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On the regularity of the solution map of the incompressible Euler equation

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Abstract. In this paper we consider the incompressible Euler equation on the Sobolev space $H^s(\mathbb{R}^n)$, $s > n/2 + 1$, and show that for any $T > 0$ its solution map $u_0 \mapsto u(T)$, mapping the initial value to the value at time T, is nowhere locally uniformly continuous and nowhere differentiable.

Contents

1. Introduction

The initial value problem for the incompressible Euler equation in $\mathbb{R}^n$, $n \geq 2$, reads as:

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\nabla p \\ \operatorname{div} u &= 0 \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

where $u(t,x) = \big(u_1(t,x), \ldots, u_n(t,x)\big)$ is the velocity of the fluid at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^n$, $u \cdot \nabla = \sum_{k=1}^n u_k \partial_k$ acts componentwise on u, ∇p is the gradient of the pressure $p(t,x)$, $\operatorname{div} u = \sum_{k=1}^n \partial_k u_k$ is the divergence of u and u_0 is the value of u at time $t = 0$ (with assumption $\operatorname{div} u_0 = 0$). The system (1.1) (going back

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to Euler [**4**]) describes a fluid motion without friction. The first equation in (1.1) reflects the conservation of momentum. The second equation in (1.1) says that the fluid motion is incompressible, i.e. that the volume of any fluid portion remains constant during the flow.
The unknowns in (1.1) are u and p. But one can express ∇p in terms of u – see Inci [**11**]. Thus the evolution of system (1.1) is completely described by u.

To state the result of this paper we have to introduce some notation. For $s \in \mathbb{R}_{\geq 0}$ we denote by $H^s(\mathbb{R}^n)$ the Hilbert space of real valued functions on $\mathbb{R}^n$ of Sobolev class s, by $H^s(\mathbb{R}^n;\mathbb{R}^n)$ the vector fields on $\mathbb{R}^n$ of Sobolev class s and by $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n) \subseteq H^s(\mathbb{R}^n;\mathbb{R}^n)$ the closed subspace consisting of divergence-free vector fields – see Adams [**1**] or Inci, Topalov, Kappeler [**9**] for details on Sobolev spaces. In particular we will often need the fact that for $n \geq 1$, $s > n/2$ and $0 \leq s' \leq s$ multiplication

$$H^s(\mathbb{R}^n) \times H^{s'}(\mathbb{R}^n) \to H^{s'}(\mathbb{R}^n), \quad (f,g) \mapsto f \cdot g \tag{1.2}$$

is a continuous bilinear map.
The notion of solution for (1.1) we are interested in are solutions which lie in $C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$ for some $T > 0$ and $s > n/2+1$. This is the space of continuous curves on $[0,T]$ with values in $H^s(\mathbb{R}^n;\mathbb{R}^n)$. To be precise we say that $u, \nabla p \in C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$ is a solution to (1.1) if

$$u(t) = u_0 + \int_0^t -(u(\tau)\cdot\nabla)u(\tau) - \nabla p(\tau)\,d\tau \quad \forall 0 \leq t \leq T \tag{1.3}$$

and $\operatorname{div} u(t) = 0$ for all $0 \leq t \leq T$ holds. As $s-1 > n/2$ we know by the Banach algebra property of $H^{s-1}(\mathbb{R}^n)$ that the integrand in (1.3) lies in $C^0\big([0,T];H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)\big)$. Due to the Sobolev imbedding and the fact $s > n/2+1$ the solutions considered here are C^1 (in the x-variable slightly better than C^1) and are thus solutions for which the derivatives appearing in (1.1) are classical derivatives. For this kind of solutions we have the following well-posedness result (it is here stated in a form which will be convenient later):

Theorem 1.1 (Kato [**12**]). Let $n \geq 2$, $s > n/2+1$ and $T > 0$. Then there is an open maximal (with respect to inclusion) neighborhood $U_T \subseteq H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ of 0 such that there is a unique solution $u \in C^0\big([0,T];H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)\big)$ of (1.1) for all $u_0 \in U_T$. Moreover the solution map

$$E_T : U_T \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n), \quad u_0 \mapsto u(T)$$

is continuous.

With this we can state the main results of this paper.

Theorem 1.2. Let $n \geq 2$, $s > n/2+1$ and $T > 0$. Then the solution map $E_T : U_T \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ is nowhere locally uniformly continuous.

Note that this means that E_T is *not* uniformly continuous on any open non-empty subset of U_T.

Corollary 1.3. The solution map E_T is nowhere locally Lipschitz.

Theorem 1.4. Let $n \geq 2$, $s > n/2+1$ and $T > 0$. Then the solution map $E_T : U_T \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ is nowhere differentiable.

Theorem 1.4 is not implied by Theorem 1.2. Indeed, for a continuous function $f : H \to \mathbb{R}$, $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space, which is nowhere locally uniformly continuous, the function $H \to \mathbb{R}$, $x \mapsto \langle x, x\rangle f(x)$ is still nowhere locally uniformly continuous, but differentiable in $x = 0$.

Related work: The question of the regularity of E_T was raised in Ebin, Marsden [**3**]. A first answer was given in Himonas, Misiolek [**8**]. Himonas and Misiolek construct a pair of sequences of solutions $(u_k)_{k\geq 1}$, $(\tilde{u}_k)_{k\geq 1}$ to (1.1) with the following property: For all $s > 0$

(i) $\big(u_k(0)\big)_{k\geq 1}$ and $\big(\tilde{u}_k(0)\big)_{k\geq 1}$ are bounded in $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\lim_{k\to\infty} ||u_k(0) - \tilde{u}_k(0)||_s = 0.$$

and there is a constant $C_s > 0$ so that

(ii) for all $0 < t < 1$

$$\liminf_{k\geq 1} ||u_k(t) - \tilde{u}_k(t)||_s \geq C_s \sin t.$$

This shows that E_T is not uniformly continuous on some bounded sets.
We should also mention the result in Kato [**13**], for the inviscid Burgers' equation

$$\partial_t u + u\partial_x u = 0,\ u(0) = u_0, \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{1.4}$$

Kato proves that for no $0 < \alpha \leq 1$ and no $t > 0$ the solution map of equation (1.4), $u_0 \mapsto u(t)$, is locally α-Hölder continuous in the Sobolev space $H^s(\mathbb{R})$, $s \geq 2$.

This paper is more or less an excerpt from the thesis Inci [**10**]. So omitted proofs or references where they can be found are given in Inci [**10**].

2. Lagrangian description

Consider now a fluid motion determined by u. If one fixes a fluid particle which at time $t = 0$ is located at $x \in \mathbb{R}^n$ and whose position at time $t \geq 0$ we denote by $\varphi(t,x) \in \mathbb{R}^n$, we get the following relation between u and φ

$$\partial_t\varphi(t,x) = u\big(t, \varphi(t,x)\big),$$

i.e. φ is the flow-map of the vectorfield u. The second equation in (1.1) translates to the well-known relation $\det(d\varphi) \equiv 1$, where $d\varphi$ is the Jacobian of φ – see Majda, Bertozzi [**15**]. In this way we get a description of system (1.1) in terms of φ. The description of (1.1) in the φ-variable is called the Lagrangian description of (1.1), whereas the description in the u-variable is called the Eulerian description of (1.1). One advantage of the Lagrangian description of (1.1) is that it leads to an ODE formulation of (1.1). This was already used in Lichtenstein [**14**] and Gunter [**6**] to get local well-posedness of (1.1).

The discussion in Section 1 shows that in this paper the state-space of (1.1) in the Eulerian description is $H^s(\mathbb{R}^n;\mathbb{R}^n)$, $s > n/2+1$. The state-space of (1.1) in the Lagrangian description is given by

$$\mathcal{D}^s(\mathbb{R}^n) = \big\{\varphi : \mathbb{R}^n \to \mathbb{R}^n \;\big|\; \varphi - \mathrm{id} \in H^s(\mathbb{R}^n;\mathbb{R}^n) \text{ and } \det d_x\varphi > 0,\ \forall x \in \mathbb{R}^n\big\}$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Due to the Sobolev imbedding and the condition $s > n/2+1$ the space of maps $\mathcal{D}^s(\mathbb{R}^n)$ consists of C^1-diffeomorphisms – see Palais [**16**] – and can be identified via $\mathcal{D}^s(\mathbb{R}^n) - \mathrm{id} \subseteq H^s(\mathbb{R}^n;\mathbb{R}^n)$ with an open subset of $H^s(\mathbb{R}^n;\mathbb{R}^n)$. Thus $\mathcal{D}^s(\mathbb{R}^n)$ has naturally a real analytic differential structure (for real analyticity we refer to Whittlesey [**17**]) with the natural identification of the tangent space

$$T\mathcal{D}^s(\mathbb{R}^n) \simeq \mathcal{D}^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n;\mathbb{R}^n).$$

Moreover it is known that $\mathcal{D}^s(\mathbb{R}^n)$ is a topological group under composition and that for $0 \leq s' \leq s$ the composition map

$$H^{s'}(\mathbb{R}^n) \times \mathcal{D}^s(\mathbb{R}^n) \to H^{s'}(\mathbb{R}^n), \quad (f,\varphi) \mapsto f \circ \varphi \tag{2.1}$$

is continuous – see Cantor [**2**] and Inci, Topalov, Kappeler [**9**]. That $\mathcal{D}^s(\mathbb{R}^n)$ is the right choice as configuration space for (1.1) in Lagrangian coordinates is justified by the fact that every $u \in C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$, $s > n/2+1$, integrates uniquely to a $\varphi \in C^1\big([0,T];\mathcal{D}^s(\mathbb{R}^n)\big)$ fullfilling

$$\partial_t \varphi(t) = u(t) \circ \varphi(t) \quad \text{for all } 0 \leq t \leq T$$

– see Fischer, Marsden [**5**] or Inci [**10**] for an alternative proof.
It turns out that one can describe system (1.1) in Lagrangian coordinates by a map, which we call the exponential map associated to (1.1). More precisely (see Inci [**11**] for the proof)

Proposition 2.1. Let $n \geq 2$ and $s > n/2+1$. Then there is an open neighborhood $U \subseteq H^s(\mathbb{R}^n;\mathbb{R}^n)$ of 0 and a real analytic map (called the exponential map associated to (1.1))

$$\exp : U \to \mathcal{D}^s(\mathbb{R}^n)$$

with the following property: For $T > 0$ let $u \in C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$ be a solution to (1.1) for some $u_0 \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ with the corresponding flow $\varphi \in C^1\big([0,T];\mathcal{D}^s(\mathbb{R}^n)\big)$ solving $\partial_t\varphi(t) = u(t)\circ\varphi(t)$ for any $0 \leq t \leq T$. Then we have

$$\varphi(t) = \exp(tu_0) \quad \forall 0 \leq t \leq T.$$

Note that we have $U \cap H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n) = U_T|_{T=1}$.

3. Vorticity

A key ingredient for the proof of Theorem 1.2 and Theorem 1.4 will be the vorticity – see Bertozzi, Majda [**15**] and Inci [**10**] for missing proofs.

Definition 3.1. Let $u = (u_1,\ldots,u_n)$ be a C^1-vector field on $\mathbb{R}^n$. Then the antisymmetric matrix

$$\Omega(u) := (\Omega_{ij})_{1\leq i,j\leq n} := (\partial_i u_j - \partial_j u_i)_{1\leq i,j\leq n}$$

is called the vorticity of u.

One can recover a divergence-free vector field from its vorticity by the Biot-Savart law.

Lemma 3.1. For $s > n/2+1$ let u be a H^s-vector field with $\operatorname{div} u = 0$ and compactly supported vorticity $\Omega := \Omega(u)$. Then we have

$$u(x) = \frac{1}{\omega_n}\int_{\mathbb{R}^n} \Omega(y) \cdot \frac{x-y}{|x-y|^n}\,dy$$

for any $x \in \mathbb{R}^n$. Here integration is done componentwise and ω_n denotes the surface area of a unit sphere in $\mathbb{R}^n$.

Recall that for $u \in H^s(\mathbb{R}^n;\mathbb{R}^n)$, $s \geq 0$, we use the norm $||\cdot||_s$ given by

$$||u||_s^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left(|\hat{u}_1(\xi)|^2 + \ldots + |\hat{u}_n(\xi)|^2\right) \, d\xi$$

where $\hat{f}$ denotes the Fourier transform of a function f. In the same way we define the norm of a matrix valued map. One than has

Lemma 3.2. Let $s > n/2+1$. Then there is a constant $C > 0$ such that we have

$$||du||_{s-1} \leq C||\Omega(u)||_{s-1}, \quad \forall u \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$$

where du denotes the Jacobian matrix of u.

A very important property of the vorticity is the following conservation law (an immediate consequence of the vorticity equation – see Inci [**10**]):

Lemma 3.3. Let $n \geq 2$ and $s > n/2+1$. Let further $u \in C^0\big([0,T]; H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$, $T > 0$, be a solution of (1.1) with $u(0) = u_0 \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$. We define

$$\Omega(t) := \Omega\big(u(t)\big) \text{ and } \varphi(t) := \exp(tu_0), \quad 0 \leq t \leq T.$$

Then we have for any $0 \leq t \leq T$

$$d\varphi(t)^\top \cdot \Omega(t) \circ \varphi(t) \cdot d\varphi(t) = \Omega(0)$$

or

$$\Omega(t) = R^{-1}_{\varphi(t)} \left((d\varphi(t)^{-1})^\top \cdot \Omega(0) \cdot d\varphi(t)^{-1}\right) \tag{3.1}$$

where R_φ denotes the map $f \mapsto f \circ \varphi$.

Note that from (3.1) we conclude that the support of the vorticity $\Omega(t)$ remains compact if $\Omega(0)$ is compact. We have the following estimate for expressions of the form (3.1).

Lemma 3.4. Let $s > n/2+1$ and $\varphi_\bullet \in \mathcal{D}^s(\mathbb{R}^n)$. Then there is $C > 0$ and a neighborhood $U \subseteq \mathcal{D}^s(\mathbb{R}^n)$ of $\varphi_\bullet$ such that

$$\frac{1}{C}||f||_{s-1} \leq ||R_\varphi^{-1}\left((d\varphi^{-1})^\top \cdot f \cdot d\varphi^{-1}\right)||_{s-1} \leq C||f||_{s-1}$$

for any $f \in H^{s-1}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ and any $\varphi \in U$.

4. Proof of Theorem 1.2 and Theorem 1.4

Before we prove the theorems, we have to make some preparation. Throughout this section we assume $n \geq 2$ and $s > n/2+1$.
First of all we can reduce the proofs to the case $T = 1$. This follows from the scaling property of (1.1). In fact, denoting $\phi = E_T|_{T=1}$ we have

$$E_T(u_0) = T^{-1}\phi(Tu_0), \quad \forall T > 0. \tag{4.1}$$

So the proof of Theorem 1.2 reduces to

Proposition 4.1. Denote by ϕ the map $E_T|_{T=1}$ and by U the domain $U_T|_{T=1}$. Then

$$\phi : U \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$$

is nowhere locally uniformly continuous.

Proof of Theorem 1.2. Follows from Proposition 4.1 and (4.1). □

In the sequel we use $C^\infty_{c,\sigma}(\mathbb{R}^n;\mathbb{R}^n)$ for the space of smooth and divergence-free vector fields with compact support, i.e.

$$C^\infty_{c,\sigma}(\mathbb{R}^n;\mathbb{R}^n) = \big\{ f \in C^\infty(\mathbb{R}^n;\mathbb{R}^n) \;\big|\; \operatorname{div} f = 0 \text{ and } \operatorname{supp} f \text{ compact}\big\}$$

where $\operatorname{supp} f$ denotes the support of f. Note that $C^\infty_{c,\sigma}(\mathbb{R}^n;\mathbb{R}^n) \subseteq H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ is dense – see Inci [**10**].

Lemma 4.2. Let $\varphi \in C^0\big([0,1];\mathcal{D}^s(\mathbb{R}^n)\big)$. Then for any $\varepsilon > 0$ there is $R > 0$ such that

$$|\varphi(t,y) - y| < \varepsilon \quad \text{and} \quad |d\varphi(t,y) - I_n| < \varepsilon$$

for any $t \in [0,1]$ and for any $y \in \mathbb{R}^n$ with $|y| \geq R$. Here $|\cdot|$ denotes the euclidean norm and I_n the $n \times n$ identity matrix.

Proof of Lemma 4.2. Note that by the Sobolev imbedding, there exists $C > 0$ such that for any $f \in H^s(\mathbb{R}^n;\mathbb{R}^n)$,

$$||f||_{C^1} \leq C||f||_s. \tag{4.2}$$

As for any element $\varphi \in \mathcal{D}^s(\mathbb{R}^n)$, $\varphi(y) - y$ is in $H^s(\mathbb{R}^n;\mathbb{R}^n)$, it then follows that for any $t_0 \in [0,1]$ there is $R_{t_0} > 0$ such that

$$|\varphi(t_0,y) - y| < \varepsilon/2 \quad \text{and} \quad |d\varphi(t_0,y) - I_n| < \varepsilon/2$$

for all $|y| \geq R_{t_0}$. Choose $\delta > 0$ in such a way that for any t in $(t_0 - \delta, t_0 + \delta) \cap [0,1]$,

$$||\varphi(t) - \varphi(t_0)||_s < \frac{\varepsilon}{2C}$$

where $C > 0$ is the imbedding constant in (4.2). Then

$$|\varphi(t,y) - y| < \varepsilon \quad \text{and} \quad |d\varphi(t,y) - I_n| < \varepsilon$$

for any $t \in (t_0 - \delta, t_0 + \delta) \cap [0,1]$ and $|y| \geq R_{t_0}$. Since we can cover $[0,1]$ with finitely many of such intervals we get the claim. $\square$

Lemma 4.3. Let $U \equiv U_T|_{T=1}$ and $u_0 \in U \cap C^\infty_{c,\sigma}(\mathbb{R}^n;\mathbb{R}^n)$. Consider the restriction of the differential of exp at u_0 to $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$,

$$d_{u_0}\exp : H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n) \to H^s(\mathbb{R}^n;\mathbb{R}^n), \quad v_0 \mapsto \partial_\varepsilon|_{\varepsilon=0}\exp(u_0 + \varepsilon v_0).$$

Then there exists $m > 0$ with the following property: For any $R > 0$ there exists $v \in C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n)$ with $|\big(d_{u_0}\exp(v)\big)(x^*)| \geq m$, $||v||_s = 1$ and support in the ball $B_1(x^*) = \{x \in \mathbb{R}^n \mid |x - x^*| < 1\}$ for some $x^* \in \mathbb{R}^n$ with $|x^*| \geq R$.

Proof of Lemma 4.3. Take $w \in C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n)$ with support in the ball $B_1(0)$ and with the properties $||w||_s = 1$ and $w(0) \neq 0$. Choose $\delta > 0$ so that $u_0 + \varepsilon w(\cdot - x^*) \in U$ for any $|\varepsilon| \leq \delta$ and any $x^* \in \mathbb{R}^n$. Now define $w_{x^*} := w(\cdot - x^*)$ where x^* will be conveniently chosen at the end of the proof. For $\varepsilon \in [-\delta,\delta]$ we denote by $u^{(\varepsilon)}$ the solution of (1.3) with initial data $u_0 + \varepsilon w_{x^*}$ and by $\varphi^{(\varepsilon)}$ the corresponding flow. By Proposition 2.1, for $(\varepsilon,t) \in [-\delta,\delta] \times [0,1]$, $\varphi^{(\varepsilon)}(t)$ is given by

$$\varphi^{(\varepsilon)}(t) = \exp\big(t(u_0 + \varepsilon w_{x^*})\big). \tag{4.3}$$

By the same Proposition

$$\varphi^{(\cdot)}(\cdot) : [-\delta,\delta] \times [0,1] \to \mathcal{D}^s(\mathbb{R}^n) \tag{4.4}$$

is C^1. Hence, denoting by I_n the $n \times n$-identity matrix,

$$d\varphi^{(\cdot)}(\cdot) - I_n : [-\delta,\delta] \times [0,1] \to H^{s-1}(\mathbb{R}^n;\mathbb{R}^{n\times n}) \tag{4.5}$$

is C^1 as well. By Lemma 3.3, the vorticity $\Omega\big(u^\varepsilon(t)\big)$ has compact support. Hence by Lemma 3.1 (Biot-Savart law) we have for all $x \in \mathbb{R}^n$ and $t \in [0,1]$, $|\varepsilon| \le \delta$

$$u^{(\varepsilon)}(t,x) = \frac{1}{\omega_n}\int_{\mathbb{R}^n} \Omega\big(u^{(\varepsilon)}(t)\big)(y)\frac{x-y}{|x-y|^n}\,dy.$$

Now using the conservation law (3.1) we get

$$u^{(\varepsilon)}(t,x) = \frac{1}{\omega_n}\int_{\mathbb{R}^n} R^{-1}_{\varphi^{(\varepsilon)}(t,y)}\Big([d\varphi^{(\varepsilon)}(t,\cdot)^\top]^{-1}\big(\Omega(u_0)+\varepsilon\Omega(w_{x^*})\big)(\cdot)[d\varphi^{(\varepsilon)}(t,\cdot)]^{-1}\Big)\frac{x-y}{|x-y|^n}\,dy.$$

Using that $\varphi^{(\varepsilon)}(t)$ is volume-preserving we have

$$u^{(\varepsilon)}(t,x) = \frac{1}{\omega_n}\int_{\mathbb{R}^n} [d\varphi^{(\varepsilon)}(t,y)^\top]^{-1}\big(\Omega(u_0)+\varepsilon\Omega(w_{x^*})\big)(y)[d\varphi^{(\varepsilon)}(t,y)]^{-1}\frac{x-\varphi^{(\varepsilon)}(t,y)}{|x-\varphi^{(\varepsilon)}(t,y)|^n}\,dy.$$

The relation $\varphi^{(\varepsilon)}(1) = \mathrm{id} + \int_0^1 u^{(\varepsilon)}(t)\circ\varphi^{(\varepsilon)}(t)\,dt$ then leads to

$$\varphi^{(\varepsilon)}(1,x) = x + I^{(\varepsilon)}(x) \tag{4.6}$$

where $I^{(\varepsilon)}(x)$ is given by

$$\frac{1}{\omega_n}\int_0^1\int_{\mathbb{R}^n} [d\varphi^{(\varepsilon)}(t,y)^\top]^{-1}\big(\Omega(u_0)+\varepsilon\Omega(w_{x^*})\big)(y)[d\varphi^{(\varepsilon)}(t,y)]^{-1}\frac{\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)}{|\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)|^n}\,dydt.$$

Write $I^{(\varepsilon)}(x)$ as the sum $I_1^{(\varepsilon)}(x) + \varepsilon I_2^{(\varepsilon)}(x)$, where $I_1^{(\varepsilon)}(x)$ is defined by

$$\frac{1}{\omega_n}\int_0^1\int_{y\in\operatorname{supp}u_0} [d\varphi^{(\varepsilon)}(t,y)^\top]^{-1}\Omega(u_0)(y)[d\varphi^{(\varepsilon)}(t,y)]^{-1}\frac{\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)}{|\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)|^n}\,dydt$$

and

$$I_2^{(\varepsilon)}(x) := \frac{1}{\omega_n}\int_0^1\int_{\mathbb{R}^n} [d\varphi^{(\varepsilon)}(t,y)^\top]^{-1}\Omega(w_{x^*})(y)[d\varphi^{(\varepsilon)}(t,y)]^{-1}\frac{\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)}{|\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)|^n}\,dydt.$$

Next we need to get an expression for

$$\big(d_{u_0}\exp(w_{x^*})\big)(x) = \partial_\varepsilon|_{\varepsilon=0}\,\varphi^{(\varepsilon)}(1,x).$$

This will be accomplished by taking the ε-derivative of the right-hand side of (4.6). But first we have to make some preparations.

Consider the curves $\varphi := \varphi^{(\varepsilon)}\big|_{\varepsilon=0} : [0,1] \to \mathcal{D}^s(\mathbb{R}^n)$ and $\varphi^{-1} : [0,1] \to \mathcal{D}^s(\mathbb{R}^n)$. As by Proposition 2.1, $U \to C^0\big([0,1];\mathcal{D}^s(\mathbb{R}^n)\big)$, $v_0 \to [t \mapsto \exp(tv_0)]$ is continuous, there exist $Q > 0$ and $\delta' > 0$ so that for any $\tilde u_0$ in the δ'-ball $B_{\delta'}(u_0)$ in $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ centered at u_0 one has

$$\max_{0\le t\le 1} ||\exp(t\tilde u_0) - \mathrm{id}\,||_s < Q \quad \text{and} \quad \max_{0\le t\le 1} ||(\exp(t\tilde u_0)^{-1} - \mathrm{id}\,||_s < Q.$$

As $||\varepsilon w_{x^*}||_s = \varepsilon$ for any $x^* \in \mathbb{R}^n$, it follows that $u_0 + \varepsilon w_{x^*} \in B_\varepsilon(u_0)$ for all $|\varepsilon| \le \delta$ and hence, by choosing δ smaller if necessary, so that $\delta \le \delta'$, one has

$$\sup_{\substack{0\le t\le 1\\ x^*\in\mathbb{R}^n\\ |\varepsilon|\le\delta}} ||\varphi^{(\varepsilon)}(t) - \mathrm{id}\,||_s < Q, \qquad \sup_{\substack{0\le t\le 1\\ x^*\in\mathbb{R}^n\\ |\varepsilon|\le\delta}} ||\big(\varphi^{(\varepsilon)}(t)\big)^{-1} - \mathrm{id}\,||_s < Q. \tag{4.7}$$

By (4.7) and the Sobolev imbedding (4.2) there exists a constant $M > 0$ with

$$|\varphi^{(\varepsilon)}(t,x) - x|,\ |d\varphi^{(\varepsilon)}(t,x)| < M \tag{4.8}$$

for any $(\varepsilon,t) \in [-\delta,\delta]\times[0,1]$, $x \in \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$. For $N \ge 1$ choose $R_N > 0$ so large that for any $z \in \mathbb{R}^n$ with $|z| \ge R_N$

$$d(z,\operatorname{supp}u_0) := \inf_{y\in\operatorname{supp}u_0} |z-y| > (N+2)M.$$

For any $y \in \operatorname{supp} u_0$ and any $x^* \in \mathbb{R}^n$ with $|x^*| \geq R_N$ we then get

$$
\begin{aligned}
(4.9)\quad |\varphi^{(\varepsilon)}(t,x^*) - \varphi^{(\varepsilon)}(t,y)| &= |(\varphi^{(\varepsilon)}(t,x^*) - x^*) + (x^* - y) + (y - \varphi^{(\varepsilon)}(t,y))| \\
&\geq |x^* - y| - |\varphi^{(\varepsilon)}(t,x^*) - x^*| - |y - \varphi^{(\varepsilon)}(t,y)| \geq NM.
\end{aligned}
$$

It then follows from (4.4)-(4.5) that by the Leibniz rule, $\partial_\varepsilon|_{\varepsilon=0} I_1^{(\varepsilon)}(x^*)$, $|x^*| \geq R_N$, can be computed by differentiating the integrand of $I_1^{(\varepsilon)}$ with respect to ε. Before doing this introduce for $0 \leq t \leq 1$ and $y \in \mathbb{R}^n$

$$J(t,y) := \partial_\varepsilon|_{\varepsilon=0}\, \varphi^{(\varepsilon)}(t,y) = \big(d_{tu_0} \exp(tw_{x^*})\big)(y).$$

In particular we then have by (4.5)

$$\partial_\varepsilon|_{\varepsilon=0}\, d\varphi^{(\varepsilon)}(t,y) = dJ(t,y) \tag{4.10}$$

and thus by the standard formula for the derivative of the inverse of a matrix

$$\partial_\varepsilon|_{\varepsilon=0} \big(d\varphi^{(\varepsilon)}(t,y)\big)^{-1} = -\big(d\varphi(t,y)\big)^{-1} dJ(t,y) \big(d\varphi(t,y)\big)^{-1}. \tag{4.11}$$

Note that for any $0 \leq t \leq 1$,

$$||J(t)||_s \leq ||d_{tu_0} \exp||\, ||tw_{x^*}||_s \leq \max_{0 \leq t \leq 1} ||d_{tu_0} \exp||$$

where $||d_{tu_0} \exp||$ is the operator norm of $d_{tu_0} \exp : H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n) \to H^s(\mathbb{R}^n;\mathbb{R}^n)$ and hence by the Sobolev imbedding (4.2), $J(t,x)$ and $dJ(t,x)$ are uniformly bounded with respect to $0 \leq t \leq 1$, $x \in \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$. For convenience let $\Delta_{x,y}\varphi(t) = \varphi(t,x) - \varphi(t,y)$. Now taking the derivative of the integrand of $I_1^{(\varepsilon)}(x^*)$ with respect to ε we get by the product rule

$$J_1(x^*) := \partial_\varepsilon|_{\varepsilon=0}\, I_1^{(\varepsilon)}(x^*) = J_{1,1}(x^*) + J_{1,2}(x^*) + J_{1,3}(x^*)$$

where $J_{1,1}(x^*)$, $J_{1,2}(x^*)$ and $J_{1,3}(x^*)$ are given by, respectively,

$$\frac{1}{\omega_n} \int_0^1 \int_{\operatorname{supp} u_0} -[d\varphi(t,y)^\top]^{-1} dJ(t,y)^\top [d\varphi(t,y)^\top]^{-1} \Omega(u_0)(y) [d\varphi(t,y)]^{-1} \frac{\Delta_{x^*,y}\varphi(t)}{|\Delta_{x^*,y}\varphi(t)|^n}\, dy\, dt,$$

$$\frac{1}{\omega_n} \int_0^1 \int_{\operatorname{supp} u_0} -[d\varphi(t,y)^\top]^{-1} \Omega(u_0)(y) [d\varphi(t,y)]^{-1} dJ(t,y) [d\varphi(t,y)]^{-1} \frac{\Delta_{x^*,y}\varphi(t)}{|\Delta_{x^*,y}\varphi(t)|^n}\, dy\, dt,$$

$$\frac{1}{\omega_n} \int_0^1 \int_{\operatorname{supp} u_0} [d\varphi(t,y)^\top]^{-1} \Omega(u_0)(y) [d\varphi(t,y)]^{-1} \cdot \left\{ \frac{\Delta_{x^*,y} J(t)}{|\Delta_{x^*,y}\varphi(t)|^n} - n \frac{\Delta_{x^*,y}\varphi(t) \cdot \Delta_{x^*,y} J(t) \Delta_{x^*,y}\varphi(t)}{|\Delta_{x^*,y}\varphi(t)|^{n+2}} \right\} dy\, dt$$

Note that the domain of integration $[0,1] \times \operatorname{supp} u_0$ of the integrals $J_{1,1}(x^*)$, $J_{1,2}(x^*)$ and $J_{1,3}(x^*)$ is compact. The integrands are uniformly bounded, independent of the choice of x^* with $|x^*| \geq R_N$. Moreover from (4.9) we see that the denominators go to infinity as $|x^*| \to \infty$. Thus for any $\rho > 0$ there exists $R'_\rho \geq R_N$ such that

$$|J_1(x^*)| < \rho \quad \forall x^* \in \mathbb{R}^n \text{ with } |x^*| \geq R'_\rho. \tag{4.12}$$

Now consider $I_2^{(\varepsilon)}(x^*)$. As a consequence of (4.7) and the Sobolev imbedding (4.2),

$$\sup_{\substack{0\le t\le 1\\ x^*\in\mathbb{R}^n\\ |\varepsilon|\le\delta}} ||\varphi^{(\varepsilon)}(t)-\mathrm{id}\,||_{C^1} \quad\text{and}\quad \sup_{\substack{0\le t\le 1\\ x^*\in\mathbb{R}^n\\ |\varepsilon|\le\delta}} ||\big(\varphi^{(\varepsilon)}(t)\big)^{-1}-\mathrm{id}\,||_{C^1}$$

are finite. Hence there is $L>0$, independent of x^*, with

$$\frac{1}{L}|x-y|\le|\varphi^{(\varepsilon)}(t,x)-\varphi^{(\varepsilon)}(t,y)|\le L|x-y| \qquad \forall x,y\in\mathbb{R}^n \tag{4.13}$$

for any $(\varepsilon,t)\in[-\delta,\delta]\times[0,1]$. When combined with (4.8) we get that the integrand of $I_2^{(\varepsilon)}(x^*)$ can be estimated uniformly for $|\varepsilon|\le\delta$ and $0\le t\le 1$ by

$$C\frac{L|x^*-y|}{(\frac{1}{L}|x^*-y|)^n}=CL^{n+1}\frac{1}{|x^*-y|^{n-1}} \tag{4.14}$$

for some constant $C>0$. As the latter bound is independent of ε one can take the limit $\varepsilon\to 0$ under the integral to get

$$\begin{aligned} J_2(x^*) &:= \partial_\varepsilon|_{\varepsilon=0}\,(\varepsilon I_2^{(\varepsilon)}(x^*))\\ &= \frac{1}{\omega_n}\int_0^1\int_{B_1(x^*)}[d\varphi(t,y)^\top]^{-1}\Omega(w_{x^*})(y)[d\varphi(t,y)]^{-1}\frac{\Delta_{x^*,y}\varphi(t)}{|\Delta_{x^*,y}\varphi(t)|^n}\,dy\,dt. \end{aligned}$$

Now the key idea is to show that the latter expression differs by a small error from

$$\frac{1}{\omega_n}\int_0^1\int_{B_1(x^*)}\Omega(w_{x^*})(y)\frac{x^*-y}{|x^*-y|^n}\,dy\,dt$$

which by the Biot-Savart law equals $w_{x^*}(x^*)=w(0)$ and hence does not vanish. To prove that the difference of $J_2(x^*)$ with the latter integral is indeed small, write $B_1(x^*)$ as the union of $B_1(x^*)\setminus B_\theta(x^*)$ and $B_\theta(x^*)$ with $0<\theta<1$ to be chosen at the end of the proof and write $J_2(x^*)$ as a sum of the corresponding integrals $J_2(x^*)=J_{2,1}^{(\theta)}(x^*)+J_{2,2}^{(\theta)}(x^*)$. First note that by the Sobolev imbedding (4.2) and the condition $s>n/2+1$, for any $y\in\mathbb{R}^n$ and for any $x^*\in\mathbb{R}^n$

$$|\Omega(w_{x^*})(y)|\le C||w_{x^*}||_s=C. \tag{4.15}$$

Using (4.14)-(4.15) we get

$$|J_{2,2}^{(\theta)}(x^*)|\le C'\int_0^1\int_{B_\theta(x^*)}\frac{1}{|x^*-y|^{n-1}}\,dy\,dt\le C''\theta \tag{4.16}$$

for some constants C',C'' independent of x^*. Note that the denominator of the integrand of $J_{2,1}^{(\theta)}(x^*)$ is bounded away from 0. Indeed, by (4.13)

$$|\Delta_{x^*,y}\varphi(t)|=|\varphi(t,x^*)-\varphi(t,y)|\ge\frac{1}{L}|x^*-y|\ge\frac{\theta}{L}$$

for all $y\in B_1(x^*)\setminus B_\theta(x^*)$. So by Lemma 4.2 and (4.15), for any fixed $0<\theta<1$, and any $\rho>0$ there exists a constant $R_\rho^{(\theta)}>0$ such that for any $x^*\in\mathbb{R}^n$ with $|x^*|\ge R_\rho^{(\theta)}$

$$\left|J_{2,1}^{(\theta)}(x^*)-\frac{1}{\omega_n}\int_{B_1(x^*)\setminus B_\theta(x^*)}\Omega(w_{x^*})(y)\frac{x^*-y}{|x^*-y|^n}\,dy\right|<\rho. \tag{4.17}$$

We now choose x^* and $0<\theta<1$ according to our needs. Write

$$w_{x^*}(x^*)=\frac{1}{\omega_n}\int_{B_1(x^*)}\Omega(w_{x^*})(y)\frac{x^*-y}{|x^*-y|^n}\,dy=w_1^{(\theta)}(x^*)+w_2^{(\theta)}(x^*)$$

where

$$w_1^{(\theta)}(x^*)=\frac{1}{\omega_n}\int_{B_1(x^*)\setminus B_\theta(x^*)}\Omega(w_{x^*})(y)\frac{x^*-y}{|x^*-y|^n}\,dy$$

and

$$w_2^{(\theta)}(x^*)=\frac{1}{\omega_n}\int_{B_\theta(x^*)}\Omega(w_{x^*})(y)\frac{x^*-y}{|x^*-y|^n}\,dy.$$

First choose $0<\theta<1$ in such a way that we have for any choice of $x^*\in\mathbb{R}^n$

$$|J_{2,2}^{(\theta)}(x^*)|<a/8 \quad\text{and}\quad |w_2^{(\theta)}(x^*)|<a/8 \tag{4.18}$$

where $a=|w(0)|$. Due to (4.16) this is possible. Then for any $R>0$ choose $x^*\in\mathbb{R}^n$ with $|x^*|\geq\max(R'_{a/8},R_{a/8}^{(\theta)},R)$ but otherwise arbitrary and let $v:=w_{x^*}$. Then by (4.12),(4.17) and (4.18)

$$\begin{aligned}|\big(d_{u_0}\exp(v)\big)(x^*)-w_{x^*}(x^*)|&=|J_1(x^*)+J_2(x^*)-w_{x^*}(x^*)|\\&=|J_1(x^*)+J_{2,1}^{(\theta)}(x^*)+J_{2,2}^{(\theta)}(x^*)-w_1^{(\theta)}(x^*)-w_2^{(\theta)}(x^*)|\\&\leq|J_1(x^*)|+|J_{2,1}^{(\theta)}(x^*)-w_1^{(\theta)}(x^*)|+|J_{2,2}^{(\theta)}(x^*)|+|w_2^{(\theta)}(x^*)|\leq a/8+a/8+a/8+a/8=a/2.\end{aligned}$$

Thus we see that $|\big(d_{u_0}\exp(v)\big)(x^*)|\geq a/2$ showing the claim with the choice $m=a/2$. $\square$

For $f_\bullet$ in $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ we denote by $B_R(f_\bullet)\subseteq H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ the open ball of radius $R>0$ with center $f_\bullet$, i.e.

$$B_R(f_\bullet)=\{f\in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)\mid ||f-f_\bullet||_s<R\}.$$

Now we can give the proof of Proposition 4.1, copied from Inci [**10**].

Proof of Proposition 4.1. It suffices to show that for any u_0 in the domain $U\subseteq H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ of ϕ there exists $R_*>0$ with $B_{R_*}(u_0)\subseteq U$ so that ϕ is not uniformly continuous on $B_R(u_0)$ for any $0<R\leq R_*$. As $s>n/2+1$, $H^s(\mathbb{R}^n;\mathbb{R}^n)\hookrightarrow C^1_0(\mathbb{R}^n;\mathbb{R}^n)$. We denote by $C>0$ the constant of this imbedding

$$||f||_{C^1}\leq C||f||_s. \tag{4.19}$$

By the continuity of the exponential map (Proposition 2.1), there exists $R_0>0$ so that $B_{R_0}(u_0)\subseteq U$ and for any $\varphi,\psi\in\exp\big(B_{R_0}(u_0)\big)$

$$||\varphi-\psi||_s<\frac{1}{C}.$$

Hence by (4.19) there is a constant $L>0$ so that for any $\varphi,\psi\in\exp\big(B_{R_0}(u_0)\big)$

$$|\varphi(x)-\psi(x)|<1\quad\text{and}\quad|\varphi(x)-\varphi(y)|<L|x-y|,\quad\forall x,y\in\mathbb{R}^n. \tag{4.20}$$

By the smoothness of the exponential map (Proposition 2.1) and Taylor's theorem, for any $v,v+h$ in an arbitrary convex subset $V\subseteq U$,

$$\exp(v+h)=\exp(v)+d_v\exp(h)+\frac{1}{2}\int_0^1(1-t)d^2_{v+th}\exp(h,h)\,dt.$$

By choosing $0<R_1\leq R_0$, smaller if necessary, we can ensure that for some $C_1>0$ and $\forall v\in B_{R_1}(u_0),h\in B_{R_1}(0)$

$$||\exp(v+h) - \exp(v) - d_v \exp(h)||_s \leq C_1 ||h||_s^2. \tag{4.21}$$

As $v \mapsto d_v \exp$ is continuous we get for some $0 < R_2 \leq R_1$ and $\forall v_1, v_2 \in B_{R_2}(u_0), h \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$.

$$||d_{v_1} \exp(h) - d_{v_2} \exp(h)||_s \leq \frac{m}{4C} ||h||_s, \tag{4.22}$$

where $m > 0$ is the constant in the statement of Lemma 4.3 and $C > 0$ given by (4.19). Finally by choosing $0 < R_3 \leq R_2$, sufficiently small, Lemma 3.4 implies that there exists $C_2 > 0$ so that

$$\frac{1}{C_2} ||f||_{s-1} \leq ||R_\varphi^{-1} \left([d\varphi^\top]^{-1} f [d\varphi]^{-1} \right) ||_{s-1} \leq C_2 ||f||_{s-1} \tag{4.23}$$

for any $f \in H^{s-1}(\mathbb{R}^n;\mathbb{R}^{n\times n})$ and any $\varphi \in \exp\big(B_{R_3}(u_0)\big)$. Now set $R_* = R_3$ and take any $0 < R \leq R_*$. By the density of $C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n)$ in $H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$, there exists $\bar{u}_0 \in C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n) \cap B_{R/4}(u_0)$. Let $\varphi_\bullet := \exp(\bar{u}_0)$ and introduce $K := \operatorname{supp} \bar{u}_0$ and

$$K' = \{ y \in \mathbb{R}^n \mid \operatorname{dist}\big(y, \varphi_\bullet(K)\big) \leq 1 \}$$

where $\operatorname{dist}\big(y, \varphi_\bullet(K)\big) = \inf_{x\in K} |y - \varphi_\bullet(x)|$ is the distance of y to the set $\varphi_\bullet(K)$. By (4.20) we see that K' has the property

$$\varphi(K) \subseteq K', \qquad \forall \varphi \in \exp\big(B_R(u_0)\big) \tag{4.24}$$

Note that $\lim_{|x|\to\infty} |\varphi_\bullet(x)| = \infty$. By Lemma 4.3 we then can choose $x^* \in \mathbb{R}^n \setminus K'$ and $v \in C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n)$ with $||v||_s = 1$ in such a way that

$$\operatorname{dist}\big(\varphi_\bullet(x^*), K'\big) > L+1 \quad \text{and} \quad |\big(d_{\bar{u}_0} \exp(v)\big)(x^*)| \geq m. \tag{4.25}$$

We set $M := |\big(d_{\bar{u}_0}\exp(v)\big)(x^*)|$ and define $v_k = \frac{R}{4k} v$, $k \geq 1$. As $||v||_s = 1$

$$||v_k||_s = \frac{R}{4k} < R/3. \tag{4.26}$$

By the definition of v_k we have $|\big(d_{\bar{u}_0}\exp(v_k)\big)(x^*)| = \delta_k := M\frac{R}{4k}$. By (4.20) for any $k \geq 1$ there is

$$0 < \rho_k < \min(\delta_k/8, 1) = \min(\frac{MR}{32k}, 1) \tag{4.27}$$

such that

$$\varphi\big(B_{\rho_k}(x^*)\big) \subseteq B_{\delta_k/8}\big(\varphi(x^*)\big) \quad \forall \varphi \in \exp\big(B_R(u_0)\big). \tag{4.28}$$

Now choose for each $k \geq 1$, a $w_k \in C^\infty_{\sigma,c}(\mathbb{R}^n;\mathbb{R}^n)$ with

$$\operatorname{supp} w_k \subseteq B_{\rho_k}(x^*) \quad \text{and} \quad ||w_k||_s = R/4 \tag{4.29}$$

and define for $k \geq 1$ the pair of initial values

$$u_{0,k} = \bar{u}_0 + w_k \quad \text{and} \quad \tilde{u}_{0,k} = u_{0,k} + v_k.$$

By our choices $(u_{0,k})_{k\geq 1}, (\tilde{u}_{0,k})_{k\geq 1} \subseteq B_R(u_0)$ and $||u_{0,k} - \tilde{u}_{0,k}||_s = ||v_k||_s \to 0$ as $k \to \infty$. Denote the diffeomorphims corresponding to $u_{0,k}, \tilde{u}_{0,k}$ by $\varphi_k, \tilde{\varphi}_k \in \mathcal{D}^s(\mathbb{R}^n)$,

$$\varphi_k = \exp(u_{0,k}) \quad \text{and} \quad \tilde{\varphi}_k = \exp(\tilde{u}_{0,k})$$

and the solutions of (1.3) corresponding to the initial values $u_{0,k}, \tilde{u}_{0,k}$ by $u_k, \tilde{u}_k : [0,1] \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$. The corresponding vorticities at time $t=0$, $\Omega_{0,k}$ and $\tilde{\Omega}_{0,k}$, and $t=1$, $\Omega_{1,k}$ and $\tilde{\Omega}_{1,k}$, are then given by

$$\begin{array}{lllll} \Omega_{0,k} & = & \Omega(u_{0,k}) & = & \Omega(\bar{u}_0) + \Omega(w_k) \\ \tilde{\Omega}_{0,k} & = & \Omega_{0,k} + \Omega(v_k) & = & \Omega(\bar{u}_0) + \Omega(w_k + v_k) \end{array} \tag{4.30}$$

and

$$\Omega_{1,k} = \Omega\big(u_k(1)\big); \quad \tilde{\Omega}_{1,k} = \Omega\big(\tilde{u}_k(1)\big).$$

Note that we have for some $C' > 0$

$$||\phi(u_{0,k}) - \phi(\tilde{u}_{0,k})||_s = ||u_k(1) - \tilde{u}_k(1)||_s \geq \frac{1}{C'}||\Omega_{1,k} - \tilde{\Omega}_{1,k}||_{s-1}. \tag{4.31}$$

We aim at estimating $||\Omega_{1,k} - \tilde{\Omega}_{1,k}||_{s-1}$ from below. By the conservation law (3.1) we have

$$\Omega_{1,k} = R^{-1}_{\varphi_k}\left([d\varphi_k^\top]^{-1}\Omega_{0,k}[d\varphi_k]^{-1}\right), \; \tilde{\Omega}_{1,k} = R^{-1}_{\tilde{\varphi}_k}\left([d\tilde{\varphi}_k^\top]^{-1}\tilde{\Omega}_{0,k}[d\tilde{\varphi}_k]^{-1}\right). \tag{4.32}$$

By (4.25) the distance of $\varphi_\bullet(x^*)$ to K' is bigger than $L+1$ and hence by (4.20)

$$\operatorname{dist}\big(\varphi(x^*), K'\big) > L \quad \text{for any } \varphi \in \exp\big(B_R(u_0)\big).$$

On the other hand by (4.20) and $\rho_k < 1$ one has

$$|\varphi(x^*) - \varphi(x)| \leq L|x^* - x| \leq L \quad \forall x \in \operatorname{supp} w_k.$$

Combining the two latter displayed inequalities one concludes that

$$\varphi\big(\operatorname{supp}(w_k)\big) \cap K' = \emptyset, \quad \forall \varphi \in \exp\big(B_R(u_0)\big). \tag{4.33}$$

As $\operatorname{supp}(w_k + v_k) \subseteq B_1(x^*)$ the same argument gives

$$\varphi\big(\operatorname{supp}(w_k + v_k)\big) \cap K' = \emptyset, \quad \forall \varphi \in \exp\big(B_R(u_0)\big). \tag{4.34}$$

By (4.24),

$$\operatorname{supp} R^{-1}_{\varphi_k}\left((d\varphi_k^\top)^{-1}\Omega(\bar{u}_0)(d\varphi_k)^\top\right) \subseteq K'$$

and

$$\operatorname{supp} R^{-1}_{\tilde{\varphi}_k}\left((d\tilde{\varphi}_k^\top)^{-1}\Omega(\bar{u}_0)(d\tilde{\varphi}_k)^\top\right) \subseteq K'.$$

From (4.33)-(4.34),

$$\varphi_k\big(\operatorname{supp}\Omega(w_k)\big) \subseteq \mathbb{R}^n \setminus K' \quad \text{and} \quad \tilde{\varphi}_k\big(\operatorname{supp}\Omega(w_k + v_k)\big) \subseteq \mathbb{R}^n \setminus K'.$$

By (4.30)-(4.32) and Lemma A.1 it then follows for a constant $\tilde{C} > 0$ that

$$\begin{aligned} &||\Omega_{1,k} - \tilde{\Omega}_{1,k}||_{s-1} = \\ &\quad \tilde{C}(||R^{-1}_{\varphi_k}\left((d\varphi_k^\top)^{-1}\Omega(\bar{u}_0)(d\varphi_k)^{-1}\right) - R^{-1}_{\tilde{\varphi}_k}\left((d\tilde{\varphi}_k^\top)^{-1}\Omega(\bar{u}_0)(d\tilde{\varphi}_k)^{-1}\right)||_{s-1}) \\ &\quad + ||R^{-1}_{\varphi_k}\left((d\varphi_k^\top)^{-1}\Omega(w_k)(d\varphi_k)^{-1}\right) - R^{-1}_{\tilde{\varphi}_k}\left((d\tilde{\varphi}_k^\top)^{-1}\Omega(w_k+v_k)(d\tilde{\varphi}_k)^{-1}\right)||_{s-1} \\ &\quad \geq \tilde{C}||R^{-1}_{\varphi_k}\left((d\varphi_k^\top)^{-1}\Omega(w_k)(d\varphi_k)^{-1}\right) - R^{-1}_{\tilde{\varphi}_k}\left((d\tilde{\varphi}_k^\top)^{-1}\Omega(w_k+v_k)(d\tilde{\varphi}_k)^{-1}\right)||_{s-1} \end{aligned} \tag{4.35}$$

We claim that, for large k,

$$\varphi_k\big(\operatorname{supp}(w_k)\big) \cap \tilde{\varphi}_k\big(\operatorname{supp}(w_k)\big) = \emptyset. \tag{4.36}$$

Indeed by the Taylor formula

$$\tilde{\varphi}_k - \varphi_k = \exp(\bar{u}_0 + w_k + v_k) - \exp(\bar{u}_0 + w_k) = d_{\bar{u}_0 + w_k}\exp(v_k) + \mathcal{R}_k$$

where $\mathcal{R}_k$ is the remainder term. Thus we can write

$$\tilde{\varphi}_k - \varphi_k = d_{\bar{u}_0}\exp(v_k) + (d_{\bar{u}_0+w_k}\exp(v_k) - d_{\bar{u}_0}\exp(v_k)) + \mathcal{R}_k. \tag{4.37}$$

We want to estimate $\tilde{\varphi}(x^*)-\varphi(x^*)$ by estimating the three terms on the right-hand side of the latter identity individually. By the Sobolev imbedding (4.19) and (4.21) we get the following estimate for $\mathcal{R}_k(x^*)\in\mathbb{R}^n$

$$|\mathcal{R}_k(x^*)|\leq C||\mathcal{R}_k||_s\leq CC_1||v_k||_s^2=CC_1\frac{R^2}{16k^2}.$$

For k sufficiently large it then follows that

$$|\mathcal{R}_k(x^*)|<\frac{\delta_k}{4}.$$

Furthermore, using (4.19) and (4.22), together with $m\leq M$ (cf (4.25))

$$\begin{aligned}&\left|\big(d_{\bar{u}_0+w_k}\exp(v_k)\big)(x^*)-\big(d_{\bar{u}_0}\exp(v_k)\big)(x^*)\right|\\&\qquad\leq C||d_{\bar{u}_0+w_k}\exp(v_k)-d_{\bar{u}_0}\exp(v_k)||_s\leq\frac{m}{4}||v_k||_s\leq\frac{MR}{16k}=\frac{\delta_k}{4}.\end{aligned}$$

Finally, for the first term on the right-hand side of (4.37) one has by definition,

$$\left|d_{\bar{u}_0}\exp(v_k)(x^*)\right|=\delta_k.$$

Combining the estimates above, (4.37) yields for k large enough

$$|\tilde{\varphi}_k(x^*)-\varphi_k(x^*)|>\frac{\delta_k}{2}.$$

By (4.28) we get for large k

$$\varphi_k\big(B_{\rho_k}(x^*)\big)\cap\tilde{\varphi}_k\big(B_{\rho_k}(x^*)\big)=\emptyset$$

showing (4.36). It leads by the triangle inequality and Lemma A.3 for some constant $\tilde{C}>0$ to the estimate

$$\begin{aligned}&||R_{\varphi_k}^{-1}\left([d\varphi_k^\top]^{-1}\Omega(w_k)[d\varphi_k]^{-1}\right)-R_{\tilde{\varphi}_k}^{-1}\left([d\tilde{\varphi}_k]^{-1}\Omega(w_k+v_k)[d\tilde{\varphi}_k]^{-1}\right)||_{s-1}\\&\quad\geq\tilde{C}(||R_{\varphi_k}^{-1}\left([d\varphi_k^\top]^{-1}\Omega(w_k)[d\varphi_k]^{-1}\right)||_{s-1}+||R_{\tilde{\varphi}_k}^{-1}\left([d\tilde{\varphi}_k^\top]^{-1}\Omega(w_k)[d\tilde{\varphi}_k]^{-1}\right)||_{s-1})\\&\qquad\qquad-||R_{\tilde{\varphi}_k}^{-1}\left([d\tilde{\varphi}_k^\top]^{-1}\Omega(v_k)[d\tilde{\varphi}_k]^{-1}\right)||_{s-1}.\end{aligned}\tag{4.38}$$

The latter term can be estimated using (4.23) by

$$||R_{\tilde{\varphi}_k}^{-1}\left([d\tilde{\varphi}_k^\top]^{-1}\Omega(v_k)[d\tilde{\varphi}_k]^{-1}\right)||_{s-1}\leq C_2||\Omega(v_k)||_{s-1}\leq C_2C'||v_k||_s\tag{4.39}$$

which by (4.26) goes to 0 for $k\to\infty$. For the first two terms on the right-hand side of the inequality (4.38) we have again by (4.23)

$$||R_{\varphi_k}^{-1}\left([d\varphi_k^\top]^{-1}\Omega(w_k)[d\varphi_k]^{-1}\right)||_{s-1}\geq\frac{1}{C_2}||\Omega(w_k)||_{s-1}\tag{4.40}$$

and

$$||R_{\tilde{\varphi}_k}^{-1}\left([d\tilde{\varphi}_k^\top]^{-1}\Omega(w_k)[d\tilde{\varphi}_k]^{-1}\right)||_{s-1}\geq\frac{1}{C_2}||\Omega(w_k)||_{s-1}.\tag{4.41}$$

Combining (4.38)-(4.41), the inequality (4.35) then leads to

$$\limsup_{k\geq1}||\Omega_{1,k}-\tilde{\Omega}_{1,k}||_{s-1}\geq\limsup_{k\geq1}\frac{2}{C_2}||\Omega(w_k)||_{s-1}.$$

We will get the result by showing that $\limsup_{k\geq1}||\Omega(w_k)||_{s-1}$ is bounded away from 0. In $H^s(\mathbb{R}^n;\mathbb{R}^n)$ the following norm

$$|||f|||_s:=||f||_{L^2}+||df||_{s-1}$$

is equivalent to the norm $||\cdot||_s$. In particular there exists $C_3 > 0$ so that for any $f \in H^s(\mathbb{R}^n;\mathbb{R}^n)$

$$\frac{1}{C_3}||f||_s \leq |||f|||_s \leq C_3||f||_s. \tag{4.42}$$

By (4.42) we thus get $|||w_k|||_s \geq \frac{1}{C_3}\frac{R}{4}$ for all $k \geq 1$. By (4.27) and (4.29)

$$\begin{aligned} ||w_k||_{L^2} \leq ||w_k||_{L^\infty} \operatorname{vol}\left(B_{\rho_k}(x^*)\right) &\leq C||w_k||_s \operatorname{vol}\left(B_{\rho_k}(x^*)\right) \\ &\leq C\frac{R}{4}\operatorname{vol}\left(B_1(0)\right)\left(\frac{MR}{16k}\right)^n. \end{aligned} \tag{4.43}$$

Hence $||w_k||_{L^2}$ goes to 0 for $k \to \infty$ implying that

$$\limsup_{k\geq 1} ||dw_k||_{s-1} \geq \frac{1}{C_3}\frac{R}{4}.$$

By Lemma 3.2

$$\limsup_{k\geq 1} ||\Omega(w_k)||_{s-1} \geq \limsup_{k\geq 1} \frac{1}{C_4}||dw_k||_{s-1} \geq \frac{1}{C_3C_4}\frac{R}{4}$$

for some constant $C_4 > 0$. By (4.31) we then conclude

$$\limsup_{k\geq 1} ||\phi(u_{0,k}) - \phi(\tilde{u}_{0,k})||_s \geq \limsup_{k\geq 1} \frac{1}{C'}||\Omega_{1,k} - \tilde{\Omega}_{1,k}||_{s-1} \geq \frac{1}{4C_3C_4}R \tag{4.44}$$

whereas $||u_{0,k} - \tilde{u}_{0,k}||_s \to 0$. As $(u_{0,k}), (\tilde{u}_{0,k})$ are in $B_R(u_0)$ this shows that ϕ is not uniformly continuous on $B_R(u_0)$. □

Finally we can give the proof of Theorem 1.4

Proof of Theorem 1.4. By (4.1) it suffices to consider the case $T = 1$, i.e. to prove that

$$\phi : U \to H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$$

is nowhere differentiable. The key ingredient is inequality (4.44). Let us reformulate it in a convenient way. Let $w \in U$. Then by the last part of the proof of Proposition 4.1 there are $R_*, C_* > 0$ with $B_{R_*}(w) \subseteq U$ satisfying the following property: for any $0 < R \leq R_*$ there are sequences $(u_{0,k})_{k\geq 1}, (\tilde{u}_{0,k})_{k\geq 1} \subseteq B_R(w)$ with

$$\lim_{k\to\infty} ||u_{0,k} - \tilde{u}_{0,k}||_s = 0 \tag{4.45}$$

and

$$||\phi(u_{0,k}) - \phi(\tilde{u}_{0,k})||_s \geq C_* R, \qquad \forall k \geq 1. \tag{4.46}$$

Assume now that ϕ is differentiable in w. For any $h \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ with $w + h \in B_{R_*}(w)$

$$\mathcal{R}(w,h) := \phi(w+h) - \phi(w) + d_w\phi(h). \tag{4.47}$$

By the definition of differentiability there is $0 < R \leq R_*$ with

$$||\mathcal{R}(w,h)||_s \leq \frac{C_*}{4}||h||_s \tag{4.48}$$

for any $h \in H^s_\sigma(\mathbb{R}^n;\mathbb{R}^n)$ with $||h||_s \leq R$. Take sequences $(u_{0,k})_{k\geq 1}, (\tilde{u}_{0,k})_{k\geq 1} \subseteq B_R(w)$ satisfying (4.45)-(4.46). We then get by (4.47)

$$\phi(u_{0,k}) = \phi(w + (u_{0,k} - w)) = \phi(w) + d_w\phi(u_{0,k} - w) + \mathcal{R}(w, u_{0,k} - w)$$

and a similar expression for $\phi(\tilde{u}_{0,k})$. Hence

$$\phi(u_{0,k}) - \phi(\tilde{u}_{0,k}) = d_w\phi(u_{0,k} - \tilde{u}_{0,k}) + \mathcal{R}(w, u_{0,k} - w) - \mathcal{R}(w, \tilde{u}_{0,k} - w).$$

and thus by (4.45), $||d_w\phi(u_{0,k} - \tilde{u}_{0,k})||_s \underset{k\to\infty}{\to} 0$ yielding

$$\limsup_{k\geq 1} ||\phi(u_{0,k}) - \phi(\tilde{u}_{0,k})||_s \leq \limsup_{k\geq 1} ||\mathcal{R}(w, u_{0,k} - w)||_s + \limsup_{k\geq 1} ||\mathcal{R}(w, \tilde{u}_{0,k} - w)||_s \leq \frac{C_*}{2} R$$

where the last inequality follows from (4.48). This is a contradiction to (4.46). Hence ϕ is not differentiable in w. As w was arbitrary the claim follows. □

Appendix A. H^s-norms

In this appendix we point out some difficulties with the H^s-norms for non-integral $s \geq 0$. For integral s we have for $f, g \in C_c^\infty(\mathbb{R}^n)$ with disjoint support

$$||f + g||_s^2 = ||f||_s^2 + ||g||_s^2$$

In particular

$$||f + g||_s \geq \frac{1}{2}(||f||_s + ||g||_s).$$

For non-integral s the non-local character of the H^s-norm makes such a statement difficult. Nevertheless we have for fixed suppports

Lemma A.1. Let $s \geq 0$ and let $K, K' \subseteq \mathbb{R}^n$ be compact disjoint subsets. There is $C > 0$ with

$$||f + g||_s \geq C(||f||_s + ||g||_s)$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$ and $\operatorname{supp} g \subseteq K'$.

Proof. Take $\psi, \psi' \in C_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ on K and $\psi' \equiv 1$ on K' and $\operatorname{supp}\psi \cap \operatorname{supp}\psi' = \emptyset$. We have

$$||f||_s = ||\psi \cdot (f + g)||_s \leq C_1||f + g||_s$$

with a constant $C_1 > 0$ just depending on ψ – see Adams [**1**]. Similarly

$$||g||_s = ||\psi' \cdot (f + g)||_s \leq C_2||f + g||_s$$

with a constant $C_2 > 0$ just depending on ψ'. Taking $C = C_1 + C_2$ gives the result. □

To handle the H^s-norm for our needs, let us recall the Slobodeckij seminorm $[\cdot]_\lambda$ for $0 < \lambda < 1$ where

$$[g]_\lambda^2 := \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2\lambda}}\, dxdy$$

We then have for $s = m + \lambda$, $m \in \mathbb{N}$, $0 < \lambda < 1$

$$\frac{1}{C}||f||_s \leq ||f||_m + \sum_{|\alpha|=m} [\partial^\alpha f]_\lambda \leq C||f||_s$$

for all $f \in H^s(\mathbb{R}^n)$ and some constant $C > 0$ independent of f – see Adams [**1**]. For the Slobodeckij seminorm we have the analog property as in Lemma A.1, i.e. for $\lambda \in (0, 1)$ and for compact disjoint $K, K' \subseteq \mathbb{R}^n$ there is a constant $C > 0$ with

$$[f + g]_\lambda \geq C([f]_\lambda + [g]_\lambda)$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$ resp. $\operatorname{supp} g \subseteq K'$. This follows from the following Lemma

LEMMA A.2. Let $n \geq 2$ and $0 < \lambda < 1$. Further let $K \subseteq \mathbb{R}^n$ be a compact subset and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$[\varphi \cdot f]_\lambda \leq C[f]_\lambda$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$.

PROOF.

$$\begin{aligned}
[\varphi \cdot f]_\lambda^2 &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\varphi(x)f(x) - \varphi(y)f(y)|^2}{|x-y|^{n+2\lambda}}\, dxdy \\
&\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\varphi(x)f(x) - \varphi(x)f(y) + \varphi(x)f(y) - \varphi(y)f(y)|^2}{|x-y|^{n+2\lambda}}\, dxdy \\
\leq \int_{\mathbb{R}^n \times \mathbb{R}^n} & 2|\varphi(x)|^2 \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\lambda}}\, dxdy + \int_{\mathbb{R}^n \times \mathbb{R}^n} 2|f(y)|^2 \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{n+2\lambda}}\, dxdy
\end{aligned}$$

Therefore we get

$$[\varphi \cdot f]_\lambda \leq C([f]_\lambda + ||f||_{L^2})$$

By a fractional Hardy type inequality – see Herbst [**7**] – we get

$$||f||_{L^2} \leq C[f]_\lambda$$

for all $f \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$. □

The following will be enough for our purpose

LEMMA A.3. Let $x, y \in \mathbb{R}^n$ centers of balls with radius $r > 0$ with $r/|x-y| \leq 1/4$. Then there is a constant $C > 0$ such that

$$||f+g||_s \geq C(||f||_s + ||g||_s)$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq B_r(x)$ and $\operatorname{supp} g \subseteq B_r(y)$.

PROOF. We have for $s = m + \lambda$, $m \in \mathbb{N}$, $0 < \lambda < 1$ by the equivalence of norms

$$\begin{aligned}
||f+g||_s &\geq C\left(||f+g||_m + \sum_{|\alpha|=m} [\partial^\alpha(f+g)]_\lambda\right) \\
&\geq C\left(\frac{1}{2}||f||_m + \frac{1}{2}||g||_m + \sum_{|\alpha|=m} [\partial^\alpha(f+g)]_\lambda\right)
\end{aligned}$$

Note that $[\cdot]_\lambda$ is translation invariant and scales like

$$[h(\mu \cdot x + t)]_\lambda = \mu^{\lambda - n/2}[h]_\lambda$$

for $\mu > 0$ and $t \in \mathbb{R}^n$. By scaling and shifting the supports $B_r(x)$ and $B_r(y)$ to the fixed supports $B_{1/2}(x_0)$ and $B_{1/2}(y_0)$ where $x_0 = (-1, 0, \ldots, 0)$ and $y_0 = (1, 0, \ldots, 0)$ and back, we get a constant $\tilde{C} > 0$ independent of x, y, r and

$$[\partial^\alpha(f+g)]_\lambda \geq \tilde{C}([\partial^\alpha f]_\lambda + [\partial^\alpha g]_\lambda), \quad \forall |\alpha| = m$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$ as above. So we get

$$||f+g||_s \geq C'(||f||_s + ||g||_s)$$

for some constant $C' > 0$. This concludes the Lemma. □

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