A topological approach to the existence and multiplicity of positive solutions of (p,q)-Laplacian systems

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ABSTRACT. In this paper we develop a new theory for the existence, localization and multiplicity of positive solutions for a class of non-variational, quasilinear, elliptic systems. In order to do this, we provide a fairly general abstract framework for the existence of fixed points of nonlinear operators acting on cones that satisfy an inequality of Harnack type. Our methodology relies on fixed point index theory. We also provide a non-existence result and an example to illustrate the theory.

CONTENTS

1. Introduction

In this paper we develop a new theory for the existence, localization and multiplicity of nonnegative weak solutions to the following Dirichlet problem for (p,q)-Laplacian systems

$$
(1.1) \qquad
\begin{cases}
-\Delta_p u = f(x,u,v) & \text{in } \Omega, \\
-\Delta_q v = g(x,u,v) & \text{in } \Omega, \\
u,v = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{1,\gamma}$ for some $\gamma \in (0,1)$, $f, g : \overline{\Omega} \times \mathbb{R}_+^2 \to \mathbb{R}_+$ are continuous functions, and $p, q > 2n/(n+1)$.

The existence of positive solutions of these types of problems has been investigated by means of different methodologies. The most common approach is topological, for example Schauder fixed point theorem was used in [12], Schaefer fixed point theorem was employed in [14], Leray-Shauder degree theory was exploited in [4, 5, 25], fixed point index on cones was applied in [16, 23] and continuation methods were used in [3]. Variational methods, making use of the Nehari manifold, have been used in [1, 21], super and subsolution methods were applied in [7, 11] and monotone techniques in [9].

The multiplicity of solutions was studied in [1, 21, 25], nonexistence was investigated in [5, 23, 25], a priori estimates were given in [3, 25], regularity results were obtained in [7] and qualitative properties of the solutions have been studied in [18].

Localization results have been given in [16], where the authors proved the existence of one positive solution in the case $p = q$, and in [4], where the authors dealt with the existence of radially symmetric solutions in a ball.

Here we develop a new method that deals with the existence, localization and multiplicity of solutions in cones, for systems of two (or more) abstract equations. This method can be applied, *as a special case*, to deal with the existence of positive solutions of the system (1.1). Our approach is purely topological and is based on an abstract Harnack-type inequality and on the fixed point index theory. In order to do this, we fully exploit the recent theory developed by Precup in [19, 20] for the case of one equation, where the author has obtained some Krasnosel'skii-type results. We remark that our results are not a trivial extension of the ones in [19, 20] to the case of systems. In fact, we fully benefit of the richer structure of the system and we improve the theory, even in the case of one equation, by allowing better constants and a more precise localization of the solutions.

In the case of the system (1.1) we obtain existence, localization, multiplicity and non-existence of positive weak solutions. We also provide an example to illustrate the theoretical results.

Our results are new and improve and complement earlier ones in literature.

2. Operator equations on Cartesian products

Let X_i $(i = 1, 2)$ be Banach spaces with norms $|\cdot|_i$ ordered by cones K_i^0 and let $\|\cdot\|_i$ be seminorms on X_i. Denote by $\leq_i$ the partial order relation associated with K_i^0. Assume that both norms and seminorms are monotone, i.e.

$$0 \leq_i u \leq_i v \ \text{ implies } \ |u|_i \leq |v|_i \ \text{ and } \ \|u\|_i \leq \|v\|_i$$

for $u, v \in X_i$. In what follows, for simplicity, we shall use the same symbols $|\cdot|$, $\|\cdot\|$, $\leq$ to denote $|\cdot|_i$, $\|\cdot\|_i$, $\leq_i$ for both $i = 1, 2$.

Let $\chi_i \in K_i^0$ be fixed such that $\|\chi_i\| > 0$. Define the cones $K_i \subset K_i^0$ by the formula

$$K_i := \left\{ u \in K_i^0 : u \geq \|u\| \chi_i \right\}$$

and assume that there exist points inside them with positive seminorms, which is equivalent to the assumption $\|\chi_i\| \leq 1$. Hence, we can choose

$$(2.1) \qquad\qquad \phi_i \in K_i, \ \ |\phi_i| = 1, \ \ \|\phi_i\| > 0.$$

In particular we may take

$$\phi_i = \frac{\chi_i}{|\chi_i|}.$$

Let us observe that the seminorm $\|\cdot\|_i$ is continuous in K_i with respect to the topology induced by the norm $|\cdot|_i$, since one has:

(2.2) $$u \geq \|u\|\chi_i, \ u \in K_i \implies \|u\| \leq \frac{1}{|\chi_i|}|u|, \ u \in K_i.$$

In what follows by *compactness* of a continuous operator we mean the relative compactness of its range. By *complete continuity* of a continuous operator we mean the relative compactness of the image of every bounded set of the domain.

We set

$$X := X_1 \times X_2, \quad K := K_1 \times K_2,$$

and we seek the fixed points of a completely continuous operator

$$N = (N_1, N_2) \colon K \to K,$$

that is $(u, v) \in K$ with $N(u, v) = (u, v)$. Note that the cone-invariance of N is equivalent to the fact that the operators N_i satisfy an *abstract weak Harnack inequality* of the type

$$N_i(u, v) \geq \|N_i(u, v)\|\chi_i, \quad (i = 1, 2).$$

We shall discuss not only existence, but also localization and multiplicity of solutions of the nonlinear equation $N(u, v) = (u, v)$. In order to do this, we utilize the Granas fixed point index, $\mathrm{ind}_C(f, U)$ (for more information on the index and its applications we refer the reader to [**6, 10**]).

The next Proposition describes some of the useful properties of the index, for details see Theorem 6.2, Chapter 12 of [**10**].

PROPOSITION 2.1. *Let C be a closed convex subset of a Banach space, $U \subset C$ be open in C and $f \colon \overline{U} \to C$ be a compact map with no fixed points on the boundary ∂U of U. Then the fixed point index has the following properties:*

(i) *(Existence) If $\mathrm{ind}_C(f, U) \neq 0$ then $\mathrm{fix}(f) \neq \emptyset$, where $\mathrm{fix}f = \{x \in \overline{U} : f(x) = x\}$.*

(ii) *(Additivity) If $\mathrm{fix}f \subset U_1 \cup U_2 \subset U$ with U_1, U_2 open in C and disjoint, then*

$$\mathrm{ind}_C(f, U) = \mathrm{ind}_C(f, U_1) + \mathrm{ind}_C(f, U_2).$$

(iii) *(Homotopy invariance) If $h \colon \overline{U} \times [0, 1] \to C$ is a compact homotopy such that $h(u, t) \neq u$ for $u \in \partial U$ and $t \in [0, 1]$ then*

$$\mathrm{ind}_C(h(., 0), U) = \mathrm{ind}_C(h(., 1), U).$$

(iv) *(Normalization) If f is a constant map, with $f(u) = u_0$ for every $u \in \overline{U}$, then*

$$\mathrm{ind}_C(f, U) = \begin{cases} 1, & \text{if } u_0 \in U \\ 0, & \text{if } u_0 \notin \overline{U}. \end{cases}$$

In particular, $\mathrm{ind}_C(f, C) = 1$ for every compact function $f : C \to C$, since f is homotopic to any $u_0 \in C$, by the convexity of C (take $h(u, t) = tf(u) + (1 - t)u_0$).

2.1. Solutions with at least one nonzero component. We begin with four theorems on the existence and localization of one solution of the operator equation $N(u,v) = (u,v)$. The first two ones assume that the operator N leaves invariant the set

$$C = \{(u,v) \in K : |u| \leq R_1, \ |v| \leq R_2\},$$

for some fixed numbers R_1, R_2.

THEOREM 2.2. *Assume that there exist numbers* $r_i > 0$ *and* $R_i > 0$ *with*

$$(2.3) \qquad\qquad r_i < \|\phi_i\|\|\chi_i\|R_i \quad (i = 1, 2),$$

such that

$$(2.4) \qquad \inf_{\substack{(u,v)\in C \\ \|u\|=r_1, \|v\|\leq r_2}} \|N_1(u,v)\| \geq \frac{r_1}{\|\chi_1\|}, \qquad \inf_{\substack{(u,v)\in C \\ \|u\|\leq r_1, \|v\|=r_2}} \|N_2(u,v)\| \geq \frac{r_2}{\|\chi_2\|},$$

and

$$(2.5) \qquad\qquad \sup_{(u,v)\in C} |N_i(u,v)| \leq R_i \quad (i = 1, 2).$$

Then N has at least one fixed point $(u,v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$ and either $\|u\| \geq r_1$ or $\|v\| \geq r_2$.

PROOF. The assumption (2.5) implies that $N(C) \subset C$. Therefore, by Proposition 2.1, $\mathrm{ind}_C(N, C) = 1$. Let

$$U := \{(u,v) \in C : \|u\| < r_1, \ \|v\| < r_2\}.$$

The boundary ∂U of U with respect to C is equal to $\partial U = A_1 \cup A_2$, where

$$\begin{aligned} A_1 &= \{(u,v) \in C : \|u\| = r_1, \ \|v\| \leq r_2\}, \\ A_2 &= \{(u,v) \in C : \|u\| \leq r_1, \ \|v\| = r_2\}. \end{aligned}$$

If there is a fixed point (u,v) of N on ∂U, then (u,v) satisfies the assertion. If not, the indices $\mathrm{ind}_C(N, U)$ and $\mathrm{ind}_C(N, C \setminus \overline{U})$ are well defined and their sum, by the additivity property of the index, is equal to one. Therefore, it suffices to prove that $\mathrm{ind}_C(N, U) = 0$. Take $h = (R_1\phi_1, R_2\phi_2) \in C$ and consider the homotopy $H : C \times [0,1] \to C$,

$$H(u,v,t) := th + (1-t)N(u,v).$$

We claim that H is fixed point free on ∂U. Since

$$(2.6) \qquad \begin{aligned} \|R_1\phi_1\| &= R_1\|\phi_1\| \geq R_1\|\phi_1\|\|\chi_1\| > r_1, \\ \|R_2\phi_2\| &= R_2\|\phi_2\| \geq R_2\|\phi_2\|\|\chi_2\| > r_2, \end{aligned}$$

we have that $(u,v) \neq h = H(u,v,1)$ for all $(u,v) \in \partial U$. It remains to show that $H(u,v,t) \neq (u,v)$ for $(u,v) \in \partial U$ and $t \in (0,1)$. Assume the contrary. Then there exists $(u,v) \in A_1 \cup A_2$ and $t \in (0,1)$ such that

$$(2.7) \qquad\qquad (u,v) = th + (1-t)N(u,v).$$

Suppose that $(u, v) \in A_1$. Then, exploiting the first coordinate of the equation (2.7), we obtain

$$u = tR_1\phi_1 + (1 - t)N_1(u, v) \geq tR_1\|\phi_1\|\chi_1 + (1 - t)\|N_1(u, v)\|\chi_1$$

$$(2.8) \quad \geq \left((tR_1\|\phi_1\| + (1 - t) \inf_{\substack{(u,v)\in C \\ \|u\|=r_1, \|v\|\leq r_2}} \|N_1(u, v)\|\right)\chi_1$$

$$\geq \left(tR_1\|\phi_1\| + (1 - t)\frac{r_1}{\|\chi_1\|}\right)\chi_1.$$

Using the monotonicity of $\|\cdot\|$ and (2.3) we obtain that

$$r_1 \geq \left(tR_1\|\phi_1\| + (1 - t)\frac{r_1}{\|\chi_1\|}\right)\|\chi_1\|$$

$$= tR_1\|\phi_1\|\,\|\chi_1\| + (1 - t)\,r_1 > tr_1 + (1 - t)\,r_1 = r_1,$$

which is impossible. Similarly we derive a contradiction if $(u, v) \in A_2$.

By the homotopy invariance of the index we obtain that $\mathrm{ind}_C(N, U) = \mathrm{ind}_C(h, U)$. From (2.6) we have $h \notin \overline{U}$, hence $\mathrm{ind}_C(N, U) = \mathrm{ind}_C(h, U) = 0$, as we wished. $\square$

REMARK 2.3. We observe that, using the relation (2.2), a lower bound for the solution in terms of the seminorm provides a lower bound for the norm of the solution, namely

$$\|u\| \geq r_1 \implies |u| \geq r_1|\chi_1|.$$

In the next result we replace, in the spirit of Lemma 4 of [**13**], the assumption (2.4) with a different one. The two conditions are not comparable and are used, in a combined way, in Theorem 2.14.

THEOREM 2.4. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

$$(2.9) \qquad\qquad r_i < \|\phi_i\|\|\chi_i\|R_i \quad (i = 1, 2),$$

such that

$$(2.10) \qquad\qquad \sup_{(u,v)\in C} |N_i(u, v)| \leq R_i \quad (i = 1, 2),$$

and

$$(2.11) \qquad \inf_{(u,v)\in A} \|N_1(u, v)\| \geq r_1 \quad or \quad \inf_{(u,v)\in A} \|N_2(u, v)\| \geq r_2,$$

where A is a subset of the set

$$U = \{(u, v) \in C : \|u\| < r_1, \ \|v\| < r_2\} \subsetneq C.$$

Then N has at least one fixed point $(u, v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$ and $(u, v) \notin A$.

PROOF. Since N is a completely continuous mapping in the bounded closed convex set C, by Schauder's fixed point theorem, it possesses a fixed point $(u, v) \in C$. We now show that the fixed point is not in A. Suppose on the contrary that $(u, v) = N(u, v)$ and $(u, v) \in A$. Suppose that the first inequality from (2.11) is satisfied. Then

$$r_1 > \|u\| = \|N_1(u, v)\| \geq r_1,$$

which is impossible. Similarly we arrive at a contradiction, if the second inequality from (2.11) is satisfied. $\square$

The next theorems do not assume the invariance condition $N(C) \subset C$ and use instead Leray-Schauder type conditions. The first result requires the Leray-Schauder condition componentwise.

THEOREM 2.5. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

$$(2.12) \qquad r_i < \|\phi_i\|\|\chi_i\|R_i \quad (i = 1, 2),$$

such that the strengthened condition (2.4)*:*

$$(2.13) \qquad \begin{aligned} &\inf_{\substack{(u,v) \in C \\ \|u\| = r_1, \|v\| \le r_2}} \left(\max \left\{ \frac{|N_1(u,v)|}{R_1}, 1 \right\} \right)^{-1} \|N_1(u,v)\| \ge \frac{r_1}{\|\chi_1\|}, \\ &\inf_{\substack{(u,v) \in C \\ \|u\| \le r_1, \|v\| = r_2}} \left(\max \left\{ \frac{|N_2(u,v)|}{R_2}, 1 \right\} \right)^{-1} \|N_2(u,v)\| \ge \frac{r_2}{\|\chi_2\|}, \end{aligned}$$

and the weakened condition (2.5)*:*

$$(2.14) \qquad \begin{aligned} &|u| = R_1, \ |v| < R_2 \ \ implies \ \ N(u,v) \ne (\lambda u, v), \ \ for \ all \ \lambda > 1; \\ &|u| < R_1, \ |v| = R_2 \ \ implies \ \ N(u,v) \ne (u, \lambda v), \ for \ all \ \lambda > 1; \\ &|u| = R_1, \ |v| = R_2 \ \ implies \ \ N(u,v) \ne (\lambda_1 u, \lambda_2 v), \\ &\qquad\qquad\qquad\qquad\qquad\qquad for \ all \ \lambda_1, \lambda_2 \ge 1 \ with \ \lambda_1 \lambda_2 > 1, \end{aligned}$$

hold. Then N has at least one fixed point $(u,v) \in K$ such that $|u| \le R_1$, $|v| \le R_2$ and either $\|u\| \ge r_1$ or $\|v\| \ge r_2$.

PROOF. Consider the retraction $\pi \colon K \to C$, $\pi(u,v) = (\pi_1(u), \pi_2(v))$, where

$$\pi_i \colon K_i \to C_i := \{u \in K_i : |u| \le R_i\}, \quad \pi_i(u) = \begin{cases} u & \text{if } |u| \le R_i \\ \frac{R_i}{|u|} u & \text{if } |u| \ge R_i. \end{cases}$$

Now we define the operator $\tilde{N}$ by the formula

$$\tilde{N}(u,v) = \pi(N(u,v)) = (\pi_1(N_1(u,v)), \pi_2(N_2(u,v))).$$

Then $\tilde{N}(C) \subset C$, i.e. $\tilde{N}$ satisfies (2.5). Now, let $(u,v) \in C$ be such that $\|u\| = r_1$, $\|v\| \le r_2$. Observe that

$$\tilde{N}_1(u,v) = \pi_1(N_1(u,v)) = \left(\max \left\{ \frac{|N_1(u,v)|}{R_1}, 1 \right\} \right)^{-1} N_1(u,v),$$

which in view of (2.13) shows that

$$\|\tilde{N}_1(u,v)\| \ge \frac{r_1}{\|\chi_1\|}.$$

A similar estimate holds for $\tilde{N}_2$, which shows that $\tilde{N}$ satisfies (2.4). By Theorem 2.2 we obtain a fixed point (u,v) of $\tilde{N}$ in the set $C \setminus U$. Therefore $\pi(N(u,v)) = (u,v)$. If $|N_1(u,v)| \le R_1$, then $\pi_1(N_1(u,v)) = N_1(u,v)$ and so $N_1(u,v) = u$. Similarly, if $|N_2(u,v)| \le R_2$, then $N_2(u,v) = v$, and the proof is complete. Otherwise, we have $|N_1(u,v)| > R_1$ or $|N_2(u,v)| > R_2$. Assume $|N_1(u,v)| > R_1$. Then

$$\pi_1(N_1(u,v)) = \frac{R_1}{|N_1(u,v)|} N_1(u,v) = u.$$

It follows that $N_1(u,v) = \frac{|N_1(u,v)|}{R_1} u$ and $|u| = R_1$ which, in view of the first implication from (2.14), is impossible if $|v| < R_2$. Hence $|v| = R_2$, which implies that

$N_2(u, v) = \lambda_2 v$ for some $\lambda_2 \geq 1$. Since $N_1(u, v) = \lambda_1 u$, where $\lambda_1 = \frac{|N_1(u,v)|}{R_1} > 1$ and $|u| = R_1$, we contradict the third condition from (2.14). Thus the inequality $|N_1(u, v)| > R_1$ is not possible. By symmetry, the inequality $|N_2(u, v)| > R_2$ is also not possible. Consequently, $\pi(N(u, v)) = N(u, v) = (u, v)$. $\qquad\square$

REMARK 2.6. Notice that under condition (2.5), (2.14) is satisfied and (2.13) reduces to (2.4).

If instead of the retraction π, we consider the retraction $\rho : K \to C$ given by

$$\rho(u, v) = \left(\max \left\{ \frac{|u|}{R_1}, \frac{|v|}{R_2}, 1 \right\} \right)^{-1} (u, v),$$

we obtain the following existence theorem under the Leray-Schauder condition acting this time, uniformly on the two components u, v.

THEOREM 2.7. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

(2.15) $$r_i < \|\phi_i\| \|\chi_i\| R_i \quad (i = 1, 2),$$

such that the strengthened condition (2.4):

(2.16)
$$\inf_{\substack{(u,v) \in C \\ \|u\| = r_1, \|v\| \leq r_2}} \left(\max \left\{ \frac{|N_1(u, v)|}{R_1}, \frac{|N_2(u, v)|}{R_2}, 1 \right\} \right)^{-1} \|N_1(u, v)\| \geq \frac{r_1}{\|\chi_1\|},$$
$$\inf_{\substack{(u,v) \in C \\ \|u\| \leq r_1, \|v\| = r_2}} \left(\max \left\{ \frac{|N_1(u, v)|}{R_1}, \frac{|N_2(u, v)|}{R_2}, 1 \right\} \right)^{-1} \|N_2(u, v)\| \geq \frac{r_2}{\|\chi_2\|},$$

and the weakened condition (2.5):

(2.17) $$N(u, v) \neq \lambda(u, v), \quad \text{for} \ (u, v) \in \partial C, \ \lambda > 1,$$

hold. Then N has at least one fixed point $(u, v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$ and either $\|u\| \geq r_1$ or $\|v\| \geq r_2$.

PROOF. The assumption (2.16) guarantees that the map $\bar{N}(u, v) = \rho(N(u, v))$, that has the property $\bar{N}(C) \subset C$, also satisfies condition (2.4). By Theorem 2.2, $\bar{N}$ has a fixed point (u, v) in $C \setminus U$. If $N(u, v) \in C$, then $\rho(N(u, v)) = N(u, v)$ and we are done. Otherwise, $|N_1(u, v)| > R_1$ or $|N_2(u, v)| > R_2$, and then $\lambda := \max \left\{ \frac{|N_1(u,v)|}{R_1}, \frac{|N_2(u,v)|}{R_2}, 1 \right\} > 1$, $N(u, v) = \lambda(u, v)$ and $(u, v) \in \partial C$, which is excluded by (2.17). $\qquad\square$

REMARK 2.8. Note that, in the case of one equation, the retractions π and ρ coincide with the one used in [**19**, Theorem 2.1], but are used differently.

2.2. Solutions with both nonzero components. In the previous results one of the components of the solution can be 0. The next three theorems avoid this situation.

THEOREM 2.9. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

(2.18) $$r_i < \|\phi_i\| \|\chi_i\| R_i \quad (i = 1, 2),$$

such that

(2.19) $$\inf_{\substack{(u,v) \in C \\ \|u\| = r_1, \ \|v\| \geq r_2}} \|N_1(u, v)\| \geq \frac{r_1}{\|\chi_1\|}, \qquad \inf_{\substack{(u,v) \in C \\ \|u\| \geq r_1, \ \|v\| = r_2}} \|N_2(u, v)\| \geq \frac{r_2}{\|\chi_2\|},$$

and

$$(2.20) \qquad \sup_{(u,v)\in C} |N_i(u,v)| \leq R_i \quad (i=1,2).$$

Then N has at least one fixed point $(u,v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$, $\|u\| \geq r_1$ and $\|v\| \geq r_2$.

PROOF. As before, the assumption (2.20) implies that $N(C) \subset C$. Thus, $\mathrm{ind}_C(N,C) = 1$. In order to finish the proof, it is sufficient to show that $\mathrm{ind}_C(N,V) = 0$, where

$$V := \{(u,v) \in C : \|u\| < r_1 \text{ or } \|v\| < r_2\}.$$

We have $\partial V = B_1 \cup B_2$, where

$$\begin{aligned} B_1 &= \{(u,v) \in C : \|u\| = r_1,\ \|v\| \geq r_2\}, \\ B_2 &= \{(u,v) \in C : \|u\| \geq r_1,\ \|v\| = r_2\}. \end{aligned}$$

If N has a fixed point $(u,v) \in \partial V$, we are finished. Thus, assume that $N(u,v) \neq (u,v)$ for $(u,v) \in \partial V$, and consider the same homotopy as in the proof of Theorem 2.2, that is

$$H(u,v,t) = th + (1-t)N(u,v), \quad \text{where} \quad h = (R_1\phi_1, R_2\phi_2).$$

We claim that H is fixed point free on ∂V. From the previous assumption we have $H(u,v,0) \neq (u,v)$ for $(u,v) \in \partial V$. Also, as in the proof of Theorem 2.2, $H(u,v,1) = h \neq (u,v)$ on ∂V. It remains to show that $H(u,v,t) \neq (u,v)$ for $(u,v) \in \partial V$ and $t \in (0,1)$. Assume the contrary. Then, there exists $(u,v) \in B_1 \cup B_2$ and $t \in (0,1)$ such that

$$(2.21) \qquad (u,v) = th + (1-t)N(u,v).$$

Suppose that $(u,v) \in B_1$. Then, exploiting the first coordinate of the equation (2.21), we obtain

$$u = tR_1\phi_1 + (1-t)N_1(u,v) \geq tR_1\|\phi_1\|\chi_1 + (1-t)\|N_1(u,v)\|\chi_1$$

$$(2.22) \qquad \geq \left(tR_1\|\phi_1\| + (1-t)\inf_{\substack{(u,v)\in C \\ \|u\|=r_1, \|v\|\geq r_2}} \|N_1(u,v)\|\right)\chi_1$$

$$\geq \left(tR_1\|\phi_1\| + (1-t)\frac{r_1}{\|\chi_1\|}\right)\chi_1$$

thus, as in the proof of Theorem 2.2, we obtain the contradiction $r_1 > r_1$. The case $(u,v) \in B_2$ is similar. Therefore $\mathrm{ind}_C(N,V) = \mathrm{ind}_C(h,V) = 0$ since in view of (2.6), $h \notin \overline{V}$. $\qquad \square$

Theorem 2.9 can be generalized in the spirit of Theorems 2.5 and 2.7.

THEOREM 2.10. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

$$(2.23) \qquad r_i < \|\phi_i\|\|\chi_i\|R_i \quad (i=1,2),$$

such that

$$(2.24) \qquad \inf_{\substack{(u,v)\in C \\ \|u\|=r_1, \|v\|\geq r_2}} \left(\max\left\{\frac{|N_1(u,v)|}{R_1}, 1\right\}\right)^{-1} \|N_1(u,v)\| \geq \frac{r_1}{\|\chi_1\|},$$

$$\inf_{\substack{(u,v)\in C \\ \|u\|\geq r_1, \|v\|=r_2}} \left(\max\left\{\frac{|N_2(u,v)|}{R_2}, 1\right\}\right)^{-1} \|N_2(u,v)\| \geq \frac{r_2}{\|\chi_2\|},$$

and

$$|u| = R_1, \ |v| < R_2 \ implies \ N(u, v) \neq (\lambda u, v), \ for \ all \lambda > 1;$$

(2.25) $$|u| < R_1, \ |v| = R_2 \ implies \ N(u, v) \neq (u, \lambda v), \ for \ all \lambda > 1;$$

$$|u| = R_1, \ |v| = R_2 \ implies \ N(u, v) \neq (\lambda_1 u, \lambda_2 v),$$
$$for \ all \lambda_1, \lambda_2 \geq 1 \ with \lambda_1 \lambda_2 > 1.$$

Then N has at least one fixed point $(u, v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$, $\|u\| \geq r_1$ and $\|v\| \geq r_2$.

PROOF. Define the retractions π_i and the operator $\tilde{N}$ as in the proof of Theorem 2.5. Then $\tilde{N}(C) \subset C$, i.e. $\tilde{N}$ satisfies (2.20). Now, let $(u, v) \in C$ be such that $\|u\| = r_1$, $\|v\| \geq r_2$. Observe that

$$\tilde{N}_1(u, v) = \pi_1(N_1(u, v)) = \left(\max \left\{ \frac{|N_1(u, v)|}{R_1}, 1 \right\} \right)^{-1} N_1(u, v),$$

which shows that

$$\|\tilde{N}_1(u, v)\| \geq \frac{r_1}{\|\chi_1\|}.$$

A similar estimate holds for $\tilde{N}_2$ which shows that $\tilde{N}$ satisfies (2.19). By Theorem 2.9 we obtain a fixed point (u, v) of $\tilde{N}$ in the set $C \backslash V$. Therefore $\pi(N(u, v)) = (u, v)$. In the proof of Theorem 2.5 it was shown that (2.25) yields $N(u, v) = (u, v)$.   □

Similarly, using the method presented in the proof of Theorem 2.7 and exploiting Theorem 2.9, we obtain the following fact.

THEOREM 2.11. *Assume that there exist numbers $r_i > 0$ and $R_i > 0$ with*

(2.26) $$r_i < \|\phi_i\| \|\chi_i\| R_i \quad (i = 1, 2),$$

such that

(2.27)
$$\inf_{\substack{(u,v) \in C \\ \|u\| = r_1, \|v\| \geq r_2}} \left(\max \left\{ \frac{|N_1(u, v)|}{R_1}, \frac{|N_2(u, v)|}{R_2}, 1 \right\} \right)^{-1} \|N_1(u, v)\| \geq \frac{r_1}{\|\chi_1\|},$$

$$\inf_{\substack{(u,v) \in C \\ \|u\| \geq r_1, \|v\| = r_2}} \left(\max \left\{ \frac{|N_1(u, v)|}{R_1}, \frac{|N_2(u, v)|}{R_2}, 1 \right\} \right)^{-1} \|N_2(u, v)\| \geq \frac{r_2}{\|\chi_2\|},$$

and

(2.28) $$N(u, v) \neq \lambda(u, v), \quad for \ (u, v) \in \partial C, \ \lambda > 1.$$

Then N has at least one fixed point $(u, v) \in K$ such that $|u| \leq R_1$, $|v| \leq R_2$, $\|u\| \geq r_1$ and $\|v\| \geq r_2$.

PROOF. Let the mapping $\bar{N}$ be defined as in the proof of Theorem 2.7. The assumption (2.27) guarantees that $\bar{N}$, having the property $\bar{N}(C) \subset C$, also satisfies condition (2.19). By Theorem 2.9, $\bar{N}$ has a fixed point (u, v) in $C \backslash V$. It was shown in the proof of Theorem 2.7 that (2.28) implies $N(u, v) = (u, v)$.   □

2.3. Multiplicity results. Recall that the seminorms $\| \cdot \|_i$ are continuous in K_i with respect to the topology induced by $| \cdot |_i$, which implies that there exist constants $c_i > 0$ such that $\|u\|_i \leq c_i |u|_i$ for all $u \in K_i$.

THEOREM 2.12. *Assume that there exist numbers ρ_i, r_i, R_i with*

$$(2.29) \qquad 0 < c_i \rho_i < r_i < \|\phi_i\| \|\chi_i\| R_i \quad (i = 1, 2),$$

such that

$$(2.30) \qquad \inf_{\substack{(u,v) \in C \\ \|u\|=r_1, \, \|v\| \geq r_2}} \|N_1(u,v)\| > \frac{r_1}{\|\chi_1\|}, \qquad \inf_{\substack{(u,v) \in C \\ \|u\| \geq r_1, \, \|v\|=r_2}} \|N_2(u,v)\| > \frac{r_2}{\|\chi_2\|},$$

$$(2.31) \qquad \sup_{(u,v) \in C} |N_i(u,v)| \leq R_i \quad (i = 1, 2),$$

and
$$(2.32)$$
$$N(u,v) \neq \lambda(u,v), \quad \text{for } \lambda \geq 1 \text{ and } (|u| = \rho_1, \, |v| \leq \rho_2 \text{ or } |u| \leq \rho_1, \, |v| = \rho_2).$$

Then N has at least three fixed points $(u_i, v_i) \in C$ $(i = 1, 2, 3)$ with

$$|u_1| \quad < \quad \rho_1, \; |v_1| < \rho_2 \quad \text{(possibly the zero solution)};$$
$$\|u_2\| \quad < \quad r_1 \text{ or } \|v_2\| < r_2; \; |u_2| > \rho_1 \text{ or } |v_2| > \rho_2 \quad \text{(possibly one component zero)};$$
$$\|u_3\| \quad > \quad r_1, \; \|v_3\| > r_2 \quad \text{(both solution components nonzero)}.$$

PROOF. Let V be as in the proof of Theorem 2.9. Strict inequalities in (2.30) guarantee that N is fixed point free on ∂V. According to the proof of Theorem 2.9 we have $\mathrm{ind}_C(N, C) = 1$, $\mathrm{ind}_C(N, V) = 0$ and therefore by the additivity property, $\mathrm{ind}_C(N, C \setminus \overline{V}) = 1$. Let

$$W := \{(u,v) \in C : |u| < \rho_1, \; |v| < \rho_2\}.$$

For every $(u,v) \in \overline{W}$, we have

$$\|u\| \leq c_1 \, |u| \leq c_1 \rho_1 < c_1 \left(\frac{1}{c_1} r_1 \right) = r_1$$

and, similarly, $\|v\| < r_2$. Hence $(u,v) \in V$, which proves that $\overline{W} \subset V$. Condition (2.32) shows that N is homotopic with zero on W. Thus $\mathrm{ind}_C(N, W) = \mathrm{ind}_C(0, W) = 1$. Then $\mathrm{ind}_C(N, V \setminus \overline{W}) = 0 - 1 = -1$. Consequently, there exist at least three fixed points of N, in W, $V \setminus \overline{W}$ and $C \setminus \overline{V}$. $\qquad\square$

If we assume the following estimates of the $\|N_i(u,v)\|$:

$$(2.33) \qquad \inf_{\substack{(u,v) \in C \\ \|u\|=r_1}} \|N_1(u,v)\| > \frac{r_1}{\|\chi_1\|}, \quad \inf_{\substack{(u,v) \in C \\ \|v\|=r_2}} \|N_2(u,v)\| > \frac{r_2}{\|\chi_2\|},$$

then we can obtain a more precise localization for the solution (u_2, v_2) in Theorem 2.12, the Figure 1 illustrates this fact.

THEOREM 2.13. *Suppose that all the assumptions of Theorem 2.12 are satisfied with the condition (2.30) replaced by (2.33). Then N has at least three fixed points*
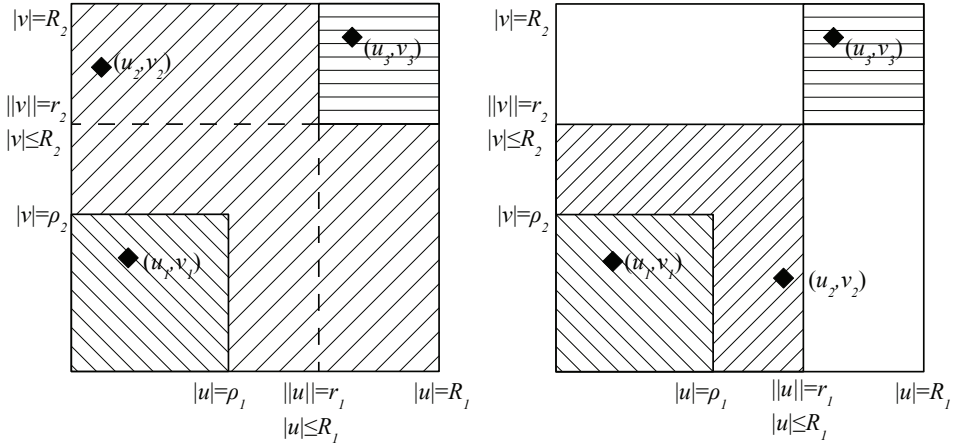
FIGURE 1. Localization of the three solutions (u_i, v_i) from Theorem 2.12 (on the left) and 2.13 (on the right).

$(u_i, v_i) \in C$ $(i = 1, 2, 3)$ *with*

$|u_1| < \rho_1$, $|v_1| < \rho_2$ *(possibly the zero solution)*;

$\|u_2\| < r_1$, $\|v_2\| < r_2$; $|u_2| > \rho_1$ *or* $|v_2| > \rho_2$ *(possibly one component zero)*;

$\|u_3\| > r_1$, $\|v_3\| > r_2$ *(both solution components nonzero)*.

PROOF. The assumption (2.33) implies both (2.4) and (2.19) and that there are no fixed points of N on ∂U and ∂V. Hence, as in the proofs of Theorems 2.2 and 2.9, the indices $\mathrm{ind}_C(N, U)$ and $\mathrm{ind}_C(N, V)$ are well defined and equal 0. An analysis similar to that in the proof of Theorem 2.12 shows that

$$\mathrm{ind}_C(N, W) = 1, \ \mathrm{ind}_C(N, U \setminus \overline{W}) = -1, \ \mathrm{ind}_C(N, C \setminus \overline{V}) = 1,$$

which completes the proof. $\qquad\square$

In order to ensure that the solution (u_1, v_1) from the theorems above is nonzero, and thereby to obtain three *nonzero* solutions, we use some additional assumptions on N.

THEOREM 2.14. *Assume that all the conditions of Theorem* 2.12 *or Theorem* 2.13 *are satisfied. Consider* $0 < \varrho_i < \|\phi_i\|\|\chi_i\|\rho_i$, $i = 1, 2$.
   (i) *If*
(2.34)
$$\inf_{\substack{(u,v) \in K, |u| \le \rho_1, |v| \le \rho_2 \\ \|u\| = \varrho_1, \|v\| \ge \varrho_2}} \|N_1(u, v)\| \ge \frac{\varrho_1}{\|\chi_1\|}, \quad \inf_{\substack{(u,v) \in K, |u| \le \rho_1, |v| \le \rho_2 \\ \|u\| \ge \varrho_1, \|v\| = \varrho_2}} \|N_2(u, v)\| \ge \frac{\varrho_2}{\|\chi_2\|}$$

and

$$|N_i(u, v)| \le \rho_i \ \text{ for } \ |u| \le \rho_1, \ |v| \le \rho_2 \quad (i = 1, 2),$$

then we can assume that the solution (u_1, v_1) *from Theorem* 2.12 *or* 2.13 *satisfies* $\|u_1\| \ge \varrho_1$ *and* $\|v_1\| \ge \varrho_2$;

(ii) *if*
(2.35)
$$\inf_{\substack{(u,v)\in K,|u|\leq\tilde{\rho}_1,|v|\leq\tilde{\rho}_2 \\ \|u\|\leq\varrho_1,\|v\|\leq\varrho_2}} \|N_1(u,v)\| \geq \varrho_1 \quad or \quad \inf_{\substack{(u,v)\in K,|u|\leq\tilde{\rho}_1,|v|\leq\tilde{\rho}_2 \\ \|u\|\leq\varrho_1,\|v\|\leq\varrho_2}} \|N_2(u,v)\| \geq \varrho_2$$

for some $\tilde{\rho}_1 \leq \rho_1$, $\tilde{\rho}_2 \leq \rho_2$, *then we can assume that the solution* (u_1,v_1) *from Theorem* 2.12 *or* 2.13 *satisfies* $\|u_1\| \geq \varrho_1$ *or* $\|v_1\| \geq \varrho_2$ *or* $|u_1| > \tilde{\rho}_1$ *or* $|v_1| > \tilde{\rho}_2$.

PROOF. (i) The inequality follows from Theorem 2.9 applied in the case of $r_i := \varrho_i$ and $R_i := \rho_i$.

(ii) From Theorem 2.4 applied in the case of $r_i := \varrho_i$, $R_i := \rho_i$ and

$$A = \{(u,v): \ \|u\| < \varrho_1, \|v\| < \varrho_2, |u| \leq \tilde{\rho}_1, |v| \leq \tilde{\rho}_2\},$$

we obtain that there are no fixed points in N in A, which ends the proof. □

The next Remark illustrates how Theorem 2.9 can be used to prove the existence of n nontrivial solutions.

REMARK 2.15. If N satisfies the conditions of Theorem 2.9 for all pairs of radii

$$r_i^j < \|\phi_i\|\|\chi_i\|R_i^j \quad \text{for} \ \ i = 1,2, \ j = 1,2,\ldots,n,$$

satisfying

$$c_i R_i^j < r_i^{j+1} \quad \text{for} \ \ i = 1,2, \ j = 1,2,\ldots,n-1,$$

then N possesses at least n nontrivial solutions (u_j, v_j) with

$$|u_j| \leq R_1^j, \ |v_j| \leq R_2^j, \ \|u_j\| \geq r_1^j, \ \|v_j\| \geq r_2^j.$$

Moreover, if the conditions

$$\inf_{\substack{(u,v)\in K,|u|\leq R_1^j,|v|\leq R_2^j \\ \|u\|=r_1^j, \ \|v\|\geq r_2^j}} \|N_1(u,v)\| > \frac{r_1^j}{\|\chi_1\|}, \qquad \inf_{\substack{(u,v)\in K,|u|\leq R_1^j,|v|\leq R_2^j \\ \|u\|\geq r_1^j, \ \|v\|=r_2^j}} \|N_2(u,v)\| > \frac{r_2^j}{\|\chi_2\|},$$

$$\sup_{(u,v)\in K,|u|\leq R_1^j,|v|\leq R_2^j} |N_i(u,v)| < R_i^j \quad (i=1,2),$$

hold, then we have $n-1$ additional solutions $(\bar{u}_j, \bar{v}_j)$, $j = 1,\ldots,n-1$ such that

$$|\bar{u}_j| < R_1^{j+1}, \ |\bar{v}_j| < R_2^{j+1}; \ |\bar{u}_j| > R_1^j \text{ or } |\bar{v}_j| > R_2^j; \ \|\bar{u}_j\| < r_1^{j+1} \text{ or } \|\bar{v}_j\| < r_2^{j+1}.$$

The first conclusion follows from Theorem 2.9 applied n times, whereas the second follows from Theorem 2.12 applied $n-1$ times.

REMARK 2.16. Assume that between the cones $K_i \subseteq K_i^0$ and the norms $|\cdot|_i$ $(i = 1,2)$ there is a strong compatibility, expressed by the following condition:
(2.36)
there exists $h_i^0 \in K_i$, $|h_i^0| = 1$, such that $h_i^0 \geq u$ for every $u \in K_i$ with $|u| \leq 1$.

Then the localization theory for fixed points of a given operator $N = (N_1, N_2) : K \to K$ can be developed without assuming the Harnack type inequality "$u \geq \|u\| \chi_i$" for the definition of K and without specifying the elements χ_i, ϕ_i. This is based on a different way of proving that the homotopy $H(u,v,t) = th + (1 - t)N(u,v)$ is fixed point free on the boundary. Here we choose $h = (R_1 h_1^0, R_2 h_2^0)$. For instance, the corresponding version of Theorem 2.2 is the following:

THEOREM 2.17. *Assume that condition* (2.36) *holds, that* $N : K \to K$ *is completely continuous and that for some numbers* $0 \le r_i < \left\|h_i^0\right\| R_i$, *the conditions*

$$(2.37) \qquad \inf_{\substack{(u,v)\in C \\ \|u\|=r_1,\|v\|\le r_2}} \|N_1(u,v)\| > r_1, \qquad \inf_{\substack{(u,v)\in C \\ \|u\|\le r_1,\|v\|=r_2}} \|N_2(u,v)\| > r_2$$

and

$$\sup_{(u,v)\in C} |N_i(u,v)| \le R_i \qquad (i=1,2)$$

hold. Then N *has at least one fixed point* $(u,v) \in K$ *such that* $|u| \le R_1$, $|v| \le R_2$ *and either* $\|u\| \ge r_1$ *or* $\|v\| \ge r_2$.

PROOF. Following the proof of Theorem 2.2 and assuming that there exists $(u,v) \in \partial U$ and $t \in (0,1)$ with $H(u,v,t) = (u,v)$, in case that $(u,v) \in A_1$, we have

$$u = N_1(u,v) + t(h - N_1(u,v)) \ge N_1(u,v).$$

Therefore, making use of the first inequality from (2.37), we obtain

$$r_1 = \|u\| \ge \|N_1(u,v)\| > r_1,$$

a contradiction. The same argument can be used if $(u,v) \in A_2$. Note that $h \notin \overline{U}$ because of the condition $r_i < \left\|h_i^0\right\| R_i$ $(i=1,2)$. $\qquad\square$

We underline that, although the Harnack inequality is not used in the above proof, it is essential to obtain the estimates from below of the type (2.37) in the applications.

REMARK 2.18. We stress that the abstract results obtained in this section can be generalized to the case of systems of more than two equations. The idea is to consider the product space $X = \Pi_{i=1}^n X_i$ of the Banach spaces X_i, endowed with the norms $|\cdot|_i$, seminorms $\|\cdot\|_i$, and the cones $K_i \subset K_i^0 \subset X_i$ such that $\phi_i \in K_i := \left\{u \in K_i^0 : u \ge \|u\|_i\chi_i\right\}$ for fixed χ_i and ϕ_i satisfying (2.1), $i = 1,2,...,n$. In this setting we are interested in the existence and localization of fixed points of a given operator $N: K \to K$, where $K = \Pi_{i=1}^n K_i$. For example, let us consider the sets

$$C = \{u \in K : |u_1|_1 \le R_1, \ldots, |u_n|_n \le R_n\},$$
$$U = \{u \in C : \|u_1\|_1 < r_1, \ldots, \|u_n\|_n < r_n\}$$

for given radii r_i, R_i with $r_i < \|\phi_i\|_i\|\chi_i\|_i R_i$, $i = 1,...,n$. If

$$\sup_{u\in C} |N_i(u)|_i \le R_i, \quad i = 1,2,...,n$$

and

$$\inf_{\substack{(u,v)\in \overline{U} \\ \|u_i\|_i=r_i}} \|N_i(u)\|_i \ge \frac{r_i}{\|\chi_i\|_i}, \quad i = 1,2,...,n$$

then N has at least one fixed point in $C \setminus U$.

As a consequence, results analogous to ones obtained later in Section 3, can be established for systems with more than two differential equations.

2.4. Case of isotone operators. Let us now turn our attention to the case when N satisfies a monotonicity condition with respect to the order induced by cones K_i^0. Precisely, assume that the operator N is *isotone*, that is

$$0 \le u \le u', \ 0 \le v \le v' \ \text{implies} \ N_1(u, v) \le N_1(u', v'), \ N_2(u, v) \le N_2(u', v'),$$

and that condition (2.36) is satisfied.

Let us examine the condition (2.4). If $(u, v) \in C$, $\|u\| = r_1$, $\|v\| \le r_2$, then $u \ge \|u\|\chi_1 = r_1\chi_1$ and $\|N_1(u, v)\| \ge \|N_1(r_1\chi_1, 0)\|$. Therefore the condition (2.4) is implied by the simpler one:

$$(2.4') \qquad \|N_1(r_1\chi_1, 0)\| \ge \frac{r_1}{\|\chi_1\|}, \quad \|N_2(0, r_2\chi_2)\| \ge \frac{r_2}{\|\chi_2\|}.$$

Similarly, the condition (2.19) is implied by the following one:

$$(2.19') \qquad \|N_1(r_1\chi_1, r_2\chi_2)\| \ge \frac{r_1}{\|\chi_1\|}, \quad \|N_2(r_1\chi_1, r_2\chi_2)\| \ge \frac{r_2}{\|\chi_2\|}.$$

Let us now examine the condition (2.5). If $|u| \le R_1$ and $|v| \le R_2$ then $u \le R_1 h_1^0$, $v \le R_2 h_2^0$ and $|N_i(u, v)| \le |N_i(R_1 h_1^0, R_2 h_2^0)|$, $i = 1, 2$. This shows that the condition

$$|N_i(R_1 h_1^0, R_2 h_2^0)| \le R_i \quad (i = 1, 2)$$

implies (2.5).

3. Applications to (p, q)-Laplacian systems

3.1. Existence results. We now turn our attention back to the existence and localization of nonnegative weak solutions of the (p, q)-Laplace system (1.1). By a *nonnegative* solution we mean a solution (u, v) with $u \ge 0$ and $v \ge 0$. A nonnegative solution (u, v) is said to be *nontrivial* if either $u \ne 0$ or $v \ne 0$, and is called *positive* if both functions u, v are different from zero, equivalently, if $u > 0$ and $v > 0$.

In order to apply the abstract theorems from the previous section, we choose $X_1 = X_2 = L^\infty(\Omega)$, and $K_1^0 = K_2^0 = K^0 := \{u \in L^\infty(\Omega) : u \ge 0\}$. Thus the two norms $|\cdot|_1, |\cdot|_2$ coincide with the usual norm of $L^\infty(\Omega)$, simply denoted by $|\cdot|$. Let $S_r : L^\infty(\Omega) \to C_0^1(\overline{\Omega})$ be the operator which assigns to any $v \in L^\infty(\Omega)$ the unique weak solution u of the problem

$$(3.1) \qquad \begin{cases} -\Delta_r u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

i.e. $S_r = (-\Delta_r)^{-1}$. It is known (see [**2**]) that S_r is well defined, completely continuous, isotone and positive. Also consider the superposition operators $F, G \colon K^0 \times K^0 \to K^0$,

$$F(u, v)(x) = f(x, u(x), v(x)), \quad G(u, v)(x) = g(x, u(x), v(x)),$$

and define the operator $N = (N_1, N_2)$ by

$$N_1(u, v) = S_p(F(u, v)), \quad N_2(u, v) = S_q(G(u, v)).$$

Note that a pair (u, v) is a nonnegative weak solution of (1.1) if and only if it is a fixed point of N in $K^0 \times K^0$. In Remark 3.11 we shall give some additional information on the regularity of the solution.

We recall a local weak Harnack inequality for nonnegative p-superharmonic functions due to Trudinger, see [**22**, Theorem 1.2], in the following form, similar to the one presented by Lindqvist, [**17**, Corollary 3.18]:

THEOREM 3.1. *For each $s \in [1, \frac{n(p-1)}{n-p})$ if $p < n$ or $s \in [1, \infty)$ if $p \geq n$ and for each $\rho > 0$ such that $B_{3\rho} \subset \Omega$, there exists a constant $C = C(n, p, s) > 0$ such that*

$$\inf_{B_\rho} u \geq C\rho^{-\frac{n}{s}} \left(\int_{B_{2\rho}} u^s dx \right)^{\frac{1}{s}}$$

for every nonnegative p-superharmonic function u in Ω.

Following a reasoning based on finite cover by balls of any compact set (see the proof of Corollary 1.2.9 in [15] or Theorem 1.3 in [19]), we obtain a variant of the Harnack inequality, stated in [20], which plays a crucial role in our investigation.

THEOREM 3.2. *For each s as in Theorem 3.1 and each compact $D \subset \Omega$, there is a constant $M = M(n, p, s, D, \Omega) > 0$ such that*

$$(3.2) \qquad \inf_D u \geq M \left(\int_D u^s dx \right)^{\frac{1}{s}}.$$

for every nonnegative p-superharmonic function u in Ω.

By means of the Trudinger-type inequality (3.2) we shall obtain the existence, localization, and multiplicity of nonnegative solutions for the problem (1.1).

Let us now consider two compact sets of nonzero Lebesgue measure $D_1, D_2 \subset \Omega$ and their characteristic functions χ_{D_1}, χ_{D_2}, which we denote by χ_1, χ_2. Since $p, q > 2n/(n+1)$, by Theorem 3.2 we obtain that there are numbers $s_1, s_1 > 1$ and constants $M_1, M_1 > 0$ such that

$$(3.3) \quad \inf_{D_1} u \geq \|u\|_1 := M_1 \left(\int_{D_1} u^{s_1} dx \right)^{\frac{1}{s_1}}, \quad \inf_{D_2} v \geq \|v\|_2 := M_2 \left(\int_{D_2} u^{s_2} dx \right)^{\frac{1}{s_2}}$$

for every nonnegative p-superharmonic function u and q-superharmonic function v. Using the natural partial order in $L^\infty(\Omega)$ we can rewrite the inequalities (3.3) in the following way:

$$(3.4) \qquad u \geq \|u\|_1 \chi_1, \quad v \geq \|v\|_2 \chi_2.$$

The nonnegativity of f and g gives that $N_1(u, v)$ and $N_2(u, v)$ are superharmonic for any $u, v \in K^0$. Therefore we obtain in a similar way as in the case of one equation studied in [20], that

$$N_1(K^0 \times K^0) \subset K_1, \quad N_2(K^0 \times K^0) \subset K_2,$$

where

$$K_i = \left\{ u \in K^0 : u \geq \|u\|_i \chi_i \right\}, \quad \chi_i = \chi_{D_i} \quad (i = 1, 2).$$

Therefore, $N : K_1 \times K_2 \to K_1 \times K_2$. In addition, the complete continuity of S_p and S_q guarantees that N is completely continuous.

REMARK 3.3. Note that the assumption (2.36) is satisfied in our context of differential equations, since the constant function $h_i^0 \equiv 1$ satisfies (2.36). Indeed, since the constant function 1 is p (q)-superharmonic in Ω, one has $h_i^0 \in K_i$. Also, the conditions $|h_i^0| = 1$ and $u \leq h_i^0$ for every $u \in K_i$ with $|u| \leq 1$ are trivially satisfied. Therefore we can use Remark 2.16. This yields that, instead of the relation (2.3) between radii, we can consider the relation

$$(3.5) \qquad r_i < \|1\|_i R_i \quad (i = 1, 2),$$

and we can improve the constants $r_i/\|\chi_i\|$ in (2.4), (2.13), (2.16), (2.19), (2.33), (2.34), replacing them by r_i.

Notice that if in our case, ϕ_i is chosen to be $\chi_i/|\chi_i| = \chi_i$, then the relation (2.3) becomes $r_i < \|1\|_i^2 R_i$, which is more restrictive than (3.5).

We can now state a result for the existence and localization of a nonnegative solution of the system 1.1 with both nonzero components (i.e. a positive solution).

THEOREM 3.4. *Let* r_1, r_2, R_1, R_2 *satisfy* (3.5). *Assume that the following conditions hold:*

(3.6)
$$\frac{\max\limits_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,R_1]\times[0,R_2]} f(x,\tau,\sigma)}{R_1^{p-1}} =: \alpha_1 \le A_p := \frac{1}{|S_p(1)|^{p-1}},$$

$$\frac{\max\limits_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,R_1]\times[0,R_2]} g(x,\tau,\sigma)}{R_2^{q-1}} =: \alpha_2 \le A_q := \frac{1}{|S_q(1)|^{q-1}},$$

and

(3.7)
$$\frac{\min\limits_{(x,\tau,\sigma)\in D_1\times[r_1,R_1]\times[0,R_2]} f(x,\tau,\sigma)}{r_1^{p-1}} =: \beta_1 > B_{1,p} := \frac{1}{\|S_p(\chi_1)\|^{p-1}},$$

$$\frac{\min\limits_{(x,\tau,\sigma)\in D_2\times[0,R_1]\times[r_2,R_2]} g(x,\tau,\sigma)}{r_2^{q-1}} =: \beta_2 > B_{2,q} := \frac{1}{\|S_q(\chi_2)\|^{q-1}}.$$

Then there exists a positive solution (u,v) *of* (1.1) *such that* $|u| \le R_1$, $|v| \le R_2$, $\|u\| \ge r_1$ *and* $\|v\| \ge r_2$.

PROOF. If $(u,v) \in C$ and $x \in \Omega$, then $0 \le u(x) \le R_1$, $0 \le v(x) \le R_2$ and therefore

$$F(u,v)(x) = f(x,u(x),v(x)) \le \alpha_1 R_1^{p-1}.$$

By the isotonicity of S_p and the monotonicity of $|\cdot|$ we have

$$|N_1(u,v)| \le |S_p(\alpha_1 R_1^{p-1})| \le \alpha_1^{1/(p-1)} R_1 |S_p(1)| \le R_1.$$

A similar estimate is true for N_2. Hence, the condition (2.20) from Theorem 2.9 is satisfied. Now, let $(u,v) \in C$ be such that $\|u\| = r_1$. For $x \in D_1$, we have $u(x) \ge \|u\| = r_1$ and, as a consequence,

$$F(u,v)(x) = f(x,u(x),v(x)) \ge \beta_1 r_1^{p-1}.$$

By the isotonicity of S_p and the monotonicity of $\|\cdot\|$ we have

$$\|N_1(u,v)\| \ge \left\|S_p(\beta_1 r_1^{p-1}\chi_1)\right\| = \beta_1^{1/(p-1)} r_1 \|S_p(\chi_1)\| > r_1.$$

A similar estimate is true for N_2. Hence, the condition (2.19) from Theorem 2.9, modified in accordance with Remark 2.16, is satisfied.

The assertion now follows from Theorem 2.9 and Remark 3.3. □

We now present the relationship between the constants that arise in Theorem 3.4.

PROPOSITION 3.5. *Let $\lambda_{1,p}$ be the first eigenvalue of the p-Laplace operator $-\Delta_p$ under the Dirichlet condition. The following relations hold*

(3.8) $$A_p \leq \lambda_{1,p} \leq B_{1,p} \quad and \quad A_q \leq \lambda_{1,q} \leq B_{2,q}.$$

PROOF. Let $u > 0$ be a positive eigenfunction corresponding to the eigenvalue $\lambda_{1,p}$. Then $-\Delta_p u = \lambda_{1,p} u^{p-1}$ and since S_p is homogeneous of degree $1/(p-1)$, we have

$$u = S_p \left(\lambda_{1,p} u^{p-1} \right) = \lambda_{1,p}^{1/(p-1)} S_p \left(u^{p-1} \right).$$

a) Assume that $|u| = 1$. Then $0 \leq u \leq 1$ and, by the isotonicity of S_p and the monotonicity of the norm, we have

$$1 = |u| = \lambda_{1,p}^{1/(p-1)} \left| S_p \left(u^{p-1} \right) \right| \leq \lambda_{1,p}^{1/(p-1)} |S_p(1)|.$$

This implies that $A_p \leq \lambda_{1,p}$.

b) Now assume that $\|u\| = 1$. Then, by the Harnack inequality, we have $u \geq \chi_{D_1}$. By the isotonicity of S_p and the monotonicity of the semi-norm we obtain

$$1 = \|u\| = \lambda_{1,p}^{1/(p-1)} \left\| S_p \left(u^{p-1} \right) \right\| \geq \lambda_{1,p}^{1/(p-1)} \|S_p(\chi_{D_1})\|.$$

This implies that $B_{1,p} \geq \lambda_{1,p}$. Similarly one can prove that $A_q \leq \lambda_{1,q} \leq B_{2,q}$. $\quad\square$

REMARK 3.6. Proposition 3.5 can be seen as an analogue of a result proved, in the context of ODEs, by Webb and Lan [**24**, Theorem 2.8]. Note that the inequalities (3.8) can be strict, see for example [**24**, Example 5.2].

THEOREM 3.7. *Let r_1, r_2, R_1, R_2 satisfy (3.5) and let $\tilde{R}_1 \leq R_1$, $\tilde{R}_2 \leq R_2$. Assume that the condition (3.6) is satisfied and that*

(3.9)
$$\frac{\min\limits_{(x,\tau,\sigma)\in D_1\times[0,\tilde{R}_1]\times[0,\tilde{R}_2]} f(x,\tau,\sigma)}{r_1^{p-1}} =: \beta_1' > B_{1,p} \quad or$$

$$\frac{\min\limits_{(x,\tau,\sigma)\in D_2\times[0,\tilde{R}_1]\times[0,\tilde{R}_2]} g(x,\tau,\sigma)}{r_2^{q-1}} =: \beta_2' > B_{2,q}.$$

Then there exists a nontrivial nonnegative solution (u, v) of (1.1) such that $|u| \leq R_1$, $|v| \leq R_2$ and

(3.10) $$\|u\| \geq r_1 \quad or \quad \|v\| \geq r_2 \quad or \quad |u| > \tilde{R}_1 \quad or \quad |v| > \tilde{R}_2.$$

In particular, if $\tilde{R}_1 = R_1$ and $\tilde{R}_2 = R_2$, then there exists a nontrivial nonnegative solution (u, v) with either $\|u\| \geq r_1$ or $\|v\| \geq r_2$.

PROOF. As in the previous proof we know that the condition (2.10) from Theorem 2.4 is satisfied. Let

$$(u, v) \in A := \left\{ (u, v): \|u\| < r_1, \|v\| < r_2, |u| \leq \tilde{R}_1, |v| \leq \tilde{R}_2 \right\}.$$

Assume that the first inequality in (3.9) holds. Then for $x \in D_1$ we have

$$F(u, v)(x) = f(x, u(x), v(x)) \geq \beta_1' r_1^{p-1}$$

and by the isotonicity of S_p and the monotonicity of $\|\cdot\|$,

$$\|N_1(u, v)\| \geq \left\| S_p(\beta_1' r_1^{p-1} \chi_1) \right\| = \beta_1'^{1/(p-1)} r_1 \|S_p(\chi_1)\| > r_1.$$

A similar estimate holds for N_2. Hence, the condition (2.11) from Theorem 2.4, modified in accordance with Remark 2.16, is satisfied. Therefore, we can apply

Theorem 2.4 and Remark 3.3, and obtain a solution $(u, v) \notin A$. Clearly, this is equivalent to (3.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

REMARK 3.8. The importance of Theorem 3.7 consists in the fact that the assumption (3.9) involves only one component of the system nonlinearity (f, g). Therefore, it allows different kinds of growth of f and g near the origin. A similar remark also applies to the following theorem.

THEOREM 3.9. *Let ρ_i, r_i, R_i satisfy $0 < \rho_i < r_i < \|1\|_i R_i$ $(i = 1, 2)$. Assume that:*

$$(3.11) \quad \frac{\displaystyle\max_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,R_1]\times[0,R_2]} f(x,\tau,\sigma)}{R_1^{p-1}} \leq A_p, \quad \frac{\displaystyle\max_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,R_1]\times[0,R_2]} g(x,\tau,\sigma)}{R_2^{q-1}} \leq A_q,$$

$$(3.12) \quad \frac{\displaystyle\max_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,\rho_1]\times[0,\rho_2]} f(x,\tau,\sigma)}{\rho_1^{p-1}} < A_p, \quad \frac{\displaystyle\max_{(x,\tau,\sigma)\in\overline{\Omega}\times[0,\rho_1]\times[0,\rho_2]} g(x,\tau,\sigma)}{\rho_2^{q-1}} < A_q,$$

$$(3.13) \quad \begin{aligned} \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_1\times[r_1,R_1]\times[0,R_2]} f(x,\tau,\sigma)}{r_1^{p-1}} &> B_{1,p}, \\[2ex] \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_2\times[0,R_1]\times[r_2,R_2]} g(x,\tau,\sigma)}{r_2^{q-1}} &> B_{2,q}. \end{aligned}$$

Then there exist three nonnegative solutions (u_i, v_i) $(i = 1, 2, 3)$ of the system (1.1) with

$$
\begin{aligned}
|u_1| \quad &< \quad \rho_1, \quad |v_1| < \rho_2 \quad \text{(possibly the zero solution)}; \\
\|u_2\| \quad &< \quad r_1, \; \|v_2\| < r_2; \; |u_2| > \rho_1 \text{ or } |v_2| > \rho_2 \quad \text{(possibly one component zero)}; \\
\|u_3\| \quad &> \quad r_1, \; \|v_3\| > r_2 \quad \text{(both solution components nonzero)}.
\end{aligned}
$$

Moreover, having given numbers $0 < \varrho_i < \|1\|\rho_i$ $(i = 1, 2)$,
$\quad$ (i) *if*

$$(3.14) \quad \begin{aligned} \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_1\times[\varrho_1,\rho_1]\times[0,\rho_2]} f(x,\tau,\sigma)}{\varrho_1^{p-1}} &> B_{1,p} \quad \text{and} \\[2ex] \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_2\times[0,\rho_1]\times[\varrho_2,\rho_2]} g(x,\tau,\sigma)}{\varrho_2^{q-1}} &> B_{2,q} \end{aligned}$$

then $\|u_1\| \geq \varrho_1$ and $\|v_1\| \geq \varrho_2$;
$\quad$ (ii) *if*

$$(3.15) \quad \begin{aligned} \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_1\times[0,\tilde{\rho}_1]\times[0,\tilde{\rho}_2]} f(x,\tau,\sigma)}{\varrho_1^{p-1}} &> B_{1,p} \quad \text{or} \\[2ex] \frac{\displaystyle\min_{(x,\tau,\sigma)\in D_2\times[0,\tilde{\rho}_1]\times[0,\tilde{\rho}_2]} g(x,\tau,\sigma)}{\varrho_2^{q-1}} &> B_{2,q} \end{aligned}$$

for some $\tilde{\rho}_1 \leq \rho_1$, $\tilde{\rho}_2 \leq \rho_2$, then $\|u_1\| \geq \varrho_1$ or $\|v_1\| \geq \varrho_2$ or $|u_1| > \tilde{\rho}_1$ or $|v_1| > \tilde{\rho}_2$.

PROOF. Observe that (2.2) implies that the constants c_i that occur in the statement of the Theorem 2.12 are equal to 1.

The assumptions (3.11), (3.12) and (3.13) imply the conditions (2.31), (2.32), (2.30), modified according to Remark 2.16. Hence, the first part of the assertion follows from Theorem 2.13, combined with Remark 3.3.

The second part follows, in a similar way, from Theorem 2.14, combined with Remark 3.3. □

REMARK 3.10. If the functions f, g do not depend on x and are monotone (increasing or decreasing) in each of the two variables u, v, then the minima and maxima in all inequalities from Theorems 3.4-3.9 are reached on some boundary point of the corresponding rectangular domain. Combining the two kinds of monotonicity on the two variables u, v, for our functions $f(u, v), g(u, v)$, we see that sixteen cases are possible. For example, in the case of Theorem 3.4, we may have the following situations:

(1) f, g increasing in both variables; then the conditions (3.6) and (3.7) read as

$$f(R_1, R_2) \leq A_p R_1^{p-1} \qquad g(R_1, R_2) \leq A_q R_2^{q-1}$$
$$f(r_1, 0) > B_{1,p} r_1^{p-1} \qquad g(0, r_2) > B_{2,q} r_2^{q-1};$$

(2) f increasing in u and decreasing in v, and g increasing in both u, v; then the conditions (3.6) and (3.7) become

$$f(R_1, 0) \leq A_p R_1^{p-1} \qquad g(R_1, R_2) \leq A_q R_2^{q-1}$$
$$f(r_1, R_2) > B_{1,p} r_1^{p-1} \qquad g(0, r_2) > B_{2,q} r_2^{q-1}.$$

REMARK 3.11. It is worth mentioning that the solutions (u, v) of the system (1.1) are continuously differentiable. This follows from the fact that $S_r(L^\infty(\Omega)) \subset C^1(\overline{\Omega})$, $r > 1$ (see [**2**]). Moreover, as a consequence of the Harnack inequality (see [**22**]), if $u \geq 0$ and $u \neq 0$, then

$$u(x) > 0 \quad \text{for} \quad x \in \Omega$$

(the same for v). Indeed, the Harnack inequality implies that the set $\{x \in \Omega : u(x) = 0\}$ is open. Being also closed in Ω, it is equal either to $\emptyset$ or to Ω. The latter is excluded by the assumption $u \neq 0$.

3.2. Non-existence results. We now present some sufficient conditions for the non-existence of positive solutions of the system (1.1).

THEOREM 3.12. *Assume that (u, v) is a nonnegative solution of the system (1.1). If one of the following conditions holds:*

(3.16) $\qquad f(x, u, v) < \lambda_{1,p} u^{p-1}$, *for every $(x, u, v) \in \Omega \times (0, \infty) \times [0, \infty)$,*

(3.17) $\qquad f(x, u, v) > \lambda_{1,p} u^{p-1}$, *for every $(x, u, v) \in \Omega \times (0, \infty) \times [0, \infty)$,*

(3.18) $\qquad f(x, u, v) > B_{1,p} u^{p-1}$, *for every $(x, u, v) \in D_1 \times (0, \infty) \times [0, \infty)$,*

then $u = 0$. Similarly, if one of the following conditions holds:

(3.19) $\qquad g(x, u, v) < \lambda_{1,q} v^{q-1}$, *for every $(x, u, v) \in \Omega \times [0, \infty) \times (0, \infty)$,*

(3.20) $\qquad g(x, u, v) > \lambda_{1,q} v^{q-1}$, *for every $(x, u, v) \in \Omega \times [0, \infty) \times (0, \infty)$,*

(3.21) $\qquad g(x, u, v) > B_{2,q} v^{q-1}$, *for every $(x, u, v) \in D_2 \times [0, \infty) \times (0, \infty)$,*

then $v = 0$.

PROOF. Let us observe that $-\Delta_p u = f(x, u, v) \geq 0$. Suppose on the contrary that $u \neq 0$. Then $u(x) > 0$ for $x \in \Omega$ (see Remark 3.11) and $\|u\| > 0$.

Assume that the inequality (3.16) holds. Then $f(x, u(x), v(x)) < \lambda_{1,p} u^{p-1}(x)$ almost everywhere in Ω and consequently

$$\lambda_{1,p} \int_\Omega u^p \leq \int_\Omega |\nabla u|^p = \int_\Omega f(x, u, v) \cdot u < \lambda_{1,p} \int_\Omega u^{p-1} u,$$

a contradiction.

Assume now that (3.17) holds. Then

$$(3.22) \qquad\qquad -\Delta_p u = \lambda_{1,p} u^{p-1} + h,$$

where the function $h(x) := f(x, u(x), v(x)) - \lambda_{1,p} u^{p-1}(x)$ is positive almost everywhere in Ω and is obviously in the space $L^\infty(\Omega)$. This contradicts the fact from [8], which states that (3.22) has no solutions.

Assume now that the inequality (3.18) holds. Set $r := \|u\| > 0$. Then, since $u \in K_1$, we have that $u(x) \geq r$ for $x \in D_1$. Therefore

$$F(u, v)(x) = f(x, u(x), v(x)) > B_{1,p} u(x)^{p-1} \geq B_{1,p} r^{p-1}.$$

By the continuity of f, u and v (see Remark 3.11) and the compactness of D_1 we deduce that $F(u, v)(x) \geq B_{1,p} r_1^{p-1}$ for some $r_1 > r$ and all $x \in D_1$. This implies that $F(u, v) \geq B_{1,p} r_1^{p-1} \chi_1$. Then, by the isotonicity of S_p, the monotonicity of the seminorm and the definition of $B_{1,p}$, we derive

$$r = \|u\| = \|N_1(u, v)\| = \|S_p F(u, v)\| \geq \|S_p(B_{1,p} r_1^{p-1})\| = B_{1,p}^{1/(p-1)} r_1 \|S_p(\chi_1)\| = r_1,$$

a contradiction.

The second assertion can be proved analogously. $\qquad\qquad\square$

COROLLARY 3.13. (i) *If one of the inequalities (3.16)-(3.21) holds, then there are no positive solutions of the system (1.1).*

(ii) *If one of the inequalities (3.16)-(3.18) holds and one of the inequalities (3.19)-(3.21) holds, then there are no nontrivial nonnegative solutions of the system (1.1).*

3.3. Example. Let

$$f(x, u, v) = \varphi(x, u, v) \cdot \frac{u^2}{4 + u^3}, \quad g(x, u, v) = \psi(x, u, v) \cdot \arctan^2 v,$$

where φ and ψ are continuous functions such that

$$a \leq \varphi(x, u, v) \leq b, \quad c \leq \psi(x, u, v) \leq d,$$

for every $(x, u, v) \in \Omega \times [0, +\infty) \times [0, +\infty)$ and some fixed constants $a, b, c, d \in (0, +\infty)$.

PROPOSITION 3.14. *Under the above assumptions, there exists a constant λ_0 (depending on a, b, c, d and Ω) such that for each $\lambda > \lambda_0$, the problem*

$$(3.23) \qquad \begin{cases} -\Delta u = \lambda f(x, u, v) & in \ \Omega, \\ -\Delta v = \lambda g(x, u, v) & in \ \Omega, \\ u, v = 0 & on \ \partial\Omega, \end{cases}$$

has at least two nontrivial nonnegative solutions.

PROOF. We shall apply Theorem 3.9. To this end, consider any compact subset $D \subset \Omega$ and put $D_1 := D$, $D_2 := D$. In order to simplify our notation we denote

$$A := A_2, \ B := B_{1,2} = B_{2,2}, \ \Phi(x) = x^2/(4 + x^3) \ \text{ and } \ \Psi(x) = \arctan^2 x.$$

Note that $\Phi(x) \le l_1 := 1/3$, $\Psi(x) \le l_2 := \pi^2/4$, Ψ is increasing, while Φ is increasing in $[0, 2]$ and decreasing in $[2, \infty)$. Therefore, if $\rho_1 < 2$ then the conditions (3.11), (3.12) and (3.13) are implied by the following ones:

$$(3.24) \qquad\qquad \lambda l_1 b \le A R_1, \quad \lambda l_2 d \le A R_2,$$

$$(3.25) \qquad\qquad \frac{\Phi(\rho_1)}{\rho_1} < \frac{A}{\lambda b}, \quad \frac{\Psi(\rho_2)}{\rho_2} < \frac{A}{\lambda d},$$

$$(3.26) \qquad\qquad \lambda a \min\{\Phi(r_1), \Phi(R_1)\} > B r_1, \quad \frac{\Psi(r_2)}{r_2} > \frac{B}{\lambda c}.$$

If we put

$$R_1(\lambda) := \lambda l_1 b/A \quad \text{and} \quad R_2(\lambda) := \lambda l_2 d/A,$$

then the condition (3.24) is satisfied and the condition (3.26) has the form

$$(3.27) \qquad \frac{R_1(\lambda)}{r_1} \min\{\Phi(r_1), \ \Phi(R_1)\} > l_1 \frac{B}{A} \frac{b}{a} =: \gamma, \quad \frac{\Psi(r_2)}{r_2} > \frac{B}{\lambda c}.$$

Fix $r_2 > 0$. There exists λ_1 such that the second inequality from (3.27), as well as $r_2 < \|1\| R_2(\lambda)$, are satisfied for $\lambda > \lambda_1$. Let us put

$$r_1 = r_1(\lambda) = 1/\sqrt{R_1(\lambda)}.$$

Then there exists $\lambda_2 \ge \lambda_1$ such that $r_1(\lambda) < \|1\| R_1(\lambda)$ for $\lambda > \lambda_2$. The first inequality from (3.27) becomes

$$(3.28) \qquad \min\left\{ \frac{R_1(\lambda)\sqrt{R_1(\lambda)}}{1 + 4R_1(\lambda)\sqrt{R_1(\lambda)}}, \ \frac{R_1^3(\lambda)}{4 + R_1^3(\lambda)} \right\} > \frac{\gamma}{\sqrt{R_1(\lambda)}}.$$

If $\lambda \to \infty$, then the left-hand side of (3.28) tends to $1/4$ and the right-hand side of (3.28) tends to 0. Therefore, there exists $\lambda_0 \ge \lambda_2$ such that (3.28) is satisfied for $\lambda > \lambda_0$. Having defined r_i and R_i for $\lambda > \lambda_0$, let us choose $\rho_1 < r_1$ and $\rho_2 < r_2$ that satisfy (3.25). This is possible because $\Phi(x) = o(x)$ and $\Psi(x) = o(x)$ in 0. By Theorem 3.9 we obtain the existence of at least two nontrivial nonnegative solutions to (3.23) for $\lambda > \lambda_0$.

Let us observe that for a fixed $\lambda > \lambda_0$, the nontrivial nonnegative solutions (u_1, v_1) and (u_2, v_2) given by Theorem 3.9, satisfy

$$|u_i| \ \le \ R_1(\lambda), \ |v_i| \le R_2(\lambda) \quad (i = 1, 2), \ \|u_1\| > r_1(\lambda), \|v_1\| > r_2,$$
$$\|u_2\| \ < \ r_1(\lambda), \ \|v_2\| < r_2 \ \text{ and } \ |u_2| > \rho_1(\lambda) \ \text{ or } \ |v_2| > \rho_2(\lambda).$$

Note that both components of the first solution are positive. $\qquad\qquad\qquad \square$

Acknowledgments

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