Dynamics of PDE, Vol.11, No.3, 269-298, 2014

Random attractors for stochastic semi-linear degenerate parabolic equations with additive noises

Wenqiang Zhao and Yangrong Li

Communicated by S.-T. Yau, received April 21, 2012.

Abstract. The existences of random attractors in $L^p(D_N) \cap L^{2p-2}(D_N)$ are proved for a class of stochastic semi-linear degenerate parabolic equations on arbitrary bounded or unbounded domains $D_N \subseteq \mathbb{R}^N$, where the leading term of the equations has the form $\mathrm{div}(\sigma(x)\nabla u)$ and the nonlinearity $f(x,u)$ satisfies some dissipative assumptions and the growth of order $p-1, p>2$. The asymptotic compactness of the corresponding random dynamical system in $L^p(D_N)$ and $L^{2p-2}(D_N)$ are established respectively by using an asymptotic a priori estimate method. Our result improves a previous result of Yang and Kloeden [**25**] concerning the existence of a compact random attractor in $L^2(D_N)$ for the same equations.

Contents

2. Introduction

The asymptotic dynamics of random dynamical systems (RDSs) have been richly developed by investigation of the random attractors ever since [**9, 16**] began their foundational works. The qualitative study of stochastic partial differential equations (SPDEs) driven by white noises is based on the theory of RDSs, see [**4**]

1991 *Mathematics Subject Classification.* 35R60; 35B40; 35B41.

Key words and phrases. Random dynamical systems, stochastic semi-linear degenerate parabolic equation, asymptotic compactness, random attractor.

and [**7**]. The existences of random attractors for RDSs defined in bounded spatial domains have been established for a wide range of SPDEs, see [**6, 8, 11, 12, 13, 14, 24, 28, 29**] and references therein. By a tail-estimate technique developed in [**19**] for the deterministic reaction-diffusion equation in unbounded domains, the asymptotic compactness of the RDSs corresponding to some concrete SPDEs defined in unbounded domains was proved and therefore the existences of random attractors for these were established in L^2 space, see [**5, 23, 20, 21, 22**].

However, the existence of random attractors on other Sobolev spaces (e.g., $L^r(r \neq 2)$ and H^1) is technically more complicate, and much less has been done, although the corresponding work was well established for many deterministic partial differential equations, such as p-Laplacian equations [**26, 27**], reaction-diffusion equations [**15, 17, 18, 31**]. Recently, [**12, 13**] made an important progress on the reaction-diffusion equations and proved the existences of random attractors in L^p space for the corresponding RDSs defined in bounded domains by asymptotic a priori estimate of the unbounded part of solutions. As a generalization of the method in [**13**], Zhao and Li [**30**] established the unique existence of random attractor in L^p space for reaction-diffusion equations defined in the unbounded domains $\mathbb{R}^N$.

In this paper, we study the existences of random attractors for the RDS generated by the solutions of a class of semi-linear degenerate parabolic equations driven by additive spatially distributed temporal noises on an arbitrary bounded or unbounded domain $D_N \subseteq \mathbb{R}^N, N \geq 2$, i.e., of equations of the form

$$du + (\lambda u - \operatorname{div}(\sigma(x)\nabla u))dt = f(x,u)dt + \sum_{j=1}^{m} h_j(x)dW_j(t), \quad x \in D_N, t \geq 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in D_N, \tag{2.2}$$

$$u(x,t)|_{\partial D_N} = 0, \quad t \geq 0, \tag{2.3}$$

where λ is a positive constant. The unknown $u = u(x,t)$ is a real valued function of $x \in D_N$ and $t \geq 0$. $h_j(1 \leq j \leq m)$ are functions on D_N. $W_j(t)(1 \leq j \leq m)$ are mutually independent two-side real-valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $f(x,u)$ is a nonlinear function satisfying some conditions which will be specified in section 3.

In order to study the asymptotic behavior of solutions to problem (1.1)-(1.3), as in [**25**], we assume that the diffusion coefficient $\sigma(x)$ satisfies the following assumptions:

$\mathcal{H}_\alpha$: when D_N is bounded, we assume that $\sigma \in L^1_{loc}(D_N)$ and $\liminf_{x\to z} |x - z|^{-\alpha}\sigma(x) > 0$ for some $\alpha \in (0,2)$ and every $z \in \overline{D_N}$;

$\mathcal{H}_\alpha^\beta$: when D_N is unbounded, we assume that σ satisfies $\mathcal{H}_\alpha$ and

$$\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$$

for some $\beta > 2$.

The assumptions $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^\beta$ indicate that the function $\sigma(x)$ is extremely irregular, i.e., first, the set $\{x|\sigma(x) = 0\}$ is finite and second, $\sigma(x)$ could be non-smooth, see [**1, 2**] for details. Under these assumptions one has the Poincaré inequality as well as the compact embedding of $D_0^{1,2}(D_N, \sigma)$ to $L^2(D_N)$.

The deterministic version of these equations has been investigated by Anh and his coworkers [**1, 2, 3**], who proved the existences of non-autonomous attractors.

Recently , Yang and Kloeden [**25**] obtained the unique existence of random attractors in $L^2(D_N)$ for the RDS corresponding to (1.1)-(1.3). In this paper we consider the existences of random attractors in $L^p(D_N) \cap L^{2p-2}(D_N)$ for the same RDS by a new asymptotic prior estimate technique. In particular, we deduce that the random attractor in $L^2(D_N)$ is actually consistent with the random attractor in $L^p(D_N) \cap L^{2p-2}(D_N)$. The results in this respect are new and appear to be optimal even in deterministic case.

This paper is organized as follows. In section 3, we present some general notions and the existence criterions of the (L^q, L^r)-random attractors for an RDS. In section 4, we obtain the corresponding RDS for stochastic semi-linear degenerate parabolic equation with additive noise. In section 5, we prove the existence of (L^2, L^p)-random attractors for corresponding RDS. In section 6, we prove the (L^2, L^{2p-2})-random attractors for the same RDS.

3. Preliminaries and abstract results

Here, we first introduce some basic notions which are relevant to our discussions, and then obtain the abstract results on the existence of bi-spaces (L^q, L^r)-random attractors for an RDS, for which the spatial domains may be bounded or unbounded and $1 < q \leq r < \infty$. A comprehensive acknowledge on RDSs please refer to [**9, 4, 7, 8**].

3.1. Preliminaries

The basic notion in RDS is a measurable dynamical system (MDS)

$$\theta \equiv (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}}),$$

which is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a group $\theta_t, t \in \mathbb{R}$, of measure preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$. A MDS θ is said to be ergodic under $\mathbb{P}$ if for any θ-invariant set $B \in \mathcal{F}$ we have either $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$, where the θ-invariant set is in the sense $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for $B \in \mathcal{F}$ and all $t \in \mathbb{R}$.

Let $(X, \|.\|_X)$ and $(Z, \|.\|_Z)$ be two separable Banach spaces with Borel σ-algebra $\mathcal{B}(X)$ and $\mathcal{B}(Z)$, respectively. The RDS is an object consisting of an MDS and a cocycle over this MDS, where the MDS is used to model the random perturbations.

Definition 3.1. An RDS on X over an MDS θ is a family of measurable mappings

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \ (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

such that for $\mathbb{P}$-a.e.$\omega \in \Omega$, the mappings $\{\varphi(t, \omega, .)\}_{t\geq 0,\omega\in\Omega}$ satisfy the cocycle property:

$$\varphi(0, \omega, .) = \mathrm{id}, \quad \varphi(t + s, \omega, .) = \varphi(t, \theta_s\omega, \varphi(s, \omega, .))$$

for all $s, t \in \mathbb{R}^+$. An RDS $\{\varphi(t, \omega, .)\}_{t\geq 0,\omega\in\Omega}$ is continuous in the meaning that the mappings $\varphi(t, \omega, .) : X \to X$ are continuous in X for all $t \in \mathbb{R}^+$ and $\mathbb{P}$-a.e.$\omega \in \Omega$.

Definition 3.2. (1) A random set $\{D(\omega)\}_{\omega\in\Omega}$ is a family of closed subsets of X indexed by ω such that for every $x \in X$ the mapping $\omega \mapsto d_X(x, D(\omega))$ is measurable with respect to $\mathcal{F}$, where for the nonempty sets $A, B \in 2^X$ we set

$$d_X(A,B) = \sup_{x\in A} \inf_{y\in B} \|x-y\|_X$$

and in particular $d_X(x,B) = d_X(\{x\},B)$.

(2) A random bounded sets $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to θ if for $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t}\|B(\theta_{-t}\omega)\|_X = 0, \quad \text{for all } \beta > 0,$$

where $\|B\|_X = \sup_{x\in B} \|x\|_X$.

(3) A random variable $\varrho(\omega) \geq 0$ is called tempered with respect to θ if for $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t}\varrho(\theta_{-t}\omega) = 0, \quad \text{for all } \beta > 0.$$

We use $\mathcal{D}_X$ to denote the collection of all tempered random subsets of X.

Definition 3.3. (1) A random set $\{K_Z(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_Z$ is called an (X,Z)-random absorbing set for RDS $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ if for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there exists $T = T(B,\omega) > 0$ such that

$$\varphi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K_Z(\omega), \quad \text{for all } t \geq T,$$

where $\varphi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) = \cup_{v_0\in B(\theta_{-t}\omega)}\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))$.

(2) An RDS $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ on Z is said to be (X,Z)-asymptotically compact if for $\mathbb{P}$-a.e.$\omega \in \Omega$, $\{\varphi(t_n,\theta_{-t_n}\omega,x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in Z whenever $t_n \to \infty$ and $x_n \in B(\theta_{-t_n}\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$.

(3) A compact random set $\{\mathcal{A}_Z(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_Z$ is said to be an (X,Z)-random attractor if the following conditions are satisfied: for $\mathbb{P}$-a.e.$\omega \in \Omega$,

(i) $\mathcal{A}_Z(\omega)$ is invariant, that is, $\varphi(t,\omega,\mathcal{A}_Z(\omega)) = \mathcal{A}_Z(\theta_t\omega)$ for all $t \geq 0$;

(ii) $\mathcal{A}_Z(\omega)$ is (X,Z)-attracting, in the sense that for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_X$,

$$\lim_{t\to\infty} d_Z(\varphi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}_Z(\omega)) = 0.$$

3.2. Abstract results

We will provide a simple and convenient criterion on the unique existence of (L^q, L^r)-random attractors for an RDS, where the spatial domains $D_N \subseteq \mathbb{R}^N, N \geq 1$, are either bounded or unbounded. In the subsequential statement, we use $\mathcal{D}_m$ to denote the collection of all tempered random subsets of L^m. In particular, for $m = 2$, $\mathcal{D}_2 = \mathcal{D}$ denotes the collection of all tempered random subsets of L^2.

Theorem 3.4. Let $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ be a continuous RDS on L^q and be an RDS on L^r over the same MDS θ, where $1 < q \leq r < \infty$. Assume that there exists a random set $\{K_q(\omega)\}_{\omega\in\Omega}$ which is an (L^q, L^q)-random absorbing set for

$\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ and $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ is (L^q,L^q)-asymptotic compact. Then the family of sets $\{\mathcal{A}_q(\omega)\}_{\omega\in\Omega}$, where

$$\mathcal{A}_q(\omega)=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K_q(\theta_{-t}\omega))}^{L^q},\quad \omega\in\Omega, \tag{3.1}$$

is a unique (L^q,L^q)-random attractor for $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ in L^q, where $\overline{A}^{L^q}$ denotes the closure of A with respect to the L^q-norm.

Furthermore, if there exists a family of sets $\{K_r(\omega)\}_{\omega\in\Omega}$ which is an (L^q,L^r)-random absorbing set for $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ and $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ is (L^q,L^r)-asymptotic compact. Then the family of sets $\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega}$, where

$$\begin{aligned}\mathcal{A}_r(\omega)&=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K_q(\theta_{-t}\omega)\cap K_r(\theta_{-t}\omega))}^{L^q}\\&=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K_q(\theta_{-t}\omega)\cap K_r(\theta_{-t}\omega))}^{L^r}\\&=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K_q(\theta_{-t}\omega))}^{L^q},\omega\in\Omega,\end{aligned}\tag{3.2}$$

is an (L^q,L^r)-random attractor for $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$.

Proof. We will work for fixed $\omega\in\Omega_0$ with $\mathbb{P}(\Omega_0)=1$. The unique existence of (L^q,L^q)-random attractor $\{\mathcal{A}_q(\omega)\}_{\omega\in\Omega}$ is followed from [**5**]. Hence it suffices to show that (3.2) is an (L^q,L^r)-random attractor. To this end, we put

$$K(\omega)=K_q(\omega)\cap K_r(\omega),\quad \text{for every fixed } \omega\in\Omega_0.$$

Then $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}_q$ and $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}_r$. Furthermore by our assumption, it follows that the family $\{K(\omega)\}_{\omega\in\Omega}$ is not only an (L^q,L^q)-absorbing set but also an (L^q,L^r) absorbing set, whence by the first result of the theorem we know that $\{\mathcal{A}_q(\omega)\}_{\omega\in\Omega}$ can be also expressed as the omega-limits set of $\{K(\omega)\}_{\omega\in\Omega}$, i.e.,

$$\mathcal{A}_q(\omega)=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K(\theta_{-t}\omega))}^{L^q},\quad \omega\in\Omega_0. \tag{3.3}$$

Put

$$\mathcal{A}_r(\omega)=\bigcap_{s\geq 0}\overline{\bigcup_{t\geq s}\varphi(t,\theta_{-t}\omega,K(\theta_{-t}\omega))}^{L^r},\quad \omega\in\Omega_0. \tag{3.4}$$

It is easy to see that:

$y\in\mathcal{A}_r(\omega)$ if and only if there exist a sequenc t_n and a sequence $x_n\in K(\theta_{-t_n}\omega)$such that

$$t_n\to\infty \text{ and } \varphi(t_n,\theta_{-t_n}\omega,x_n)\xrightarrow{\|\cdot\|_{L^r}} y \text{ as } n\to\infty. \tag{3.5}$$

Since $\{K_r(\omega)\}_{\omega\in\Omega}$ is (L^q,L^r)-random absorbing and $\{K(\omega)\}_{\omega\in\Omega}\in\mathcal{D}_q$, then we obtain that for $x_n\in K(\theta_{-t_n}\omega)$,

$$\varphi(t_n,\theta_{-t_n}\omega,x_n)\in K_r(\omega),\quad \text{for } n \text{ large enough},$$

whereas by (3.5) we deduce that $\mathcal{A}_r(\omega) \subset \overline{K_r(\omega)}^{L^r}$ for $\omega \in \Omega_0$. Therefore

$$\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_r,$$

that is to say, $\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega}$ defined in (3.4) is tempered in L^r space. Furthermore, from our assumptions and by the same arguments as in [**9**], we can show that $\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega}$ is nonempty, compact in L^r and (L^q, L^r)-attracting for φ. Hence, we only need to verify the invariance property of $\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega}$, i.e., $\varphi(t, \omega, \mathcal{A}_r(\omega)) = \mathcal{A}_r(\theta_t\omega)$ for all $t \geq 0$ and $\omega \in \Omega_0$. For this purpose, it is sufficient to prove that $\mathcal{A}_r(\omega) = \mathcal{A}_q(\omega)$ for every $\omega \in \Omega_0$.

Given $x \in \mathcal{A}_q(\omega)$, by (3.1), we have that

$$\text{there exist sequences } t_n \text{ and } x_n \in K_q(\theta_{-t_n}\omega) \text{ such that } t_n \to \infty \text{ and}$$

$$\varphi(t_n, \theta_{-t_n}\omega, x_n) \xrightarrow{\|\cdot\|_{L^q}} x \text{ as } n \to \infty. \tag{3.6}$$

Since $\{K_q(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_q$, then by our assumption of (L^q, L^r)-asymptotic compactness, there exists $y \in L^r$ such that, up to a subsequence,

$$\varphi(t_n, \theta_{-t_n}\omega, x_n) \xrightarrow{\|\cdot\|_{L^r}} y \text{ as } n \to \infty. \tag{3.7}$$

Note that both L^q and L^r are continuous embedding into the distribution functions space $\mathscr{D}'(D_N)$ on $\mathscr{D}(D_N)$, where $D_N \subseteq \mathbb{R}^N$ is bounded or unbounded. Then by the uniqueness of limits we get $x = y$. It remains to show that $y \in \mathcal{A}_r(\omega)$. Note that $\{K_q(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_q$, and $\{K(\omega)\}_{\omega\in\Omega}$ is also (L^q, L^r)-random absorbing, then we know that for $x_n \in K_q(\theta_{-t_n}\omega)$, there exists $T(K_q, \omega) > 0$ such that for all $t \geq T(K_q, \omega)$,

$$y_n = \varphi(t, \theta_{-t}\theta_{-(t_n-t)}\omega, x_n(\theta_{-t}\theta_{-(t_n-t)}\omega)) \in K(\theta_{-(t_n-t)}\omega), \tag{3.8}$$

where the subsequences t_n and x_n are in (3.7). Moreover, by the cocycle property of φ, for $t_n \geq t \geq T(K_q, \omega)$,

$$\begin{aligned}
&\varphi(t_n, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega))\\
&= \varphi(t_n - t + t, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega))\\
&= \varphi(t_n - t, \theta_{-(t_n-t)}\omega, \varphi(t, \theta_{-t_n}\omega, x_n(\theta_{-t_n}\omega)))\\
&= \varphi(t_n - t, \theta_{-(t_n-t)}\omega, \varphi(t, \theta_{-t}\theta_{-(t_n-t)}\omega, x_n(\theta_{-t}\theta_{-(t_n-t)}\omega))).
\end{aligned} \tag{3.9}$$

Put $t'_n = t_n - t$. Then $y_n \in K(\theta_{-t'_n}\omega)$, where y_n is in (3.8). It follows from (3.7)-(3.9) that

$$\varphi(t'_n, \theta_{-t'_n}\omega, y_n) \xrightarrow{\|\cdot\|_r} y \quad \text{as } n \to \infty, \tag{3.10}$$

whereas by (3.5), we get $y \in \mathcal{A}_r(\omega)$ which prove the inclusion relation $\mathcal{A}_q(\omega) \subseteq \mathcal{A}_r(\omega)$ for $\omega \in \Omega_0$.

On the other hand, if $x \in \mathcal{A}_r(\omega)$, by (3.5), there exist $t_n \to \infty$ and $x_n \in K(\theta_{-t_n}\omega)$ such that

$$\varphi(t_n, \theta_{-t_n}\omega, x_n) \xrightarrow{\|\cdot\|_{L^r}} x \quad \text{as } n \to \infty. \tag{3.11}$$

Note that $K(\omega) \subset K_q(\omega)$ for $\omega \in \Omega_0$, and by our assumption that φ is (L^q, L^q)-asymptotically compact. Then there exists $y \in L^q$ such that, up to a subsequence,

$$\varphi(t_n, \theta_{-t_n}\omega, x_n) \xrightarrow{\|\cdot\|_{L^q}} y \quad \text{as } n \to \infty. \tag{3.12}$$

By (3.11) and (3.12), we have $x = y$. But from (3.6) and (3.12), $y \in \mathcal{A}_q(\omega)$, whence $\mathcal{A}_r(\omega) \subseteq \mathcal{A}_q(\omega)$ which proves that $\mathcal{A}_q(\omega) = \mathcal{A}_r(\omega)$ for $\omega \in \Omega_0$ and thus $\{\mathcal{A}_r(\omega)\}_{\omega\in\Omega}$ is invariant as required.

We present the following result which can be employed easily to prove the existence of the (L^q, L^r)-random attractor for a concrete RDS.

Theorem 3.5. Let $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ be a continuous RDS on L^q and be an RDS on L^r over the same MDS θ, where $1 < q \leq r < \infty$. Assume that $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ possesses an (L^q, L^q)-random attractor. Then $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ admits an (L^q, L^r)-random attractor provided that

(i) $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ has an (L^q, L^r)-random absorbing set $\{K_1(\omega)\}_{\omega\in\Omega}$;

(ii) For any $\varepsilon > 0$ and every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_q$, there exist positive random constants $c = c(\omega), M = M(\varepsilon, B, \omega)$ and $T = T(\varepsilon, B, \omega)$ such that, for all $t \geq T$,

$$\sup_{u_0(\omega)\in B(\omega)} \int_{D_N(|\varphi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|\geq M)} |\varphi(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))|^r dx \leq c\varepsilon. \tag{3.13}$$

Proof. It suffices to show that $\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$ is (L^q, L^r)-asymptotically compact, that is, for $\mathbb{P}$-a.e.$\omega \in \Omega$, the sequence $\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega))$ has a convergent subsequence in $L^r(D_N)$ provided that $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}_q$ and $u_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$. By a standard argument we can show that

$$\{\varphi(t,\omega,.)\}_{t\geq 0,\omega\in\Omega}$$

is (L^q, L^q)-asymptotically compact, and then there exists $\xi \in L^q(D_N)$ and a subsequence of

$$\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)),$$

which is still denoted by

$$\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)),$$

such that,

$$\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) \to \xi \ \text{ strongly in } L^q(D_N). \tag{3.14}$$

Then there exist $T_1 = T_1(\varepsilon, B, \omega)$ and $\overline{N}_1 = \overline{N}_1(\varepsilon, B, \omega)$ such that $t_n, t_{n'} \geq T_1$ with $n, n' \geq \overline{N}_1$,

$$\|\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))\|^q_{L^q(D_N)} < \varepsilon. \tag{3.15}$$

For every $\varepsilon > 0$, by assumption (ii), there exist $T_2 = T_2(\varepsilon, B, \omega) > 0$, $M = M(\varepsilon, B, \omega)$ and $\overline{N}_2 = \overline{N}_2(\varepsilon, B, \omega)$ such that $t_n, t_{n'} \geq T_2$ with $n, n' \geq \overline{N}_2$,

$$\int_{D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\geq M)} |\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega))|^r dx < \varepsilon, \tag{3.16}$$

and

$$\int_{D_N(|\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|\geq M)} |\varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx < \varepsilon. \tag{3.17}$$

Consider that

$$\begin{aligned}
&\|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))\|_r^r\\
&=\int_{D_N}|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx\\
&\leq (\int_{D_N^1}+\int_{D_N^2}+\int_{D_N^3}+\int_{D_N^4})|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\\
&-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx,
\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}
D_N^1&=D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\leq M)\\
&\cap D_N(|\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|\leq M),\\
D_N^2&=D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\geq M)\\
&\cap D_N(|\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|\leq M),\\
D_N^3&=D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\leq M)\\
&\cap D_N(|\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|\geq M),\\
D_N^4&=D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\geq M)\\
&\cap D_N(|\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|\geq M).
\end{aligned}$$

It is obvious that $D_N \subseteq D_N^1 \cup D_N^2 \cup D_N^3 \cup D_N^4$. Put $T=\max\{T_1,T_2\}$. Then from (3.15) we find that, for all $t_n, t_{n'} \geq T$,

$$\begin{aligned}
&\int_{D_N^1}|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx\\
&\leq (2M)^{r-q}\int_{D_N^1}|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\\
&-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|^q dx\\
&\leq (2M)^{r-q}\|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))\\
&-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))\|_{L^q(D_N)}^q \leq c\varepsilon.
\end{aligned}\tag{3.19}$$

Observe that

$$|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|$$

$$\leq 2|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|$$

in D_N^2, whereas by (3.16) we obtain that, for all $t_n, t_{n'} \geq T$,

$$\begin{aligned}
&\int_{D_N^2}|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))-\varphi(t_{n'},\theta_{-t_{n'}}\omega,u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx\\
&\leq 2^r\int_{D_N(|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|\geq M)}|\varphi(t_n,\theta_{-t_n}\omega,u_{0,n}(\theta_{-t_n}\omega))|^r dx \leq c\varepsilon.
\end{aligned}\tag{3.20}$$

Similarly by (3.17) we get that, for all $t_n, t_{n'} \geq T$,

$$\begin{aligned}
&\int_{D_N^3} |\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx \\
&\leq 2^r \int_{D_N(|\varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))| \geq M)} |\varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx \leq c\varepsilon.
\end{aligned} \tag{3.21}$$

By Hölder's inequality, $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. This together with (3.16)-(3.17) imply that, for all $t_n, t_{n'} \geq T$,

$$\begin{aligned}
&\int_{D_N^4} |\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx \\
&\leq 2^{r-1}\Big(\int_{D_N(|\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega))| \geq M)} |\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega))|^r dx \\
&\quad + \int_{D_N(|\varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))| \geq M)} |\varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))|^r dx\Big) \leq c\varepsilon.
\end{aligned} \tag{3.22}$$

Then, it follows from (3.19)-(3.22) that, for all $t_n, t_{n'} \geq T$,

$$\|\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) - \varphi(t_{n'}, \theta_{-t_{n'}}\omega, u_{0,n'}(\theta_{-t_{n'}}\omega))\|^r_{L^r(D_N)} \leq c\varepsilon, \tag{3.23}$$

whence it follows from (3.23) that the subsequence in (3.15) is also a Cauchy sequence in space $L^r(D_N)$. By the completeness of $L^r(D_N)$, there exists a function $\eta \in L^r(D_N)$ such that

$$\varphi(t_n, \theta_{-t_n}\omega, u_{0,n}(\theta_{-t_n}\omega)) \to \eta \ \text{ strongly in } L^r(D_N).$$

Note that both $L^q(D_N)$ and $L^r(D_N)$ are continuously embedded into the distributed space $\mathscr{D}'(D_N)$. Then $\eta = \xi$. This completes the proof. □

4. The stochastic semi-linear degenerate parabolic equation with additive noise

For convenience, we set

$$Au = -\text{div}(\sigma(x)\nabla u).$$

Then A is a positive and self-adjoint linear operator with domains defined by

$$\text{Dom}(A) = \{u \in D_0^{1,2}(D_N, \sigma) : Au \in L^2(D_N)\},$$

where $D_0^{1,2}(D_N, \sigma)$ is a Hilbert space with respect to the scalar product

$$(u, v)_\sigma = \int_{D_N} \sigma(x)\nabla u.\nabla v dx,$$

and therefore is the closure of $C_0^\infty(D_N)$ with respect to the norm

$$\|u\|_{D_0^{1,2}(D_N,\sigma)} = (\int_{D_N} \sigma(x)|\nabla u|^2 dx)^{\frac{1}{2}}.$$

Furthermore, we define $\mathrm{D}^m(A) = \{u \in D_0^{1,2}(D_N, \sigma) : Au \in L^m(D_N)\}$.

In this section, we will show the generation of an RDS corresponding to stochastic semi-linear degenerate parabolic equation with additive noise on the bounded or unbounded domains $D_N \subseteq \mathbb{R}^N, N \geq 2$, i.e. to equations of the form

$$du + (\lambda u + Au)dt = f(x,u)dt + \sum_{j=1}^{m} h_j(x)dW_j(t), \quad x \in D_N, t \geq 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \quad x \in D_N, \tag{4.2}$$

$$u(x,t)|_{\partial D_N} = 0, \quad t \geq 0, \tag{4.3}$$

where the functions $h_j \in L^2(D_N) \cap \mathrm{Dom}(A) \cap \mathrm{D}^p(A) \cap \mathrm{D}^{2p-2}(A) \cap L^\infty(D_N)$; The function $f(x,u)$ in (4.1) satisfies the following conditions: for $x \in D_N, u \in \mathbb{R}$,

$$f(x,u)u \leq -\alpha_1|u|^p + \phi_1(x), \tag{4.4}$$

$$|f(x,u)| \leq \alpha_2|u|^{p-1} + \phi_2(x), \tag{4.5}$$

$$\Big|\frac{\partial f}{\partial x}(x,u)\Big| \leq \phi_3(x), \quad \frac{\partial f}{\partial u}(x,u) \leq \beta, \tag{4.6}$$

where α_1, α_2 and β are positive constants and $p > 2$. $\phi_1 \in L^1(D_N) \cap L^\infty(D_N)$, $\phi_2 \in L^2(D_N) \cap L^{p'}(D_N) \cap L^{2p-2}(D_N)$, $\phi_3 \in L^2(D_N)$, with $\frac{1}{p'} + \frac{1}{p} = 1$. $W(t) = (W_1(t), ..., W_m(t))$ are pairwise independent two-sided real-valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = \mathbf{0}\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and $\mathbb{P}$ is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we identify $W(t)$ with

$$W(t) = W(t,\omega) = (W_1(t,\omega), W_2(t,\omega), ..., W_m(t,\omega)) = \omega(t), \quad t \in \mathbb{R}.$$

Define the Wiener time shift by

$$\theta_t\omega(s) = \omega(s+t) - \omega(t), \quad \omega \in \Omega,\ t,\ s \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$ is an ergodic MDS.

We now employ the approach similar to [**25**] to translate equation (4.1) by one change of variables into a deterministic system with a random parameter. To this end, on the probability space defined above we introduce the process, for $j = 1, 2, ..., m$,

$$t \to z_j(\theta_t\omega_j) = -\lambda \int_{-\infty}^{0} e^{\lambda s}(\theta_t\omega_j)(s)ds, \quad t \in \mathbb{R},$$

where λ is the positive number in (4.1). It is easy to check that $z_j(\theta_t\omega_j)$ solves the Itó differential equation

$$dz_j + \lambda z_j dt = dW_j(t), \quad j = 1, 2, ..., m, \tag{4.7}$$

see also [**10**]. Indeed, by the θ-invariance of $\mathbb{P}$, $t \to z_j(\theta_t\omega_j)$ is a stationary process which is called stationary Ornstein-Uhlenbeck process. In particular, the random variable $z_j(\theta_t\omega_j)$ is continuous in t for $\mathbb{P}$-a.e.$\omega \in \Omega$; $|z_j(\omega_j)|$ is tempered and therefore by Proposition 4.3.3 in [**4**] there exists a tempered variable $\varrho(\omega) > 0$ such that, for $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^2 + |z_j(\theta_t\omega_j)|^p + |z_j(\theta_t\omega_j)|^{2p-2} + |z_j(\theta_t\omega_j)|^{3p-4}) \leq \varrho(\theta_t\omega), \tag{4.8}$$

and

$$\varrho(\theta_t\omega) \leq e^{\frac{\lambda}{2}|t|}\varrho(\omega), \quad t \in \mathbb{R}. \tag{4.9}$$

Put $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$. Then by (4.7) we have

$$dz + \lambda z dt = \sum_{j=1}^{m} h_j dW_j(t).$$

We let $v(t) = u(t) - z(\theta_t\omega)$, where u is a solution of problem (4.1)-(4.3). Then we can consider the following evolution equation with random parameter but without white noise:

$$\frac{dv(t)}{dt} + \lambda v(t) + Av(t) = f(x, v(t) + z(\theta_t\omega)) - Az(\theta_t\omega), \quad x \in D_N, t \geq 0, \tag{4.10}$$

$$v(x, 0) = v_0(\omega) = u_0 - z(\omega), \quad x \in D_N, \tag{4.11}$$

$$v(x, t)|_{\partial D_N} = 0, \quad t \geq 0. \tag{4.12}$$

By a standard Galerkin approximation method, we can show that for all $v_0 \in L^2(D_N)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, the equations (4.10)-(4.12) with f satisfying (4.4)-(4.6) admits a unique solution $v(t, \omega, v_0) \in C([0, T]; L^2(D_N)) \cap L^2((0, T); D_0^{1,2}(D_N, \sigma)) \cap L^p((0, T); L^p(D_N))$ with $v_0 = v_0(\omega) = v(0, \omega, v_0)$ for every $T \geq 0$. Furthermore $v(t, \omega, v_0)$ is continuous with respect to the initial value v_0 in $L^2(D_N)$. Let $u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega)$. Then $u(t, \omega, u_0)$ is the solution to the problem (4.1)-(4.3) in certain sense.

Let $v(t, \omega, v_0)$ be solution to (4.10)-(4.12). If we define two family of mappings φ and $\psi : \mathbb{R}^+ \times \Omega \times L^2(D_N) \to L^2(D_N)$ respectively by

$$\varphi(t, \omega, v_0) = v(t, \omega, v_0), \tag{4.13}$$

$$\psi(t, \omega, u_0) = u(t, \omega, u_0) = v(t, \omega, u_0 - z(\omega)) + z(\theta_t\omega), \tag{4.14}$$

for all $t \geq 0, \omega \in \Omega$ and $v_0, u_0 \in L^2(D_N)$, then φ is a continuous RDS on $L^2(D_N)$ associated with (4.10)-(4.12) and hence ψ is a continuous RDS on $L^2(D_N)$ associated with (4.1)-(4.3).

The unique existence of random attractor for this RDS ψ on $L^2(D_N)$ has been obtained in [**25**], which states

Theorem 4.1.(see[**25**].) Assume that (4.4)-(4.6) hold. Then the random dynamical system ψ defined in (4.14) admits a unique $(L^2(D_N), L^2(D_N))$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is tempered random set in space $L^2(D_N)$.

5. The random attractor in $L^p(D_N)$

5.1. Uniform estimates of solutions

In this subsection, we give some estimates of the solutions corresponding to (4.10)-(4.12).

For convenience, we sometimes abbreviate

$$v(t, \omega, v_0(\omega)) = v(t),$$

where $v(t, \omega, v_0(\omega))$ is the solution to (4.10)-(4.12). mes(A) denotes the measure of the measurable subset $A \subseteq D_N$. The generic constants c or c_i used in our discussions may be different in the context but independent of ε.

Note that by the interpolation inequality we know that the coefficients $h_j(j = 1, ..., m) \in L^p \cap L^{2p-2} \cap L^{3p-4}$ and $\phi_1 \in L^{\frac{p}{2}} \cap L^{p-1}$ and $\phi_2 \in L^p$.

Lemma 5.1. Assume that (4.4)-(4.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for every $u_0(\omega) \in B(\omega)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there exists $T = T(B, \omega) > 0$ such that for all $t \geq T$ and $s \in [t, t+1]$,

$$\|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2_{L^2(D_N)} \leq c(1 + \varrho(\omega)),$$

where $v_0(\omega) = u_0(\omega) - z(\omega) =$ and $\varrho(\omega)$ is in (4.8).

Proof. Multiplying (4.10) with v and then integrating over D_N, by a similar arguments as Lemma 6.1 in [**25**], we can show that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2_{L^2(D_N)} + \lambda\|v(t)\|^2_{L^2(D_N)} + \|v(t)\|^2_{D_0^{1,2}(D_N,\sigma)}\\ + \alpha_1\|u\|^p_{L^p(D_N)} \leq p_1(\theta_t\omega) + c,\end{aligned} \tag{5.1}$$

where

$$p_1(\theta_t\omega) = c\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^p + |z_j(\theta_t\omega_j)|^2). \tag{5.2}$$

Due to (4.9) it shows that for $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$p_1(\theta_\tau\omega) \leq ce^{\frac{1}{2}\lambda|\tau|}\varrho(\omega), \quad \tau \in \mathbb{R}. \tag{5.3}$$

By using the Gronwall' lemma to (5.1) we get that for all $s \geq 0$,

$$\|v(s, \omega, v_0(\omega))\|^2_{L^2(D_N)} \leq e^{-\lambda s}\|v_0(\omega)\|^2_{L^2(D_N)} + \int_0^s e^{\lambda(\tau-s)}p_1(\theta_\tau\omega)d\tau + \frac{c}{\lambda}. \tag{5.4}$$

Working with ω instead of $\theta_{-t-1}\omega$, along with (5.3), we get that, for $s \in [t, t+1]$,

$$\begin{aligned}
&\|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2_{L^2(D_N)}\\
&\leq e^{-\lambda s}\|v_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + \int_0^s e^{\lambda(\tau-s)}p_1(\theta_{\tau-t-1}\omega)d\tau + \frac{c}{\lambda}\\
&\leq e^{\lambda}e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + \int_0^{t+1} e^{\lambda(\tau-t)}p_1(\theta_{\tau-t-1}\omega)d\tau + \frac{c}{\lambda}\\
&\leq e^{\lambda}e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + \int_{-t-1}^0 e^{\lambda(\tau+1)}p_1(\theta_\tau\omega)d\tau + \frac{c}{\lambda}\\
&\leq e^{\lambda}\Big(e^{-\lambda(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + c\int_{-t-1}^0 e^{\frac{\lambda}{2}\tau}\varrho(\omega)d\tau\Big) + \frac{c}{\lambda}\\
&\leq e^{\lambda}\Big(2e^{-\lambda(t+1)}(\|u_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + \|z(\theta_{-t-1}\omega)\|^2_{L^2(D_N)}) + \frac{2c}{\lambda}\varrho(\omega)\Big) + \frac{c}{\lambda}.
\end{aligned} \tag{5.5}$$

Since $\|z(\omega)\|^2_{L^2(D_N)}$ is also tempered and $u_0(\omega) \in B(\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, then there exists $T = T(B, \omega) > 0$ such that for all $t \geq T$,

$$e^{-\lambda(t+1)}(\|u_0(\theta_{-t-1}\omega)\|^2_{L^2(D_N)} + \|z(\theta_{-t-1}\omega)\|^2_{L^2(D_N)}) \leq c(1 + \varrho(\omega)), \tag{5.6}$$

and therefore it follows from (5.5)-(5.6) that for all $t \geq T$,

$$\|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^2_{L^2(D_N)} \leq c(1 + \varrho(\omega)),$$

which completes the proof. $\square$

Lemma 5.2. Assume that and (4.4)-(4.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for every $u_0(\omega) \in B(\omega)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there exists $T = T(B, \omega) > 0$ such that for all $t \geq T$,

$$\int_t^{t+1} \|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{L^p(D_N)}^p ds \leq c(1 + \varrho(\omega)),$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$ and $\varrho(\omega)$ is in (4.8).

Proof. Firstly, let T be the same number as in Lemma 5.1 and $t \geq T$. We replace t by s in (5.1) and then integrate with respect to s over intervals $[t, t+1]$ to find that for all $t \geq T$,

$$\alpha_1 \int_t^{t+1} \|v(s, \omega, v_0(\omega))\|_{L^p(D_N)}^p ds \leq \int_t^{t+1} p_1(\theta_s\omega) ds + \|v(t, \omega, v_0(\omega))\|_{L^2(D_N)}^2 + c. \tag{5.7}$$

Working with $\theta_{-t-1}\omega$ instead of ω in (5.7), association with Lemma 5.1, it yields that for all $t \geq T$,

$$\begin{aligned}
&\alpha_1 \int_t^{t+1} \|v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{L^p(D_N)}^p ds \\
&\leq \int_t^{t+1} p_1(\theta_{s-t-1}\omega) ds + \|v(t, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|_{L^2(D_N)}^2 + c \\
&\leq \int_t^{t+1} p_1(\theta_{s-t-1}\omega) ds + c(1 + \varrho(\omega)) \\
&= \int_{-1}^{0} p_1(\theta_s\omega) ds + c(1 + \varrho(\omega)) \\
&\leq c(1 + \varrho(\omega)) \quad (\text{by } (5.3)).
\end{aligned}$$

This completes the proof. $\square$

The following lemma shows the existence of an (L^2, L^p)-random absorbing set for the RDS defined in φ in (4.13).

Lemma 5.3. Assume that (4.4)-(4.6) hold. Then there exists a random ball $\{K_p(\omega)\}_{\omega\in\Omega}$ centered at 0 with random radius $\Big\{c(1+\varrho(\omega))\Big\}^{\frac{1}{p}}$ such that $\{K_p(\omega)\}_{\omega\in\Omega}$ is an (L^2, L^p)-random absorbing set for the RDS φ in $\mathcal{D}_p$, where c is a deterministic positive constant.

Proof. We multiply (4.10) with $|v|^{p-2}v$ and then integrate over D_N to obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|v\|_{L^p(D_N)}^p + \lambda\|v\|_{L^p(D_N)}^p + \int_{D_N} Av|v|^{p-2}v dx \\
&\qquad = \int_{D_N} f(x, v + z(\theta_t\omega))|v|^{p-2}v dx - \int_{D_N} Az(\theta_t\omega)|v|^{p-2}v dx,
\end{aligned} \tag{5.8}$$

where,

$$\begin{aligned}
&\int_{D_N} Av|v|^{p-2}vdx \\
&= \int_{D_N} \sigma(x)\nabla v.\nabla(|v|^{p-2}v)dx \\
&= (p-2)\int_{D_N} \sigma(x)|v|^{p-4}v^2|\nabla v|^2dx + \int_{D_N} \sigma(x)|v|^{p-2}|\nabla v|^2dx \geq 0. \qquad (5.9)
\end{aligned}$$

Then we estimate the nonlinearity in (5.8). By our assumptions (4.4)-(4.5), it is easy to see that

$$f(x, v + z(\theta_t\omega))v \leq -\frac{1}{2}\alpha_1|u|^p + c(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2) + \phi_1 + \frac{1}{2}\phi_2^2,$$

where $u = v + z(\theta_t\omega)$. By the Hölder's inequality in series form, $|u|^p \geq 2^{1-p}|v|^p - |z(\theta_t\omega)|^p$, it gives that

$$f(x, v + z(\theta_t\omega))v \leq -\frac{\alpha_1}{2^p}|v|^p + c(|z(\theta_t\omega)|^p + |z(\theta_t\omega)|^2) + \phi_1 + \frac{1}{2}\phi_2^2, \qquad (5.10)$$

whence by using the Young's inequality four times we have

$$\begin{aligned}
&f(x, v + z(\theta_t\omega))|v|^{p-2}v \\
&\leq -\frac{\alpha_1}{2^p}|v|^{2p-2} + c|z(\theta_t\omega)|^p|v|^{p-2} + c|z(\theta_t\omega)|^2|v|^{p-2} + \phi_1|v|^{p-2} + \frac{1}{2}\phi_2^2|v|^{p-2} \\
&\leq -\frac{\alpha_1}{2^p}|v|^{2p-2} + \frac{\alpha_1}{2^{p+1}}|v|^{2p-2} + c|z(\theta_t\omega)|^{2p-2} + \frac{\lambda}{2}|v|^p + c|z(\theta_t\omega)|^p \\
&+ \frac{\lambda}{4}|v|^p + c\phi_1^{\frac{p}{2}} + \frac{\lambda}{4}|v|^p + c\phi_2^p \\
&\leq -\frac{\alpha_1}{2^{p+1}}|v|^{2p-2} + \lambda|v|^p + c(|z(\theta_t\omega)|^{2p-2} + |z(\theta_t\omega)|^p) + c(\phi_1^{\frac{p}{2}} + \phi_2^p). \qquad (5.11)
\end{aligned}$$

Therefore by (5.11) the nonlinearity has the following estimate:

$$\begin{aligned}
&\int_{D_N} f(x, v + z(\theta_t\omega))|v|^{p-2}vdx \\
&\leq -\frac{\alpha_1}{2^{p+1}}\|v\|^{2p-2}_{L^{2p-2}(D_N)} + \lambda\|v\|^p_{L^p(D_N)} + c\|z(\theta_t\omega)\|^{2p-2}_{L^{2p-2}(D_N)} \\
&+ c\|z(\theta_t\omega)\|^p_{L^p(D_N)} + c(\|\phi_1\|^{\frac{p}{2}}_{L^{\frac{p}{2}}(D_N)} + \|\phi_2\|^p_{L^p(D_N)}). \qquad (5.12)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big|\int_{D_N} Az(\theta_t\omega)|v|^{p-2}vdx\Big| &\leq \frac{\alpha_1}{2^{p+2}}\int_{D_N}|v|^{2p-2}dx + c\int_{D_N}|Az(\theta_t\omega)|^2dx \\
&= \frac{\alpha_1}{2^{p+2}}\|v\|^{2p-2}_{L^{2p-2}(D_N)} + c\|Az(\theta_t\omega)\|^2_{L^2(D_N)}. \qquad (5.13)
\end{aligned}$$

Thus it follows from (5.8)-(5.9) and (5.12)-(5.13) that for all $t \geq 0$,

$$\begin{aligned}
&\frac{d}{dt}\|v\|^p_{L^p(D_N)} + \frac{\alpha_1 p}{2^{p+2}}\|v\|^{2p-2}_{L^{2p-2}(D_N)} \\
&\leq c(\|z(\theta_t\omega)\|^{2p-2}_{L^{2p-2}(D_N)} + \|z(\theta_t\omega)\|^p_{L^p(D_N)} + \|Az(\theta_t\omega)\|^2_{L^2(D_N)}) + c_0. \qquad (5.14)
\end{aligned}$$

Note that $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z_j(\theta_t\omega_j)$ and $h_j \in \mathrm{Dom}(A) \cap L^p(D_N) \cap L^{2p-2}(D_N)$. Then the right hand side of (5.14) is controlled by

(5.15)
$$c\Big(\sum_{j=1}^{m}|z_j(\theta_t\omega_j)|^{2p-2} + \sum_{j=1}^{m}|z_j(\theta_t\omega_j)|^{p} + \sum_{j=1}^{m}|z_j(\theta_t\omega_j)|^{2}\Big) + c_0 = p_2(\theta_t\omega) + c_0.$$

By (4.8)-(4.9) we have, for all $t \in \mathbb{R}$,

$$p_2(\theta_t\omega) \le c\varrho(\omega)e^{\frac{1}{2}\lambda|t|}. \tag{5.16}$$

It follows from (5.14)-(5.15) that

$$\frac{d}{dt}\|v\|^p_{L^p(D_N)} + \frac{\alpha_1 p}{2^{p+2}}\|v\|^{2p-2}_{L^{2p-2}(D_N)} \le p_2(\theta_t\omega) + c_0. \tag{5.17}$$

Working with l instead of t in (5.17) and integrating with respect to l from $\tau(t \le \tau \le t+1/2)$ to $s(t+1/2 \le s \le t+1)$, it yields that

(5.18)
$$\|v(s,\omega,v_0(\omega))\|^p_{L^p(D_N)} \le \int_\tau^s p_2(\theta_l\omega)dl + \|v(\tau,\omega,v_0(\omega))\|^p_{L^p(D_N)} + c_0(s-\tau).$$

By replacing ω by $\theta_{-t-1}\omega$ in (5.18) and then integrating with respect to τ from t to $t+1/2$, we obtain that, for all $s \in [t+1/2, t+1]$,

$$\begin{aligned} &\|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^p_{L^p(D_N)} \\ &\le \int_t^{t+1} p_2(\theta_{l-t-1}\omega)dl + \int_t^{t+1}\|v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^p_{L^p(D_N)}d\tau + c_0. \end{aligned} \tag{5.19}$$

Hence by employing Lemma 5.2, association with (5.16), it follows from (5.19) that, for all $t \ge T$ and $s \in [t+1/2, t+1]$,

$$\begin{aligned} \|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^p_{L^p(D_N)} &\le \int_{-1}^{0} p_2(\theta_l\omega)dl + c(1+\varrho(\omega)) + c_0 \\ &\le c\varrho(\omega)\int_{-1}^{0} e^{-\frac{1}{2}\lambda l}dl + c(1+\varrho(\omega)) + c_0 \\ &\le c(1+\varrho(\omega)), \end{aligned} \tag{5.20}$$

where $T = T(B,\omega)$ is in Lemma 5.2 and $v_0(\omega) + z(\omega) = u_0(\omega) \in B(\omega)$. Then by (5.20) for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there is $T' = T(B,\omega) + 1$ such that, for all $t \ge T'$,

$$\varphi(t,\theta_{-t}\omega, B(\theta_{-t}\omega) - z(\theta_{-t}\omega)) \subseteq K_p(\omega),$$

where

$$K_p(\omega) = \{v \in L^p(D_N) : \|v\|_{L^p(D_N)} \le \{c(1+\varrho(\omega))\}^{\frac{1}{p}}\}.$$

That is, $\{K_p(\omega)\}_{\omega\in\Omega}$ is an $(L^2(D_N), L^p(D_N))$-random absorbing set in $\mathcal{D}_p$ for the RDS φ, which completes the proof. $\square$

We give an unform estimate of the unbounded part of the modulus $|\varphi|$ in the topology of space $L^p(D_N)$. We start with some auxiliary lemmas.

Lemma 5.4. Assume (4.4)-(4.6) hold. Let $u_0(\omega) \in B(\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega} \in$

$\mathcal{D}$. Then for every $\varepsilon > 0$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there exist $T = T(B,\omega) > 0$ and $M = M(\varepsilon, B, \omega)$ such that for all $t \geq T$ and $s \in [t, t+1]$,

$$\mathrm{mes}(D_N|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M) \leq \varepsilon.$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$.

Proof. By Lemma 5.1, for every $u_0(\omega) \in B(\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exists a constant $T = T(B,\omega)$ such that for all $t \geq T$ and $s \in [t, t+1]$,

$$\int_{D_N} |\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^2 dx \leq c(1 + \varrho(\omega)), \tag{5.21}$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$. On the other hand, for any fixed $s \in [t, t+1]$ and positive number $M = M(\omega)$,

$$\begin{aligned} &\int_{D_N} |\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^2 dx \\ &\geq \int_{D_N(|\varphi(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M)} |\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^2 dx \\ &\geq M^2 \mathrm{mes}(D_N(|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M). \end{aligned} \tag{5.22}$$

It follows from (5.21)-(5.22) that

$$\mathrm{mes}(D_N(|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M) \leq \frac{c(1 + \varrho(\omega))}{M^2}, \quad s \in [t, t+1]. \tag{5.23}$$

Hence for any $\varepsilon > 0$, we deduce from (5.23) that, for all $s \in [t, t+1]$ and $t \geq T$,

$$\mathrm{mes}(D_N|\varphi(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M) \leq \varepsilon,$$

provided that $M > (\frac{c(1+\varrho(\omega))}{\varepsilon})^{\frac{1}{2}}$. $\square$

Lemma 5.5. Assume that (4.4)-(4.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for $\mathbb{P}$-a.e.$\omega \in \Omega$ and any $\varepsilon > 0$, there exist random constants $c = c(\omega)$, $T = T(\varepsilon, B, \omega) > 0$ and $M = M(\varepsilon, B, \omega)$ such that for all $t \geq T$,

$$\sup_{u_0(\omega)\in B(\omega)} \int_{D_N(|\varphi(t,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))| \geq M)} |\varphi(t, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^2 dx \leq c\varepsilon,$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$.

Proof. From Theorem 4.1 there exists a compact random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ for RDS ψ in $L^2(D_N)$. Then for $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\mathbb{P}$-a.e.$\omega \in \Omega$, there holds

$$\lim_{t\to\infty} d(\psi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \tag{5.24}$$

where d is the Hausdorff semi-distance in $L^2(D_N)$. Working with $\theta_{-1}\omega$ instead of ω in (5.24), it yields that

$$\lim_{t\to\infty} d(\psi(t, \theta_{-t-1}\omega, B(\theta_{-t-1}\omega)), \mathcal{A}(\theta_{-1}\omega)) = 0. \tag{5.25}$$

Then (5.25) and along with (4.14) implies that there exists $T_1 = T_1(\varepsilon, B, \omega) > 0$ such that for all $t \geq T_1$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\varphi(t, \theta_{-t-1}\omega, B(\theta_{-t-1}\omega) - z(\theta_{-t-1}\omega)) + z(\theta_{-1}\omega) \subseteq N_\varepsilon(\mathcal{A}(\theta_{-1}\omega)),$$

where $N_\varepsilon(\mathcal{A}(\theta_{-1}\omega))$ is the $(\frac{\varepsilon}{8})^{\frac{1}{2}}$-neighborhood of $\mathcal{A}(\theta_{-1}\omega)$ in $L^2(D_N)$. From the compactness of $\mathcal{A}(\theta_{-1}\omega)$, we deduce that

$$\bigcup_{t\geq T_1}\Big(\varphi(t,\theta_{-t-1}\omega, B(\theta_{-t-1}\omega)-z(\theta_{-t-1}\omega))+z(\theta_{-1}\omega)\Big)$$

has a finite $(\frac{\varepsilon}{8})^{\frac{1}{2}}$-net. Thus by Lemma 2.5 in [**26**], there exists $M_1 = M_1(\varepsilon, B, \omega)$ such that for all $t \geq T_1$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\begin{aligned}&\sup_{u_0(\omega)\in B(\omega)}\int_{D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M_1)}|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\\ &\quad+z(\theta_{-1}\omega)|^2dx\leq c\varepsilon,\end{aligned}\tag{5.26}$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$. Note that from Lemma 4.4, there exist $T_2 = T_2(\varepsilon, B, \omega)$ and $M_2 = M_2(\varepsilon, B, \omega)$ such that for all $t \geq T_2$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\mathrm{mes}(D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M_2))\leq\varepsilon.$$

Put $T = \max\{T_1, T_2\}$ and $M = \max\{M_1, M_2\}$. Observer that $h_j \in L^2(D_N)$. Then by Hölder's inequality, along with (5.27) below and (5.3), we infer that for all $t \geq T$,

$$\begin{aligned}&\int_{D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-1}\omega))|\geq M)}|z(\theta_{-1}\omega)|^2dx\\ &\leq\sum_{j=1}^m|z_j(\theta_{-1}\omega)|^2\int_{D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-1}\omega))|\geq M)}|h_j|^2dx\\ &\leq c\varepsilon\sum_{j=1}^m|z_j(\theta_{-1}\omega)|^2\leq c\varrho(\omega)\varepsilon,\end{aligned}$$

from which and by an utilization of the inequality $|a+b|^2 \geq \frac{|a|^2}{2} - |b|^2$ to (5.26), it yields that

$$\sup_{u_0(\omega)\in B(\omega)}\int_{D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M)}|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^2dx\leq c\varepsilon,$$

where $v_0(\omega)) = u_0 - z(\omega)$. □

Lemma 5.6. Assume that (4.4)-(4.6) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for $\mathbb{P}$-a.e.$\omega \in \Omega$ and any $\varepsilon > 0$, there exist $c = c(\omega), T = T(\varepsilon, B, \omega) > 0$ and $M = M(\varepsilon, B, \omega)$ such that for all $t \geq T_1$,

$$\sup_{u_0(\omega)\in B(\omega)}\int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\geq M)}|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^pdx\leq c\varepsilon,$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$.

Proof. For any fixed $\varepsilon > 0$, if $\mathrm{g} \in L^l(D_N)$ then there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for any $e \subset D_N$ with $\mathrm{mes}(e) < \delta_1$,

$$\int_e|\mathrm{g}|^ldx<\varepsilon.\tag{5.27}$$

As $h_j \in L^p(D_N)\cap \mathrm{Dom}(A)\cap L^{2p-2}(D_N)$ for $j=1,2,...,m$, there exists $\delta_2=\delta_2(\varepsilon)>0$ such that for any $e\subset D_N$ with $\mathrm{mes}(e)<\delta_2$,

$$\int_e (|h_j(x)|^{2p-2}+|h_j(x)|^p+|h_j(x)|^2+|Ah_j(x)|^2)dx < \frac{\varepsilon}{m^{p-1}\varrho(\omega)}. \tag{5.28}$$

From Lemma 5.4, we know that for every $u_0(\omega)\in B(\omega)$, there exist positive constants $T_1=T_1(\varepsilon,B,\omega)$ and $M_1=M_1(\varepsilon,B,\omega)$ such that for all $t\geq T_1$ and $s\in[t,t+1]$,

$$\mathrm{mes}(D_N(|\varphi(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M_1))<\min\{\varepsilon,\delta_1,\delta_2\}, \tag{5.29}$$

where $v_0(\omega)=u_0(\omega)-z(\omega)$. On the other hand, by Lemma 5.5, there exists $T_2=T_2(\varepsilon,B,\omega)$ and $M_2=M_2(\varepsilon,B,\omega)$ such that for all $t\geq T_2$,

$$\int_{D_N(|\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M_2)} |\varphi(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^2dx\leq c\varepsilon. \tag{5.30}$$

By our assumption (4.4), we can choose $M_3=M_3(\varepsilon,B,\omega)>0$ such that for all $x\in D_N$,

$$f(x,u)\leq 0 \quad \text{if } u\geq M_3. \tag{5.31}$$

We let

$$E=E(\omega)=\max_{-1\leq s\leq 0}\|z(\theta_s\omega)\|_{L^\infty(D_N)}. \tag{5.32}$$

Then by our assumption $h_j\in L^\infty(D_N)$, together with (4.8)-(4,9), we have $E(\omega)$ is finite for $\mathbb{P}$-a.e.$\omega\in\Omega$. Let now

$$M=M(\omega)=\max\{M_1,M_2,M_3\}+E(\omega),\quad T=\max\{T_1,T_2\}. \tag{5.33}$$

Then by replacing ω by $\theta_{t+1}\omega$ we have

$$M(\theta_{t+1}\omega)=\max\{M_1(\theta_{t+1}\omega),M_2(\theta_{t+1}\omega),M_3(\theta_{t+1}\omega)\}+E(\theta_{t+1}\omega).$$

For these positive constants $M(\omega)$ and T, (5.29)-(5.31) hold when $t\geq T$, and therefore (5.27)-(5.28) hold for $e=D_N(|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|\geq M)$, $s\in[t,t+1]$.

For fixed $t\in\mathbb{R}$ and $\omega\in\Omega$, define

$$(v(s)-M(\theta_{t+1}\omega))_+=\begin{cases} v(s)-M(\theta_{t+1}\omega), & \text{if } v(s)\geq M(\theta_{t+1}\omega),\\ 0, & \text{if } v(s)\leq M(\theta_{t+1}\omega).\end{cases}$$

Multiplying (4.10) with $(v(s)-M(\theta_{t+1}\omega))_+$ and then integrating over D_N, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\|(v(s)-M(\theta_{t+1}\omega))_+\|^2_{L^2(D_N)}\\&+\lambda\int_{D_N}v(s)(v(s)-M(\theta_{t+1}\omega))_+dx+\int_{D_N}Av(s)(v(s)-M(\theta_{t+1}\omega))_+dx\\&=\int_{D_N}f(x,v(s)+z(\theta_s\omega))(v(s)-M(\theta_{t+1}\omega))_+dx\\&-\int_{D_N}Az(\theta_s\omega)(v(s)-M(\theta_{t+1}\omega))_+dx,\end{aligned} \tag{5.34}$$

where

$$\int_{D_N}Av(s)(v(s)-M(\theta_{t+1}\omega))_+dx\geq 0, \tag{5.35}$$

$$\lambda \int_{D_N} v(s)(v(s) - M(\theta_{t+1}\omega))_+ dx \geq \lambda \|(v(s) - M(\theta_{t+1}\omega))_+\|^2_{L^2(D_N)}. \tag{5.36}$$

By Young's inequality, the second term on the right hand side of (5.34) is bounded by

$$\lambda \|(v(s) - M(\theta_{t+1}\omega))_+\|^2_{L^2(D_N)} + \frac{1}{4\lambda} \int_{D_N(v(s) \geq M(\theta_{t+1}\omega))} |Az(\theta_s\omega)|^2) dx. \tag{5.37}$$

Then by (5.34)-(5.37) we find that

$$\begin{aligned}
&\frac{d}{ds} \|(v(s) - M(\theta_{t+1}\omega))_+\|^2_{L^2(D_N)} \\
&\leq 2 \int_{D_N} f(x, v(s) + z(\theta_s\omega))(v(s) - M(\theta_{t+1}\omega))_+ dx \\
&+ \frac{1}{2\lambda} \int_{D_N(v(s) \geq M(\theta_{t+1}\omega))} |Az(\theta_s\omega)|^2 dx
\end{aligned} \tag{5.38}$$

We integrate (5.38) with respect to s from t to $t+1$ to yield that

$$\begin{aligned}
&- \int_t^{t+1} 2 \int_{D_N} f(x, v(s) + z(\theta_s\omega))(v(s) - M(\theta_{t+1}\omega))_+ dx ds \\
&\leq \frac{1}{2\lambda} \int_t^{t+1} \int_{D_N(v(s,\omega,v_0(\omega)) \geq M(\theta_{t+1}\omega))} |Az(\theta_s\omega)|^2 dx ds \\
&+ \|(v(t, \omega, v_0(\omega)) - M(\theta_{t+1}\omega))_+\|^2_{L^2(D_N)}.
\end{aligned} \tag{5.39}$$

Denote

$$D_i(s, t+1) = D_N(v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \geq iM(\omega)), \quad i = 1, 2, 4, 8.$$

Then $D_1(s, t+1) \supseteq D_2(s, t+1)$. Replacing ω by $\theta_{-t-1}\omega$ in (5.39), we see that

$$\begin{aligned}
&- 2 \int_t^{t+1} \int_{D_1(s,t+1)} f(x, v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \\
&+ z(\theta_{s-t-1}\omega))(v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M(\omega))_+ dx ds \\
&\leq c_1 \int_t^{t+1} \int_{D_1(s,t+1)} |Az(\theta_{s-t-1}\omega)|^2 dx ds + \\
&\|(v(t, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M(\omega))_+\|^2_{L^2(D_N)}.
\end{aligned} \tag{5.40}$$

Note that $s - t - 1 \in [-1, 0]$ for $s \in [t, t+1]$. Then it follows from (5.32) that $v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) + z(\theta_{s-t-1}\omega) \geq M(\omega) - E(\omega) \geq M_3$ on $D_1(s, t+1)$ for $s \in [t, t+1]$. This along with (5.31) implies that, for $s \in [t, t+1]$,

$$f(x, v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) + z(\theta_{s-t-1}\omega)) \leq 0, \quad \text{on } D_1(s, t+1). \tag{5.41}$$

By (5.41) and the fact that $2(v - M) \geq v$ for $v \geq 2M$, it yields that, for $s \in [t, t+1]$,

$$\begin{aligned}
&\int_{D_1(s,t+1)} 2f(x, v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \\
&+ z(\theta_{s-t-1}\omega))(v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M(\omega))_+ dx \\
&\leq \int_{D_2(s,t+1)} f(x, v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \\
&+ z(\theta_{s-t-1}\omega)) v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) dx.
\end{aligned} \tag{5.42}$$

But by (5.10),

$$f(x, v(s) + z(\theta_s\omega))v \le -\frac{\alpha_1}{2^p}|v(s)|^p + c(|z(\theta_s\omega)|^p + |z(\theta_s\omega)|^2) + \phi_1 + \frac{1}{2}\phi_2^2. \tag{5.43}$$

It follows from (5.43) that the right hand side of (5.42) is bounded by

$$\begin{aligned}
& - \frac{\alpha_1}{2^p}\int_{D_2(s,t+1)} |v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^p dx \\
& + c\int_{D_2(s,t+1)} (|z(\theta_{s-t-1}\omega)|^p + |z(\theta_{s-t-1}\omega)|^2)dx \\
& + \int_{D_2(s,t+1)} (\phi_1 + \frac{1}{2}\phi_2^2)dx.
\end{aligned} \tag{5.44}$$

Then from (5.40) and (5.44) we deduce that

$$\begin{aligned}
& \int_t^{t+1}\int_{D_2(s,t+1)} |v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^p dxds \\
& \le c_1\int_t^{t+1}\int_{D_1(s,t+1)} (|z(\theta_{s-t-1}\omega)|^p + |z(\theta_{s-t-1}\omega)|^2 + |Az(\theta_{s-t-1}\omega)|^2)dxds \\
& + c_2\int_t^{t+1}\int_{D_1(s,t+1)} (\phi_1 + \frac{1}{2}\phi_2^2)dxds \\
& + c_3\|(v(t, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - M(\omega))_+\|^2_{L^2(D_N)}.
\end{aligned} \tag{5.45}$$

Since $\phi_1 \in L^1(D_N)$ and $\phi_2 \in L^2(D_N)$ then by (5.29) and (5.27) we have, for all $t \ge T$,

$$c_2\int_t^{t+1}\int_{D_1(s,t+1)} (\phi_1 + \frac{1}{2}\phi_2^2)dxds \le c\varepsilon. \tag{5.46}$$

The Hölder's inequality implies that

$$\Big|\sum_{j=1}^m h_j z_j(\theta_{\sigma-t-1}\omega)\Big|^p \le m^{p-2}\sum_{j=1}^m |h_j|^p \sum_{j=1}^m |z_j(\theta_{s-t-1}\omega)|^p,$$

whereas it follows from (5.28)-(5.29) that, for all $t \ge T$,

$$
\begin{aligned}
& c_1 \int_t^{t+1} \int_{D_1(s,t+1)} (|z(\theta_{s-t-1}\omega)|^p + |z(\theta_{s-t-1}\omega)|^2 + |Az(\theta_{s-t-1}\omega)|^2) dxds \\
& \leq c_1 m^{p-2} \int_t^{t+1} \int_{D_1(s,t+1)} \Big(\sum_{j=1}^m |h_j|^p \sum_{j=1}^m |z_j(\theta_{s-t-1}\omega_j)|^p \\
& + \sum_{j=1}^m |h_j|^2 \sum_{j=1}^m |z_j(\theta_{s-t-1}\omega_j)|^2 \\
& + \sum_{j=1}^m |Ah_j|^2 \sum_{j=1}^m |z_j(\theta_{s-t-1}\omega_j)|^2 \Big) dxds \\
& \leq \frac{c_1 \varepsilon}{\varrho(\omega)} \int_t^{t+1} (\sum_{j=1}^m |z_j(\theta_{s-t-1}\omega_j)|^p + 2 \sum_{j=1}^m |z_j(\theta_{s-t-1}\omega_j)|^2) ds \\
& \leq \frac{2c_1 \varepsilon}{\varrho(\omega)} \int_t^{t+1} p_1(\theta_{s-t-1}\omega) ds \leq \frac{2c_1\varepsilon}{\varrho(\omega)} \int_{-1}^0 \varrho(\omega) e^{-\frac{1}{2}\lambda s} ds \leq c\varepsilon,
\end{aligned} \tag{5.47}
$$

where $p_1(\theta_s\omega)$ is in (5.2). Then by (5.30) and (5.46)-(5.47), the inequality (5.45) can be expressed in a simple form, i.e., for all $t \geq T$,

$$
\int_t^{t+1} \int_{D_2(s,t+1)} |v(s, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))|^p dxds \leq c\varepsilon, \tag{5.48}
$$

where $D_2(s,t+1) = D_N(v(s,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) \geq 2M(\omega))$.

We then take the inner product of (4.10) with $(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1}$ and integrate over D_N to find that

$$
\begin{aligned}
& \frac{1}{p}\frac{d}{ds} \|(v(s) - 2M(\theta_{t+1}\omega))_+\|_{L^p(D_N)}^p \\
& + \lambda \int_{D_N} v(s)(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx \\
& + \int_{D_N} Av(s)(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx \\
& = \int_{D_N} f(x, v(s) + z(\theta_s\omega))(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx \\
& - \int_{D_N} Az(\theta_s\omega)(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx,
\end{aligned} \tag{5.49}
$$

where

$$
\lambda \int_{D_N} v(s)(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx \geq \lambda \|(v(s) - 2M(\theta_{t+1}\omega))_+\|_{L^p(D_N)}^p, \tag{5.50}
$$

$$
\int_{D_N} Av(s)(v(s) - 2M(\theta_{t+1}\omega))_+^{p-1} dx \geq 0. \tag{5.51}
$$

But by Young's inequality, we deduce that

$$\Big|\int_{D_N} Az(\theta_s\omega)(v(s)-2M(\theta_{t+1}\omega))_+^{p-1})dx\Big| \le \lambda\|(v(s)-2M(\theta_{t+1}\omega))_+\|^p_{L^p(D_N)} + \frac{1}{4\lambda}\int_{D_N(v(s)\ge 2M(\theta_{t+1}\omega))}|Az(\theta_s\omega)|^p)dx. \tag{5.52}$$

Thus from (5.49)-(5.52) we get that, for all $s\ge 0$,

$$\begin{aligned}&\frac{d}{ds}\|(v(s)-2M(\theta_{t+1}\omega))_+\|^p_{L^p(D_N)}\\ &\le p\int_{D_N} f(x,v(s)+z(\theta_s\omega))\\ &(v(s)-2M(\theta_{t+1}\omega))_+^{p-1}dx\\ &+\frac{p}{4\lambda}\int_{D_N(v(s)\ge 2M(\theta_{t+1}\omega))}|Az(\theta_s\omega)|^p dx,\end{aligned} \tag{5.53}$$

Integrating (5.53) with respect to s from $\tau(t\le\tau\le t+1/2)$ to $s(t+1/2\le s\le t+1)$ we get that

$$\begin{aligned}&\|(v(s)-2M(\theta_{t+1}\omega))_+\|^p_{L^p(D_N)}\\ &\le p\int_\tau^s\int_{D_N} f(x,v(\sigma)+z(\theta_\sigma\omega))(v(\sigma)-2M(\theta_{t+1}\omega))_+^{p-1}dxd\sigma\\ &+\frac{p}{4\lambda}\int_\tau^s\int_{D_N(v(\sigma)\ge 2M(\theta_{t+1}\omega))}|Az(\theta_\sigma\omega)|^p dxd\sigma\\ &+\|(v(\tau,\omega,v_0(\omega))-M(\theta_{t+1}\omega))_+\|^p_{L^p(D_N)}.\end{aligned} \tag{5.54}$$

Working with $\theta_{-t-1}\omega$ instead of ω in (5.54) then integrating with respect to τ over intervals $[t,t+1/2]$ it yields that

$$\begin{aligned}&\|(v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\\ &-2M(\omega))_+\|^p_{L^p(D_N)} \le \frac{p}{4\lambda}\int_t^{t+1}\int_{D_2(s,t+1)}|Az(\theta_{s-t-1}\omega)|^p dxds\\ &+\int_t^{t+1/2}\|(v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-2M(\omega))_+\|^p_{L^p(D_N)}d\tau,\end{aligned} \tag{5.55}$$

where we use the fact that $f(x,v(\sigma,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))+z(\theta_{\sigma-t-1}\omega))\le 0$ on $D_2(\sigma,t+1)$. By a similar argument as (5.47), we can show that, for all $t\ge T$,

$$\frac{p}{4\lambda}\int_t^{t+1}\int_{D_2(s,t+1)}|Az(\theta_{s-t-1}\omega)|^p dxds \le c\varepsilon. \tag{5.56}$$

Therefore, by (5.48) and (5.55)-(5.56) we find that, for all $t\ge T$ and $s\in[t+1/2,t+1]$,

$$\|(v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-2M(\omega))_+\|^p_{L^p(D_N)}\le c\varepsilon, \tag{5.57}$$

and then we deduce that, for all $t\ge T+1$,

$$\int_{D_4(t,t)}|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^p dx\le c\varepsilon, \tag{5.58}$$

where $D_4(t,t)=D_N(v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\ge 4M(\omega))$. Repeating the same arguments above, working with $(v(s)+M(\theta_{t+1}\omega))_-$ and $|(v(s)+M(\theta_{t+1}\omega))_-|^{p-2}(v(s)+$

$M(\theta_{t+1}\omega))_-$ instead of $(v(s)-M(\theta_{t+1}\omega))_+$ and $(v(s)-M(\theta_{t+1}\omega))_+^{p-1}$, respectively, where $(v(s)+M((\theta_{t+1}\omega))_-$ is the negative part of $v(s)+M(\theta_{t+1}\omega)$, we can deduce that, for all $t \geq T+1$,

$$\int_{D_{-4}(v(t,t)} |v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)|^p dx \leq c\varepsilon, \tag{5.59}$$

where $D_{-4}(t,t) = D_N(v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \leq -4M(\omega))$. Therefore, it follows from (5.58) and (5.59) that for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exist $T(\varepsilon, B, \omega) = T+1$ and $M' = 4M$ such that for all $t \geq T(\varepsilon, B, \omega)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\sup_{u_0(\omega)\in B(\omega)} \int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\geq M')} |\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^p dx < c\varepsilon, \tag{5.60}$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$. This concludes our result. $\square$

5.2. The (L^2, L^p)-random attractor

From Lemma 5.3 and Lemma 5.6 the RDS φ associated with the solutions to (4.10)-(4.12) satisfies the assumptions (i) and (ii) of Theorem 3.5, and therefore admits an (L^2, L^p)-random attractor. We show that this holds true for the RDS ψ generated by the original problem (4.1)-(4.3).

Theorem 5.7. Assume that (4.4)-(4.6) hold. Then the RDS ψ generated by (4.1)-(4.3) admits a unique $(L^2(D_N), L^p(D_N))$-random attractor $\{\mathcal{A}_p(\omega)\}_{\omega\in\Omega}$. Furthermore, $\mathcal{A}_p(\omega) = \mathcal{A}(\omega)$ for $\omega \in \Omega$, where $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is the $(L^2(D_N), L^2(D_N))$-random attractor.

Proof. It suffices to show that the RDS ψ defined in (4.14) satisfies Theorem 3.5. By (4.14),

$$\psi(t,\omega,u_0(\omega)) = \varphi(t,\omega,u_0(\omega) - z(\omega)) + z(\theta_t\omega). \tag{5.61}$$

Then by Lemma 5.3, for t large enough,

$$\begin{aligned}
&\|\psi(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^p_{L^p(D_N)} \\
&= \|\varphi(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega) - z(\theta_{-t}\omega)) + z(\omega)\|^p_{L^p(D_N)} \\
&\leq 2^{p-1}(\|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^p_{L^p(D_N)} + \|z(\omega)\|^p_{L^p(D_N)}) \\
&\leq c2^{p-1}(1+\varrho(\omega)) + 2^{p-1}\|z(\omega)\|^p_{L^p(D_N)} \\
&\leq c(1+\varrho(\omega)),
\end{aligned} \tag{5.62}$$

which shows the existence of a closed $(L^2(D_N), L^p(D_N))$-random absorbing set for RDS ψ. On the other hand, by Lemma 5.6, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and any $\varepsilon > 0$, there exist $c = c(\omega), T = T(\varepsilon, B, \omega) > 0$ and $M = M(\varepsilon, B, \omega)$ such that for all $t \geq T$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\sup_{u_0(\omega)\in B(\omega)} \int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\geq M)} |\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^p dx < c\varepsilon, \tag{5.63}$$

where $v_0(\omega) = u_0(\omega) - z(\omega)$. But by (5.61) we see that

$$D_N(|\psi(t,\theta_{-t}\omega, u_0(\theta_{-t}\omega))| \geq M+F) \subset D_N(|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))| \geq M), \tag{5.64}$$

where $F = \|z(\omega)\|_{L^\infty(D_N)}$. Hence, by (5.63) and (5.64), using Lemma 5.4, we get that, for all $t \geq T$,

$$\begin{aligned}
&\int_{D_N(|\psi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|\geq M+F)} |\psi(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))|^p dx \\
&\leq 2^{p-1}\Big(\int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\geq M)} |v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^p dx \\
&+\int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\geq M)} |z(\omega)|^p dx\Big) \\
&\leq 2^{p-1}(c\varepsilon + F^p \mathrm{mes}(D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))| \geq M))) \leq c\varepsilon,
\end{aligned}$$

which show that the assumptions (ii) of Theorem 3.5 is fulfilled. This ends the proof. □

6. The random attractor in $L^{2p-2}(D_N)$

In this section, we will prove the existence of $(L^2(D_N), L^{2p-2}(D_N))$-random attractor. For this purpose, at first, we will give some a priori estimates of the RDS φ associated with (4.10)-(4.12) in L^{2p-2} space.

Lemma 6.1. Assume that (4.4)-(4.6) hold. Then there exists a random ball $\{K_{2p-2}(\omega)\}_{\omega\in\Omega}$ centered at 0 with random radius $\Big\{c(1+\varrho(\omega))\Big\}^{\frac{1}{2p-2}}$ such that $\{K_{2p-2}(\omega)\}_{\omega\in\Omega}$ is an (L^2, L^{2p-2})-random absorbing set for RDS φ in $\mathcal{D}_{2p-2}$, where c is a deterministic positive constant.

Proof. Replacing t by s in (5.17) and then integrating with respect to s from $t+1/2$ to $t+1$ we get that

$$\int_{t+1/2}^{t+1} \|v(s)\|^{2p-2}_{L^{2p-2}(D_N)} ds \leq c\int_{t+1/2}^{t+1} p_2(\theta_s)ds + c\|v(t+1/2)\|^p_{L^p(D_N)} + \frac{1}{2}c_0. \tag{6.1}$$

Working with $\theta_{-t-1}\omega$ instead of ω in (6.1), then using (5.20) we find that there exists $T = T(\varepsilon, B, \omega) > 0$ such that for all $t \geq T$,

$$\begin{aligned}
&\int_{t+1/2}^{t+1} \|v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^{2p-2}_{L^{2p-2}(D_N)} ds \\
&\leq c\int_{t+1/2}^{t+1} p_2(\theta_{s-t-1})ds + c\|v(t+1/2,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^p_{L^p(D_N)} + \frac{1}{2}c_0 \\
&\leq c\int_{-1/2}^{0} p_2(\theta_s)ds + c(1+\varrho(\omega)) + \frac{1}{2}c_0 \leq c(1+\varrho(\omega)).
\end{aligned}\tag{6.2}$$

Multiplying (4.10) with v^{2p-3} we obtain that

$$\begin{aligned}
\frac{1}{2p-2}\frac{d}{dt}\int_{D_N} v^{2p-2}dx + \lambda\int_{D_N} v^{2p-2}dx \leq& \int_{D_N} f(x, v+z(\theta_t\omega))v^{2p-3}dx \\
&-\int_{D_N} Az(\theta_t\omega)v^{2p-3}dx.
\end{aligned}\tag{6.3}$$

By careful calculations, we deduce that

$$
\begin{aligned}
&\Big|\int_{D_N} f(x, v+z(\theta_t\omega))v^{2p-4}v dx\Big| \\
&\le -\frac{\alpha_1}{2^{p+1}}\int_{D_N}|v|^{3p-4}dx + \frac{\lambda}{2}\int_{D_N} v^{2p-2}dx \\
&+ c\int_{D_N}(z(\theta_t\omega)^{3p-4} + z(\theta_t\omega)^{2p-2})dx + c\int_{D_N}(\phi_1^{p-1}+\phi_2^{2p-2})dx,
\end{aligned}
\tag{6.4}
$$

and

$$
\Big|\int_{D_N} Az(\theta_t\omega)v^{2p-3}dx\Big| \le \frac{\lambda}{2}\int_{D_N} v^{2p-2}dx + c\int_{D_N}|Az(\theta_t\omega)|^{2p-2}dx. \tag{6.5}
$$

Therefore by (6.3)-(6.5) we get that

$$
\frac{d}{dt}\int_{D_N} v^{2p-2}dx \le c\int_{D_N}(z(\theta_t\omega)^{3p-4} + z(\theta_t\omega)^{2p-2} + |Az(\theta_t\omega)|^{2p-2})dx + c. \tag{6.6}
$$

Note that $h_j \in L^{3p-4}(D_N)\cap L^{2p-2}(D_N)\cap \mathrm{D}^{2p-2}(A)$. Then the right hand side of (6.6) is controlled by

$$
c\sum_{j=1}^{m}(|z_j(\theta_t\omega_j)|^{3p-4} + |z_j(\theta_t\omega_j)|^{2p-2}) + c = p_3(\theta_t\omega) + c. \tag{6.7}
$$

Furthermore,

$$
p_3(\theta_s\omega) \le ce^{\frac{1}{2}\lambda|s|}\varrho(\omega), \quad s\in\mathbb{R}. \tag{6.8}
$$

By integrating (6.6) from $\tau(t+1/2\le\tau\le t+1)$ to $t+1$ we obtain that, with (6.2),

$$
\begin{aligned}
&\|v(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^{2p-2}_{L^{2p-2}(D_N)} \\
&\le \int_\tau^{t+1} p_3(\theta_{\tau-t-1}\omega)d\tau + \|v(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^{2p-2}_{L^{2p-2}(D_N)} + \frac{1}{2}c \\
&\le \int_{t+1/2}^{t+1}\|v(\tau,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega))\|^{2p-2}_{L^{2p-2}(D_N)}d\tau + c(1+\varrho(\omega)) \le c(1+\varrho(\omega)),
\end{aligned}
\tag{6.9}
$$

for all $t\ge T$, which completes the proof. □

Lemma 6.2. Assume that (4.4)-(4.6) hold. Let $B=\{B(\omega)\}_{\omega\in\Omega}\in\mathcal{D}$. Then for $\mathbb{P}$-a.e.$\omega\in\Omega$ and any $\varepsilon>0$, there exist $c=c(\omega)$, $T=T(\varepsilon,B,\omega)>0$ and $M=M(\varepsilon,B,\omega)$ such that for all $t\ge T$,

$$
\sup_{u_0(\omega)\in B(\omega)}\int_{D_N(|\varphi(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|\ge M)}|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))|^{2p-2}dx < c\varepsilon,
$$

where $v_0(\omega)=u_0(\omega)-z(\omega)$.

Proof. Integrating (5.53) with respect to s from $t+1/2$ to $t+1$ to yield that

$$\begin{aligned}0 \le p &\int_{t+1/2}^{t+1}\int_{D_N} f(x, v(s,\omega,v_0(\omega)) + z(\theta_s\omega))\\&\times (v(s,\omega,v_0(\omega)) - 2M(\theta_{t+1}\omega))_+^{p-1} dxds\\&+ \frac{p}{4\lambda}\int_{t+1/2}^{t+1}\int_{D_N(v(s)\ge 2M(\theta_{t+1}\omega))} |Az(\theta_{s-t-1}\omega)|^p dxds\\&+ \|(v(t+1/2,\omega,v_0(\omega)) - 2M(\theta_{t+1}\omega))_+\|^p_{L^p(D_N)}.\end{aligned}$$

Replacing ω with $\theta_{-t-1}\omega$ in the above inequality and connection with (5.56) and (5.57) we get that there exists $T = T(\varepsilon, B, \omega)$ such that for all $t \ge T$,

$$\begin{aligned}0 \le p &\int_{t+1/2}^{t+1}\int_{D_2(s,t+1)} f(x, v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)) + z(\theta_{s-t-1}\omega))\times\\&(v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)) - 2M(\omega))_+^{p-1} dxds + c\varepsilon.\end{aligned}\tag{6.10}$$

Note that

$$\begin{aligned}&\int_{D_2(s,t+1)} 2^{p-1} f(x, v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\\&+ z(\theta_{s-t-1}\omega))(v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)) - 2M(\omega))_+^{p-1} dx\\&\le \int_{D_4(s,t+1)} f(x, v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\\&+ z(\theta_{s-t-1}\omega)) v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))^{p-1} dx,\end{aligned}\tag{6.11}$$

and by (5.11),

$$\begin{aligned}&f(x, v + z(\theta_t\omega))|v|^{p-2}v \le -\frac{\alpha_1}{2^{p+1}}|v|^{2p-2} + \lambda|v|^p + c(|z(\theta_t\omega)|^{2p-2}\\&+ |z(\theta_t\omega)|^p) + c(\phi_1^{\frac{p}{2}} + \phi_2^p).\end{aligned}\tag{6.12}$$

Then it follows from (6.10)-(6.12) that, for all $t \ge T$,

$$\begin{aligned}\int_{t+1/2}^{t+1}&\int_{D_4(s,t+1)} |v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^{2p-2} dxds\\&\le c_1\int_t^{t+1}\int_{D_4(s,t+1)} |v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))|^p dxds\\&+ c_2\int_t^{t+1}\int_{D_4(s,t+1)} (|z(\theta_{s-t-1}\omega)|^{2p-2} + |z(\theta_{s-t-1}\omega)|^p) dxds\\&+ c_3\int_t^{t+1}\int_{D_4(s,t+1)} c(\phi_1^{\frac{p}{2}} + \phi_2^p) dxds + c\varepsilon\\&\le c\varepsilon.\end{aligned}\tag{6.13}$$

Multiplying (4.10) with $(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}$ we obtain that

$$\frac{1}{2p-2}\frac{d}{ds}\|(v(s)-4M(\theta_{t+1}\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)}+\lambda\int_{D_N}v(s)(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx$$
$$+\int_{D_N}Av(s)(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx$$
$$=\int_{D_N}f(x,v(s)+z(\theta_s\omega))(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx$$
$$-\int_{D_N}Az(\theta_s\omega)(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx. \tag{6.14}$$

Consider that

$$\Big|\int_{D_N}Az(\theta_s\omega)(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx\Big|$$
$$\leq\lambda\int_{D_N}(v(s)-4M(\theta_{t+1}\omega))_+^{2p-2}dx+$$
$$\frac{1}{4\lambda}\int_{D_N(v(s)\geq 4M(\theta_{t+1}\omega))}|Az(\theta_s\omega)|^{2p-2}dx, \tag{6.15}$$

and

$$\lambda\int_{D_N}v(s)(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx\geq\lambda\int_{D_N}(v(s)-4M(\theta_{t+1}\omega))_+^{2p-2}dx. \tag{6.16}$$

It follows from (6.14)-(6.16) that

$$\frac{d}{ds}\|(v(s)-4M(\theta_{t+1}\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)}$$
$$=(2p-2)\int_{D_N}f(x,v(s)+z(\theta_s\omega))(v(s)-4M(\theta_{t+1}\omega))_+^{2p-3}dx$$
$$+\frac{2p-2}{4\lambda}\int_{D_N(v(s)\geq 4M(\theta_{t+1}\omega))}|Az(\theta_s\omega)|^{2p-2}dx. \tag{6.17}$$

By integrating (6.17) with respect to s from $\tau(t+1/2\leq\tau\leq t+1)$ to $t+1$ we see that

$$\|(v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-4M((\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)}$$
$$\leq c\int_\tau^{t+1}\int_{D_4(s,t+1)}f(x,v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))+z(\theta_{s-t-1}\omega))\times$$
$$(v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-4M(\omega))_+^{2p-3}dxds$$
$$+c\int_\tau^{t+1}\int_{D_4(s,t+1)}|Az(\theta_{s-t-1}\omega)|^{2p-2}dxds$$
$$+\|(v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-4M(\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)}$$
$$\leq c\int_t^{t+1}\int_{D_4(s,t+1)}|Az(\theta_{s-t-1}\omega)|^{2p-2}dxds+$$
$$\|(v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))-4M(\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)}.$$

Integrating the above with respect to τ from $t+1/2$ to $t+1$, employing (6.13), it produces that, for all $t \geq T$

$$\|(v(t+1, \theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)) - 4M(\omega))_+\|^{2p-2}_{L^{2p-2}(D_N)} \leq c\varepsilon,$$

where T is in (6.10), and then we deduce that, for all $t \geq T+1$,

$$\int_{D_8(t,t)} |v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^{2p-2} dx \leq c\varepsilon. \tag{6.18}$$

Repeating the same arguments above, working with $(v(s) + M(\theta_{t+1}))_-^{2p-3}$ instead of $(v(s) - M(\theta_{t+1}))_+^{2p-3}$, we can deduce that, for all $t \geq T+1$,

$$\int_{D_{-8}(v(t,t)} |v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^{2p-2} dx \leq c\varepsilon, \tag{6.19}$$

where $D_{-8}(t,t) = D_N(v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega)) \leq -8M)$. Therefore, (6.18) and (6.19) imply that for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, there exist $T(\varepsilon, B, \omega) = T+1$ and $M' = 8M$ such that for all $t \geq T(\varepsilon, B, \omega)$ and $\mathbb{P}$-a.e.$\omega \in \Omega$,

$$\sup_{u_0(\omega)\in B(\omega)} \int_{D_N(|\varphi(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))| \geq M')} |\varphi(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))|^{2p-2} dx < c\varepsilon,$$

which completes the proof. $\Box$

By a similar argument as Theorem 5.7, we can show the following:

Theorem 6.3. Assume that (4.4)-(4.6) hold. Then the RDS ψ generated by (4.1)-(4.3) has a unique $(L^2(D_N), L^{2p-2}(D_N))$-random attractor $\{\mathcal{A}_{2p-2}(\omega)\}_{\omega\in\Omega}$. Furthermore, $\mathcal{A}_{2p-2}(\omega) = \mathcal{A}(\omega)$ for $\omega \in \Omega$, where $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is the $(L^2(D_N), L^2(D_N))$-random attractor.

Remark 6.4. Although the RDS ψ associated with (4.1)-(4.3) is only continuous in $L^2(D_N)$, we also obtain the existences of random attractors in $L^p(D_N)$ and $L^{2p-2}(D_N)$, respectively. Furthermore, by the interpolation inequality, we can immediately deduce that the RDS ψ also has a unique random attractor in the space $L^r(D_N)$ (where $r \in [2, 2p-2]$), and $\mathcal{A}_r(\omega) = \mathcal{A}(\omega)$ for $\omega \in \Omega$.

Acknowledgments:
This work was supported by the China NSF Grant (No.11271388), the Science and Technology Funds of Chongqing Educational Commission (No.KJ130703), the Basis and Frontier Research Project of CSTC (No.cstc2014jcyjA00035).

References

[1] Anh, C. T., Bao, T. Q. Pullback attractors for a non-autonomous semi-linear degenerate parabolic equation. Glasg. Math. J., 52: 537-554 (2010)
[2] Anh, C. T., Hung, P. Q. Global existence and long-time behavior of solutions to a class of degenerate parabolic equations. Ann. Polon. Math., 93: 217-230 (2008)
[3] Anh, C. T., Chuong, N. M., Ke, T. D. Global attractors for the m-semiflow degenerated by a quasilinear degenerate parabolic equation. J. Math. Anal. Appl., 363: 444-453 (2010)
[4] Arnold, L. Random Dynamical System. Springer-Verlag, Berlin, 1998

[5] Bates, P. W., Lu, K., Wang, B. Random attractors for stochastic reaction-diffusion equations on unbounded domains. J. Differential Equations, 246(2): 845-869 (2009)
[6] Caraballo, T., Langa, J. A. Stability and random attractors for a reaction-diffusion equation with multiplicative noise. Discrete contin. Dyn. Syst., 6: 875-892 (2000)
[7] Chueshov, I. Monotone Random Systems Theory and Applications. Springer-Verlag, 2002
[8] Crauel, H., Debussche, A., Flandoli, F. Random attractors. J. Dynam.Differential Equations, 9: 307-341 (1997)
[9] Crauel, H.,Flandoli, F. Attractors for random dynamical systems. Probab. Theory Related Fields, 100: 365-393 (1994)
[10] Da Prato, G., Zabczyk, J. Stochastic Equations in Infinite Dimensions. Cambridge Univ. Press, Cambridge, 1992
[11] Flandoli, F., Schmalfuß, B. Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise. Stoch. Stoch. Rep., 59: 21-45 (1996)
[12] Li, J., Li, Y. R., Wang, B. Random attractors of reaction-diffusion equations with multiplicative noise in L^p. Appl. Math. Compu., 215: 3399-3407 (2010)
[13] Li, Y. R., Guo, B. L. Random attractors for quasi-continuous random dynamical systems and applications to stochastic reaction-diffusion equations. J. Differential Equations, 245: 1775-1800 (2008)
[14] Li, Y. R., Guo, B. L. Random attractors of Boussinesq equations with multiplicative noise. Acta Math. Sinica, English Series, 25: 481-490 (2009)
[15] Łukaszewicz, G. On pullback attractors in L^p for nonautonomous reaction-diffusion equations. Nonlinear Anal., 73(2): 350-357 (2010)
[16] Schmalfuß, B. Backward cocycle and attractors of stochastic differential equations, in: V. Reitmann, T. Riedrich, N. Koksch (Eds.), International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior, Technische Universität, Dresden, pp: 185-192 (1992)
[17] Song, H. T., Zhong, C. K. Attractors of non-autonomous reaction-diffusion equations in L^p. Nonlinear Anal., 68(7): 1890-1897 (2008)
[18] Sun, C. Y., Zhong, C. K. Attractors for the semi-linear reaction-diffusion equation with distribution derivatives in unbounded domains. Nonlinear Anal., 63(1): 49-65 (2005)
[19] Wang, B. X. Attractors for reaction-diffusion equations in unbounded domains. Physica D, 128(1): 41-52 (1999)
[20] Wang, B. X. Random attractors for the FitzHugh-Nagumo system on unbounded domains. Nonlinear Anal., 71(7-8): 2811-2828 (2009)
[21] Wang, B. X. Asymptotic behavior of stochastic wave equaitons with critical exponents on $\mathbb{R}^3$. Trans. Amer. Math. Soc., 363: 3639-3663 (2011)
[22] Wang, B. X. Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains. J. Differential Equations, 246(6): 2506-2537 (2009)
[23] Wang, Z. J., Zhou, S. F. Random attractors for stochastic reaction-diffusion equations with multiplicative noise on unbounded domains. J. Math. Anal. Appl., 384(1): 160-172 (2011)
[24] Wang, G. L., Guo, B. L., Li, Y. R. The asymptotic behavior of the stochastic Ginzburg-Landou equation with additive noise. Appl. Math. Compu., 198(2): 849-857 (2008)
[25] Yang, M. H., Kloeden, P. E. Random attractors for stochastic semi-linear degenerate parabolic equations. Nonlinear Analysis: Real World Applications, 12(5): 2811-2821 (2011)
[26] Yang, M. H., Sun, C. Y., Zhong, C. K. Existence of a global attractor for a p-Laplacian equation in $\mathbb{R}^n$. Nonlinear Anal., 66(1): 1-13 (2007)
[27] Yang, M. H., Sun, C. Y., Zhong, C. K. Global attractor for p-Laplacian equation. J. Math. Anal. Appl., 327(2): 1130-1142 (2007)
[28] Zhao, W. Q. Asymptotic behavior of stochastic p-Laplacian-type equation with multiplicative noise. Boundary Value Problems, 2012: 61(2012)
[29] Zhao, W. Q., Li, Y. R. Asymptotic behavior of two-dimensional stochastic magneto-hydrodynamics equations with additive noises. J. Math. Phys., 52: 072701(2011)
[30] Zhao, W. Q., Li, Y. R. (L^2, L^p)-random attractors for stochastic reaction-diffusion equation on unbounded domains. Nonlinear Anal., 75(2): 485-502(2012)
[31] Zhong, C. K., Yang, M. H., Sun, C. Y. The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations. J. Differential Equations, 223(2): 367-399 (2006)

School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, China

E-mail address: zhaomq@ctbu.edu.cn

School of Mathematics and Statistics, Southwest China University, Chongqing 400715, China

E-mail address: liyr@swu.edu.cn