Dynamics of PDE, Vol.12, No.1, 13-42, 2015

Nodal and multiple solutions for nonlinear elliptic equations involving a reaction with zeros

Leszek Gasiński and Nikolaos S. Papageorgiou

Communicated by Yuncheng You, received September 16, 2012.

Abstract. We consider a nonlinear Dirichlet problem driven by a nonhomogeneous differential operator and with a Carathéodory reaction. Imposing control on the growth of the reaction only near zero and assuming that it has two constant sign z-dependent zeros, we prove two multiplicity theorem producing three nontrivial smooth solutions, one positive, the second negative and the third nodal. Then for the particular case of $(p, 2)$-equations and assuming that the reaction is $(p-1)$-superlinear near $\pm\infty$, without satisfying the Ambrosetti-Rabinowitz condition, we show that the problem has at least six nontrivial smooth solutions.

Contents

1991 *Mathematics Subject Classification.* Primary 35J20; Secondary 35J60, 35J92, 58E05.

Key words and phrases. Local minimizers, critical groups, constant sign and nodal solutions, truncations, multiplicity theorems, extremal solutions.

The research was supported by the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement No. 295118, the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under grant no. W111/7.PR/2012 and the National Science Center of Poland under Maestro Advanced Project no. DEC-2012/06/A/ST1/00262.

1. Introduction

Let $\Omega \in \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Dirichlet problem:

$$\begin{cases} -\operatorname{div} a\big(\nabla u(z)\big) \;=\; f\big(z, u(z)\big) \quad \text{in } \Omega, \\ \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Here $a\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a continuous, strictly monotone map which satisfies certain other regularity conditions. The precise hypotheses on a are formulated in $H(a)_1$ (see Section 3). These conditions incorporate as special cases and unify the treatment of some important differential operators, such as the p-Laplacian and the $(p.q)$-Laplacian which is important in problems of mathematical physics. The reaction $f(z,\zeta)$ is a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous on $\mathbb{R}$). We assume that $f(z,\cdot)$ has a positive and a negative zeros, which in general depend on $z \in \Omega$. Restricting the behaviour of $f(z,\cdot)$ near zero, but without any control on $f(z,\cdot)$ near $\pm\infty$, we show that the problem has at least three nontrivial smooth solutions, the first positive, the second negative and the third nodal (i.e., sign changing). When the differential operator is the Dirichlet p-Laplacian, a somewhat complementary situation was studied by Bartsch-Liu-Weth [**4**]. In [**4**], the reaction $f(z,\zeta)$ belong in $C(\overline{\Omega} \times \mathbb{R})$ and has two constant zeros, i.e., there exist $c_- < 0 < c_+$, such that

$$f(z, c_+) \;\leqslant\; 0 \;\leqslant\; f(z, c_+) \quad \forall z \in \overline{\Omega}.$$

Assuming that $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$ and satisfies the well known Ambrosetti-Rabinowitz condition, they show the existence of a nodal solution, without imposing any condition on $f(z,\cdot)$ near zero (see Theorem 1.1 of [**4**]). We should also mention the work of Iturriaga-Massa-Sánchez-Ubilla [**22**], where the authors produce positive solutions for a parametric problem with a $(p-1)$-superlinear reaction possessing variable zeros. Finally for other problems with a nonhomogeneous operator we refer to Gasiński-Papageorgiou [**18, 19, 20**].

Our approach combines variational methods (critical point theory) with truncation and comparison techniques and Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical tools which will be employed in this paper.

2. Mathematical Background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). For a given $\varphi \in C^1(X)$, we say that it satisfies the *Cerami condition*, if the following is true:

> "Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\big\{\varphi(x_n)\big\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
>
> $$\big(1 + \|x_n\|\big)\varphi'(x_n) \;\longrightarrow\; 0 \quad \text{in } X^*,$$
>
> admits a strongly convergent subsequence."

This compactness-type condition, is in general weaker than the usual Palais-Smale condition. Nevertheless, it suffices to prove a deformation theorem and from it obtain minimax characterizations of certain values of φ. In particular, we can state the following theorem known in the literature as the *mountain pass theorem.*

THEOREM 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the Cerami condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \varrho > 0$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} \ < \ \inf\{\varphi(x) : \ \|x - x_0\| = \varrho\} \ = \ \eta_\varrho,$$

and

$$c \ = \ \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma \ = \ \big\{\gamma \in C\big([0,1]; X\big) : \ \gamma(0) = x_0, \ \gamma(1) = x_1\big\},$$

then $c \geqslant \eta_\varrho$ and c is a critical value of φ.

In the analysis of problem (1.1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will also use the Banach space

$$C_0^1(\overline{\Omega}) \ = \ \big\{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\big\}.$$

This is an ordered Banach space with positive cone

$$C_+ \ = \ \big\{u \in C_0^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\big\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ \ = \ \left\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\},$$

where $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

Our method of proof uses a result relating local $C_0^1(\overline{\Omega})$-minimizers and local $W_0^{1,p}(\Omega)$-minimizers ($1 < p < +\infty$). The result that we will state is essentially due Motreanu-Papageorgiou [**27**] and generalizes earlier ones due to Brézis-Nirenberg [**7**] and García Azorero-Manfredi-Peral Alonso [**14**]. To state the result, we need to impose some conditions on the map a. So, let $h \in C^1(0, +\infty)$ be such that

$$\text{(2.1)} \qquad \begin{cases} 0 < \dfrac{th'(t)}{h(t)} \leqslant c_0 \text{ for all } t > 0, \\[2ex] c_1 t^{p-1} \leqslant h(t) \leqslant c_2(t^{q-1} + t^{p-1}) \text{ for all } t > 0, \end{cases}$$

with some $c_0, c_1, c_2 > 0$. We consider the following conditions on a:

<u>$H(a)$:</u> $a \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N) \cap C(\mathbb{R}^N; \mathbb{R}^N)$, $a(0) = 0$ and
(i) there exists $c_3 > 0$, such that

$$\|\nabla y\| \ \leqslant \ c_3 \frac{h(\|y\|)}{\|y\|} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(ii) we have

$$\big(\nabla a(y)\xi, \ \xi\big)_{\mathbb{R}^N} \ \geqslant \ \frac{h(\|y\|)}{\|y\|}\|\xi\|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \ \xi \in \mathbb{R}^N.$$

Eventually in order to deal with problem (1.1), we will strengthen the above hypotheses. Let G be the primitive of a, defined by

$$G'(y) \ = \ a(y) \quad \forall y \in \mathbb{R}^N \quad \text{and} \quad G(0) \ = \ 0.$$

Also, let $f_0\colon \Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ be a Carathéodory function with subcritical growth in $\zeta\in\mathbb{R}$, i.e.,

$$\big|f_0(z,\zeta)\big| \ \leqslant\ a(z)+c|\zeta|^{r-1} \quad \text{for almost all } z\in\Omega,\ \text{all } \zeta\in\mathbb{R},$$

with

$$1\ <\ r\ <\ p^*\ =\ \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p<N,\\ +\infty & \text{if} \quad p\geqslant N. \end{cases}$$

We set

$$F_0(z,\zeta)\ =\ \int_0^\zeta f_0(z,s)\,ds$$

and consider the C^1-functional $\psi_0\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$, defined by

$$\psi_0(u)\ =\ \int_\Omega G\big(\nabla u(z)\big)\,dz-\int_\Omega F_0\big(z,u(z)\big)\,dz \quad \forall u\in W_0^{1,p}(\Omega). \tag{2.2}$$

Proposition 2.2. *If hypotheses $H(a)$ hold, $\psi_0\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$ is defined by (2.2) and $u_0\in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_0, i.e., there exists $\varrho_0>0$, such that*

$$\psi_0(u_0)\ \leqslant\ \psi_0(u_0+h) \quad \forall h\in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})}\leqslant\varrho_0,$$

then $u_0\in C_0^{1,\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$ and u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of ψ_0, i.e., there exists $\varrho_1>0$, such that

$$\psi_0(u_0)\ \leqslant\ \psi_0(u_0+h) \quad \forall h\in W_0^{1,p}(\Omega),\ \|h\|\leqslant\varrho_1.$$

Remark 2.3. In Motreanu-Papageorgiou [**27**], the result is stated for the bigger space $W^{1,p}(\Omega)$ and for its subspace $C^1(\overline{\Omega})$. Also, the hypotheses on a are a little more restrictive. However, a careful inspection of their proof reveals that it remains valid under the more general conditions $H(a)$, using this time the regularity result of Lieberman [**25**, p. 320].

Let $A\colon W_0^{1,p}(\Omega)\longrightarrow W^{-1,p'}(\Omega)=W_0^{1,p}(\Omega)^*$ (where $\frac{1}{p}+\frac{1}{p'}=1$) be the nonlinear map defined by

$$\big\langle A(u),y\big\rangle\ =\ \int_\Omega \big(a(\nabla u),\nabla y\big)_{\mathbb{R}^N}dz \quad \forall u,y\in W_0^{1,p}(\Omega). \tag{2.3}$$

From Gasiński-Papageorgiou [**16**, Proposition 3.5], we have the following result.

Proposition 2.4. *If hypotheses $H(a)$ hold and $A\colon W_0^{1,p}(\Omega)\longrightarrow W^{-1,p'}(\Omega)$ is defined by (2.3), then A is maximal monotone and of type $(S)_+$, i.e., if $u_n\longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\big\langle A(u_n),\ u_n-u\big\rangle\ \leqslant\ 0,$$

then $u_n\longrightarrow u$ in $W_0^{1,p}(\Omega)$.

Particular cases of A are the maps $A_p\colon W_0^{1,p}(\Omega)\longrightarrow W^{-1,p'}(\Omega)$ (with $1<p<+\infty$), defined by

$$\big\langle A_p(u),y\big\rangle\ =\ \int_\Omega \|\nabla u\|^{p-2}\big(\nabla u,\nabla y\big)_{\mathbb{R}^N}\,dz \quad \forall u,y\in W_0^{1,p}(\Omega). \tag{2.4}$$

They correspond to the p-Laplacian differential operator. If $p=2$, then we write $A_2=A\in\mathcal{L}\big(H_0^1(\Omega);H^{-1}(\Omega)\big)$.

As we already mentioned, a particular case of the differential operator

$$W_0^{1,p}(\Omega) \in u \longmapsto \operatorname{div} a(\nabla u),$$

is the Dirichlet p-Laplacian, defined by

$$\Delta_p u = \operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big) \quad \forall u \in W_0^{1,p}(\Omega),$$

with $1 < p < +\infty$. We know (see e.g., Gasiński-Papageorgiou [**15**]) that the operator $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$ has a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which is isolated and admits the following variational characterization

$$\widehat{\lambda}_1(p) = \left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.5}$$

In (2.5) the infimum is realized on the corresponding one dimensional eigenspace. In what follows, by $\widehat{u}_{1,p}$ we denote the L^p-normalized (i.e., $\|\widehat{u}_{1,p}\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p)$. Nonlinear regularity theory (see e.g., Gasiński-Papageorgiou [**15**, pp. 737-738]) and the nonlinear maximum principle of Vázquez [**29**] imply that $\widehat{u}_{1,p} \in \operatorname{int} C_+$. Since the spectrum $\sigma(p)$ of the operator $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$ is closed and $\widehat{\lambda}_1(p) > 0$ is isolated, the second eigenvalue

$$\widehat{\lambda}_2(p) = \inf\big\{\widehat{\lambda} \in \sigma(p) : \widehat{\lambda} > \widehat{\lambda}_1(p)\big\}$$

is also well defined. Let

$$\begin{aligned} \partial B_1^{L^p} &= \big\{u \in L^p(\Omega) : \|u\|_p = 1\big\}, \\ M &= W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}. \end{aligned}$$

From Cuesta-de Figueiredo-Gossez [**9**], we have the following variational characterization of $\widehat{\lambda}_2(p)$.

PROPOSITION 2.5. *If*

$$\widehat{\Gamma} = \big\{\widehat{\gamma} \in C\big([-1,1]; M\big) : \widehat{\gamma}(-1) = -\widehat{u}_{1,p},\ \widehat{\gamma}(1) = \widehat{u}_{1,p}\big\},$$

then

$$\widehat{\lambda}_2(p) = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \big\|\nabla\widehat{\gamma}(t)\big\|_p^p.$$

Next, we recall a few basic things about critical groups (Morse theory). So, if X is a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, then we introduce the following sets:

$$\begin{aligned} \varphi^c &= \big\{x \in X : \varphi(x) \leqslant c\big\}, \\ K_\varphi &= \big\{x \in X : \varphi'(x) = 0\big\}, \\ K_\varphi^c &= \big\{x \in K_\varphi : \varphi(x) = c\big\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in X$ of φ with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k\big(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{x_0\}\big) \quad \forall k \geqslant 0,$$

where U is a neighbourhood of x_0, such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology theory implies that the above definition is independent of the particular choice of the neighbourhood U.

Now suppose that $\varphi \in C^1(X)$ satisfies the Cerami condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) \;=\; H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The second deformation theorem (see e.g., Gasiński-Papageorgiou [**15**, Theorem 5.1.33, p. 628]), implies that this definition of critical groups at infinity is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that K_φ is finite and set

$$M(t,x) \;=\; \sum_{k\geqslant 1} \operatorname{rank} C_k(\varphi, x) t^k \quad \forall t \in \mathbb{R},\ x \in K_\varphi$$

and

$$P(t,\infty) \;=\; \sum_{k\geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{x\in K_\varphi} M(t,x) \;=\; P(t,\infty) + (1+t)Q(t), \tag{2.6}$$

where

$$Q(t) \;=\; \sum_{k\geqslant 0} \beta_k t^k$$

is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Let $g, \widehat{g} \in L^\infty(\Omega)$. We write $g \prec \widehat{g}$, if for every compact set $K \subseteq \Omega$, we can find $\varepsilon > 0$, such that

$$g(z) + \varepsilon \;\leqslant\; \widehat{g}(z) \quad \text{for almost all } z \in K.$$

Evidently, if $g, \widehat{g} \in C(\Omega)$ and $g(z) < \widehat{g}(z)$ for all $z \in \Omega$, then $g \prec \widehat{g}$.

A straightforward modification of the proof of Proposition 2.6 of Arcoya-Ruiz [**3**] in order to accommodate the Laplace differential operator, gives the following result.

Proposition 2.6. *If $\xi \geqslant 0$, $g, \widehat{g} \in L^\infty(\Omega)$ with $g \prec \widehat{g}$, $u, v \in C_0^1(\overline{\Omega})$ are solutions of*

$$\begin{cases} -\Delta_p u(z) - \lambda\Delta u(z) + \xi\big|u(z)\big|^{p-2}u(z) \;=\; g(z) & \text{in } \Omega, \\ -\Delta_p v(z) - \lambda\Delta v(z) + \xi v(z)^{p-1} \;=\; \widehat{g}(z) & \text{in } \Omega, \end{cases}$$

with $\lambda \geqslant 0$ and $v \in \operatorname{int} C_+$, then $v - u \in \operatorname{int} C_+$.

In this paper, for every $u \in W_0^{1,p}(\Omega)$, we set $\|u\| = \|\nabla u\|_p$ (by virtue of the Poincaré inequality). We mention that the notation $\|\cdot\|$ is also used to denote the norm of $\mathbb{R}^N$. No confusion is possible, since it will always be clear from the context which one is used. For every $\zeta \in \mathbb{R}$, we set $\zeta^+ = \max\{\zeta, 0\}$ and $\zeta^- = \max\{-\zeta, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$, we define

$$u^+(\cdot) \;=\; u(\cdot)^+ \quad \text{and} \quad u^-(\cdot) \;=\; u(\cdot)^-.$$

We know that $u^+, u^- \in W_0^{1,p}(\Omega)$, $|u| = u^+ + u^-$ and $u = u^+ - u^-$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, if $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function (for example, h is a Carathéodory function), then we set

$$N_h(u)(\cdot) \;=\; h\big(\cdot, u(\cdot)\big) \quad \forall u \in W_0^{1,p}(\Omega).$$

3. Three Nontrivial Solutions

We introduce the hypotheses on the map $y \longmapsto a(y)$ of the differential operator. We consider a function $h \in C^1(0,+\infty)$ which satisfies the growth condition in (2.1), namely

$$\begin{cases} 0 < \dfrac{th'(t)}{h(t)} \leqslant c_0 \text{ for all } t > 0, \\ \\ c_1 t^{p-1} \leqslant h(t) \leqslant c_2(t^{q-1} + t^{p-1}) \text{ for all } t > 0, \end{cases} \tag{3.1}$$

for some $c_0, c_1, c_2 > 0$, $1 < q < p$.

The new stronger hypotheses on a are the following:

$\underline{H(a)_1}$: $a(y) = a_0(\|y\|)y$ for any $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, $a(0) = 0$ and

(i) $a_0 \in C^1(0,+\infty)$, $\lim\limits_{t\to 0^+} ta_0(t) = 0$ and $\lim\limits_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} = c > -1$;

(ii) there exists $c_3 > 0$, such that

$$\|\nabla a(y)\| \leqslant c_3 \frac{h(\|y\|)}{\|y\|} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) we have

$$\big(\nabla a(y)\xi,\ \xi\big)_{\mathbb{R}^N} \geqslant \frac{h(\|y\|)}{\|y\|}\|\xi\|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N;$$

(iv) if

$$G_0(t) = \int_0^t a_0(s)s\,ds,$$

then there exist $\mu \in (1,q]$ (see (3.1)) and $\tau \in (1,p]$, such that

$$\text{the map } t \longmapsto G_0(t^{\frac{1}{\tau}}) \text{ is convex on } (0,+\infty) \quad \text{and} \quad \lim_{t\to 0^+} \frac{G_0(t)}{t^\mu} = 0.$$

Remark 3.1. Evidently G_0 is strictly convex and strictly increasing. Let

$$G(y) = G_0(\|y\|) \quad \forall y \in \mathbb{R}^N.$$

Then G is convex, $G(0) = 0$ and

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)y = a(y) \quad \forall y \in \mathbb{R}^N \setminus \{0\}.$$

Therefore G is the primitive of a. The convexity of G implies that

$$G(y) \leqslant \big(a(y), y\big) \quad \forall y \in \mathbb{R}^N. \tag{3.2}$$

Hypotheses $H(a)_1$, together with (3.1) and (3.2) lead to the following lemma, which summarize the important properties of the map a.

Lemma 3.2. *If hypotheses $H(a)_1$ hold, then*

(a) *the map $y \longmapsto a(y)$ is maximal monotone and strictly monotone;*

(b) *there exists $c_4 > 0$, such that*

$$\|a(y)\| \leqslant c_4\big(\|y\|^{q-1} + \|y\|^{p-1}\big) \quad \forall y \in \mathbb{R}^N;$$

(c) *we have*

$$\big(a(y),\ y\big)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}\|y\|^p \quad \forall y \in \mathbb{R}^N$$

(where $c_1 > 0$ is as in (2.1)).

The above lemma and the integral form of the mean value theorem, lead at once to the following growth estimates for the primitive G.

COROLLARY 3.3. *If hypotheses $H(a)_1$ hold, then*

$$\frac{c_1}{p(p-1)}\|y\|^p \leqslant G(y) \leqslant c_5\big(\|y\|^q + \|y\|^p\big) \quad \forall y \in \mathbb{R}^N,$$

for some $c_5 > 0$.

EXAMPLE 3.4. The following maps $a(y)$ satisfy hypotheses $H(a)_1$:
(a) $a(y) = \|y\|^{p-2}y$, with $1 < p < +\infty$. This map corresponds to the p-Laplacian differential operator Δ_p.
(b) $a(y) = \|y\|^{p-2}y + \lambda\|y\|^{q-2}y$, with $2 \leqslant q < p < +\infty$, $\lambda \geqslant 0$. This map corresponds to the (p,q)-Laplace differential operator, namely the operator, defined by

$$W_0^{1,p}(\Omega) \ni u \longmapsto \Delta_p u + \lambda\Delta_q u.$$

This operator is important in quantum physics (see Benci-Fortunato-Pisani [**5**]).
(c) $a(y) = \big(1 + \|y\|^2\big)^{\frac{p-2}{2}} y$ with $1 < p < +\infty$;
(d) $a(y) = \|y\|^{p-2}y + c\dfrac{\|y\|^{p-2}y}{1+\|y\|^p}$, with $2 \leqslant p < +\infty$, $0 < c < \dfrac{4p}{(p-1)^2}$.
(e) $a(y) = \|y\|^{p-2}y + \ln\big(1 + \|y\|^2\big)y$, with $2 \leqslant p < +\infty$.
(f) $a(y) = \begin{cases} \|y\|^{p-2}y + c_1\|y\|^{r-2}y & \text{if} \quad \|y\| \leqslant 1, \\ \|y\|^{p-2}y + c_2\|y\|^{q-2}y & \text{if} \quad \|y\| > 1, \end{cases}$ with $1 < q < p < r < +\infty$ and $c_1(p + r - 4) = c_2(p + q - 4)$.

The hypotheses on the reaction $f(z,\zeta)$ are the following:

$\underline{H(f)_1}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and
(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$, such that

$$\big|f(z,\zeta)\big| \leqslant a_\varrho(z) \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho;$$

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, such that

$$w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \forall z \in \overline{\Omega},$$

$$f\big(z, w_+(z)\big) \leqslant f\big(z, w_-(z)\big) \quad \text{for almost all } z \in \Omega$$

and

$$A(w_-) \leqslant 0 \leqslant A(w_+) \quad \text{in } W^{-1,p'}(\Omega);$$

(iii) if

$$F(z,\zeta) = \int_0^\zeta f(z,s)\,ds$$

and $\mu \in (1,p)$ is as in $H(a)(iv)$, then there exists $\delta_0 > 0$, such that

$$0 < f(z,\zeta)\zeta \leqslant \mu F(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } 0 < |\zeta| \leqslant \delta_0$$

and $\operatorname*{ess\,inf}_\Omega F(\cdot,\delta_0) > 0$;
(iv) there exits constants $\widehat{c}_0, \widehat{c}_1 > 0$ and $s \leqslant \tau \leqslant p \leqslant r < p^*$ with $s \neq r$, such that

$$f(z,\zeta)\zeta \geqslant \widehat{c}_0|\zeta|^s - \widehat{c}_1|\zeta|^r \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}.$$

REMARK 3.5. Note that no control is imposed on $f(z,\cdot)$ for $|\zeta|$ large. In particular $f(z,\cdot)$ can be supercritical or even exponential. If there exist constants $c_- < 0 < c_+$, such that

$$f(z,c_+) \leqslant 0 \leqslant f(z,c_-) \quad \text{for almost all } z \in \Omega,$$

then hypothesis $H(f)_1(ii)$ is satisfied with $w_+(z) = c_+$ and $w_-(z) = c_-$ for all $z \in \overline{\Omega}$. This is the setting in Bartsch-Liu-Weth [**4**]. Hypothesis $H(f)(iii)_1$ implies that

$$c_6|\zeta|^\mu \leqslant F(z,\zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } |\zeta| \leqslant \delta_0, \tag{3.3}$$

with some $c_6 > 0$.

From (3.3), we see that $F(z,\cdot)$ is "concave" near zero.

First we produce two constant sign smooth solutions.

PROPOSITION 3.6. *If hypotheses $H(a)_1$ and $H(f)_1$ hold, then problem (1.1) has at least two nontrivial constant sign smooth solutions*

$$u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 \in -\operatorname{int} C_+.$$

PROOF. First we produce the nontrivial positive smooth solution. To this end, we introduce the following truncation of $f(z,\cdot)$:

$$\widehat{f}_+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ f(z,\zeta) & \text{if } 0 \leqslant \zeta \leqslant w_+(z), \\ f\big(z,w_+(z)\big) & \text{if } w_+(z) < \zeta. \end{cases} \tag{3.4}$$

This is a Carathéodory function. Let

$$\widehat{F}_+(z,\zeta) = \int_0^\zeta \widehat{f}(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}_+(u) = \int_\Omega G\big(\nabla u(z)\big)\,dz - \int_\Omega \widehat{F}_+\big(z,u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear from (3.4) that $\widehat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we show that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$, such that

$$\widehat{\varphi}_+(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}_+(u) = \widehat{m}_+. \tag{3.5}$$

By virtue of hypothesis $H(a)(iv)$, for a given $\varepsilon > 0$, we can find $\delta_1 = \delta_1(\varepsilon) \in (0,\delta_0]$, such that

$$G_0(t) \leqslant \varepsilon t^\mu \quad \forall t \in [0,\delta_1],$$

so

$$G(y) \leqslant \varepsilon \|y\|^\mu \quad \forall y \in \mathbb{R}^N,\ \|y\| \leqslant \delta_1. \tag{3.6}$$

Let $\widetilde{u} \in \operatorname{int} C_+$ and let $t \in (0,1)$ be small, such that

$$t\widetilde{u}(z) \leqslant \delta_2 \quad \text{and} \quad \big\|\nabla(t\widetilde{u})(z)\big\| \leqslant \delta_2 \quad \forall z \in \overline{\Omega}, \tag{3.7}$$

with $\delta_2 = \min\{\delta_1, c_+\}$. Then, we have

$$\begin{aligned}\widehat{\varphi}_+(t\widetilde{u}) &= \int_\Omega G\big(\nabla(t\widetilde{u})\big)\,dz - \int_\Omega \widehat{F}_+(z, t\widetilde{u})\,dz \\ &\leqslant t^\mu\big(\varepsilon\|\nabla\widetilde{u}\|_\mu^\mu - c_6\|\widetilde{u}\|_\mu^\mu\big)\end{aligned}$$

(see (3.3), (3.4), (3.6) and (3.7)). Choosing $\varepsilon \in \big(0, \frac{c_6\|\widetilde{u}\|_\mu^\mu}{\|\nabla\widetilde{u}\|_\mu^\mu}\big)$, we see that

$$\widehat{\varphi}_+(t\widetilde{u}) \;<\; 0,$$

so

$$\widehat{\varphi}_+(u_0) \;=\; \widehat{m}_+ \;<\; 0 \;=\; \widehat{\varphi}_+(0)$$

(see (3.5)), hence $u_0 \neq 0$.

From (3.5), we have

$$\widehat{\varphi}'_+(u_0) \;=\; 0,$$

so

$$A(u_0) \;=\; N_{\widehat{f}_+}(u_0). \tag{3.8}$$

On (3.8), first we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\big\langle A(u_0),\; -u_0^-\big\rangle \;=\; 0$$

(see (3.4)), so

$$\frac{c_1}{p-1}\|\nabla u_0^-\|_p^p \;\leqslant\; 0$$

(see Lemma 3.2(c)), hence $u_0 \geqslant 0$, $u_0 \neq 0$.

Next on (3.8), we act with $(u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}\big\langle A(u_0),\; (u_0 - w_+)^+\big\rangle &= \int_\Omega \widehat{f}_+(z, u_0)(u_0 - w_+)^+\,dz \\ &= \int_\Omega f(z, w_+)(u_0 - w_+)^+\,dz \\ &\leqslant \big\langle A(w_+),\; (u_0 - w_+)^+\big\rangle\end{aligned}$$

(see (3.4) and hypothesis $H(f)_1(ii)$), so

$$\int_{\{u_0 > w_+\}} \big(a(\nabla u_0) - a(\nabla w_+),\; \nabla u_0 - \nabla w_+\big)_{\mathbb{R}^N} dz \;\leqslant\; 0,$$

thus

$$\big|\{u_0 > w_+\}\big|_N \;=\; 0$$

(see Lemma 3.2(a)), hence $u_0 \leqslant w_+$.

So, we have proved that

$$u_0 \;\in\; [0, w_+] \quad\text{and}\quad u_0 \neq 0,$$

where $[0, w_+] \;=\; \big\{u \in W_0^{1,p}(\Omega):\; 0 \leqslant u(z) \leqslant w_+(z) \text{ for almost all } z \in \Omega\big\}$. Then, by virtue of (3.4), equation (3.8) becomes:

$$A(u_0) \;=\; N_f(u_0),$$

so

$$\begin{cases} -\operatorname{div} a\big(\nabla u_0(z)\big) \;=\; f\big(z, u_0(z)\big) & \text{for almost all } z \in \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{3.9}$$

Hence u_0 solves (1.1). From Ladyzhenskaya-Uraltseva [**24**, p. 286], we have that $u_0 \in L^\infty(\Omega)$. Then applying the regularity result of Lieberman [**25**, p. 320], we have that $u_0 \in C_+ \setminus \{0\}$. Let $\varrho_0 = \|u_0\|_\infty$. By virtue of hypothesis $H(f)_1(iv)$ and since $u_0 \geqslant 0$, we have

$$\operatorname{div} a\big(\nabla u_0(z)\big) \;\leqslant\; \widehat{c}_1 \varrho_0^{r-p} u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega,$$

so $u_0 \in \operatorname{int} C_+$ (see Pucci-Serrin [**28**, p. 120]).

For the nontrivial negative solution, we introduce the following truncation of $f(z,\cdot)$

$$\widehat{f}_-(z,\zeta) \;=\; \begin{cases} f\big(z, w_-(z)\big) & \text{if } \;\zeta < w_-(z), \\ f(z,\zeta) & \text{if } \;w_-(z) \leqslant \zeta \leqslant 0, \\ 0 & \text{if } \;0 < \zeta. \end{cases}$$

This is a Carathéodory function. We set

$$\widehat{F}_-(z,\zeta) \;=\; \int_0^\zeta \widehat{f}_-(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_- \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\varphi}_-(u) \;=\; \int_\Omega G\big(\nabla u(z)\big)\,dz - \int_\Omega \widehat{F}_-\big(z,u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Working as above, this time with the functional $\widehat{\varphi}_-$, we produce a solution $v_0 \in -\operatorname{int} C_+$ of (1.1). $\square$

In fact we can show that problem (1.1) has extremal nontrivial solutions of constant sign, namely there is a smallest nontrivial positive solution and a biggest nontrivial negative solution. To this end, we consider the following auxiliary Dirichlet problem:

$$\text{(3.10)} \qquad \begin{cases} -\operatorname{div} a\big(\nabla u(z)\big) \;=\; \widehat{c}_0 \big|u(z)\big|^{s-2} u(z) - \widehat{c}_1 \big|u(z)\big|^{r-2} u(z) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Recall that $s \leqslant \tau \leqslant p \leqslant r < p^*$ with $s \neq r$.

Proposition 3.7. *If hypotheses $H(a)_1$ hold, then problem (3.10) has a unique nontrivial positive solution $u_* \in \operatorname{int} C_+$ and due to the oddness of (3.10), $v_* = -u_* \in -\operatorname{int} C_+$ is the unique nontrivial negative solution.*

Proof. First we establish the existence of a nontrivial positive solution. To this end, we consider the C^1-functional $\psi_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\psi_+(u) \;=\; \int_\Omega G\big(\nabla u(z)\big)\,dz - \frac{\widehat{c}_0}{s}\|u^+\|_s^s + \frac{\widehat{c}_1}{r}\|u^+\|_r^r \quad \forall u \in W_0^{1,p}(\Omega).$$

Using Corollary 3.3 and since $r \geqslant \tau \geqslant s$, with $r \neq s$, we infer that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_* \in W_0^{1,p}(\Omega)$, such that

$$\text{(3.11)} \qquad \psi_+(u_*) \;=\; \inf_{u \in W_0^{1,p}(\Omega)} \psi_+(u) \;=\; m_+^*.$$

Since $s \leqslant p \leqslant r$, $s \neq r$, as in the proof of Proposition 3.6, we have that

$$\psi_+(u_*) \;=\; m_+^* \;<\; 0 \;=\; \psi_+(0),$$

hence $u_* \neq 0$.

From (3.11), we have

$$\psi'_+(u_*) \;=\; 0,$$

so

$$A(u_*) \;=\; \widehat{c}_0 (u_*^+)^{s-1} - \widehat{c}_1 (u_*^+)^{r-1}. \tag{3.12}$$

On (3.12) we act with $-u_*^- \in W_0^{1,p}(\Omega)$ and using Lemma 3.2(c), we obtain $u_* \geqslant 0$, $u_* \neq 0$. Hence

$$A(u_*) \;=\; \widehat{c}_0 u_*^{s-1} - \widehat{c}_1 u_*^{r-1},$$

so

$$\begin{cases} -\operatorname{div} a\big(\nabla u_*(z)\big) \;=\; \widehat{c}_0 u_*(z)^{s-1} - \widehat{c}_1 u_*(z)^{r-1} & \text{in } \Omega, \\ u_*|_{\partial\Omega} = 0. \end{cases}$$

Therefore u_* solves (3.10). Moreover, the nonlinear regularity theory (see Ladyzhenskaya-Uraltseva [**24**, p. 286] and Lieberman [**25**, p. 320]), implies that $u_* \in C_+ \setminus \{0\}$. Invoking the nonlinear maximum principle of Pucci-Serrin [**28**, p. 120], we have that $u_* \in \operatorname{int} C_+$.

Next we check the uniqueness of this positive solutions. For this purpose, we introduce the following integral functional $\sigma_+ \colon L^1(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\sigma_+(u) \;=\; \begin{cases} \int_\Omega G(\nabla u^{\frac{1}{\tau}})\,dz & \text{if } u \geqslant 0,\; u^{\frac{1}{\tau}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom}\sigma_+$ and $t \in [0,1]$. Let $w \in \big(tu_1 + (1-t)u_2\big)^{\frac{1}{\tau}}$. Also, let $y_1 = u_1^{\frac{1}{\tau}}$ and $y_2 = u_2^{\frac{1}{\tau}}$. Using Hölder inequality, as in Lemma 1 of Diaz-Saa [**11**] (see also Lemma 4 of Benguria-Brézis-Lieb [**6**]), we have

$$\big\|\nabla w(z)\big\| \;\leqslant\; \big(t\big\|\nabla y_1(z)\big\|^\tau + (1-\tau)\big\|\nabla y_2(z)\big\|^\tau\big)^{\frac{1}{\tau}} \quad \text{for almost all } z \in \Omega.$$

Recall that $G_0(\cdot)$ is increasing. So, we have

$$G_0\big(\big\|\nabla w(z)\big\|\big) \;\leqslant\; G_0\Big(\big(t\big\|\nabla y_1(z)\big\|^\tau + (1-t)\big\|\nabla y_2(z)\big\|^\tau\big)^{\frac{1}{\tau}}\Big).$$

By virtue of hypothesis $H(a)_1(iv)$, we have that

$$\begin{aligned} & G_0\Big(\big(t\big\|\nabla y_1(z)\big\|^\tau + (1-t)\big\|\nabla y_2(z)\big\|^\tau\big)^{\frac{1}{\tau}}\Big) \\ \leqslant\;& tG_0\big(\big\|\nabla y_1(z)\big\|\big) + (1-t)G_0\big(\big\|\nabla y_2(z)\big\|\big) \quad \text{for almost all } z \in \Omega. \end{aligned}$$

But recall that $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. Therefore

$$G\big(\nabla w(z)\big) \;\leqslant\; tG\big(\nabla u_1(z)^{\frac{1}{\tau}}\big) + (1-t)G\big(\nabla u_2(z)^{\frac{1}{\tau}}\big) \quad \text{for almost all } z \in \Omega,$$

so σ_+ is convex.

Also, using Fatou's lemma, we see that σ_+ is lower semicontinuous. Moreover, $\operatorname{dom}\sigma_+ \neq \emptyset$, i.e., $\sigma_+ \in \Gamma_0\big(L^1(\Omega)\big)$ (see Gasiński-Papageorgiou [**15**, p. 488]).

Now, let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of (3.10). Then, from the first part of the proof, we have $u \in \operatorname{int} C_+$. Also

$$u^\tau \geqslant 0, \quad (u^\tau)^{\frac{1}{\tau}} = u \in W_0^{1,p}(\Omega)$$

and so $u^\tau \in \operatorname{dom}\sigma_+$. Let $h \in C_0^1(\overline{\Omega})$. Then $u^\tau + \lambda h \in \operatorname{int} C_+$ for all $\lambda \in (-1,1)$ small and this implies that $u^\tau + \lambda h \in \operatorname{dom}\sigma_+$. Hence the Gâteaux derivative of σ_+ at u^τ in the direction h exists and in addition, via the chain rule, we have

$$\sigma_+'(u^\tau)(h) \;=\; \int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{\tau-1}} h\,dz. \tag{3.13}$$

In a similar fashion, if $y \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (3.10), then $y \in \operatorname{int} C_+$ and we have

$$\sigma_+'(y^\tau)(h) \;=\; \int_\Omega \frac{-\operatorname{div} a(\nabla y)}{y^{\tau-1}} h\,dz. \tag{3.14}$$

The convexity of σ_+ implies the monotonicity of σ_+'. Hence

$$\begin{aligned} 0 \;&\leqslant\; \big\langle \sigma_+'(u^\tau) - \sigma_+'(y^\tau),\; u^\tau - y^\tau \big\rangle_{L^1} \\ &=\; \int_\Omega \left(\frac{-\operatorname{div} a(\nabla u)}{u^{\tau-1}} + \frac{\operatorname{div} a(\nabla y)}{y^{\tau-1}}\right)(u^\tau - y^\tau)\,dz \\ &=\; \int_\Omega \left(\frac{\widehat{c}_0 u^{s-1} - \widehat{c}_1 u^{r-1}}{u^{\tau-1}} - \frac{\widehat{c}_0 y^{s-1} - \widehat{c}_1 y^{r-1}}{y^{\tau-1}}\right)(u^\tau - y^\tau)\,dz \\ &=\; \int_\Omega \left[\widehat{c}_0\left(\frac{1}{u^{\tau-s}} - \frac{1}{y^{\tau-s}}\right) - \widehat{c}_1\big(u^{r-\tau} - y^{r-\tau}\big)\right]dz \;\leqslant\; 0 \end{aligned}$$

(see (3.13), (3.14), (3.10)), since the map $\zeta \longmapsto \widehat{c}_0 \frac{1}{\zeta^{r-s}} - \widehat{c}_1 \zeta^{r-\tau}$ is strictly decreasing on $(0,+\infty)$ (recall that $s \leqslant \tau \leqslant r$ with $s \neq r$). Therefore, it follows that $u = y$ and this proves the uniqueness of $u_* \in \operatorname{int} C_+$.

The oddness of (3.10) implies that $v_* = -u_* \in -\operatorname{int} C_+$ is the unique nontrivial negative solution of (3.10). □

Using this proposition, we can establish the existence of extremal nontrivial constant sign solutions for problem (1.1).

PROPOSITION 3.8. *If hypotheses $H(a)_1$ and $H(f)_1$ hold, then problem (1.1) has a smallest nontrivial positive solution $u_+ \in \operatorname{int} C_+$ and a biggest nontrivial negative solution $v_- \in -\operatorname{int} C_+$.*

PROOF. Let $\mathcal{Y}_+$ be the set of nontrivial positive solutions of problem (1.1) in the order interval $[0, w_+]$. From Proposition 3.6 and its proof, we know that

$$\mathcal{Y}_+ \;\neq\; \emptyset \quad \text{and} \quad \mathcal{Y}_+ \;\subseteq\; \operatorname{int} C_+.$$

Claim. If $y \in \mathcal{Y}_+$, then $u_* \leqslant y$ with $u_* \in \operatorname{int} C_+$ as in Proposition 3.7.

Let $\beta_+ : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be the Carathéodory function, defined by

$$\beta_+(z,\zeta) \;=\; \begin{cases} 0 & \text{if } \;\zeta < 0, \\ \widehat{c}_0 \zeta^{s-1} - \widehat{c}_1 \zeta^{r-1} & \text{if } \;0 \leqslant \zeta \leqslant y(z), \\ \widehat{c}_0 y(z)^{s-1} - \widehat{c}_1 y(z)^{r-1} & \text{if } \;y(z) < \zeta. \end{cases} \tag{3.15}$$

We set

$$B_+(z,\zeta) \;=\; \int_0^\zeta \beta_+(z,s)\,ds$$

and consider the C^1-functional $\xi_+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\xi_+(u) \;=\; \int_\Omega G\big(\nabla u(z)\big) - \int_\Omega B_+\big(z, u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Evidently, ξ_+ is coercive (see (3.15) and Corollary 3.3). Also, it is sequentially weakly lower semicontinuous. Therefore, by the Weierstrass theorem, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$, such that

$$\xi_+(\widehat{u}) \;=\; \inf\big\{\xi_+(u):\; u \in W_0^{1,p}(\Omega)\big\}.$$

As in the proof of Proposition 3.6, using hypotheses $H(a)_1(iv)$ and $H(f)_1(iii)$, we have

$$\xi_+(\widehat{u}) \;<\; 0 \;=\; \xi_+(0),$$

hence $\widehat{u} \neq 0$. Also, we have

$$\xi_+'(\widehat{u}) \;=\; 0,$$

so

$$A(\widehat{u}) \;=\; N_{\beta_+}(\widehat{u}). \tag{3.16}$$

Acting on (3.16) with $-\widehat{u}^- \in W_0^{1,p}(\Omega)$ and with $(\widehat{u}-y)^+ \in W_0^{1,p}(\Omega)$, we show that

$$\widehat{u} \;\in\; [0,y],$$

where $[0,y] = \big\{u \in W_0^{1,p}(\Omega):\; 0 \leqslant u(z) \leqslant y(z) \text{ for almost all } z \in \Omega\big\}$ and $\widehat{u} \neq 0$ (see the proof of Proposition 3.6). Therefore, (3.16) becomes

$$A(\widehat{u}) \;=\; \widehat{c}_0\widehat{u}^{s-1} - \widehat{c}_1\widehat{u}^{r-1}$$

(see (3.15)), so

$$\widehat{u} \text{ is a nontrivial solution of (3.10)}$$

and

$$\widehat{u} \;=\; u_* \;\in\; \operatorname{int} C_+$$

(see Proposition 3.4) and so $u_* \leqslant y$. This proves the Claim.

Now, let $C \subseteq \mathcal{Y}_*$ be a chain (i.e., a totally ordered subset of $\mathcal{Y}_+$). From Dunford-Schwartz [**12**, p. 336], we know that we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq C$, such that

$$\inf C \;=\; \inf_{n\geqslant 1} u_n.$$

We have

$$A(u_n) \;=\; N_f(u_n) \quad \text{and} \quad u_* \;\leqslant\; u_n \;\leqslant\; w_+ \quad \forall n \geqslant 1 \tag{3.17}$$

(see the Claim), so

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Therefore, by passing to a suitable subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega),\\ u_n &\longrightarrow u \quad \text{in } L^p(\Omega). \end{aligned}$$

Acting on (3.17) with $u_n - u \in W_0^{1,p}(\Omega)$ and passing to the limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} \big\langle A(u_n),\; u_n - u\big\rangle \;=\; 0,$$

so

$$u_n \;\longrightarrow\; u \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.4). Hence, passing to the limit as $n \to +\infty$ in (3.17), we have

$$A(u) \;=\; N_f(u) \quad \text{and} \quad v_* \;\leqslant\; u \;\leqslant\; w_+,$$

so

$$u \in \mathcal{Y}_+ \quad \text{and} \quad u = \inf C.$$

Since C was an arbitrary chain in $\mathcal{Y}_+$, from the Kuratowski-Zorn lemma, we infer that $\mathcal{Y}_+$ has a minimal element $u_+ \in \mathcal{Y}_+$. As in Gasiński-Papageorgiou [**17**, Lemma 3.2], using the monotonicity of a (see Lemma 3.2(a)), we have that $\mathcal{Y}_+$ is downward directed, i.e., if $u_1, u_2 \in \mathcal{Y}_+$, then we can find $u \in \mathcal{Y}_+$, such that $u \leqslant u_1$ and $u \leqslant u_2$. This implies that $u_+ \in \operatorname{int} C_+$ is the smallest nontrivial positive solution of (1.1).

Let $\mathcal{Y}_-$ be the set of nontrivial negative solution of (1.1). Again we have

$$\mathcal{Y}_- \neq \emptyset \quad \text{and} \quad \mathcal{Y}_- \subseteq -\operatorname{int} C_+$$

(see Proposition 3.6 and its proof). Moreover, in this case $\mathcal{Y}_-$ is upward directed, i.e., if $v_1, v_2 \in \mathcal{Y}_-$, then we can find $v \in \mathcal{Y}_-$, such that $v_1 \leqslant v$ and $v_2 \leqslant v$ (see Gasiński-Papageorgiou [**17**, Lemma 3.3]). So, using $v_* = -u_* \in -\operatorname{int} C_+$ and reasoning as above, we obtain the biggest nontrivial negative solution $v_- \in -\operatorname{int} C_+$ of problem (1.1). ☐

These extremal solutions lead to a nodal solution.

PROPOSITION 3.9. *If hypotheses $H(a)_1$ and $H(f)_1$ hold, then problem (1.1) has a nodal (i.e., sign changing) solution $y_0 \in C_0^1(\overline{\Omega})$.*

PROOF. Let $u_1 \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of problem (1.1) produced in Proposition 3.8. We introduce the following truncation of the reaction $f(z, \cdot)$:

$$h(z,\zeta) = \begin{cases} f\big(z, v_-(z)\big) & \text{if } \zeta < v_-(z), \\ f(z,\zeta) & \text{if } v_-(z) \leqslant \zeta \leqslant u_+(z), \\ f\big(z, u_+(z)\big) & \text{if } u_+(z) < \zeta. \end{cases} \tag{3.18}$$

This is a Carathéodory function. We set

$$H(z,\zeta) = \int_0^\zeta h(z,s)\, ds$$

and consider the C^1-functional $\vartheta\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\vartheta(u) = \int_\Omega G\big(\nabla u(z)\big)\, dz - \int_\Omega H\big(z, u(z)\big)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, let

$$h_\pm(z,\zeta) = h(z, \pm\zeta^\pm) \quad \text{and} \quad H_\pm(z,\zeta) = \int_0^\zeta h_\pm(z,s)\, ds$$

and consider the C^1-functionals $\vartheta_\pm\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\vartheta_\pm(u) = \int_\Omega G\big(\nabla u(z)\big)\, dz - \int_\Omega H_\pm\big(z, u(z)\big)\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As before (see the proof of Proposition 3.6), using (3.18), we show that

$$K_\vartheta \subseteq [v_-, u_+], \qquad K_{\vartheta_+} \subseteq [0, u_+], \qquad K_{\vartheta_-} \subseteq [v_-, 0].$$

The extremality of v_- and v_+ implies that

$$K_\vartheta \subseteq [v_-, u_+], \qquad K_{\vartheta_+} = \{0, u_+\}, \qquad K_{\vartheta_-} = \{v_-, 0\}. \tag{3.19}$$

Claim. u_+ and v_- are local minimizers of ϑ.

It is clear from (3.18) and Corollary 3.3, that ϑ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$, such that

$$\vartheta_+(\widetilde{u}) = \inf_{u \in W_0^{1,p}(\Omega)} \vartheta_+(u).$$

As in the proof of Proposition 3.6, we show that

$$\vartheta_+(\widetilde{u}) < 0 = \vartheta_+(0),$$

hence $\widetilde{u} \neq 0$.

Since $\widetilde{u} \in K_{\vartheta_+} \setminus \{0\}$, it follows that $\widetilde{u} = u_+$ (see (3.19)). Note that $\vartheta|_{C_+} = \vartheta_+|_{C_+}$. Hence $\widetilde{u} = u_+ \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of ϑ and so by virtue of Proposition 2.2, it is also a local $W_0^{1,p}(\Omega)$-minimizer of ϑ. Similarly, for $v_- \in -\operatorname{int} C_+$, using this time the functional ϑ_-. This proves the Claim.

Without any loss of generality, we may assume that $\vartheta(v_-) \leqslant \vartheta(u_+)$ (the analysis is similar, if the opposite inequality holds). By virtue of the Claim, as in Aizicovici-Papageorgiou-Staicu [**1**, Proposition 29] (see also Gasiński-Papageorgiou [**17**, proof of Theorem 3.4]), we can find $\varrho \in (0,1)$ small, such that

(3.20)
$$\vartheta(v_-) \leqslant \vartheta(u_+) < \inf\{\vartheta(u) : \|u - u_+\| = \varrho\} = \eta_\varrho \quad \text{and} \quad \|v_- - u_+\| > \varrho.$$

Since ϑ is coercive, it satisfies the Cerami condition. This fact and (3.20) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $y_0 \in W_0^{1,p}(\Omega)$, such that

$$y_0 \in K_\vartheta \quad \text{and} \quad \eta_\varrho \leqslant \vartheta(y_0). \tag{3.21}$$

From (3.20) and (3.21), it follows that $y_0 \notin \{v_-, u_+\}$ and $y_0 \in [v_-, u_+]$ (see (3.19)). So y_0 solves problem (1.1). Since y_0 is a critical point of ϑ of mountain pass type, we have

$$C_1(\vartheta, y_0) \neq 0. \tag{3.22}$$

On the other hand, using hypothesis $H(f)_1(iii)$ and $(2)'$ and reasoning as in the proof of Proposition 2.1 of Jiu-Su [**23**], we have

$$C_k(\vartheta, 0) = 0 \quad \forall k \geqslant 0. \tag{3.23}$$

Comparing (3.23) and (3.24), we see that $y_0 \neq 0$. Since $y_0 \in [v_-, u_+] \setminus \{0, v_-, u_+\}$ (see (3.19)), by virtue of the extremality of v_- and u_+, we have that y_0 is nodal and $y_0 \in C_0^1(\overline{\Omega})$ (by the nonlinear regularity theory). ☐

Therefore, we can state the following multiplicity theorem for problem (1.1).

THEOREM 3.10. *If hypotheses $H(a)_1$ and $H(f)_1$ hold, then problem (1.1) has at least three nontrivial smooth solutions:*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in C_0^1(\overline{\Omega}) \textit{ nodal with } v_0 \leqslant y_0 \leqslant u_0.$$

In the above theorem, the reaction $f(z,\cdot)$ exhibits a "concave" behaviour zero (see hypothesis $H(f)_1(iii)$). We can relax this condition at the expense of introducing a more restrictive growth condition on a. More precisely, the new hypotheses on the map $y \longmapsto a(y)$ and the function $(z,\zeta) \longmapsto f(z,\zeta)$ are the following:

$\underline{H(a)_2}$: $a(y) = a_0\big(\|y\|\big)y$ for any $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, $a(0) = 0$ and
(i) $a_0 \in C^1(0, +\infty)$, $\lim\limits_{t\to 0^+} ta_0(t) = 0$ and $\lim\limits_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} = c > -1$;
(ii) there exists $c_7 > 0$, such that

$$\big\|\nabla a(y)\big\| \;\leqslant\; c_7\|y\|^{p-2} \quad \forall y \in \mathbb{R}^N \setminus \{0\};$$

(iii) there exists $c_8 > 0$, such that

$$\big(\nabla a(y)\xi,\ \xi\big)_{\mathbb{R}^N} \;\geqslant\; c_8\|y\|^{p-2}\|\xi\|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N;$$

(iv) if

$$G_0(t) \;=\; \int_0^t a_0(s)s\,ds,$$

then we have that

$$\text{the map } t \longmapsto G_0(t^{\frac{1}{p}}) \text{ is convex on } (0, +\infty).$$

Remark 3.11. Evidently, the new growth conditions $H(a)_2(ii), (iii)$ and (iv) are more restrictive than the corresponding hypotheses in $H(a)_1$. In particular they exclude from consideration, the (p, q)-Laplace differential operator and the generalized p-mean curvature differential operator. However the other examples of maps $a(y)$ still satisfy $H(a)_2$ (see Example 3.4).

The new growth conditions $H(a)_2(ii)$ and (iii), imply that

$$\frac{c_8}{p-1}\|y\|^p \;\leqslant\; \big(a(y), y\big)_{\mathbb{R}^N} \;\leqslant\; \frac{c_7}{p-1}\|y\|^p \quad \forall y \in \mathbb{R}^N. \tag{3.24}$$

From (3.24) and the integral form of the mean value theorem, we obtain

$$\frac{c_8}{p(p-1)}\|y\|^p \;\leqslant\; G(y) \;\leqslant\; \frac{c_7}{p(p-1)}\|y\|^p \quad \forall y \in \mathbb{R}^N. \tag{3.25}$$

The new hypotheses on the function $f(z, \zeta)$ are the following:

$\underline{H(f)_2}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and
(i) for every $\varrho > 0$, there exists $a_\varrho \in L^\infty(\Omega)_+$, such that

$$\big|f(z, \zeta)\big| \;\leqslant\; a_\varrho(z) \quad \text{for almost all } z \in \Omega,\ \text{all } |\zeta| \leqslant \varrho;$$

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, such that

$$w_-(z) \;\leqslant\; c_- \;<\; 0 \;<\; c_+ \;\leqslant\; w_+(z) \quad \forall z \in \overline{\Omega},$$

$$f\big(z, w_+(z)\big) \;\leqslant\; 0 \;\leqslant\; f\big(z, w_-(z)\big) \quad \text{for almost all } z \in \Omega$$

and

$$A(w_-) \;\leqslant\; 0 \;\leqslant\; A(w_+) \quad \text{in } W^{-1,p'}(\Omega);$$

(iii) if

$$F(z, \zeta) \;=\; \int_0^\zeta f(z, s)\,ds,$$

then there exist $\widehat{\eta} > \widehat{\lambda}_2(p)$ and $\delta_0 > 0$, such that

$$\frac{c_7\widehat{\eta}}{p(p-1)}|\zeta|^p \;\leqslant\; F(z, \zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } |\zeta| \leqslant \delta_0;$$

(iv) there exists constants $\widehat{c}_0, \widehat{c}_1 > 0$ and $r \in (p, p^*)$, such that

$$f(z, \zeta)\zeta \;\geqslant\; \widehat{c}_0|\zeta|^p - \widehat{c}_1|\zeta|^r \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R}.$$

REMARK 3.12. Evidently hypothesis $H(f)_2(iii)$ permits reactions $f(z,\cdot)$ which are $(p-1)$-linear near zero. This was not possible under hypothesis $H(f)_1(iii)$. Of course hypothesis $H(f)_2(iii)$ still allows for nonlinearities with concave term near zero. Therefore hypothesis $H(f)_2(iii)$ is more general than $H(f)_2(iii)$.

A careful inspection of their proofs, reveals that Propositions 3.6, 3.7 and 3.8 remain valid under the new hypotheses. However, the proof of the existence of a nodal solution (see Proposition 3.9) changes.

PROPOSITION 3.13. *If hypotheses $H(a)_2$ and $H(f)_2$ hold, then problem (1.1) has a nodal solution $y_0 \in C_0^1(\overline{\Omega})$.*

PROOF. Let $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ be the two extremal nontrivial constant sign solutions of (1.1). Using them, we introduce $h(z,\zeta)$ (see (3.18)) and then consider the C^1-functionals $\vartheta, \vartheta_\pm \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ (see the proof of Proposition 3.9). As before, $u_+ \in \operatorname{int} C_+$ and $v_- \in -\operatorname{int} C_+$ are both local minimizers of ϑ and so (3.20) remains valid. Since ϑ is coercive, it satisfies the Cerami condition. Therefore, applying the mountain pass theorem (see Theorem 2.1), we obtain $y_0 \in W_0^{1,p}(\Omega)$, such that

$$y_0 \in K_\vartheta \setminus \{u_+, v_-\} \quad \text{and} \quad \vartheta(y_0) \;=\; \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \vartheta\big(\gamma(t)\big), \tag{3.26}$$

where

$$\Gamma \;=\; \big\{\gamma \in C\big([0,1]; W_0^{1,p}(\Omega)\big) :\; \gamma(0) = v_-,\; \gamma(1) = u_+\big\}.$$

Evidently y_0 solves problem (1.1). It remains to show that y_0 is nontrivial and then y_0 is the desired nodal solution of (1.1). To show the nontriviality of y_0, we will use the minimax expression in (3.26). According to this minimax expression, it suffices to produce a path $\gamma_* \in \Gamma$, such that $\vartheta|_{\gamma_*} < 0$.

To this end, recall that

$$\partial B_1^{L^p} \;=\; \big\{u \in L^p(\Omega) :\; \|u\|_p = 1\big\} \quad \text{and} \quad M \;=\; W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}.$$

We set

$$M_c \;=\; M \cap C_0^1(\overline{\Omega}).$$

We endow M with the relative $W_0^{1,p}(\Omega)$-topology and M_c with the relative $C_0^1(\overline{\Omega})$-topology. Then M_c is dense in M and so also the space $C\big([-1,1]; M_c\big)$ is dense in $C\big([-1,1]; M\big)$. Let

$$\widehat{\Gamma} \;=\; \big\{\widehat{\gamma} \in C\big([0,1]; M\big) :\; \widehat{\gamma}(-1) = -\widehat{u}_{1,p},\; \widehat{\gamma}(1) = \widehat{u}_{1,p}\big\}$$

and

$$\widehat{\Gamma}_c \;=\; \big\{\widehat{\gamma} \in C\big([0,1]; M_c\big) :\; \widehat{\gamma}(-1) = -\widehat{u}_{1,p},\; \widehat{\gamma}(1) = \widehat{u}_{1,p}\big\}.$$

Clearly $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$. Invoking Proposition 2.5, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$, such that

$$\big\|\nabla\widehat{\gamma}_0(t)\big\|_p^p \;<\; \widehat{\eta} \quad \forall t \in [-1,1]. \tag{3.27}$$

Since $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ and $u_+ \in \operatorname{int} C_+$, $v_- \in -\operatorname{int} C_+$, we can find $\lambda \in (0,1)$ small, such that

$$v_-(z) \;\leqslant\; \lambda\widehat{\gamma}_0(t) \;\leqslant\; u_+(z) \quad \text{and} \quad \lambda\big|\widehat{\gamma}_0(t)(z)\big| \;\leqslant\; \delta_0 \quad \forall z \in \overline{\Omega},\; t \in [-1,1]. \tag{3.28}$$

So, for all $t \in [-1,1]$, we have

$$\vartheta\big(\lambda\widehat{\gamma}_0(t)\big) \;=\; \int_\Omega G\big(\lambda\nabla\widehat{\gamma}_0(t)\big)\,dz - \int_\Omega F\big(z, \lambda\widehat{\gamma}_0(t)\big)\,dz$$

$$\begin{aligned} &\leqslant \frac{c_7}{p(p-1)}\lambda^p\|\nabla\widehat{\gamma}_0(t)\|_p^p - \frac{c_7\widehat{\eta}}{p(p-1)}\lambda^p\|\widehat{\gamma}_0(t)\|_p^p \\ &= \frac{c_7\lambda^p}{p(p-1)}\big(\|\nabla\widehat{\gamma}_0(t)\|_p^p - \widehat{\eta}\big) < 0 \end{aligned}$$

(see (3.28), (3.25), (3.27), hypothesis $H(f)_2(iii)$ and recall that $\|\widehat{\gamma}_0(t)\|_p = 1$ for all $t \in [-1,1]$). So, if we set $\gamma_0 = \lambda\widehat{\gamma}_0$, then γ_0 is a continuous path in $W_0^{1,p}(\Omega)$ which connects $-\lambda\widehat{u}_{1,p}$ and $\lambda\widehat{u}_{1,p}$, such that

$$\vartheta\big|_{\gamma_0} < 0. \tag{3.29}$$

Next, we will produce a continuous path in $W_0^{1,p}(\Omega)$ which connects $\lambda\widehat{u}_{1,p}$ and u_+ and along which the functional ϑ is strictly negative. To this end let

$$\alpha = \inf_{W_0^{1,p}(\Omega)} \vartheta_+ = \vartheta_+(u_+) < 0 = \vartheta_+(0)$$

(see the proof of Proposition 3.9). By virtue of the second deformation theorem (see Gasiński-Papageorgiou [**15**, p. 628]), we can find a deformation

$$h\colon [0,1]\times(\vartheta_+^0 \setminus K_{\vartheta_+}^0) \longrightarrow K_{\vartheta_+}^0,$$

such that

$$h(t,\cdot)\big|_{K_{\vartheta_+}^\alpha} = id\big|_{K_{\vartheta_+}^\alpha}$$

(note that $K_{\vartheta_+}^a = \{u_+\}$ and $K_{\vartheta_+}^0 = \{0\}$; see (3.19)),

$$h\big(1, \vartheta_+^0 \setminus K_{\vartheta_+}^a\big) \subseteq \vartheta_+^a \tag{3.30}$$

and

$$\vartheta_+\big(h(t,u)\big) \leqslant \vartheta_+\big(h(s,u)\big) \quad \forall s,t\in[0,1], \text{ with } s\leqslant t \text{ and all } u\in\vartheta_+^0\setminus K_{\vartheta_+}^0. \tag{3.31}$$

Let

$$\gamma_+(t) = h\big(t,\lambda\widehat{u}_{1,p}\big)^+$$

(note that by (3.29), we have $\vartheta(\lambda\widehat{u}_{1,p}) = \vartheta_+(\lambda\widehat{u}_{1,p}) < 0$). Then, we have

$$\gamma_+(0) = \lambda\widehat{u}_{1,p}, \quad \gamma(1) = u_+$$

(see (3.30)) and

$$\vartheta_+\big(\gamma_+(t)\big) \leqslant \vartheta_+\big(\lambda\widehat{u}_{1,p}\big) < 0 \quad \forall t\in[0,1]$$

(see (3.31) and (3.29)). Therefore γ_+ is a continuous path in $W_0^{1,p}(\Omega)$ which connects $\lambda\widehat{u}_{1,p}$ and u_+ and such that

$$\vartheta_+\big|_{\gamma_+} < 0.$$

If

$$W_+ = \big\{u\in W_0^{1,p}(\Omega):\ u(z)\geqslant 0 \text{ for almost all } z\in\Omega\big\},$$

then

$$\vartheta_+\big|_{W_+} = \vartheta\big|_{W_+}$$

and $\gamma_+\big([0,1]\big)\subseteq W_+$. So, it follows that

$$\vartheta\big|_{\gamma_+} < 0. \tag{3.32}$$

In a similar fashion, we produce a continuous path γ_- in $W_0^{1,p}(\Omega)$ which connects $-\lambda\widehat{u}_{1,p}$ and v_- and such that

$$\vartheta\big|_{\gamma_-} < 0. \tag{3.33}$$

We concatenate $\gamma_-, \gamma_0, \gamma_+$ and produce $\gamma_* \in \Gamma$, such that

$$\vartheta\big|_{\gamma_*} < 0$$

(see (3.29), (3.32) and (3.33)). This proves that $y_0 \neq 0$ (see (3.26)) and so $y_0 \in C_0^1(\overline{\Omega})$ is a nodal solution of (1.1). $\square$

So, we can state the following multiplicity theorem for problem (1.1).

THEOREM 3.14. *If hypotheses $H(a)_2$ and $H(f)_2$ hold, then problem (1.1) has at least three nontrivial smooth solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in C_0^1(\overline{\Omega}) \ \textit{nodal}.$$

4. $(p,2)$-Equations

In the previous section, the multiplicity results proved, did not impose any growth condition on the reaction $f(z,\cdot)$ near $\pm\infty$. In this section, for the particular case of $(p,2)$-equations (i.e., $a(y) = \|y\|^{p-2}y+\lambda y$, $\lambda \geqslant 0$, $2 < p < +\infty$) and assuming that $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition, we will extend Theorem 3.10 producing six nontrivial smooth solutions. However, we are unable to determine the sign of the sixth solution. Our multiplicity result in this section, can be viewed as a partial extension of the result of Bartsch-Liu-Weth [**4**], since here we deal with $(p,2)$-equation which is nonhomogeneous and $f(z,\cdot)$ need not satisfy the Ambrosetti-Rabinowitz condition. However, in contrast to [**4**], here we restrict the behaviour of $f(z,\cdot)$ near zero.

So, now the problem under consideration is the following:

$$\text{(4.1)} \qquad \begin{cases} -\Delta_p u(z) - \lambda \Delta u(z) \ = \ f\big(u(z)\big) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ 2 < p < +\infty, \ \lambda \geqslant 0. \end{cases}$$

We introduce the following hypotheses for the reaction f:

$H(f)_3$: $f \in C^1(\mathbb{R})$, $f(0) = 0$ and

(i) there exist $c > 0$ and $r \in (p, p^*)$, such that

$$\big|f'(\zeta)\big| \ \leqslant \ c\big(1 + |\zeta|^{r-2}\big) \quad \text{for all } \zeta \in \mathbb{R};$$

(ii) there exist constants $c_- < 0 < c_+$, such that $f(c_-) = 0 = f(c_+)$;

(iii) if

$$F(\zeta) \ = \ \int_0^\zeta f(s)\,ds,$$

then there exist $\tau \in \big((r-p)\max\{\frac{N}{p}, 1\},\ p^*\big)$ and $\beta_0 > 0$, such that

$$\lim_{\zeta \to \pm\infty} \frac{F(\zeta)}{|\zeta|^p} \ = \ +\infty \quad \text{and} \quad 0 \ < \ \beta_0 \ \leqslant \ \liminf_{\zeta \to \pm\infty} \frac{f(\zeta)\zeta - pF(\zeta)}{|\zeta|^\tau};$$

(iv) there exist $\mu \in (1,2)$ and $\delta_0 > 0$, such that

$$0 \ < \ f(\zeta)\zeta \ \leqslant \ \mu F(\zeta) \quad \forall \zeta \in [-\delta_0, \delta_0];$$

(v) there exist constants $\widehat{c}_0, \widehat{c}_1 > 0$ and $1 < s \leqslant 2 \leqslant d$ with $s \neq 2$, such that

$$f(\zeta)\zeta \ \geqslant \ \widehat{c}_0|\zeta|^s - \widehat{c}_1|\zeta|^d \quad \forall \zeta \in \mathbb{R}.$$

REMARK 4.1. Hypothesis $H(f)_3(iii)$ implies that the primitive F is p-superlinear near $\pm\infty$. This is the case, if f is $(p-1)$-superlinear near $\pm\infty$, i.e.,

$$\lim_{\zeta\to\pm\infty} \frac{f(\zeta)}{|\zeta|^{p-2}\zeta} = \pm\infty.$$

However, note that we do not use the Ambrosetti-Rabinowitz condition, which says that there exist $q > p$ and $M > 0$, such that

$$0 < qF(\zeta) \leqslant f(\zeta)\zeta \quad \forall |\zeta| \geqslant M.$$

Instead, we use a weaker condition. Similar conditions can be found in Costa-Magalhães [**8**] and Fei [**13**]. Hypothesis $H(f)_3(i)$ implies that for every $\varrho > 0$, we can find $\xi_\varrho > 0$, such that the maps $\zeta \longmapsto f(\zeta) + \xi_\varrho |\zeta|^{p-2}\zeta$ and $\zeta \longmapsto f(\zeta) + \xi_\varrho \zeta$ are nondecreasing on $[-\varrho, \varrho]$.

In this case

$$a(y) = \|y\|^{p-2}y + \lambda y,$$

which as we already pointed out satisfies hypotheses $H(a)_1$. Moreover, hypotheses $H(f)_3$ are a particular version of hypotheses $H(f)_1$. So, Theorem 3.10 remains valid and we have the following result.

PROPOSITION 4.2. *If hypotheses $H(f)_3$ hold, then problem (4.1) has at least three nontrivial smooth solutions:*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+, \quad y_0 \in C_0^1(\overline{\Omega}) \text{ nodal.}$$

Concerning the solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, we can have the following additional information, which will be important in the sequel.

LEMMA 4.3. *If hypotheses $H(f)_3$ hold and $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, are the two nontrivial constants sign solutions from Proposition 4.2, then*

$$u_0(z) < c_+ \quad \text{and} \quad c_- < v_0(z) \quad \forall z \in \overline{\Omega}.$$

PROOF. First suppose that $\lambda > 0$. Then, we have

$$A_p(u_0) + \lambda A(u_0) - N_f(u_0) = 0 = A_p(c_+) + \lambda A(c_+) - N_f(c_+)$$

(see (2.4) and hypothesis $H(f)_3(iii)$). Recall that in this case

$$a(y) = \|y\|^{p-2}y + \lambda y,$$

so

$$\nabla a(y) = \|y\|^{p-2}\left(I + (p-2)\frac{y \otimes y}{\|y\|^2}\right) + \lambda I$$

and thus

$$\big(\nabla a(y)\xi,\ \xi\big) \geqslant \lambda \|\xi\|^2 \quad \forall y \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}.$$

So, we can apply the tangency principle of Pucci-Serrin [**28**, p.35] and infer that

$$u_0(z) < c_+ \quad \forall z \in \overline{\Omega}$$

(see hypothesis $H(f)_3(ii)$ and recall that $u_0|_{\partial\Omega} = 0$). Similarly, we show that

$$c_- < v_0(z) \quad \forall z \in \overline{\Omega}.$$

Now, suppose that $\lambda = 0$. Then, we can apply Theorem 1.4 of Damascelli-Sciunzi [**10**] (see also Lucia-Prashanth [**26**]) and again, we obtain that

$$u_0(z) < c_+ \quad \text{and} \quad c_- < v_0(z) \quad \forall z \in \overline{\Omega}.$$

$\square$

Using this lemma, we can now produce two more constant sign smooth solutions.

PROPOSITION 4.4. *If hypotheses $H(f)_3$ hold, then problem (4.1) has two more constant sign smooth solutions*

$$\widehat{u} \in \operatorname{int} C_+ \quad \textit{and} \quad \widehat{v} \in -\operatorname{int} C_+,$$

such that

$$\widehat{u} - u_0 \in \operatorname{int} C_+ \quad \textit{and} \quad v_0 - \widehat{v} \in \operatorname{int} C_+.$$

PROOF. First we produce the second nontrivial positive smooth solution. To this end, we introduce the following truncation of f:

$$k_+(z,\zeta) \;=\; \begin{cases} f\big(u_0(z)\big) & \text{if} \quad \zeta < u_0(z), \\ f(\zeta) & \text{if} \quad u_0(z) \leqslant \zeta. \end{cases} \tag{4.2}$$

This is a Carathéodory function. We set

$$K_+(z,\zeta) \;=\; \int_0^\zeta k_+(z,s)\,ds$$

and consider the C^1-function $\widehat{\psi}_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\psi}_+(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{2}\|\nabla u\|_2^2 - \int_\Omega K_+\big(z,u(z)\big) \quad \forall u \in W_0^{1,p}(\Omega).$$

Claim 1. $\widehat{\psi}_+$ satisfies the Cerami condition.

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence, such that

$$\big|\widehat{\psi}_+(u_n)\big| \;\leqslant\; M_1 \quad \forall n \geqslant 1, \tag{4.3}$$

for some $M_1 > 0$ and

$$\big(1 + \|u_n\|\big)\widehat{\psi}'_+(u_n) \;\longrightarrow\; 0 \quad \text{in } W^{-1,p'}(\Omega) \quad \text{as } n \to +\infty. \tag{4.4}$$

From (4.4), we have

$$\big|\big\langle \widehat{\psi}'_+(u_n), h\big\rangle\big| \;\leqslant\; \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega),$$

with $\varepsilon_n \searrow 0$, so

$$\left|\big\langle A_p(u_n), h\big\rangle + \lambda\big\langle A(u_n), h\big\rangle - \int_\Omega k_+(z,u_n)h\,dz\right| \;\leqslant\; \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \quad \forall n \geqslant 1. \tag{4.5}$$

In (4.5), first we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n^-\|_p^p + \lambda\|\nabla u_n^-\|_2^2 \;\leqslant\; M_2 \quad \forall n \geqslant 1,$$

for some $M_2 > 0$ and so

$$\text{the sequence } \{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{4.6}$$

From (4.3) and (4.6), we have

$$\|\nabla u_n^+\|_p^p + \frac{\lambda p}{2}\|\nabla u_n^+\|_2^2 - \int_\Omega pK_+(z,u_n^+)\,dz \;\leqslant\; M_3 \quad \forall n \geqslant 1, \tag{4.7}$$

for some $M_3 > 0$. Also, if in (4.5), we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$-\|\nabla u_n^+\|_p^p - \lambda\|\nabla u_n^+\|_2^2 + \int_\Omega k_+(z,u_n^+)u_n^+\,dz \;\leqslant\; M_4 \quad \forall n \geqslant 1, \tag{4.8}$$

for some $M_4 > 0$ (see (4.2)).

Adding (4.7) and (4.8), we obtain

$$\lambda\Big(\frac{p}{2} - 1\Big)\|\nabla u_n^+\|_2^2 + \int_\Omega \big(k_+(z, u_n^+)u_n^+ - pK_+(z, u_n^+)\big)\, dz \;\leqslant\; M_5,$$

with $M_5 = M_3 + M_4$, so

$$\int_\Omega \big(k_+(z, u_n^+)u_n^+ - K_+(z, u_n^+)\big)\, dz \;\leqslant\; M_5 \quad \forall n \geqslant 1. \tag{4.9}$$

From hypothesis $H(f)_3(iii)$ and (4.2), (4.9), we infer that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{4.10}$$

First, suppose that $N \neq p$. It is clear from hypothesis $H(f)_3(iii)$ that we may assume that $\tau \leqslant r < p^*$. So, we can find $t \in [0, 1]$, such that

$$\frac{1}{r} \;=\; \frac{1-t}{\tau} + \frac{t}{p^*}.$$

Invoking the interpolation inequality (see Gasiński-Papageorgiou [**15**, p. 905]), we have

$$\|u_n^+\|_r \;\leqslant\; \|u_n^+\|_\tau^{1-t}\|u_n^+\|_{p^*}^t,$$

so

$$\|u_n^+\|_r^r \;\leqslant\; M_6\|u_n^+\|^{tr} \quad \forall n \geqslant 1, \tag{4.11}$$

$M_6 \geqslant 0$ (see (4.10) and use the Sobolev embedding theorem). From hypothesis $H(f)_3(i)$, we have

$$f(\zeta)\zeta \;\leqslant\; c_6\big(1 + |\zeta|^r\big) \quad \forall \zeta \in \mathbb{R}, \tag{4.12}$$

for some $c_6 > 0$. From (4.5), with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we have

$$\|\nabla u_n^+\|_p^p + \lambda\|\nabla u_n^+\|_2^2 - \int_\Omega k_+(z, u_n^+)u_n^+\, dz \;\leqslant\; M_7 \quad \forall n \geqslant 1,$$

for some $M_7 > 0$ (see (4.2)), so

$$\|\nabla u_n^+\|_p^p \;\leqslant\; M_8\big(1 + \|u_n^+\|^{tr}\big) \quad \forall n \geqslant 1, \tag{4.13}$$

for some $M_8 > 0$ (see (4.2), (4.11) and (4.12)). The hypothesis on τ (see $H(f)_3(iii)$) implies that $tr < p$. Hence, from (4.13), it follows that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{4.14}$$

From (4.14) and (4.6), it follows that

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{4.15}$$

If $N = p$, then by definition $p^* = +\infty$, while the Sobolev embedding theorem implies that $W_0^{1,p}(\Omega) \subseteq L^q(\Omega)$ for all $q \in [1, +\infty)$. Then in the above argument we need to replace $p^* = +\infty$ by $q > r$ large, such that $tr = \frac{q(r-\tau)}{q-\tau} < p$. Again, we reach (4.15).

By virtue of (4.15) and by passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{4.16}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{4.17}$$

In (4.5) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (4.16). Then

$$\lim_{n\to+\infty} \big(\langle A_p(u_n), u_n - u\rangle + \lambda\langle A(u_n), u_n - u\rangle\big) \;=\; 0,$$

thus

$$\limsup_{n\to+\infty} \big(\langle A_p(u_n), u_n - u\rangle + \lambda\langle A(u), u_n - u\rangle\big) \;\leqslant\; 0$$

(exploiting the monotonicity of A), so

$$\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle\rangle \;\leqslant\; 0$$

and hence

$$u_n \;\longrightarrow\; u \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.4).

This shows that $\widehat{\psi}_+$ satisfies the Cerami condition. So, we have proved Claim 1.

Claim 2. We may assume that $u_0 \in \operatorname{int} C_+$ is a local minimizer of $\widehat{\psi}_+$.

To this end, we consider the following truncation of $k_+(z,\cdot)$:

$$\widehat{k}_+(z,\zeta) \;=\; \begin{cases} k_+(z,\zeta) & \text{if } \zeta < c_+, \\ k_+(z,c_+) & \text{if } c_+ \leqslant \zeta. \end{cases} \tag{4.18}$$

This is a Carathéodory function. Let

$$\widehat{K}_+(z,\zeta) \;=\; \int_0^\zeta \widehat{k}_+(z,s)\,ds$$

and consider the C^1-functional $\widehat{\gamma}_+ : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\widehat{\gamma}_+(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{2}\|\nabla u\|_2^2 - \int_\Omega \widehat{K}_+\big(z,u(z)\big)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear from (4.18) and (4.2) that $\widehat{\gamma}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$, such that

$$\widehat{\gamma}_+(\widehat{u}_0) \;=\; \inf_{u\in W_0^{1,p}(\Omega)} \widehat{\gamma}_+(u),$$

so

$$\widehat{\gamma}_+'(\widehat{u}_0) \;=\; 0$$

and hence

$$A_p(\widehat{u}_0) + \lambda A(\widehat{u}_0) \;=\; N_{\widehat{k}_+}(\widehat{u}_0). \tag{4.19}$$

On (4.19), we act with $(u_0 - \widehat{u}_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A_p(\widehat{u}_0),\; (u_0 - \widehat{u}_0)^+\rangle + \lambda\langle A(\widehat{u}_0),\; (u_0 - \widehat{u}_0)^+\rangle \\ =\; &\int_\Omega \widehat{k}_+(z,\widehat{u}_0)(u_0 - \widehat{u}_0)^+\,dz \\ =\; &\int_\Omega k_+(z,\widehat{u}_0)(u_0 - \widehat{u}_0)^+\,dz \\ =\; &\int_\Omega f(u_0)(u_0 - \widehat{u}_0)^+\,dz \\ =\; &\langle A_p(u_0), (u_0 - \widehat{u}_0)^+\rangle + \lambda\langle A(u_0), (u_0 - \widehat{u}_0)^+\rangle \end{aligned}$$

(see (4.18), (4.2) and recall that $u_0 \leqslant w_+$), so

$$\int_{\{u_0>\widehat{u}_0\}} \big(\|\nabla u_0\|^{p-2}\nabla u_0 - \|\nabla \widehat{u}_0\|^{p-2}\nabla \widehat{u}_0,\ \nabla u_0 - \nabla \widehat{u}_0\big)_{\mathbb{R}^N}\, dz + \lambda \big\|\nabla (u_n - \widehat{u}_0)^+\big\|_2^2 \;=\; 0$$

and hence

$$\big|\{u_0 > \widehat{u}_0\}\big|_N \;=\; 0, \quad \text{so } u_0 \;\leqslant\; \widehat{u}_0.$$

Also, on (4.19) we act with $(\widehat{u}_0 - c_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \big\langle A_p(\widehat{u}_0),\ (\widehat{u}_0 - c_+)^+\big\rangle + \lambda\big\langle A(\widehat{u}_0),\ (\widehat{u}_0 - c_+)^+\big\rangle \\ = {} & \int_\Omega \widehat{k}_+(z,\widehat{u}_0)(\widehat{u}_0 - c_+)^+\, dz \\ = {} & \int_\Omega k_+(z,w_+)(\widehat{u}_0 - c_+)^+\, dz \\ = {} & \int_\Omega f(w_+)(\widehat{u}_0 - c_+)^+\, dz \\ \leqslant {} & \big\langle A_p(w_+), (\widehat{u}_0 - c_+)^+\big\rangle + \lambda\big\langle A(w_+), (\widehat{u}_0 - c_+)^+\big\rangle \end{aligned}$$

(see (4.18), (4.2) and hypothesis $H(f)_3(ii)$), so

$$\int_{\{\widehat{u}_0>c_+\}} \|\nabla \widehat{u}_0\|^p\, dz + \big\|\nabla(\widehat{u}_0 - c_+)^+\big\|_2^2 \;\leqslant\; 0$$

and thus

$$\big|\{\widehat{u}_0 > c_+\}\big|_N \;=\; 0, \quad \text{so } \widehat{u}_0 \;\leqslant\; c_+.$$

So, we have proved that

$$\widehat{u}_0 \;\in\; [u_0, c_+],$$

where $[u_0, c_+] = \big\{u \in W_0^{1,p}(\Omega):\ u_0(z) \leqslant \widehat{u}_0(z) \leqslant c_+ \text{ for almost all } z \in \Omega\big\}$. This means that (4.19) becomes

$$A_p(\widehat{u}_0) + \lambda A(\widehat{u}_0) \;=\; N_f(\widehat{u}_0)$$

(see (4.18) and (4.2)), so

$$\begin{cases} -\Delta_p \widehat{u}_0(z) - \lambda\Delta\widehat{u}_0(z) \;=\; f\big(\widehat{u}_0(z)\big) & \text{a.e. in } \Omega, \\ \widehat{u}_0|_{\partial\Omega} = 0 \end{cases}$$

and hence $\widehat{u}_0 \in \operatorname{int} C_+$ (nonlinear regularity; see Lieberman [**25**]) is a solution of (4.1).

If $\widehat{u}_0 \neq u_0$, then this is the desired second nontrivial positive smooth solution of (4.1) and so we are done.

Therefore, we may assume that $\widehat{u}_0 = u_0$. Note that

$$\widehat{\gamma}_+\big|_{[0,c_+]} \;=\; \widehat{\psi}_+\big|_{[0,c_+]}$$

and from Lemma 4.3, we know that $u_0(z) < c_+$ for all $z \in \overline{\Omega}$, while $u_0 \in \operatorname{int} C_+$. Therefore $u_0 = \widehat{u}_0$ is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\psi}_+$. Invoking Proposition 2.2, we infer that u_0 is also a local $W_0^{1,p}(\Omega)$-minimizer of $\widehat{\psi}_+$. This proves Claim 2.

As above, we can check that

$$K_{\widehat{\psi}_+} \;\subseteq\; [u_0),$$

where

$$[u_0) = \{u \in W_0^{1,p}(\Omega) : u_0(z) \leqslant u(z) \text{ for almost all } z \in \Omega\}.$$

Also, we may assume that $u_0 \in K_{\widehat{\psi}_+}$ (see Claim 2) is isolated. Indeed, if this is not the case, then we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq K_{\widehat{\psi}_+}$, such that $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$. Then $\{u_n\}_{n\geqslant 1} \subseteq \operatorname{int} C_+$ (nonlinear regularity; see Lieberman [**25**]) are distinct nontrivial positive solutions of (4.1) and so we are done. By virtue of Claim 2, as in Aizicovici-Papageorgiou-Staicu [**1**, proof of Proposition 29] (see also Gasiński-Papageorgiou [**17**, proof of Theorem 3.4]), we can find $\varrho \in (0,1)$ small, such that

$$\widehat{\psi}_+(u_0) < \inf\{\widehat{\psi}_+(u) : \|u-u_0\| = \varrho\} = \widehat{\eta}_\varrho^+. \tag{4.20}$$

Moreover, hypothesis $H(f)_3(iii)$ implies that, if $u \in \operatorname{int} C_+$, then

$$\widehat{\psi}_+(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{4.21}$$

Then Claim 1 and (4.20), (4.21) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$, such that

$$\widehat{u} \in K_{\widehat{\psi}_+} \subseteq [u_0) \quad \text{and} \quad \widehat{\psi}_+(u_0) < \widehat{\eta}_\varrho^+ \leqslant \widehat{\psi}_+(\widehat{u}).$$

Hence, we have

$$A_p(\widehat{u}) + \lambda A(\widehat{u}) = N_f(\widehat{u})$$

(see (4.2)), so $\widehat{u} \in \operatorname{int} C_+$ solves (4.1) and $\widehat{u} \neq u_0$, $u_0 \leqslant \widehat{u}$.

Suppose that $\lambda > 0$. Then for $\varrho = \|\widehat{u}\|_\infty$, we choose $\xi_\varrho > 0$, such that the maps $\zeta \longmapsto f(\zeta) + \xi_p|\zeta|^{p-2}\zeta$ and $\zeta \longmapsto f(\zeta) + \xi_\varrho\zeta$ are both nondecreasing on $[-\varrho, \varrho]$. then

$$\begin{aligned}
&-\Delta_p u_0(z) - \lambda\Delta u_0(z) + \xi_\varrho u_0(z)^{p-1} \\
&= f\big(u_0(z)\big) + \xi_\varrho u_0(z)^{p-1} \\
&\leqslant f\big(\widehat{u}(z)\big) + \xi_\varrho \widehat{u}(z)^{p-1} \\
&= -\Delta_p \widehat{u}(z) - \lambda\Delta\widehat{u}(z) + \xi_\varrho\widehat{u}(z) \quad \text{for almost all } z \in \Omega
\end{aligned} \tag{4.22}$$

(since $u_0 \leqslant \widehat{u}$). The tangency principle of Pucci-Serrin [**28**, p.35] implies that

$$u_0(z) < \widehat{u}(z) \quad \forall z \in \Omega.$$

Since $u_0, \widehat{u} \in \operatorname{int} C_+$, from (4.22) and Proposition 2.6, we infer that

$$\widehat{u} - u_0 \in \operatorname{int} C_+.$$

Now, suppose that the map $\lambda = 0$. Recall that the map $\zeta \longmapsto f(\zeta) + \xi_\varrho\zeta$ is nondecreasing on $[-\varrho, \varrho]$. Then, we have

$$\begin{aligned}
&-\Delta_p u_0(z) + \xi_\varrho u_0(z) \\
&= f\big(u_0(z)\big) + \xi_\varrho u_0(z) \\
&\leqslant f\big(\widehat{u}(z)\big) + \xi_\varrho \widehat{u}(z) \\
&= -\Delta_p \widehat{u}(z) + \xi_\varrho\widehat{u}(z) \quad \text{for almost all } z \in \Omega.
\end{aligned} \tag{4.23}$$

Invoking Theorem 1.4 of Damascelli-Sciunzi [**10**], we infer that

$$u_0(z) < \widehat{u}(z) \quad \forall z \in \Omega$$

and so from (4.22) (with $\lambda = 0$) and Proposition 2.6, once again we infer that

$$\widehat{u} - u_0 \in \operatorname{int} C_+.$$

Similarly, using this time $v_0 \in -\operatorname{int} C_+$ and $c_- < 0$, as above we produce a second nontrivial negative solution $\widehat{v} \in -\operatorname{int} C_+$ with $v_0 - \widehat{v} \in \operatorname{int} C_+$. ☐

Let $\varphi\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the energy functional for problem (4.1), defined by

$$\varphi(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{2}\|\nabla u\|_2^2 - \int_\Omega F\big(u(z)\big) \quad \forall u \in W_0^{1,p}(\Omega).$$

Then $\varphi \in C^2\big(W_0^{1,p}(\Omega)\big)$.

Using hypothesis $H(f)_3(iii)$, with minor modifications in the proof of Proposition 2.6 in Aizicovici-Papageorgiou-Staicu [**2**], we obtain the following result.

PROPOSITION 4.5. *If hypotheses $H(f)_3$ hold, then*

$$C_k(\varphi, \infty) \;=\; 0 \quad \forall k \geqslant 0.$$

Now, we are ready for the complete multiplicity theorem for problem (4.1).

THEOREM 4.6. *If hypotheses $H(f)_3$ hold, then problem (4.1) has at least six nontrivial smooth solutions:*

$$\begin{aligned} &u_0,\ \widehat{u} \;\in\; \operatorname{int} C_+, \quad \textit{with } \widehat{u} - u_0 \;\in\; \operatorname{int} C_+,\\ &v_0,\ \widehat{v} \;\in\; -\operatorname{int} C_+, \quad \textit{with } v_0 - \widehat{v} \;\in\; \operatorname{int} C_+,\\ &y_0 \;\in\; C_0^1(\overline{\Omega}) \textit{ nodal and } \widehat{y} \;\in\; C_0^1(\overline{\Omega}) \setminus \{0\}. \end{aligned}$$

PROOF. From Theorem 3.10 and Proposition 4.4, we already have five nontrivial smooth solutions:

$$\begin{aligned} &u_0,\ \widehat{u} \;\in\; \operatorname{int} C_+, \quad \text{with } \widehat{u} - u_0 \;\in\; \operatorname{int} C_+,\\ &v_0,\ \widehat{v} \;\in\; -\operatorname{int} C_+, \quad \text{with } v_0 - \widehat{v} \;\in\; \operatorname{int} C_+,\\ &y_0 \;\in\; C_0^1(\overline{\Omega}) \text{ nodal with } v_0 \;\leqslant\; y_0 \;\leqslant\; u_0. \end{aligned}$$

Moreover, reasoning as in the proof of Proposition 4.4, using Proposition 2.6, the tangency principle of Pucci-Serrin [**28**, p.35] and Theorem 1.4 of Damascelli-Sciunzi [**10**], we obtain

$$u_0 - y_0 \;\in\; \operatorname{int} C_+ \quad \text{and} \quad y_0 - v_0 \;\in\; \operatorname{int} C_+.$$

We assume that K_φ is finite or otherwise we have an infinity of nontrivial smooth solutions. Let $\widehat{\psi}_+, \widehat{\psi}_-\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functionals introduced in the proof of Proposition 4.4.

Claim. We have

$$C_k(\varphi, \widehat{u}) \;=\; C_k(\widehat{\psi}_+, \widehat{u}) \quad \text{and} \quad C_k(\varphi, \widehat{v}) \;=\; C_k(\widehat{\psi}_-, \widehat{v}) \quad \forall k \geqslant 0.$$

We consider the homotopy

$$h_+(t,u) \;=\; t\varphi(u) + (1-t)\widehat{\psi}_+(u) \quad \forall (t,u) \in [0,1] \times W_0^{1,p}(\Omega).$$

We show that we can find $\varrho \in (0,1)$ small, such that

$$K_{(h_+)_t} \cap \overline{B}_\varrho(\widehat{u}) \;=\; \{\widehat{u}\} \quad \forall t \in [0,1],$$

where

$$(h_+)_t(\cdot) \;=\; h_+(t,\cdot) \quad \text{and} \quad \overline{B}_\varrho(\widehat{u}) \;=\; \big\{u \in W_0^{1,p}(\Omega):\ \|u - \widehat{u}\| \leqslant \varrho\big\}.$$

Arguing by contradiction, suppose that no such $\varrho > 0$ can be found. Then there exist two sequences $\{t_n\}_{n\geqslant 1} \subseteq [0,1]$ and $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$, such that

$$t_n \;\longrightarrow\; t, \quad u_n \;\longrightarrow\; \widehat{u} \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad (h_+)'_{t_n}(u_n) \;= 0 \quad \forall n \geqslant 1.$$

We have

$$A_p(u_n) + \lambda A(u_n) \;=\; t_n N_f(u_n) + (1-t_n) N_{k_+}(u_n),$$

so

$$\begin{cases} -\Delta_p u_n(z) - \lambda \Delta u_n(z) \;=\; t_n f\big(u_n(z)\big) + (1-t_n) k_+\big(z, u_n(z)\big) & \text{a.e. in } \Omega, \\ u_n|_{\partial\Omega} = 0. \end{cases}$$

From Ladyzhenskaya-Uraltseva [**24**, p. 286], we know that we can find $M_9 > 0$, such that

$$\|u_n\|_\infty \;\leqslant\; M_9 \quad \forall n \geqslant 1.$$

From this estimate and the regularity result of Lieberman [**25**, p. 320], we can find $\alpha \in (0,1)$ and $M_{10} > 0$, such that

$$u_n \;\in\; C_0^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \;\leqslant\; M_{10} \quad \forall n \geqslant 1.$$

This estimate and the compactness of the embedding $C_0^{1,\alpha}(\overline{\Omega}) \subseteq C_0^1(\overline{\Omega})$, imply that, passing to a subsequence if necessary, we may assume that

$$u_n \;\longrightarrow\; \widehat{u} \quad \text{in } C_0^1(\overline{\Omega}),$$

so

$$u_n \in [u_0) \quad \forall n \geqslant n_0$$

for some $n_0 \geqslant 1$ (since $\widehat{u} - u_0 \in \operatorname{int} C_+$). Thus K_φ is infinite, a contradiction.

Invoking the homotopy invariance property of critical groups, we have

$$C_k(\varphi, \widehat{u}) \;=\; C_k(\widehat{\psi}_+, \widehat{u}) \quad \forall k \geqslant 0.$$

Similarly, we show that

$$C_k(\varphi, \widehat{v}) \;=\; C_k(\widehat{\psi}_-, \widehat{v}) \quad \forall k \geqslant 0.$$

But from the proof of Proposition 4.4, we know that $\widehat{u}$ is a critical point of $\widehat{\psi}_+$ of mountain pass type. Therefore

$$C_1(\widehat{\psi}_+, \widehat{u}) \;\neq\; 0,$$

so

$$C_1(\varphi, \widehat{u}) \;\neq\; 0$$

and thus

$$C_k(\varphi, \widehat{u}) \;=\; \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.24}$$

(see Gasiński-Papageorgiou [**21**, Theorem 4.1]).

Similarly, we have

$$C_k(\varphi, \widehat{v}) \;=\; \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0. \tag{4.25}$$

Since $\varphi|_{[0,c_+]} = \widehat{\varphi}_+|_{[0,c_+]}$ (see (3.4), with $w_+ = c_+$) and $u_0 \in \operatorname{int} C_+$, $u_0(z) < c_+$ for all $z \in \overline{\Omega}$ (see Lemma 4.3), we infer that u_0 is in fact a local $C_0^1(\overline{\Omega})$-minimizer (see the proof of Proposition 3.6). Invoking Proposition 2.2, we infer that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ. Similarly, we show that $v_0 \in -\operatorname{int} C_+$ is a local minimizer of φ. Therefore

$$C_k(\varphi, u_0) \;=\; C_k(\varphi, v_0) \;=\; \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{4.26}$$

Recall that y_0 is a critical point of mountain pass type for the functional ϑ (see the proof of Proposition 3.9). Since $\vartheta|_{[v_0,u_0]} = \varphi|_{[v_0,u_0]}$ (see (3.18)) and without any loss of generality, assume that u_0, v_0 are the extremal nontrivial constant sign solutions of (4.1) (see Proposition 3.8) and $u_0 - y_0 \in \operatorname{int} C_+$, $y_0 - v_0 \in \operatorname{int} C_+$ (see the beginning of this proof), we have as above

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0 \tag{4.27}$$

(see Gasiński-Papageorgiou [**21**]). Hypothesis $H(f)_3(iv)$ and Proposition 2.1 of Jiu-Su [**23**], imply that

$$C_k(\varphi, 0) = 0 \quad \forall k \geqslant 0. \tag{4.28}$$

Finally, from Proposition 4.5, we have

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0. \tag{4.29}$$

Suppose that $K_\varphi = \{0, u_0, v_0, y_0, \widehat{u}, \widehat{v}\}$. From the Morse relation (2.6) with $t = -1$ and (4.24)-(4.29), we have

$$2(-1)^0 + 3(-1)^1 = 0,$$

a contradiction. So, we can find $\widehat{y} \in K_\varphi$, $\widehat{y} \notin \{0, u_0, v_0, y_0, \widehat{u}, \widehat{v}\}$. Then $\widehat{y} \in C_0^1(\overline{\Omega})$ (nonlinear regularity) solves problem (4.1). □

References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, *Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints*, Mem. Amer. Math. Soc., vol. 196 (2008).

[2] S. Aizicovici, N.S. Papageorgiou, V. Staicu, *On a p-superlinear Neumann p-Laplacian equations*, Topol. Methods Nonlinear Anal., **34** (2009), 111–130.

[3] D. Arcoya, D. Ruiz, *The Ambrosetti-Prodi problem for the p-Laplace operator*, Comm. Partial Differential Equations, **31** (2006), 849–865.

[4] T. Bartsch, Z. Liu, T. Weth, *Nodal solutions of a p-Laplacian equations*, Proc. London Math. Soc., **91** (2005), 129–152.

[5] V. Benci, D. Fortunato, L. Pisani, *Solitons like solutions of a Lorentz invariant equation in dimension* 3, Rev. Math. Phys., **10** (1998), 315–344.

[6] R. Benguria, H. Brézis, H. Lieb, *The Thomas-Fermi-von Weizaäcker theory of atoms and molecules*, Comm. Math. Phys., **79** (1981), 167–180.

[7] H. Brézis, L. Nirenberg, *H^1 versus C^1 local minimizers*, C. R. Acad. Sci. Paris Sér. I Math., **317** (1993), 465–472.

[8] D.G. Costa, C.A. Magalhães, *Existence results for perturbations of the p-Laplacian*, Nonlinear Anal., **24** (1995), 409–418.

[9] M. Cuesta, D.G. de Figueiredo, J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations, **159** (1999), 212–238.

[10] L. Damascelli, B.Sciunzi, *Harnack inequalities, maximum, and comparison principles and regularity of positive solutions of m-Laplace equations*, Calc. Var. Partial Differential Equations, **25** (2005), 139–159.

[11] J.I. Diaz, J.E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math., **305** (1987), 521–524.

[12] N. Dunford and J.T. Schwartz, Linear Operators. I. General Theory, Volume 7 of Pure and Applied Mathematics, Wiley, New York, 1958.

[13] G. Fei, *On periodic solutions of superquadratic Hamiltonian systems*, Electron. J. Differential Equations, **2002** (2002), 1–12.

[14] J. García Azorero, J. Manfredi, I. Peral Alonso, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Commun. Contemp. Math., **2** (2000), 385–404.

[15] L. Gasiński, N.S. Papageorgiou, Nonlinear Analysis, Chapman and Hall/ CRC Press, Boca Raton, FL, 2006.

[16] L. Gasiński and N.S. Papageorgiou, *Existence and multiplicity of solutions for Neumann p-Laplacian-type equations*, Adv. Nonlinear Stud., **8** (2008) 843–870.

[17] L. Gasiński, N.S. Papageorgiou, *Nodal and multiple constant sign solutions for resonant p-Laplacian equations with a nonsmooth potential*, Nonlinear Anal., **71** (2009), 5747–5772.

[18] L. Gasiński, N.S. Papageorgiou, *Nonhomogeneous nonlinear Dirichlet problems with a p-superlinear reaction*, Abstr. Appl. Anal., **2012**, 1–28, Article ID 918271.
[19] L. Gasiński, N.S. Papageorgiou, *Multiple solutions for nonlinear coercive problems with a nonhomogeneous differential operator and a nonsmooth potential*, Set-Valued Anal., **20** (2012), 417–443.
[20] L. Gasiński, N.S. Papageorgiou, *On generalized logistic equations with a non-homogeneous differential operator*, Dyn. Syst., **29** (2014), 190–207.
[21] L. Gasiński, N.S. Papageorgiou, *Multiplicity theorems for $(p, 2)$-equations*, submitted.
[22] L. Iturriaga, E. Massa, J. Sánchez, P. Ubilla, *Positive solutions of the p-Laplacian involving a superlinear nonlinearity with zeros*, J. Differential Equations, **248** (2010), 309–327.
[23] Q.-S. Jiu, J.-B. Su, *Existence and Multiplicity Results for Dirichlet Problems with p-Laplacian*, J. Math. Anal. Appl., **281** (2003), 587–601.
[24] O.A. Ladyzhenskaya, N. Uraltseva, Linear and Quasilinear Elliptic Equations, Vol. 46 of Mathematics in Science and Engineering, Academic Press, New York, 1968.
[25] G.M. Lieberman, *The natural generalizations of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations*, Comm. Partial Differential Equations, **16** (1991), 311–361.
[26] M. Lucia, S. Prashanth, *Strong comparison principle for solutions of quasilinear equations*, Proc. Amer. Math. Soc., **132** (2004), 1005–1011.
[27] D. Motreanu, N.S. Papageorgiou, *Multiple solutions for nonlinear Neumann problems driven by a nonhomogeneous differential operators*, Proc. Amer. Math. Soc., **139** (2011), 3527–3535.
[28] P. Pucci, J. Serrin, The Maximum Principle, Birkhäuser Verlag, Basel, 2007.
[29] J.L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., **12** (1984), 191–202.

Jagiellonian University, Faculty of Mathematics and Computer Science, ul. Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: `Leszek.Gasinski@ii.uj.edu.pl`

National Technical University, Department of Mathematics Zografou Campus, Athens 15780, Greece
E-mail address: `npapg@math.ntua.gr`