\mu_n - (\pi n)^4 + 2p_0(\pi n)^2 + \frac{\|p\|^2 - p_0^2}{2} \right) = -\frac{\|p\|^2 - p_0^2}{4} - \frac{q(0) + q(1)}{4}.$$

The trace formula (1.11) is proved in Theorem 3.3. Now we deduce this result immediately from Theorem 1.1.

Proof of Corollary 1.2. Put $q = p'' + p^2 - P \in \mathscr{H}_2^0$ in Theorem 1.1, then $H = h^2 - P$ and $\mu_n = \alpha_n^2 - P$. Substituting these identities into (1.8) we obtain (1.11). ∎

In order to prove Corollaries 1.3, 1.5 ii) we need the following results.

Lemma 4.1. *Let* $(p, q) \in \mathscr{H}_2 \times \mathscr{H}_0^0$. *Then there exists* $N = N(\|p\|_{\mathscr{H}_2}, \|q\|_{\mathscr{H}_0})$ *such that the operator* H *has exactly* N *eigenvalues, counting with multiplicities, in the disc* $\{|\lambda| < \pi^4(N + \frac{1}{2})^4\}$ *and for each integer* $n > N$ *it has exactly one simple eigenvalue in the domain* $\{|\lambda^{1/4} - \pi n| < \frac{\pi}{4}\}$. *There are no other eigenvalues.*

Proof. The proof is based on Rouché's theorem and repeats the arguments from the proof of Lemma 2.5 in [**BK6**]. ∎

Proof of Corollary 1.3. Let $(p, q) \in \mathscr{H}_{4,per} \times \mathscr{H}_{2,per}$. Introduce the function $F(\tau) = \sum_{n=1}^N \mu_n(\tau), \tau \in \mathbb{T}$, where N is given in Lemma 4.1, and the resolvents

$R_\tau(\lambda) = (H_\tau - \lambda)^{-1}$. Then we have

$$F(\tau_1) - F(\tau_2) = \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \operatorname{Tr}\left(R_{\tau_1}(\lambda) - R_{\tau_2}(\lambda)\right) d\lambda,$$

$$\mu_n(\tau_1) - \mu_n(\tau_2) = \frac{1}{2\pi i} \oint_{\ell_n} \lambda \operatorname{Tr}\left(R_{\tau_1}(\lambda) - R_{\tau_2}(\lambda)\right) d\lambda \qquad \forall \quad n > N,$$

where the contours Γ_N and ℓ_n are given by

$$\Gamma_N = \left\{\lambda \in \mathbb{C} : |\lambda|^{\frac{1}{4}} = \pi\left(N + \tfrac{1}{2}\right)\right\}, \quad \ell_n = \left\{\lambda \in \mathbb{C} : \{|\lambda^{\frac{1}{4}} - \pi n| = \tfrac{\pi}{4}\}\right\}.$$

Using the identities

$$R_{\tau_1}(\lambda) - R_{\tau_2}(\lambda) = R_{\tau_1}(\lambda)\left(H_{\tau_2} - H_{\tau_1}\right)R_{\tau_2}(\lambda)$$

$$= R_{\tau_1}(\lambda)\left(\partial(p_{\tau_2} - p_{\tau_1})\partial + q_{\tau_2} - q_{\tau_1}\right)R_{\tau_2}(\lambda),$$

where $f_\tau = f(\cdot + \tau)$, we obtain

$$\left|F^{(k)}(\tau_1) - F^{(k)}(\tau_2)\right| \leqslant C\left(\|p_{\tau_2}^{(k+1)} - p_{\tau_1}^{(k+1)}\|_\infty + \|p_{\tau_2}^{(k)} - p_{\tau_1}^{(k)}\|_\infty + \|q_{\tau_2}^{(k)} - q_{\tau_1}^{(k)}\|_\infty\right)$$

for some constant $C > 0$ and $k = 0, 1$, where $f^{(0)} = f, f^{(k)} = \frac{d^k f}{d\tau^k}$. These estimates imply $F \in C^1(\mathbb{T})$. The similar arguments show that $\mu_n \in C^1(\mathbb{T})$ for all $n > N$. The trace formula (1.8) yields (1.13). ∎

Proof of Corollary 1.5 ii). Repeating the arguments from the proof of Corollary 1.3 we deduce that the functions $\sum_{n=1}^N \nu_n(\tau)$ and $\nu_n(\tau), n > N$, belong to $C^1(\mathbb{T})$. The trace formula (1.15) yields (1.16). ∎

5. Asymptotics of J_j

In this Section we will consider the integrals $J_j, j = 1, ..., 5$ given by

$$(5.1) \qquad J_j(k) = \frac{1}{\pi i j} \int_{\gamma_k} z \operatorname{Tr}(r_0(z)p)^j dz, \qquad k \in \mathbb{N},$$

see (3.4), and prove Lemma 3.2. Now we will determine asymptotics of the sequence J_1. Introduce the functions

$$s_n = \sqrt{2}\sin \pi n x, \qquad c_n = \sqrt{2}\cos \pi n x, \qquad n \in \mathbb{N}.$$

Proof of identity (3.5) in Lemma 3.2. Substituting the identity

$$\operatorname{Tr} p r_0(z) = \sum_{n=1}^\infty \frac{(ps_n, s_n)}{(\pi n)^2 - z},$$

into (5.1) we obtain

$$J_1(k) = \frac{1}{\pi i} \int_{\gamma_k} z \operatorname{Tr} p\left(r_0(z)\right) dz$$

$$= \frac{1}{\pi i} \int_{\gamma_k} z \sum_{n=1}^\infty \frac{(ps_n, s_n)}{(\pi n)^2 - z} dz = -2 \sum_{n=1}^k (\pi n)^2 (ps_n, s_n).$$

Using the identities

$$(ps_n, s_n) = p_0 - \widehat{p}_{cn} = p_0 - \frac{p'(1) - p'(0) - \widehat{(p'')}_{cn}}{(2\pi n)^2},$$

we obtain (3.5). ∎

Introduce the coefficients

$$\varkappa_n = (p, c_n), \qquad n \in \mathbb{N}.$$

Then

$$(5.2) \qquad (ps_n, s_m) = \frac{\varkappa_{n-m} - \varkappa_{m+n}}{\sqrt{2}} \qquad \forall \ m, n \in \mathbb{N}.$$

The integration by parts gives

$$\varkappa_n = \frac{1}{(\pi n)^2} \Big(\sqrt{2}\big(p'(0) - (-1)^n p'(1)\big) + (p'', c_n) \Big) \qquad \forall \ n \in \mathbb{N},$$

which yields the estimate

$$(5.3) \qquad |\varkappa_n| \leqslant \frac{C}{n^2} \qquad \forall \ n \in \mathbb{N},$$

where

$$C = \frac{1}{\pi^2} \Big(\sqrt{2}\big(|p'(0)| + |p'(1)|\big) + \max_{n \in \mathbb{N}} |(p'', c_n)| \Big).$$

Now we determine asymptotics of the sequences $J_j(k), j = 3, 4, 5$.

Proof of asymptotics (3.7) in Lemma 3.2. Let $k \in \mathbb{N}$. Identity (5.1) gives

$$(5.4) \qquad J_3 = \frac{1}{3\pi i} \int_{\gamma_k} z \operatorname{Tr} \big(r_0(z)p\big)^3 dz = \frac{A_1 + A_2}{3\pi i},$$

where

$$A_1 = \int_{\gamma_k} z \operatorname{Tr} \Big(pr_0(z)\big(r_0(z)p\big)^2 \Big) dz,$$

$$A_2 = - \int_{\gamma_k} z \operatorname{Tr} \Big([p, r_0(z)]\big(r_0(z)p\big)^2 \Big) dz,$$

$[a, b] = ab - ba$. Moreover, the integration by parts gives

$$A_2 = - \int_{\gamma_k} z \operatorname{Tr} \Big(r_0(z)[p, h_0]r_0^2(z)pr_0(z)p \Big) dz = - \int_{\gamma_k} z \operatorname{Tr} \Big([p, h_0]r_0'(z)\big(pr_0(z)\big)^2 \Big) dz$$

$$= -\frac{1}{3} \int_{\gamma_k} z \operatorname{Tr} \Big([p, h_0]r_0(z)\big(pr_0(z)\big)^2 \Big)' dz = \frac{1}{3} \int_{\gamma_k} \operatorname{Tr} \Big([p, h_0]r_0(z)\big(pr_0(z)\big)^2 \Big) dz.$$

Let $k \to \infty$. The identity $[p, h_0] = -2p'\partial - p''$ and the uniform on γ_k estimate

$$\Big| \operatorname{Tr} \Big(p'' r_0(z)\big(pr_0(z)\big)^2 \Big) \Big| \leqslant \|p''\|_\infty \|p\|_\infty^2 \|r_0(z)\|_2^3 = O(k^{-3})$$

give

$$(5.5) \qquad A_2 = -\frac{2}{3} \int_{\gamma_k} \operatorname{Tr} \Big(r_0(z)\big(pr_0(z)\big)^2 p'\partial \Big) dz + O(k^{-1}).$$

Using the estimate $\|\partial r_0(z)\|_2 = O(1)$ we obtain

$$(5.6) \qquad \Big| \operatorname{Tr} \Big(r_0(z)\big(pr_0(z)\big)^2 p'\partial \Big) \Big| \leqslant \|\partial r_0(z)\|_2 \|p\|_\infty^2 \|p'\|_\infty \|r_0(z)\|_2^2 = O(k^{-2})$$

uniformly on γ_k. The estimate $\|r_0(z)\| \leqslant k^{-\alpha}$ for $|\operatorname{Im} z| \geqslant k^\alpha, \alpha > 0$, gives

$$(5.7) \qquad \Big| \operatorname{Tr} \Big(r_0(z)\big(pr_0(z)\big)^2 p'\partial \Big) \Big| \leqslant \|\partial r_0(z)\|_2 \|p\|_\infty^2 \|p'\|_\infty \|r_0(z)\|^2 = O(k^{-2\alpha})$$

as $|\operatorname{Im} z| \geqslant k^\alpha$. The length of part $|\operatorname{Im} z| < k^\alpha$ of the contour γ_k is $O(k^\alpha)$, then estimate (5.6) shows that its contribution to the integral (5.5) is $O(k^{-2+\alpha})$. Moreover, estimate (5.7) yields that the contribution of the rest part of the contour γ_k to the integral (5.5) is $O(k^{-2\alpha+2})$. Let $\alpha = \frac{4}{3}$. Then (5.5) gives

$$(5.8) \qquad A_2 = O(k^{-\frac{2}{3}}).$$

Furthermore,

$$A_1 = \int_{\gamma_k} z \operatorname{Tr}\left(pr_0^2(z)pr_0(z)p\right) dz = \int_{\gamma_k} z \operatorname{Tr}\left(pr_0'(z)pr_0(z)p\right) dz$$

$$= \frac{1}{2} \int_{\gamma_k} z \left(\operatorname{Tr}\left(pr_0(z)\right)^2 p\right)' dz = -\frac{1}{2} \int_{\gamma_k} \operatorname{Tr}\left(\left(pr_0(z)\right)^2 p\right) dz,$$

where we have used the integration by parts. Using the identity

$$\operatorname{Tr}\left(\left(pr_0(z)\right)^2 p\right) = \sum_{m,n=1}^{\infty} \frac{\alpha_{mn}}{(z - z_m)(z - z_n)}$$

where

$$(5.9) \qquad z_n = (\pi n)^2, \qquad \alpha_{mn} = (p^2 s_m, s_n)(p s_m, s_n),$$

we obtain

$$A_1 = -\frac{1}{2} \int_{\gamma_k} \sum_{m,n=1}^{\infty} \frac{\alpha_{mn}}{(z - z_m)(z - z_n)} dz = \sum_{m=1}^{k} \sum_{n=k+1}^{\infty} \frac{\alpha_{mn}}{z_n - z_m}$$

$$= -\frac{1}{2\pi^2} \sum_{m=1}^{k} \sum_{n=k+1}^{\infty} \frac{\alpha_{mn}}{n^2 - m^2}.$$

Estimates

$$\frac{1}{n^2 - m^2} \leqslant \frac{1}{(k+1)^2 - k^2} \leqslant \frac{1}{2k} \qquad \forall \ \ 1 \leqslant m \leqslant k, \ n \geqslant k+1,$$

imply

$$(5.10) \qquad |A_1| \leqslant \frac{1}{2\pi^2 k} \sum_{m=1}^{k} \sum_{n=k+1}^{\infty} |\alpha_{mn}|.$$

Substituting (5.2) into (5.9) we obtain

$$\alpha_{mn} = \frac{1}{2}(\widetilde{\varkappa}_{n-m} - \widetilde{\varkappa}_{n+m})(\varkappa_{n-m} - \varkappa_{n+m}), \qquad \text{where} \quad \widetilde{\varkappa}_n = (p^2, c_n),$$

which yields

$$\sum_{m=1}^{k} \sum_{n=k+1}^{\infty} |\alpha_{mn}| \leqslant 2 \sum_{m,n=1}^{\infty} |\widetilde{\varkappa}_m||\varkappa_n| < \infty.$$

Then estimate (5.10) gives $A_1 = O(k^{-1})$ as $k \to \infty$. This asymptotics and (5.8) and (5.4) give the first asymptotics in (3.7).

We show the second asymptotics in (3.7). Identities (5.1) imply

$$(5.11) \qquad |J_4| \leqslant \frac{1}{4\pi} \int_{\gamma_k} |z| |\operatorname{Tr}(r_0(z)p)^4| |dz|.$$

Let $k \to \infty$. Asymptotics (3.1) gives

$$(5.12) \qquad |\operatorname{Tr}(r_0(z)p)^4| \leqslant \|r_0(z)\|_2^4 \|p\|_\infty^4 = O(k^{-4})$$

on the contours γ_k. Using the estimate $\|r_0(z)\| \leqslant k^{-\beta}$ for $|\operatorname{Im} z| \geqslant k^\beta, \beta > 0$, we obtain

$$(5.13) \qquad |\operatorname{Tr}(r_0(z)p)^4| \leqslant \|r_0(z)\|^4 \|p\|_\infty^4 = O(k^{-4\beta})$$

as $|\operatorname{Im} z| \geqslant k^\beta$. Since the length of the part $|\operatorname{Im} z| < k^\beta$ of the contour γ_k is $O(k^\beta)$, estimate (5.12) shows that its contribution to the integral (5.11) is $O(k^{-2+\beta})$. Moreover, estimate (5.13) yields that the contribution of the rest part of the contour is $O(k^{-4\beta+4})$. Let $\beta = \frac{6}{5}$. Then (5.11) yields the second asymptotics in (3.7).

We prove the third asymptotics in (3.7). Asymptotics (3.1) gives

$$\left| \operatorname{Tr}\left(r_0(z)p\right)^5 \right| \leqslant \|r_0(z)\|_2^5 \|p\|_\infty^5 = O(k^{-5})$$

as $k \to \infty$ on the contours γ_k. Then identities (5.1) imply

$$|J_5(k)| \leqslant \frac{1}{5\pi} \int_{\gamma_k} |z| \left| \operatorname{Tr}\left(r_0(z)p\right)^5 \right| |dz| = O(k^{-1})$$

as $k \to \infty$, which yields the last asymptotics in (3.7). $\blacksquare$

Introduce the coefficients

$$(5.14) \qquad a_{mn} = (ps_n, s_m)^2 = \frac{(\varkappa_{n-m} - \varkappa_{m+n})^2}{2}, \qquad m, n \in \mathbb{N},$$

where we have used (5.2). In order to determine the asymptotics of $J_2(k)$ we need the following preliminary results.

Lemma 5.1. *Let $p \in \mathscr{H}_3$. Then*
i) The following identity holds true:

$$(5.15) \qquad J_2 = A_1 - A_2,$$

where
(5.16)

$$A_1 = k\|p\|^2 - \sum_{n=1}^k (\widehat{p^2})_{cn}, \qquad A_2 = \sum_{m=1}^k \sum_{n=k+1}^\infty a_{mn}\theta_{mn}, \qquad \theta_{mn} = \frac{n^2 + m^2}{n^2 - m^2}.$$

ii) The coefficient A_2 satisfies:

$$(5.17) \qquad A_2 = \frac{2k+1}{4} B_0 + B_1 + B_2 + B_3,$$

where
(5.18)

$$B_0 = \sum_{\ell=1}^k \varkappa_\ell^2, \quad B_1 = \sum_{m=1}^k \sum_{n=m+k+1}^\infty a_{mn}\theta_{mn},$$

$$B_2 = \frac{1}{2} \sum_{m=1}^k \sum_{n=k+1}^{m+k} (\varkappa_{m+n}^2 - 2\varkappa_{n-m}\varkappa_{m+n})\theta_{mn}, \quad B_3 = \frac{1}{4} \sum_{\ell=1}^k \varkappa_\ell^2 \sum_{n=k+1}^{k+\ell} \frac{\ell}{2n-\ell}.$$

Moreover,

$$B_0(k) = \|p\|^2 - p_0^2 + O(k^{-3}),$$

$$(5.19) \qquad B_1(k) = O(k^{-2}), \qquad B_2(k) = O(k^{-1}), \qquad B_3(k) = O(k^{-1})$$

as $k \to \infty$ uniformly on any bounded subset of $\mathscr{H}_2$.

  iii) The sequence $A_2(k)$ satisfies the asymptotics

$$(5.20) \qquad A_2(k) = (2k+1)\frac{\|p\|^2 - p_0^2}{4} + O(k^{-1})$$

as $k \to \infty$ uniformly on any bounded subset of $\mathscr{H}_2$.

Proof. i) Substituting the identity

$$\mathrm{Tr}\left(r_0(z)p\right)^2 = \sum_{m,n=1}^{\infty} \frac{a_{mn}}{(z-z_n)(z-z_m)}, \qquad z_n = (\pi n)^2,$$

into (5.1) we obtain

$$J_2 = \frac{1}{2\pi i}\int_{\gamma_k} z\,\mathrm{Tr}\left(r_0(z)p\right)^2 dz = \frac{1}{2\pi i}\int_{\gamma_k} z \sum_{m,n=1}^{\infty} \frac{a_{mn}}{(z-z_n)(z-z_m)}dz$$

$$(5.21) \qquad = \frac{1}{4\pi i}\int_{\gamma_k}\sum_{m,n=1}^{\infty} a_{mn}\Big(\frac{1}{z-z_n} + \frac{1}{z-z_m} + \frac{z_n + z_m}{(z-z_n)(z-z_m)}\Big)dz$$

$$= F_1 + F_2 + F_3,$$

where

$$F_1 = \sum_{m,n=1}^{k} a_{mn} + \frac{1}{2}\sum_{\substack{m,n=1:\\ m\neq n}}^{k} a_{mn}(z_n + z_m)\Big(\frac{1}{z_m - z_n} + \frac{1}{z_n - z_m}\Big) = \sum_{m,n=1}^{k} a_{mn},$$

$$F_2 = \frac{1}{2}\sum_{m=1}^{k}\sum_{n=k+1}^{\infty} a_{mn}\Big(1 + \frac{z_n + z_m}{z_m - z_n}\Big),$$

$$F_3 = \frac{1}{2}\sum_{n=1}^{k}\sum_{m=k+1}^{\infty} a_{mn}\Big(1 + \frac{z_n + z_m}{z_n - z_m}\Big) = F_2.$$

Using these identities and (5.21), we obtain (5.15), where A_2 is defined by the second identity in (5.16) and

$$A_1 = \sum_{m=1}^{k}\sum_{n=1}^{\infty} a_{mn} = \sum_{m=1}^{k}\sum_{n=1}^{\infty}(ps_m, s_n)^2.$$

The Parseval identity $\sum_{n=1}^{\infty}(f, s_n)^2 = \|f\|^2$ gives

$$A_1 = \sum_{m=1}^{k}\|ps_m\|^2 = \sum_{m=1}^{k}(\|p\|^2 - (\widehat{p^2})_{cm}),$$

which shows that A_1 satisfies the first identity in (5.16).

  ii) Definition (5.14) and the definition of A_2 in (5.16) give

$$(5.22) \qquad A_2 = \frac{1}{2}\sum_{m=1}^{k}\sum_{n=k+1}^{m+k} \varkappa_{n-m}^2 \theta_{mn} + B_1 + B_2,$$

and using the new variable $n - m = \ell$ we obtain

$$\sum_{m=1}^{k} \sum_{n=k+1}^{m+k} \varkappa_{n-m}^2 \theta_{mn} = \sum_{m=1}^{k} \sum_{\ell=k+1-m}^{k} \varkappa_\ell^2 \frac{(m+\ell)^2 + m^2}{\ell(\ell+2m)}$$

$$= \sum_{\ell=1}^{k} \varkappa_\ell^2 \sum_{m=k+1-\ell}^{k} \left(\frac{m}{\ell} + \frac{m+\ell}{\ell+2m} \right).$$

Due to the identities

$$\sum_{m=k+1-\ell}^{k} \frac{m}{\ell} = \frac{2k+1-\ell}{2},$$

$$\sum_{m=k+1-\ell}^{k} \frac{m+\ell}{\ell+2m} = \sum_{n=k+1}^{k+\ell} \frac{n}{2n-\ell} = \frac{\ell}{2} + \frac{1}{2} \sum_{n=k+1}^{k+\ell} \frac{\ell}{2n-\ell},$$

we get

(5.23) $$\sum_{m=1}^{k} \sum_{n=k+1}^{m+k} \varkappa_{n-m}^2 \theta_{mn} = \frac{2k+1}{2} \sum_{\ell=1}^{k} \varkappa_\ell^2 + \frac{1}{2} \sum_{\ell=1}^{k} \varkappa_\ell^2 \sum_{n=k+1}^{k+\ell} \frac{\ell}{2n-\ell}.$$

Substituting this identity into (5.22) we obtain (5.17).

We will prove the first asymptotics in (5.19). The Parseval identity $\|f_0\|^2 + \sum_{n=1}^{\infty} (f, c_n)^2 = \|f\|^2$ implies

$$\sum_{\ell=1}^{\infty} \varkappa_\ell^2 = \|p\|^2 - p_0^2.$$

Estimate (5.3) gives

$$\left| \sum_{\ell=k+1}^{\infty} \varkappa_\ell^2 \right| \leqslant C^2 \sum_{\ell=k+1}^{\infty} \frac{1}{n^4} \leqslant C^2 \int_k^{\infty} \frac{dx}{x^4} = \frac{C^2}{3k^3}.$$

Then the definition of B_0 in (5.18) yields the first asymptotics in (5.19).

We will prove the second asymptotics in (5.19). We have the estimates

$$\theta_{mn} = 1 + \frac{2m^2}{n^2 - m^2} \leqslant 1 + \frac{2k^2}{(m+k+1)^2 - m^2} \leqslant 3, \quad 1 \leqslant m \leqslant k \leqslant n - m - 1.$$

Moreover, definition (5.14) implies

$$a_{mn} \leqslant \varkappa_{m-n}^2 + \varkappa_{m+n}^2 \quad \forall \; m, n \in \mathbb{N}.$$

Then the definition of B_1 in (5.18) and estimate (5.3) give

$$0 \leqslant B_1 \leqslant 3 \sum_{m=1}^{k} \sum_{n=m+k+1}^{\infty} (\varkappa_{n-m}^2 + \varkappa_{m+n}^2) = 3 \left(\sum_{m=1}^{k} \sum_{\ell=k+1}^{\infty} \varkappa_\ell^2 + \sum_{m=1}^{k} \sum_{\ell=2m+k+1}^{\infty} \varkappa_\ell^2 \right)$$

$$\leqslant 6 \sum_{m=1}^{k} \sum_{\ell=k+1}^{\infty} \varkappa_\ell^2 = 6k \sum_{\ell=k+1}^{\infty} \varkappa_\ell^2 \leqslant 6kC^2 \sum_{\ell=k+1}^{\infty} \frac{1}{n^4} \leqslant 6kC^2 \int_k^{\infty} \frac{dx}{x^4} = \frac{2C^2}{k^2},$$

which yields the second asymptotics in (5.19).

We will prove the third asymptotics in (5.19). The definition of B_2 in (5.18) and estimate (5.3) imply

$$|B_2| \leqslant \frac{1}{2} \sum_{m=1}^{k} \sum_{n=k+1}^{m+k} |\varkappa_{m+n}|(|\varkappa_{m+n}| + 2|\varkappa_{n-m}|)\theta_{mn}$$

$$\leqslant \frac{C^2}{2} \sum_{m=1}^{k} \sum_{n=k+1}^{m+k} \frac{1}{(n-m)^2}\left(\frac{1}{(n+m)^2} + \frac{2}{(n-m)^2}\right)\theta_{mn}.$$

Using the estimates

$$\frac{1}{(n-m)^2}\left(\frac{1}{(n+m)^2} + \frac{2}{(n-m)^2}\right)\theta_{mn} = \frac{(3n^2 + 3m^2 + 2mn)(n^2 + m^2)}{(n-m)^3(n+m)^5}$$

$$\leqslant \frac{3(n^2 + m^2)^2}{(n-m)^3(n+m)^5} \leqslant \frac{3}{(n-m)^3(n+m)}, \qquad 1 \leqslant m < n,$$

we obtain

$$|B_2| \leqslant \frac{3C^2}{2} \sum_{m=1}^{k} \sum_{n=k+1}^{m+k} \frac{1}{(n-m)^3(n+m)} = \frac{3C^2}{2} \sum_{m=1}^{k} \sum_{\ell=k+1-m}^{k} \frac{1}{\ell^3(\ell + 2m)}$$

$$= \frac{3C^2}{2} \sum_{\ell=1}^{k} \frac{1}{\ell^3} \sum_{m=k+1-\ell}^{k} \frac{1}{\ell + 2m} \leqslant \frac{3C^2}{2(k+2)} \sum_{\ell=1}^{k} \frac{1}{\ell^3} \sum_{m=k+1-\ell}^{k} 1 = \frac{3C^2}{2(k+2)} \sum_{\ell=1}^{k} \frac{1}{\ell^2},$$

which yields the third asymptotics in (5.19).

We will prove the fourth asymptotics in (5.19). The definition of B_3 in (5.18) and the estimate

$$\sum_{n=k+1}^{k+\ell} \frac{1}{2n-\ell} \leqslant \frac{\ell}{k+1} \qquad \forall \quad 1 \leqslant \ell \leqslant k,$$

imply

$$0 \leqslant B_3 \leqslant \frac{1}{4(k+1)} \sum_{\ell=1}^{k} \ell^2 \varkappa_\ell^2 \leqslant \frac{C^2}{4(k+1)} \sum_{\ell=1}^{k} \frac{1}{\ell^2}.$$

where we have used (5.3). This asymptotics yields the fourth asymptotics in (5.19).

iii) Asymptotics (5.19) and identity (5.17) yield (5.20). ∎

Remark. If p is a trigonometric polynomial, then $\varkappa_n = 0$ for all $n \in \mathbb{N}$ large enough, hence $B_1 = B_2 = 0$ for all $k \in \mathbb{N}$ large enough.

Proof of asymptotics (3.6) in Lemma 3.2. Substituting the definition of A_1 in (5.16) and asymptotics (5.20) into (5.15) we obtain (3.6). ∎

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