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A regularity result for a linear elliptic equation with Hardy-type potential

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Abstract. We consider a linear elliptic problem whith Dirichlet boundary conditions, with a potential term $b(x)u$ where the potential function b behaves as $\frac{1}{\mathrm{dist}^2(x,\partial\Omega)}$ close to the boundary. We study the effect of this potential term on the H^2 regularity of the solution of the problem. An application to a stationnary Fokker-Planck-Smoluchowski equation for FENE models of diluted polymers is given.

Contents

1. Introduction

We consider Ω an open, bounded and regular set in $\mathbb{R}^{n+1}$, with $n \in \mathrm{N}$, and $u \in H_0^1(\Omega)$ a solution of the equation

$$-\Delta u + b(x)u = f \quad \text{in } \mathcal{D}'(\Omega). \tag{1.1}$$

In the above f is an element of $H^{-1}(\Omega)$ and the function b is positive, regular and "behaves as $\frac{1}{\mathrm{dist}^2(x,\partial\Omega)}$ for x close to $\partial\Omega$".

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The goal of this paper is to study the influence of the term $b(x)u$ on the H^2 - regularity of the solution of the problem.
Suppose for simplicity that the function b is given by the expression:

$$b(x) = \frac{b^*}{\text{dist}^2(x, \partial\Omega)}$$

where $b^* > 0$ is a constant (more general forms of b are given in Section 3). We prove that, under the supplementary hypothesis $b^* > \frac{3}{4}$, the solution u belongs to H^2 for any $f \in L^2(\Omega)$, and we also have an appropriate inequality. Moreover, we prove that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, that is, $u \in H_0^2(\Omega)$.

The H^2 regularity of u in the case $b \equiv 0$ is a classical result; nevertheless, remark that the regularity result presented in this paper is not an obvious consequence of this classical regularity, since in general the term $b(x)u$ is not an element of $L^2(\Omega)$ under the hypothesis $u \in H_0^1(\Omega)$.

To the best of our knowledge, the H^2 regularity of the solution of such a problem was never studied in the past. For a related work, we mention the paper [**1**] where a problem like (1.1) is considered, with a function b of the form $b(x) = \frac{b^*}{\|x\|^2}$ with $b^* \in \mathbb{R}$ and $0 \in \Omega$. In that paper the authors suppose that b^* is negative with small enough absolute value and they study the $W^{1,p}$ regularity of u under L^q regularity hypothesis on f, with $p, q \geq 1$.

Our paper is organized as follows: In Section 2 we consider the one dimensional case (n = 0) while in Section 3 the case $n \in \mathbb{N}^*$ is treated. The proof for $n \in \mathbb{N}^*$ is based on the result obtained in the one dimensional case. Notice that these results are stated in a more general setting than described in the beginning of this section. In Section 4 we give an application to a stationnary Fokker-Planck-Smoluchowski equation for FENE models of diluted polymers; this was the initial motivation for the result obtained in this paper.

2. The one dimensional case

In this section we denote $\Omega =]0, a[$ with $a > 0$ a given number.
Let $f \in L^2(\Omega)$, $u \in L^2(\Omega)$ and $b \in \mathbb{R}$ satisfying

$$-u''(x) + \frac{b}{x^2}u(x) = f(x) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.1}$$

Our goal is to prove a H^2 - regularity result for u.

The main result of this section is the following:

THEOREM 2.1. *Suppose that $b > \frac{3}{4}$ and let us denote*

$$\alpha \equiv \alpha(b) = \frac{1}{2}(1 + \sqrt{1 + 4b}).$$

Then for any $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$ satisfying (2.1) *we have:*

$$u \in H^2(\Omega) \quad \textit{and} \quad u(0) = u'(0) = 0.$$

We also have the estimate

$$\|u\|_{H^2(\Omega)} \leq C_1 \|u\|_{L^2(\Omega)} + C_2 \|f\|_{L^2(\Omega)} \tag{2.2}$$

where

$$C_1 = C_1(a, b) = 1 + \frac{a}{\sqrt{2}(2\alpha - 1)}\left(\frac{\alpha}{\sqrt{2\alpha - 3}} + \frac{\alpha - 1}{\sqrt{2\alpha + 1}}\right) + \frac{4\alpha(\alpha - 1)}{(2\alpha + 1)(2\alpha - 3)} +$$

$$+\frac{\alpha}{2(2\alpha-1)}\left(1+\sqrt{\frac{2\alpha+1}{2\alpha-3}}\right)\left(\frac{\alpha-1}{\sqrt{2\alpha-3}}+\frac{a}{\sqrt{2\alpha-1}}\right)$$

and

$$C_2 = C_2(a,b) = 1+\frac{\alpha\sqrt{2\alpha+1}}{a^2}\left(\frac{a}{\sqrt{2\alpha-1}}+\frac{\alpha-1}{\sqrt{2\alpha-3}}\right).$$

Proof. We solve the equation (2.1) by using the change of variables

$$x = e^t \quad \text{with } t \in]-\infty, \log a[.$$

Since

$$\frac{d^2u}{dx^2} = \frac{1}{x^2}\frac{d^2u}{dt^2} - \frac{1}{x^2}\frac{du}{dt}$$

we obtain from (2.1):

$$\frac{d^2u}{dt^2} - \frac{du}{dt} - bu = -e^{-2t}f. \tag{2.3}$$

We easily find that the general solutions of (2.3) is given by

$$u(t) = e^{\alpha t}\left[\beta_1 - \frac{1}{2\alpha-1}\int_{\log(a)}^{t} e^{(2-\alpha)t'}f(t')\,dt'\right]$$
$$+e^{(1-\alpha)t}\left[\beta_2 + \frac{1}{2\alpha-1}\int_{-\infty}^{t} e^{(1+\alpha)t'}f(t')\,dt'\right]$$

with $\beta_1, \beta_2 \in \mathbb{R}$ arbitrary. Now passing in the variable x we obtain

$$\begin{aligned} u(x) &= x^{\alpha}\left[\beta_1 + \frac{1}{2\alpha-1}\int_x^a f(x')(x')^{1-\alpha}\,dx'\right] \\ &\quad +x^{1-\alpha}\left[\beta_2 + \frac{1}{2\alpha-1}\int_0^x f(x')(x')^{\alpha}\,dx'\right]. \end{aligned} \tag{2.4}$$

The expression of u can be written in the form

$$u(x) = \beta_1 x^{\alpha} + \frac{1}{2\alpha-1}x^{\alpha}u_1(x) + \frac{1}{2\alpha-1}x^{1-\alpha}u_2(x) + \beta_2 x^{1-\alpha}$$

with

$$u_1(x) = \int_x^a f(x')(x')^{1-\alpha}\,dx'$$

and

$$u_2(x) = \int_0^x f(x')(x')^{\alpha}\,dx'.$$

From the hypothesis $b > \frac{3}{4}$ we deduce

$$\alpha > \frac{3}{2}, \tag{2.5}$$

then the function $\beta_1 x^{\alpha}$ is an element of $H^2(\Omega)$. Let us prove that $x^{\alpha}u_1, x^{1-\alpha}u_2 \in L^2(\Omega)$. We are the following inequalities:

$$|u_1(x)| \le \left[\int_x^a (x')^{2-2\alpha}\,dx'\right]^{1/2} \|f\|_{L^2(\Omega)}$$

which gives

$$|u_1(x)| \le \frac{x^{3/2-\alpha}}{\sqrt{2\alpha-3}}\|f\|_{L^2(\Omega)}, \tag{2.6}$$

and we obtain in the same manner

$$|u_2(x)| \leq \frac{x^{1/2+\alpha}}{\sqrt{2\alpha+1}}\|f\|_{L^2(\Omega)}. \tag{2.7}$$

We deduce from the above inequalities that $x^\alpha u_1, x^{1-\alpha}u_2 \in L^2(\Omega)$ with

$$\|x^\alpha u_1\|_{L^2(\Omega)} + \|x^{1-\alpha}u_2\|_{L^2(\Omega)} \leq \frac{a^2}{2}\left(\frac{1}{\sqrt{2\alpha+1}} + \frac{1}{\sqrt{2\alpha-3}}\right)\|f\|_{L^2(\Omega)}. \tag{2.8}$$

Since u must be an element of $L^2(\Omega)$ we necessarily have $\beta_2 = 0$, that is, u is given by

$$u(x) = \beta_1 x^\alpha + \frac{1}{2\alpha-1}x^\alpha u_1(x) + \frac{1}{2\alpha-1}x^{1-\alpha}u_2(x). \tag{2.9}$$

We easily compute

$$u'(x) = \alpha\beta_1 x^{\alpha-1} + \frac{\alpha}{2\alpha-1}x^{\alpha-1}u_1(x) + \frac{1-\alpha}{2\alpha-1}x^{-\alpha}u_2(x) \tag{2.10}$$

and

$$u''(x) = -f(x) + \alpha(\alpha-1)\beta_1 x^{\alpha-2} + \frac{\alpha(\alpha-1)}{2\alpha-1}\left[x^{\alpha-2}u_1(x) + x^{-1-\alpha}u_2(x)\right]. \tag{2.11}$$

With the help of (2.6) and (2.7) we obtain:

$$|u'(x)| \leq \alpha|\beta_1|x^{\alpha-1} + \left[\frac{\alpha}{2\alpha-1}\frac{1}{\sqrt{2\alpha-3}} + \frac{\alpha-1}{2\alpha-1}\frac{1}{\sqrt{2\alpha+1}}\right]x^{1/2}\|f\|_{L^2(\Omega)}, \quad \forall\, x \in \Omega$$

and we deduce that $u' \in L^2(\Omega)$ with

$$\|u'\|_{L^2(\Omega)} \leq |\beta_1|\frac{\alpha}{\sqrt{2\alpha-1}}a^{\alpha-1/2} + \frac{a}{\sqrt{2}(2\alpha-1)}\left(\frac{\alpha}{\sqrt{2\alpha-3}} + \frac{\alpha-1}{\sqrt{2\alpha+1}}\right)\|f\|_{L^2(\Omega)}. \tag{2.12}$$

Now using the Hardy inequalities (see for exemple Lemma 6.2.1 of [**7**]), we infer that $x^{\alpha-2}u_1,\ x^{-1-\alpha}u_2 \in L^2(\Omega)$ and

$$\|x^{\alpha-2}u_1\|_{L^2(\Omega)} \leq \frac{2}{2\alpha-3}\|f\|_{L^2(\Omega)}$$

$$\|x^{-1-\alpha}u_2\|_{L^2(\Omega)} \leq \frac{2}{2\alpha+1}\|f\|_{L^2(\Omega)}.$$

Then from (2.11) we deduce that $u'' \in L^2(\Omega)$ and

$$\|u''\|_{L^2(\Omega)} \leq |\beta_1|\frac{\alpha(\alpha-1)}{\sqrt{2\alpha-3}}a^{\alpha-3/2} + \left[1 + \frac{4\alpha(\alpha-1)}{(2\alpha+1)(2\alpha-3)}\right]\|f\|_{L^2(\Omega)} \tag{2.13}$$

□

3. The general dimensional case

In this section we consider $n \in \mathbb{N}^*$ and we denote by Ω an open bounded domain included in $\mathbb{R}^{n+1}$, with boundary of class C^2.

Let us denote $\Gamma = \partial\Omega$; for any $x \in \Gamma$ we denote $\nu \equiv \nu(x) \in \mathbb{R}^{n+1}$ the normal vector in x to Γ oriented to the interior of Ω. For any $\epsilon > 0$ we denote

$$\Sigma_\epsilon = \{x \in \Omega;\ \ dist(x,\Gamma) < \epsilon\}.$$

It is well-known that there exists $\epsilon_0 > 0$ small enough such that for any $\epsilon \in]0, \epsilon_0[$ we have

$$\Sigma_\epsilon = \{x + s\nu(x),\ x \in \Gamma,\ s \in]0, \epsilon[\}.$$

We also denote for any $\epsilon \in]0, \epsilon_0[$

$$\tilde{\Gamma}_\epsilon = \{x + \epsilon\nu(x),\ x \in \Gamma\}.$$

Let us now give $a > 0$ and consider $f \in L^2(\Sigma_a)$ and $u \in H^1(\Sigma_a)$ satisfying

$$\begin{cases} -\Delta u + b(x)u = f & \text{in} \quad H^{-1}(\Sigma_a) \\ u = 0 & \text{for} \quad x \in \Gamma \end{cases} \tag{3.1}$$

In the above $b : \Sigma_a \to \mathbb{R}$ is a given regular enough function which "behaves as $\frac{1}{\mathrm{dist}^2(x,\Gamma)}$ for x close" to Γ. For simplicity we suppose that there exists a function $b_0 : \Gamma \to \mathbb{R}$ with $b_0 \in C^2(\Gamma)$, such that

$$b(x + s\nu(x)) = \frac{b_0(x)}{s^2}, \quad \forall\, x \in \Gamma,\ \forall\, s \in]0, \min\{\epsilon_0, a\}[. \tag{3.2}$$

In all this section we denote by C a generic positive constant.
We have the following preliminary result:

LEMMA 3.1. *Let us consider $f \in L^2(\Sigma_a)$ and $u \in H^1(\Sigma_a)$ satisfying* (3.1).
a) *For any $a_1 \in]0, a[$ we have*

$$\|u\|_{H^1(\Sigma_{a_1})} \leq C \left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.3}$$

b) *For any a_1, a_2 with $0 < a_1 < a_2 < a$, we have $u \in H^2(\Sigma_{1,2})$ and*

$$\|u\|_{H^2(\Sigma_{1,2})} \leq C \left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.4}$$

where we denoted $\Sigma_{1,2} = \Sigma_{a_2} - \overline{\Sigma_{a_1}}$.

PROOF. **a)** We consider a cut-off function $\varphi \in C^\infty(\overline{\Sigma_a})$ such that $\varphi \equiv 1$ on Σ_{a_1} and $\varphi \equiv 0$ on $\partial(\Sigma_a) - \Gamma$. It is clear that $\tilde{u} \equiv u\varphi$ satisfies

$$\begin{cases} -\Delta \tilde{u} + b(x)\tilde{u} = \tilde{f} & \text{in} \quad \mathcal{D}'(\Sigma_a) \\ \tilde{u} = 0 & \text{for} \quad x \in \partial\Sigma_a \end{cases} \tag{3.5}$$

where $\tilde{f} = f\varphi - 2\nabla u \cdot \nabla\varphi - u\Delta\varphi$. We have that $\tilde{f}$ is an element of $H^{-1}(\Sigma_a)$ and

$$\|\tilde{f}\|_{H^{-1}(\Sigma_a)} \leq c \left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.6}$$

From the Hardy inequality

$$\left\| \frac{v}{dist(x,\Gamma)} \right\|_{L^2(\Sigma_a)} \leq c\|v\|_{H^1(\Sigma_a)}, \quad \forall v \in H^1_0(\Sigma_a) \tag{3.7}$$

we easily deduce that the problem (3.5) has an unique solution $\tilde{u} \in H^1_0(\Sigma_a)$ for any $\tilde{f} \in H^{-1}(\Sigma_a)$ and

$$\|\tilde{u}\|_{H^1(\Sigma_a)} \leq c\|\tilde{f}\|_{H^{-1}(\Sigma_a)}.$$

With the help pf (3.6) we obtain the expected result.
b) This part is obvious by interior regularity.

□

We can now state the following result

THEOREM 3.2. *Suppose that $b_0(x) > \frac{3}{4}\ \forall\, x \in \Gamma$. Then for any $f \in L^2(\Sigma_a)$ and $u \in H^1(\Sigma_a)$ satisfying* (3.1) *we have $u \in H^2(\Sigma_a)$ with*

$$\|u\|_{H^2(\Sigma_a)} \leq C \left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.8}$$

where $C \geq 0$ is a constant independent on u and f.
We also have

$$\frac{\partial u}{\partial \nu} = 0 \quad on \ \Gamma. \tag{3.9}$$

PROOF. From Lemma 3.1 **b)** it suffices to prove that $u \in H^2(\Sigma_{s_0})$ for $s_0 > 0$ small enough.
Let us consider the open sets $\Omega_1, \Omega_2, \cdots \Omega_N$ such that

$$\Sigma_{s_0} \subset \Omega_1 \cup \Omega_2 \cdots \Omega_N$$

and $\varphi_1, \varphi_2, \cdots \varphi_N$ a partition of unity associated to $\Omega_1, \Omega_2, \cdots \Omega_N$, with $\varphi_k \in \mathcal{D}(\Omega_k)$ and

$$0 \leq \varphi_k \leq 1, \qquad \sum_{k=1}^{N} \varphi_k = 1 \ \text{ on } \ \Sigma_{s_0}.$$

Let us denote for any $k \in \{1, 2, \cdots N\}$:

$$\tilde{\Omega}_k = \Omega_k \cap \Sigma_{s_0}, \qquad \Gamma_k = \Omega_k \cap \Gamma$$

and

$$u_k = u\,\varphi_k.$$

Since $u = \sum_{k=1}^{N} u_k$, to obtain the regularity result it suffices to prove that $u_k \in H^2(\tilde{\Omega}_k)$ and to obtain appropriate estimates for $\|u_k\|_{H^2(\tilde{\Omega}_k)}$.
It is clear that

$$\left\{ \begin{array}{ccc} -\Delta u_k + b u_k = f_k & \text{in} & \tilde{\Omega}_k \\ u_k = 0 & \text{on} & x \in \partial\tilde{\Omega}_k - \tilde{\Gamma}_{s_0} \end{array} \right. \tag{3.10}$$

where we denoted

$$f_k = f\varphi - 2\nabla u \cdot \nabla \varphi_k - u\Delta\varphi_k.$$

We can suppose that

$$\tilde{\Omega}_k = \{x + s\nu(x), \ \ x \in \Gamma_k, \ \ s \in [0, a]\}.$$

We also suppose that every set Γ_k is an n-dimensional C^2 - manifold which can be written in the following manner:

$$\Gamma_k = \{g^k(t), \quad t \in T_k\}$$

where T_k is a bounded regular open set in $\mathbb{R}^n$ and $g^k : T_k \to \mathbb{R}^{n+1}$ is an injective and C^2 - function (for exemple Γ_k can be defined as a C^2 - graph in an appropriate local coordinates system).
We now introduce the vectors in $\mathbb{R}^{n+1}$:

$$\tau_j \equiv \tau_j(t) = \frac{\partial g^k}{\partial t_j}(t), \quad j = 1, \cdots n$$

These vectors are tangent to the manifold Γ_k and we suppose that $\tau_1(t), \cdots \tau_n(t)$ are independent in $\mathbb{R}^{n+1}$ for any $t \in T_k$. Let us define

$$\tilde{\nu} \equiv \tilde{\nu}(t) = \tau_1 \wedge \tau_2 \cdots \wedge \tau_n$$

(the vectorial product in $\mathbb{R}^{n+1}$) where we recall that $\tilde{\nu}_j = (-1)^{k+1} \det(\mathbf{M}_j)$ with $\mathbf{M}_j$ the $n \times n$ matrix obtained from $\mathbf{M}$ by suppresing the line j, where $\mathbf{M}$ is the

$(n+1)\times n$ matrix whose l-column is τ_l.
It is clear that $\tilde{\nu}$ is normal to Γ_k and we suppose that ν is given by

$$\nu = (-1)^n \frac{\tilde{\nu}}{d}$$

where we denote $d \equiv d(t) = \|\tilde{\nu}(t)\| > 0$ (since $\tilde{\nu}(t) \neq 0$).

Let us denote $Q = T_k \times [0, s_0]$. We suppose that $s_0 > 0$ is small enough such that the function $\theta : Q \to \tilde{\Omega}_k$ given by

$$\theta(t,s) = g(t) + s\nu(t)$$

is injective. It is easy to see that the Jacobian matrix of θ is the matrix $\mathbf{A} \equiv \mathbf{A}(t,s)$ given by

$$\mathbf{A}(t,s) = \mathbf{A}_0(t) + s\mathbf{J}(t)$$

where $\mathbf{A}_0$ and $\mathbf{J}_0$ are written by columns as

$$\mathbf{A}_0 \equiv \mathbf{A}_0(t) = (\tau_1(t)\cdots\tau_n(t) \;\; \nu(t))$$

and

$$\mathbf{J}_0 \equiv \mathbf{J}_0(t) = \left(\frac{\partial \nu}{\partial t_1}(t) \cdots \frac{\partial \nu}{\partial t_n}(t) \;\; 0 \right).$$

We also have

$$\det(\mathbf{A}_0) = \sum_{j=1}^{n+1} (-1)^{n+j+1} \nu_j \det(\mathbf{M}_j) = \sum_{j=1}^{n+1} (-1)^n \nu_j \tilde{\nu}_j = d(t) > 0, \quad \forall\, t \in T_k.$$

By continuity we deduce that there exists $d_0 > 0$ such that

$$\det(\mathbf{A}(t,s)) \geq d_0 \quad \text{on} \quad Q$$

for s_0 small enough. Then the function θ is a diffeomorphisme between Q and $\tilde{\Omega}_k$.
Let us denote $\mathbf{B} \equiv \mathbf{B}(t,s) = \mathbf{A}^{-1}$. It is well-known that the laplacian operator on $\tilde{\Omega}_k$ is given in coordinates (t,s) by

$$\Delta_x v = L(v)$$

where we set

$$L(v) = \frac{1}{\det \mathbf{A}} \nabla_{t,s} \cdot \left[(\det \mathbf{A}) \mathbf{B}\mathbf{B}^T \nabla_{t,s} v \right] \tag{3.11}$$

Then the equation (3.10) is written in coordinates (t,s) under the form

$$\begin{cases} -L(u_k) + \frac{b_0(t)}{s^2} u_k = f_k & \text{in } Q \\ u_k = 0 & \text{on } \partial Q - \{s = s_0\}. \end{cases} \tag{3.12}$$

It is clear that for s_0 small enough, we have for any $m \in \mathrm{N}$ that
$v \in H^m(\tilde{\Omega}_k) \Longleftrightarrow v \in H^m(Q)$ with equivalence of norms.
Let us observe that

$$\mathbf{A}_0^T \mathbf{A}_0 = \begin{pmatrix} \mathbf{R}(t) & 0 \\ 0 & 1 \end{pmatrix} \tag{3.13}$$

where $\mathbf{R} \equiv \mathbf{R}(t)$ is the $n \times n$ invertible matrix given by

$$\mathbf{R}_{ij}(t) = \tau_i(t) \cdot \tau_j(t), \quad i,j = 1, \cdots n.$$

Since $\mathbf{B}\mathbf{B}^T = (\mathbf{A}^T\mathbf{A})^{-1}$ we deduce

$$\mathbf{B}\mathbf{B}^T = \left(\mathbf{A}_0^T \mathbf{A}_0\right)^{-1} + sO(1)$$

where we denote in all this paper by O(1) terms which are regular enough on $\overline{\Sigma_{s_0}}$. We remark that

$$\det \mathbf{A} = d(t) + sO(1).$$

By a direct calculus we can prove that the operator L can be written in the form

$$L(v) = \frac{\partial^2 v}{\partial s^2} + L_0(v) + L_1(v) + sL_2(v) \tag{3.14}$$

where

$$L_0(v) = \frac{1}{d}\nabla_t \cdot (d\mathbf{R}\nabla_t v)$$

L_1 is a first order linear differential operator in (t,s)
L_2 is a second order linear differential operator in (t,s).
The goal is now to prove that $f_k, L_0(u_k), L_1(u_k)$ and $sL_2(u_k)$ are in $L^2(Q)$, to obtain appropiate L^2- estimates for these expressions and to conclude using the results of Theorem 2.1.
From Lemma 3.1 we deduce that

$$\|u_k\|_{H^1(Q)} \leq C\left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.15}$$

and

$$\|u_k\|_{H^{3/2}(\{s=s_0\})} \leq C\left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right] \tag{3.16}$$

which allows to obtain $f_k,\ L_1(u_k) \in L^2(Q)$ with

$$\|f_k\|_{L^2(Q)} + \|L_1(u_k)\|_{L^2(Q)} \leq C\left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}.\right] \tag{3.17}$$

Let us now observe that for any v we have

$$L(sv) = sL(v) + L_3(v) \tag{3.18}$$

and

$$L_2(sv) = sL_2(v) + L_4(v) \tag{3.19}$$

with L_3, L_4 first order linear differential operators in (t,s). Then multiplying (3.12) by s and using (3.18) we obtain

$$\begin{cases} -L(s\,u_k) = s\,f_k - \frac{b_0(t)}{s}u_k - L_3(u_k) & \text{in } Q \\ s\,u_k = 0 & \text{on } \partial Q - \{s=s_0\}. \end{cases} \tag{3.20}$$

Now we have the following Hardy inequality

$$\left\|\frac{v}{s}\right\|_{L^2(Q)} \leq C\|v\|_{H^1(Q)} \quad \forall\, v \in H^1(Q) \text{ with } v=0 \text{ on } \{s=0\} \tag{3.21}$$

which implies

$$\left\|\frac{v}{s^2}\right\|_{H^{-1}(Q)} \leq C\|v\|_{H^1(Q)} \quad \forall\, v \in H^1(Q) \text{ with } v=0 \text{ on } \{s=0\}. \tag{3.22}$$

From (3.15), (3.16) and (3.21) we deduce by classical regularity that
$s\,u_k \in H^2(Q)$. Then using (3.19) we deduce that $s\,L_2(u_k) \in L^2(Q)$ and

$$\|s\,L_2(u_k)\|_{L^2(Q)} \leq C\left[\|u\|_{L^2(\Sigma_a)} + \|f\|_{L^2(\Sigma_a)}\right]. \tag{3.23}$$

On the other hand, we remark that for any $j = 1, \cdots n$ we have

$$\frac{\partial}{\partial t_j}L(v) = L\left(\frac{\partial v}{\partial t_j}\right) + L_5(v)$$

with L_5 a second order linear differential operators in (t,s). Now deriving (3.12) with respect to t_j we deduce

(3.24)
$$\begin{cases} -L\left(\frac{\partial u_k}{\partial t_j}\right)+\frac{b_0(t)}{s^2}\frac{\partial u_k}{\partial t_j}=\frac{\partial f_k}{\partial t_j}+L_5(u_k)-\frac{u_k}{s^2}\frac{\partial b_0}{\partial t_j} & \text{in } Q \\ \frac{\partial u_k}{\partial t_j}=0 & \text{on } \partial Q-\{s=s_0\}. \end{cases}$$

Denoting by g_k the right-hand part of the first equation of (3.24) we prove, with the help of (3.22), that $g_k\in H^{-1}(Q)$ and

$$\|g_k\|_{H^{-1}(Q)}\le C\left[\|u\|_{L^2(\Sigma_a)}+\|f\|_{L^2(\Sigma_a)}\right].$$

We now use the fact that the operator $-Lv+\frac{b_0}{s^2}v$ is an isomorphisme from $H_0^1(Q)$ to $H^{-1}(Q)$ and that $\frac{\partial u_k}{\partial t_j}\in H^{1/2}(\{s=s_0\})$ with

$$\left\|\frac{\partial u_k}{\partial t_j}\right\|_{H^{1/2}(\{s=s_0\})}\le C\left[\|u\|_{L^2(\Sigma_a)}+\|f\|_{L^2(\Sigma_a)}\right]$$

as a consequence of Lemma 3.1. Then for any $j=1,\cdots n$ we deduce that $\frac{\partial u_k}{\partial t_j}\in H^1(Q)$, and we obtain $L_0(u_k)\in L^2(Q)$ with

$$\|L_0(u_k)\|_{L^2(Q)}\le C\left[\|u\|_{L^2(\Sigma_a)}+\|f\|_{L^2(\Sigma_a)}\right]. \tag{3.25}$$

Now the equation (3.12) can be written in the form

$$\begin{cases} -\frac{\partial^2 u_k}{\partial s^2}+\frac{b_0(t)}{s^2}u_k=h_k & \text{in } Q \\ u_k(s=0)=0 \end{cases} \tag{3.26}$$

with

$$h_k=f_k+L_1(u_k)+sL_2(u_k)+L_0(u_k).$$

From (3.17), (3.23) and (3.25) we deduce that $h_k\in L^2(Q)$ and

$$\|h_k\|_{L^2(Q)}\le C\left[\|u\|_{L^2(\Sigma_a)}+\|f\|_{L^2(\Sigma_a)}\right] \tag{3.27}$$

which allows to write $h_k(t,\cdot)\in L^2(]0,s_0[)$ a.e. $t\in T$.
Since $u_k(t,\cdot)\in L^2(]0,s_0[)$ we can apply Theorem 2.1 and deduce that $u_k(t,\cdot)\in H^2(]0,s_0[)$ a.e. $t\in T$. We also have a.e. $t\in T_k$:

$$\left\|\frac{\partial^2 u_k(t,\cdot)}{\partial s^2}\right\|_{L^2(]0,s_0[)}\le C_1(s_0,\alpha)\|h_k(t,\cdot)\|_{L^2(]0,s_0[)}+C_2(s_0,\alpha)\|u_k(t,\cdot)\|_{L^2(]0,s_0[)}$$

with $\alpha\equiv\alpha(t)=\frac{1}{2}\left(1+\sqrt{b_0(t)}\right)$ and C_1, C_2 given in Theorem 2.1. We also obtain $\frac{\partial u_k}{\partial s}(t,0)=0$ a.e. $t\in T_k$ which gives (3.9), where we use the fact that $\mathbf{B}(t,0)\nu=(0,\cdots,0,1)^T$. Integrating in T_k and using (3.27) we easily obtain that u belongs to $H^2(\Sigma_a)$ and satisfies (3.8). □

4. An application to a FENE model for diluted polymers

We consider the stationary Fokker-Planck-Smoluchowski equation in $\mathbb{R}^d$, $d=2$ or $d=3$, which comes from the modelisation of the diluted polymers where the molecules are considered as elastic springs (**FENE** models, see for exemple [**3**], [**4**], [**5**] and [**6**]). We suppose that

(1) The length of the molecules are no larger than a physical constant supposed equal to 1 by normalization.

(2) The gradient of the velocity of the fluid is a constant traceless $d\times d$ matrix denoted by $\mathbf{G}$.
(3) The force in the elastic springs is given by

$$F(x)=\frac{2\delta x}{1-\|x\|^2},\quad \forall x\in B$$

where we denote $\delta>0$ a physical constant and $B=B(0,1)$ the ball in $\mathbb{R}^d$ centered in 0 with radius 1.

We search for a density probability ψ defined on B solution of

$$\left\{\begin{array}{lll} -\Delta\psi-\nabla\cdot[F(x)\psi-\mathbf{G}x\psi]=0 & \text{for} & x\in B\\ \frac{\partial\psi}{\partial x}+[F(x)\psi-\mathbf{G}x\psi]\cdot x=0 & \text{for} & x\in\partial B\end{array}\right.\tag{4.1}$$

Remark that $F(x)=\nabla\phi(x)$ with $\phi(x)=-\delta\log{(1-\|x\|^2)}$, then the problem (4.1) can be written in the form

$$\left\{\begin{array}{lll} -\nabla\cdot\left[M\nabla\left(\frac{\psi}{M}\right)\right]+\nabla\cdot(\mathbf{G}x\psi)=0 & \text{for} & x\in B\\ \left[-M\nabla\left(\frac{\psi}{M}\right)+\mathbf{G}x\psi\right]\cdot x=0 & \text{for} & x\in\partial B\end{array}\right.\tag{4.2}$$

where we denote $M(x)=(1-\|x\|^2)^\delta$. This equation has to be completed by the conditions:

$$\psi\geq 0\tag{4.3}$$

$$\int_B\psi(x)\,dx=q\tag{4.4}$$

with $q>0$ a given constant.

Let us now introduce the following functional spaces

$$L^2_M\equiv L^2_M(B):=\left\{\varphi\in L^1_{\text{loc}}(B),\ \int_B\frac{\varphi^2}{M}\,dx<\infty\right\}$$

$$H^1_M\equiv H^1_M(B):=\left\{\varphi\in L^1_{\text{loc}}(B),\ \int_B\left[\frac{\varphi^2}{M}+M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2\right]\,dx<\infty\right\}.$$

Then the variational formulation of the problem (4.2) is: find $\psi\in H^1_M$ such that

$$\int_B\left[M\nabla\left(\frac{\psi}{M}\right)\cdot\nabla\left(\frac{\varphi}{M}\right)-\mathbf{G}x\psi\cdot\nabla\left(\frac{\varphi}{M}\right)\right]\,dx=0\quad\forall\varphi\in H^1_M.\tag{4.5}$$

The existence and uniqueness of a solution of (4.5) satisfying also (4.3) and (4.4) was given in [**2**] and [**3**].

The goal of this section is to give a supplementary regularity result for ψ. Let us begin by the following preliminary result:

LEMMA 4.1. *For any $\delta>1$ we have*

$$H^1_M=\left\{\varphi\in L^1_{loc}(B),\ \frac{\varphi}{\sqrt{M}}\in H^1_0(B)\right\}$$

and there exist constants $0<c_1<c_2$ such that

$$c_1\left\|\frac{\varphi}{\sqrt{M}}\right\|_{H_1(B)}\leq\|\varphi\|_{H^1_M}\leq c_2\left\|\frac{\varphi}{\sqrt{M}}\right\|_{H_1(B)},\quad\forall\varphi\in H^1_M.\tag{4.6}$$

Proof. Let us consider $\varphi \in H^1_M$ arbitrary. We have

$$\nabla\left(\frac{\varphi}{\sqrt{M}}\right) = \sqrt{M}\nabla\left(\frac{\varphi}{M}\right) - \frac{\nabla M}{2M^{3/2}}\varphi.$$

Since $\nabla M = -2\delta x M^{1-1/\delta}$ we obtain

$$\left\|\frac{\varphi}{\sqrt{M}}\right\|^2_{H^1(B)} \leq \int_B \frac{\varphi^2}{M} + 2\int_B M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2 + 2\delta^2\int_B \frac{\|x\|^2}{M^{1+2/\delta}}\varphi^2$$

Now using Theorem 6.2.5 of [**7**] (see also the inclusion (3.10) of [**3**]) we deduce that $\frac{\varphi}{\sqrt{M}} \in H^1(B)$ and that the first inequality of (4.6) is satisfied.
On the other hand, from the density of $\mathcal{D}(B)$ in H^1_M (see Remark 3.7 of [**6**]) we deduce that there exists a sequence $\varphi_k \in \mathcal{D}(B)$ such that $\varphi_k \to \varphi$ in H^1_M. Then

$$\frac{\varphi_k}{\sqrt{M}} \to \frac{\varphi}{\sqrt{M}} \quad \text{in } H^1(B) \quad \text{with } \frac{\varphi_k}{\sqrt{M}} \in \mathcal{D}(B)$$

and this implies $\frac{\varphi}{\sqrt{M}} \in H^1_0(B)$.
Let us now consider $v \in H^1_0(B)$ and denote $\varphi = \sqrt{M}v$. We have

$$\int_B \frac{\varphi^2}{M} + \int_B M\left|\nabla\left(\frac{\varphi}{M}\right)\right|^2 = \int_B v^2 + \int_B |\nabla v|^2 + \frac{1}{4}\int_B \left|\frac{\nabla M}{M}\right|^2 v^2.$$

With the help of the Hardy inequality we deduce that $\varphi \in H^1_M$ and obtain the second inequality of (4.6). $\square$

Then using the changes $\psi = \sqrt{M}f$ and $\varphi = \sqrt{M}g$, the problem (4.5) can be written in the equivalent form: find $f \in H^1_0(B)$ such that

$$\int_B \left[M\nabla\left(\frac{f}{\sqrt{M}}\right)\cdot\nabla\left(\frac{g}{\sqrt{M}}\right) - \mathbf{G}x\sqrt{M}f\cdot\nabla\left(\frac{g}{\sqrt{M}}\right)\right]dx = 0 \quad \forall\, g \in H^1_0(B).$$

By an elementary calculus, the above equality writes

$$\int_B \left\{\nabla f\cdot\nabla g - \mathbf{G}xf\cdot\nabla g + \left[(\delta^2-2\delta)M^{-2/\delta} - (\delta^2+(n-2)\delta+\delta\mathbf{G}x\cdot x)M^{-1/\delta}\right]fg\right\}$$

then $f \in H^1_0(B)$ satisfies the problem

$$-\Delta f + \mathbf{G}x\cdot\nabla f + \left[\frac{\delta^2-2\delta}{(1-\|x\|^2)^2} - \frac{\delta^2+(n-2)\delta+\delta\mathbf{G}x\cdot x}{1-\|x\|^2}\right]f = 0 \quad \text{in } H^{-1}(B).$$

Let us now write

$$\frac{1}{(1-\|x\|^2)^2} = \frac{1}{(1+\|x\|)^2}\frac{1}{(1-\|x\|)^2} = \frac{1}{4}\frac{1}{(1-\|x\|)^2} + \frac{3+\|x\|}{4(1+\|x\|)^2}\frac{1}{1-\|x\|}$$

and observe that $\operatorname{dist}(x,\partial B) = 1-\|x\|, \ \forall x \in B$. Since $\frac{f}{1-\|x\|} \in L^2(B)$ by Hardy inequality, we deduce that f satisfies

$$-\Delta f + \frac{\delta^2-2\delta}{4\operatorname{dist}^2(x,\partial B)}f = h \quad \text{in } H^{-1}(B)$$

with $h \in L^2(B)$.
Then the result of Theorem 3.2 applies with $b_0 = \frac{\delta^2-2\delta}{4}$ provided that $\frac{\delta^2-2\delta}{4} > \frac{3}{4} \iff \delta > 3$ and we get $f \in H^2(B)$ and $\frac{\partial f}{\partial\nu} = 0$ on ∂B.
We then proved the following regularity result:

Proposition 4.2. Under the hypothesis $\delta > 3$ the solution ψ of (4.5) satisfies

$$\frac{\psi}{\sqrt{M}} \in H_0^2(B).$$

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