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Log-Lipschitz continuity of the vector field on the attractor of certain parabolic equations

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Abstract. We discuss various issues related to the finite-dimensionality of the asymptotic dynamics of solutions of parabolic equations. In particular, we study the regularity of the vector field on the global attractor associated with these equations. We show that if the linear term associated with certain dissipative partial differential equations is log-Lipschitz continuous on the global attractor $\mathcal{A}$, then $\mathcal{A}$ lies within a small neighbourhood of a smooth manifold, given as a Lipschitz graph over a finite number of Fourier modes. In this case, $\mathcal{A}$ can be shown to have zero Lipschitz deviation and, therefore, there are linear maps L into finite-dimensional spaces, whose inverses restricted to $L\mathcal{A}$ are Hölder continuous with an exponent arbitrarily close to one. Finally, we use an argument due to Kukavica (2007; *Proc. Amer. Math. Soc.* **135** 2415-2421) to prove that the linear term associated with a class of parabolic equations, that includes the 2D Navier-Stokes equations, is 1-log-Lipschitz continuous.

Contents

1. Introduction

The existence of global attractors with finite upper box-counting dimension for a wide class of dissipative equations (see Babin and Vishik [**2**], Foias and Temam

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[**15**], Hale [**17**], Temam [**44**], for example) strongly suggests that it might be possible to construct a system of ordinary differential equations whose asymptotic dynamics reproduces the dynamics on the original attractor. However, because of the complexity of the flow on the attractor $\mathcal{A}$ and its irregular structure, the finite dimensionality of $\mathcal{A}$ alone is not immediately sufficient to guarantee the existence of such a system of ordinary differential equations.

Indeed, the existence of an ordinary differential equation with analogous asymptotic dynamics has only been proved for dissipative partial differential equations that possess an inertial manifold, i.e. a finite-dimensional, positively invariant Lipschitz manifold that attracts all orbits exponentially (Foias, Sell and Temam [**13, 14**]; Constantin and Foias [**6**]; Constantin *et al.* [**7**]; Foias, Manley and Temam [**12**]; and Temam [**44**], for more details). All the methods available in the literature construct inertial manifolds as graphs of functions from a finite-dimensional eigenspace associated with the low Fourier modes into the complementary infinite-dimensional eigenspace corresponding to the high Fourier modes, under a certain 'spectral gap condition'. Unfortunately, this sufficient condition is quite restrictive, and there are many equations, such as the 2D Navier–Stokes equations, that do not satisfy it. (Adopting a different approach, Kukavica [**21, 22**] has been able to show that the global attractors of certain dissipative equations of the form $u_t - u_{xx} + f(x, u, u_x)$ in one space dimension, which do not satisfy the spectral gap condition, still lie in a Lipschitz graph over a finite number of Fourier modes.)

In the cases in which an inertial manifold has not been shown to exist, other approaches have been explored to reconstruct the dynamics on the attractor within a finite-dimensional system. Eden et al. [**9**] were the first to consider explicitly the problem of projecting a dissipative partial differential equation into a Euclidean space of sufficiently high dimension, and obtaining a finite system of ordinary differential equations which reproduce the dynamics on the attractor $\mathcal{A}$.

Romanov [**39**] (see also [**40, 41**]) discussed the problem of a finite-dimensional description of the asymptotic behaviour of dissipative equations more abstractly. He defined the dynamics on the attractor $\mathcal{A}$ to be[1] 'Lipschitz finite-dimensional' if there exists a bi-Lipschitz map $\Pi : \mathcal{A} \to \mathbb{R}^N$, for some N, and an ordinary differential equation with a Lipschitz vector field on $\mathbb{R}^N$ such that the dynamics on $\mathcal{A}$ and $\Pi(\mathcal{A})$ are conjugated under Π. He then showed that this property is equivalent to the attractor being contained in a finite-dimensional Lipschitz manifold, given as a graph over a sufficiently large number of Fourier modes. Hence, his definition and that of an inertial manifold are much more similar than they first appear. In Section 3 we give a concise proof of this result, and give a possible alternative definition of what it might mean for the asymptotic dynamics to be 'finite-dimensional'.

To illustrate the problem of constructing a finite set of ordinary differential equations that reproduces the dynamics on the global attractor, consider a governing equation $\dot{u} = \mathcal{G}(u)$ defined on a Hilbert space H. Suppose there exists a linear map $L : H \to \mathbb{R}^N$ that is injective on $\mathcal{A}$. In order to study the smoothness of the embedded equation on $\mathcal{X} = L\mathcal{A}$,

$$\dot{x} = h(x) = L\mathcal{G}L^{-1}(x), \quad x \in \mathcal{X}, \tag{1}$$

[1] We have added the word 'Lipschitz' here to emphasise that Lipschitz continuity 'wherever possible' forms a key part of Romanov's definition; contrast this with our weaker definition, Definition 3.3.

one needs to consider the continuity of the vector field on $\mathcal{A}$ and the continuity of the inverse of the embedding L restricted to $\mathcal{X}$.

If one would like a system of ordinary differential equations with unique solutions that generates a flow $\{S_t\}$, then the embedded vector field h in $\mathcal{X}$ does not need to be Lipschitz; it is sufficient for h to be α-log-Lipschitz with any exponent $0 \leq \alpha \leq 1$, i.e. there exists a $C > 0$ and a $\delta > 0$ such that

$$|h(x) - h(y)| \leq C|x-y| \log(-|x-y|)^{\alpha} \quad \text{for all} \quad |x-y| < \delta$$

(Osgood's criterion then guarantees the uniqueness of solutions, since the integral $\int_0^{\delta} \frac{1}{r(-\log r)^{\alpha}}$ diverges, see [**18**]). Given such a function defined on $\mathcal{X}$, one can readily extend $h : \mathcal{X} \to \mathbb{R}^N$ to 1-log-Lipschitz function $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ using a standard extension result (see McShane [**26**], for example).

Hence, one needs to show that there exist

(i): an exponent $\gamma > 0$ such that the vector field on the attractor $\mathcal{A}$ is γ-log-Lipschitz in H, and

(ii): an exponent $\eta > 0$ such the inverse of linear embedding $L : H \to \mathbb{R}^N$ is η-log-Lipschitz when restricted to $\mathcal{X}$,

with γ and η such that $\gamma + \eta \leq 1$.

The regularity of the embedding L (i.e. the continuity properties of L^{-1}) has been discussed in a variety of papers (see Mañé [**25**], Ben-Artzi *et al.* [**4**], Eden et al. [**9, 10**], Foias and Olson [**11**], Hunt and Kaloshin [**20**], Olson and Robinson [**30**], Robinson [**36**] for more details), with the strongest current result [**37, 38**] providing an η-log-Lipschitz embedding for any $\eta > 1/2$ when the Assouad dimension[2] of the set of differences $\mathcal{A} - \mathcal{A}$ is finite.

Under this condition, we would therefore require the vector field $\mathcal{G}$, restricted to $\mathcal{A}$, to be γ-log-Lipschitz for some exponent $\gamma < 1/2$. (Under these two assumptions, on the Assouad dimension of $\mathcal{A} - \mathcal{A}$ and the continuity of the vector field, Pinto de Moura et al. [**33**] show that there is an ordinary differential equation in some $\mathbb{R}^k$ that has unique solutions and reproduces the dynamics on $\mathcal{A}$, i.e. that the dynamics are 'finite-dimensional' in the sense of our Definition 3.3). We therefore focus our attention in this paper on what we can say about the regularity of $\mathcal{G}$ (on $\mathcal{A}$) in a general semilinear parabolic problem, and how this relates to other properties of the attractor.

In Section 4, we prove that if the the linear term is γ-log-Lipschitz continuous, then there exists a family of Lipschitz manifolds $\mathcal{M}_N$ such that the distance between the N-dimensional manifold $\mathcal{M}_N$ and the attractor $\mathcal{A}$ is exponentially small in N. It is interesting to note that this result does not rely explicitly on a dynamical argument, but is a consequence of the regularity of the functions that lie on the attractor. We then recall (after Pinto de Moura and Robinson [**32**]) that this result implies that one can obtain linear embeddings of the $\mathcal{A}$ into some $\mathbb{R}^N$ that have Hölder continuous inverse and whose exponent can be made arbitrarily close to one by choosing an embedding space of sufficiently high dimension.

In Section 5, we show that for certain dissipative partial differential equations, including the 2D Navier–Stokes equations, the linear term (and hence the whole vector field) is 1-log-Lipschitz continuous on the attractor, using methods developed by Kukavica [**23**]. Of course, improvement of this result is required if one is to

[2]For a comprehensive treatment of the Assouad dimension see Luukkainen [**24**], Olson [**29**], or Robinson [**37**].

follow the 'finite-dimensional programme' outlined above, unless one can guarantee the existence of a bi-Lipschitz embedding of $\mathcal{A}$ into some $\mathbb{R}^N$.

2. Notation and general setting

Consider a semilinear parabolic equation written as an abstract evolution equation of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = F(u) \tag{2}$$

in a separable real Hilbert space H with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We suppose that A is an unbounded positive self-adjoint linear operator with compact inverse and dense domain $D_H(A) \subset H$. For each $\alpha \geq 0$, we denote by $D_H(A^\alpha)$ the domain of A^α in H, i.e.

$$D_H(A^\alpha) = \{u : A^\alpha u \in H\};$$

these are Hilbert spaces with inner product $(u,v)_\alpha = (A^\alpha u, A^\alpha v)$ and norm $\|u\|_\alpha = \|A^\alpha u\|$. We know that for $\alpha > \beta$, the embedding $D_H(A^\alpha) \subset D_H(A^\beta)$ is dense and continuous such that

$$\|u\|_\beta \leq c(\alpha,\beta)\|u\|_\alpha, \quad \text{for } u \in D_H(A^\alpha) \tag{3}$$

(see Henry [**19**] or Sell and You [**43**], for details). Moreover, we assume that for some $\alpha \in [0,1)$ the nonlinear term F is locally Lipschitz from $D_H(A^\alpha)$ into H. Hence, for $u, v \in D_H(A^\alpha)$,

$$\big\|F(u) - F(v)\big\| \quad \leq \quad K(R)\big\|u - v\big\|_\alpha, \quad \text{with} \quad \|u\|_\alpha, \|v\|_\alpha \leq R, \tag{4}$$

where K is a constant depending only on R. This abstract setting includes, among others, the 2D Navier–Stokes equations and the original Burgers equation with Dirichlet boundary values (see Eden *et al.* [**9**] or Temam [**44**] for example).

Since A is a self-adjoint densely defined operator and its inverse is compact, H has an orthonormal basis $\{w_j\}_{j\in\mathbb{N}}$ consisting of eigenfunctions of A such that

$$Aw_j = \lambda_j w_j \quad \text{for all} \quad j \in \mathbb{N}$$

with $0 < \lambda_1 \leq \lambda_2, \ldots$ and $\lambda_j \to \infty$ as $j \to \infty$. For $n \in \mathbb{N}$ fixed, define the finite-dimensional orthogonal projections P_n and their orthogonal complements Q_n by

$$P_n u = \sum_{j=1}^{n} (u, w_j) w_j \quad \text{and} \quad Q_n u = \sum_{j=n+1}^{\infty} (u, w_j) w_j.$$

Hence, we can write $u = P_n u + Q_n u$, for all $u \in H$. The orthogonal projections P_n and Q_n are bounded on the Hilbert spaces $D_H(A^\alpha)$, for any $\alpha > 0$ (see (3)). Notice that $P_n H = P_n D_H(A^\alpha) \subset D_H(A^\alpha)$, since $P_n H$ is a finite-dimensional subspace generated by the eigenvectors of A corresponding to the first n eigenvalues of A. These spectral projections commute with the operators e^{-At} for $t > 0$, i.e., $P_n \mathrm{e}^{-At} = \mathrm{e}^{-At} P_n$ and $Q_n \mathrm{e}^{-At} = \mathrm{e}^{-At} Q_n$. Moreover, we have the following estimate

$$\|\mathrm{e}^{-At} Q_n u\|_\alpha \quad \leq \quad \sup_{j \geq n+1} \big\{\lambda_j^\alpha \mathrm{e}^{-\lambda_j t}\big\} \|Q_n u\| \leq b_{n,\alpha}(t) \|Q_n u\| \leq b_{n,\alpha}(t)\|u\|,$$

where

$$b_{n,\alpha}(t) = \begin{cases} \left(\dfrac{\mathrm{e}t}{\alpha}\right)^{-\alpha} & \text{for} \quad 0 < t \leq \alpha/\lambda_{n+1} \\[2ex] \lambda_{n+1}^\alpha \mathrm{e}^{-\lambda_{n+1} t} & \text{for} \quad t \geq \alpha/\lambda_{n+1}. \end{cases}$$

Therefore,

$$\left\| A^{\alpha} \mathrm{e}^{-At} Q_n \right\|_{\mathscr{L}(H,H)} \leq b_{n,\alpha}(t). \tag{5}$$

Within this general setting, one can prove the local existence and uniqueness of solutions of (2) (see Henry [**19**] for details). In particular, it follows from Henry [**19**, Lemma 3.3.2] that the solution of the nonlinear equation (2), with initial condition $u(t_0) = u_0 \in D_H(A^{\alpha})$, is given by the variation of constants formula

$$u(t) = \mathrm{e}^{-A(t-t_0)} u_0 + \int_{t_0}^{t} \mathrm{e}^{-A(t-s)} F\big(u(s)\big)\ \mathrm{d}s, \tag{6}$$

for $t > t_0$.

Thus, we can define $\{\Phi_t\}_{t\geq 0}$ to be the semigroup in $D_H(A^{\alpha})$ generated by (2) such that, for any initial condition $u_0 \in D_H(A^{\alpha})$, there exists a unique solution given by $u(t; u_0) = \Phi_t u_0$. We assume that this system is dissipative, i.e. that there exists a compact invariant absorbing set in $D_H(A^{\alpha})$. It follows from standard results that (2) possesses a global attractor $\mathcal{A}$, the maximal compact invariant set in $D_H(A^{\alpha})$ that uniformly attracts the orbits of all bounded sets (see Babin and Vishik [**2**], Hale [**17**], Robinson [**34**], Temam [**44**]).

3. Finite-dimensionality of flows

As discussed in the Section 1, inertial manifolds are a convenient, although indirect, method to obtain a system of ordinary differential equations that reproduces the asymptotic dynamics on the global attractor. Romanov considered in [**39**] a more general definition of what it means for a system to be asymptotically finite-dimensional. We will see that this definition implies the existence of a Lipschitz manifold that contains the attractor, but does not require it to be exponentially attracting. Romanov defined the dynamics on a global attractor $\mathcal{A}$ to be *Lipschitz finite-dimensional* if for some $N \geq 1$ there exist:

- **(i):** an ordinary differential equation $\dot{x} = \mathcal{H}(x)$ with a Lipschitz vector field $\mathcal{H}(x)$ in $\mathbb{R}^N$,
- **(ii):** a corresponding flow $\{S_t\}$ on $\mathbb{R}^N$ and
- **(iii):** a bi-Lipschitz embedding $\Pi : \mathcal{A} \to \mathbb{R}^N$, such that $\Pi(\Phi_t u) = S_t \Pi(u)$ for any $u \in \mathcal{A}$ and $t \geq 0$.

(Note that nothing is required of the attractor of $\{S_t\}$, which may be much larger than $\Pi\mathcal{A}$.)

It follows from this definition that the evolution operators Φ_t are injective on $\mathcal{A}$ for $t > 0$. If we set $\Phi_{-t} = \Pi^{-1} S_{-t} \Pi$, then we see that in fact Φ_t is Lipschitz on $\mathcal{A}$ even for $t < 0$. Hence, we obtain a Lipschitz flow $\{\Phi_t\}$ defined on $\mathcal{A}$ for all $t \in \mathbb{R}$. In particular, there exist $C \geq 1$ and $\mu > 0$ such that

$$\left\| \Phi_t u - \Phi_t v \right\|_{\alpha} \leq C \left\| u - v \right\|_{\alpha} \mathrm{e}^{\mu |t|}, \tag{7}$$

for every $t \in \mathbb{R}$.

Considering the general Banach space case, Romanov [**39**] proved that the finite-dimensionality of the dynamics on the attractor $\mathcal{A}$ is equivalent to five different criteria. In this paper we single out one of these, and recall here Romanov's proof that if the attractor $\mathcal{A}$ has 'finite-dimensional dynamics' in Romanov's sense, then it must lie on a finite-dimensional manifold, defined as the graph of a Lipschitz function over $P_n H$, for some $n < \infty$. It seems worth reproducing the elegant proof here, since our setting is much simpler and so the argument becomes much more transparent. These results are also discussed in the review paper by Zelik [**46**].

THEOREM 3.1. *(Romanov [**39**]) If the dynamics on $\mathcal{A}$ is finite-dimensional, then given any γ with $\alpha \leq \gamma < 1$ there exists an n_0 such that for any $n \geq n_0$*

$$\big\|Q_n(u-v)\big\|_\gamma \leq c\big\|P_n(u-v)\big\|_\alpha \quad \text{for all} \quad u, v \in \mathcal{A}, \tag{8}$$

where $c = c(\mathcal{A}, n, \gamma, \alpha)$.

PROOF. First consider the variation of constants formula (6) with $t = 0$ and $u(0) = u \in \mathcal{A}$. If we apply the projection operator Q_n to both sides of (6), then

$$Q_n u = Q_n \mathrm{e}^{At_0} u(t_0) + \int_{t_0}^0 \mathrm{e}^{As} Q_n F\big(u(s)\big)\ \mathrm{d}s.$$

Now, since the compact set $\mathcal{A}$ is bounded in $D_H(A^\alpha)$ and $u(t) \in \mathcal{A}$, it follows from (5) that $\lim_{t_0 \to -\infty} \|Q_n \mathrm{e}^{At_0} u(t_0)\|_\alpha = 0$. Consequently, letting t_0 tend to $-\infty$ we obtain

$$Q_n u = \int_{-\infty}^0 \mathrm{e}^{As} Q_n F(\Phi_s u)\ \mathrm{d}s,$$

which converges in $D_H(A^\alpha)$. It follows from (7) that, for $u, v \in \mathcal{A}$,

$$\begin{aligned}
\big\|Q_n u - Q_n v\big\|_\gamma &\leq \int_{-\infty}^0 \Big\|\mathrm{e}^{As} Q_n\big(F(\Phi_s u) - F(\Phi_s v)\big)\Big\|_\gamma\ \mathrm{d}s \\
&\leq K \int_{-\infty}^0 \Big\|A^\gamma \mathrm{e}^{As} Q_n\Big\|_{op} \big\|\Phi_s u - \Phi_s v\big\|_\alpha \mathrm{d}s \\
&\leq KC\big\|u-v\big\|_\alpha \int_{-\infty}^0 \Big\|A^\gamma \mathrm{e}^{As} Q_n\Big\|_{op} \mathrm{e}^{\mu|s|}\mathrm{d}s.
\end{aligned}$$

Using estimate (5) with $t = -s$, we find that

$$\big\|Q_n u - Q_n v\big\|_\gamma \leq KC\big\|u-v\big\|_\alpha \int_{-\infty}^0 b_{n,\gamma}(-s)\mathrm{e}^{-\mu s}\mathrm{d}s$$

from which we obtain the inequality

$$\big\|Q_n u - Q_n v\big\|_\gamma \leq \vartheta_n \big\|u-v\big\|_\alpha, \tag{9}$$

where

$$\vartheta_n := \frac{1}{KC}\left\{\left(\frac{\mathrm{e}}{\gamma}\right)^{-\gamma}\left(\frac{\gamma}{\lambda_{n+1}}\right)^{1-\gamma}\frac{1}{1-\gamma} + \frac{\lambda_{n+1}^\gamma}{\lambda_{n+1}-\mu}\ \mathrm{e}^{\frac{-\gamma(\lambda_{n+1}-\mu)s}{\lambda_{n+1}}}\right\},$$

can be obtained by simple algebraic manipulation.

Note that, since $\gamma < 1$ and λ_{n+1} tends to infinity as $n \to \infty$, one can choose n sufficiently large to ensure that $\vartheta_n \lambda_{n+1}^{\alpha-\gamma} < 1$. Since $P_n + Q_n = I$, it follows that

$$\begin{aligned}
\|Q_n(u_0 - v_0)\|_\gamma &\leq \vartheta_n \|P_n(u_0 - v_0)\|_\alpha + \vartheta_n \|Q_n(u_0 - v_0)\|_\alpha \\
&\leq \vartheta_n \|P_n(u_0 - v_0)\|_\alpha + \vartheta_n \lambda_{n+1}^{\alpha-\gamma} \|Q_n(u_0 - v_0)\|_\gamma,
\end{aligned}$$

whence

$$\|Q_n(u_0 - v_0)\|_\gamma \leq \frac{\vartheta_n}{1 - \vartheta_n \lambda_{n+1}^{\alpha-\gamma}} \|P_n(u_0 - v_0)\|_\alpha.$$

□

Under the assumption that the non-linear term F is in $C^2(D_H(A^\alpha), H)$, Romanov [**39**] showed that the finite-dimensionality of $\mathcal{A}$ implies that the vector field $\mathcal{G}(u) = -Au + F(u)$ is Lipschitz[3]. However, it is not clear how to adapt his argument to prove that A is Lipschitz. Here we give a simple argument that shows that finite-dimensionality implies that the operator A^β is Lipschitz in $\mathcal{A}$, provided that $\alpha + \beta < 1$.

COROLLARY 3.2. *If the dynamics on $\mathcal{A}$ is finite-dimensional, then, for β with $\alpha + \beta < 1$, A^β is Lipschitz on $\mathcal{A}$, i.e.*

$$\left\|A^\beta(u-v)\right\|_\alpha \le M\left\|u-v\right\|_\alpha, \quad \text{for all} \quad u, v \in \mathcal{A},$$

where α is given by (4).

PROOF. It follows from Theorem 3.1 that, for all $u, v \in \mathcal{A}$,

$$\begin{aligned}\left\|A^\beta(u-v)\right\|_\alpha &= \left\|u-v\right\|_{\alpha+\beta} \le \left\|P_n(u-v)\right\|_{\alpha+\beta} + \left\|Q_n(u-v)\right\|_{\alpha+\beta} \\ &\le \left(\lambda_n^\beta + \frac{\vartheta_n}{1-\vartheta_n\lambda_{n+1}^{-\beta}}\right)\left\|P_n(u-v)\right\|_\alpha \le M\left\|u-v\right\|_\alpha.\end{aligned}$$

□

Note, however, that the requirement in Romanov's definition that $\mathcal{A}$ admits a bi-Lipschitz embedding into some $\mathbb{R}^N$ is very strong and unlikely to be satisfied in general. A sensible way to weaken this definition would be to relax the bi-Lipschitz assumption and assume the embedded vector field $\mathcal{H}$ to be just log-Lipschitz, but the argument would not work in this case.

Another possible option would be to remove the assumption that the flow is generated by an ODE. The following is a reasonable minimal definition of what it might mean for the dynamics on the attractor to be finite-dimensional.

DEFINITION 3.3. *The dynamics on a global attractor $\mathcal{A}$ is* finite-dimensional *if, for some $N \ge 1$, there exist an embedding $\Pi : \mathcal{A} \to \mathbb{R}^N$ that is injective on $\mathcal{A}$, a flow $\{S_t\}$ in $\mathbb{R}^N$ with an invariant set $\mathcal{X}$, such that the dynamics on $\mathcal{A}$ and $\mathcal{X}$ are conjugate under Π via $\Pi(\Phi_t u) = S_t\Pi(u)$, for any $u \in \mathcal{A}$ and $t \ge 0$.*

(One could strengthen this definition somewhat by requiring $\mathcal{X}$ to be the attractor of $\{S_t\}$, but as remarked above this is not contained in Romanov's original definition.) However, even in this weak sense, it is still an open problem whether the finite-dimensionality of the global attractor $\mathcal{A}$ implies that the dynamics on $\mathcal{A}$ is finite-dimensional.

4. Smoothness of the linear term

In the last section, we showed that if the dynamics on the attractor is finite-dimensional, then A^β is Lipschitz on $\mathcal{A}$ provided that $\alpha + \beta < 1$, where α is given by (4). It is relatively easy to show that the converse is also true.

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[3] If $F \in C^2\big(D_H(A^\alpha), H\big)$, then it follows from Henry [**19**, Corollary 3.4.6] that the map $(u_0, t) \mapsto u(t)$ is also in $C^2\big(\mathbb{R}^+ \times D_H(A^\alpha), D_H(A^\alpha)\big)$. Hence, the function $(u_0, t) \mapsto \mathrm{d}u(t)/\mathrm{d}t$ is C^1 with respect to (u_0, t). Since $\mathrm{d}u(t)/\mathrm{d}t = \mathcal{G}\big(u(t)\big)$, for a fixed time (we choose $t = 1$), the map $u_0 \mapsto \mathcal{G}\big(u(1)\big)$ is also a C^1-function and, consequently, a Lipschitz function. The finite dimensionality of the dynamics on $\mathcal{A}$ implies that the map $u_0 \mapsto u(1)$ is bi-Lipschitz on $\mathcal{A}$. And, therefore, the map $u(1) \mapsto \mathcal{G}\big(u(1)\big)$ is Lipschitz continuous.

Proposition 4.1 (Robinson [**35**]). *Suppose that A^β is Lipschitz continuous on the attractor from $D_H(A^\alpha)$ into itself, i.e.*

$$\|A^\beta u - A^\beta v\|_\alpha \leq M\|u-v\|_\alpha \quad \text{for all} \quad u,v\in\mathcal{A}$$

for some $M>0$. Then, the attractor is a subset of a Lipschitz manifold given as a graph over $P_N H$ for some N.

In this section, we will show that if the linear term A on $\mathcal{A}$ is log-Lipschitz continuous, then there exists of a family of Lipschitz manifolds $\mathcal{M}_N$ such that

$$\operatorname{dist}(\mathcal{M}_N,\mathcal{A}) \leq c\mathrm{e}^{-k\lambda_{N+1}},$$

where $\mathcal{M}_N$ is an N-dimensional manifold and c and k are positive constants. This result has a close relationship with the concept of an approximate inertial manifold. Introduced by [**12**], these are finite-dimensional Lipschitz manifolds whose neighbourhood contains the global attractor $\mathcal{A}$ (cf. [**8**] and [**42**] for the construction of explicit families that are also of 'exponential order', as here). The following result is obtained without appealing directly to any dynamical property.

Proposition 4.2. *Suppose that, for some constants $\gamma>0$ and $C>0$,*

$$\big\|A(u-v)\big\| \leq C\|u-v\|\log\big(M_1^2/\|u-v\|^2\big)^\gamma, \quad \text{for all} \quad u,v\in\mathcal{A}, \tag{10}$$

where $M_1 \geq 4\sup_{u\in\mathcal{A}}\|u\|$. Then, for each $n>0$, there exists a Lipschitz function $\Phi_n : P_nH \to Q_nH$,

$$\big\|\Phi_n(p_1)-\Phi_n(p_2)\big\| \leq \|p_1-p_2\| \quad \text{for all} \quad p_1,p_2\in P_nH,$$

such that $\mathcal{A}$ lies within a $2M_1^2\mathrm{e}^{-\{\lambda_{n+1}^2/2C^2\}^{1/2\gamma}}$-neighbourhood of the graph Φ_n,

$$\mathbf{G}[\Phi_n] = \{u\in H : u = p+\Phi_n(p), p\in P_nH\}.$$

The proof of this result uses an argument similar to the one developed in [**12**], and can also be used to prove Proposition 4.1.

Proof. Let $w=u-v$, for $u,v\in\mathcal{A}$. We can split $w=P_nw+Q_nw$, and observe that

$$\begin{aligned}\|Aw\|^2 &= \big\|A(P_nw+Q_nw)\big\|^2 = \big\|A(P_nw)\big\|^2 + \big\|A(Q_nw)\big\|^2\\ &\geq \lambda_{n+1}^2\|Q_nw\|^2.\end{aligned}$$

It follows from (13) that

$$\begin{aligned}\|Aw\|^2 &\leq C^2\|w\|^2\Big[\log\big(M_1^2/\|w\|^2\big)\Big]^{2\gamma}\\ &\leq C^2\big(\|P_nw\|^2+\|Q_nw\|^2\big)\Big[\log\big(M_1^2/\|Q_nw\|^2\big)\Big]^{2\gamma}.\end{aligned}$$

Since $\log\big(M_1^2/\|Q_nw\|^2\big)>1$,

$$\frac{\lambda_{n+1}^2\|Q_nw\|^2}{\Big[\log\big(M_1^2/\|Q_nw\|^2\big)\Big]^{2\gamma}} \leq C^2\|P_nw\|^2 + C^2\|Q_nw\|^2.$$

Consider a subset Y of $\mathcal{A}$ that is maximal for the relation

$$\big\|Q_n(u-v)\big\| \leq \big\|P_n(u-v)\big\| \quad \text{for all} \quad u,v\in Y. \tag{11}$$

Note that if the P_n components of u and v agree, so that $P_n u = P_n v$, then $Q_n u = Q_n v$. Hence, for every $u \in Y$, we can define uniquely $\phi_n(P_n u) = Q_n u$ such that $u = P_n u + \phi_n(P_n u)$. Moreover, it follows from (11) that

$$\big\|\phi_n(p_1) - \phi_n(p_2)\big\| \le \|p_1 - p_2\| \quad \text{for all} \quad p_1, p_2 \in P_n Y.$$

Standard results (see Wells and Williams [**45**], for example) allow one to extend ϕ_n to a function $\Phi_n : P_n H \to Q_n H$, that satisfies the same Lipschitz bound.

Now, if $u \in \mathcal{A}$ but $u \notin Y$, it follows that

$$\big\|Q_n(u - v)\big\| \ge \big\|P_n(u - v)\big\|,$$

for some $v \in Y$. Thus, if $w = u - v$, then

$$\frac{\lambda_{n+1}^2 \|Q_n w\|^2}{\Big[\log\big(M_1^2/\|Q_n w\|^2\big)\Big]^{2\gamma}} \le 2C^2 \|Q_n w\|^2.$$

Hence,

$$\|Q_n w\|^2 \le M_1^2 \mathrm{e}^{-\{\lambda_{n+1}^2/2C^2\}^{1/2\gamma}},$$

which implies that

$$\begin{aligned} \|w\|^2 = \|P_n w\|^2 + \|Q_n w\|^2 &\le 2\|Q_n w\|^2 \\ &\le 2M_1^2 \mathrm{e}^{-\{\lambda_{n+1}^2/2C^2\}^{1/2\gamma}}. \end{aligned}$$

Therefore,

$$\operatorname{dist}\big(u, \mathbf{G}[\Phi_n]\big) \le 2M_1^2 \mathrm{e}^{-\{\lambda_{n+1}^2/2C^2\}^{1/2\gamma}}. \tag{12}$$

□

Given $\alpha > 0$, one can replace the linear term A by A^α and obtain the same result with $\lambda_{n+1}^{2\alpha}$ instead of just λ_{n+1}^2. Therefore, if there exist constants $C > 0$ and $M_0 \ge 4 \sup_{u \in \mathcal{A}} \|u\|$ such that

$$\|A^{1/2} w\| \le C\|w\| \log\big(M_0^2/\|w\|^2\big)^{1/2}$$

(as obtained in [**23**]), then there exists a family of Lipschitz manifolds $\mathcal{M}_n$ such that

$$\operatorname{dist}(\mathcal{M}_n, \mathcal{A}) \le 2M_0^2 \mathrm{e}^{-\lambda_{n+1}/2C^2}.$$

The existence of a family of approximating Lipschitz manifolds for a dissipative equation of the form of (2) implies that the global attractor $\mathcal{A}$ has zero Lipschitz deviation, a concept which we will define below. Introduced by Olson and Robinson [**30**] and refined in [**32**], the Lipschitz deviation is a variant of the 'thickness exponent' of [**20**], and measures how well a compact set X in a Hilbert space H can be approximated by graphs of Lipschitz functions (with prescribed Lipschitz constant) defined over a finite-dimensional subspace of H.

Definition 4.3. *(Olson and Robinson* [**30**]*) Let X be a compact subset of a real Hilbert space H. Let $\delta_m(X, \epsilon)$ be the smallest dimension of a linear subspace $U \subset H$ such that*

$$\operatorname{dist}(X, \mathbf{G}_U[\Phi]) < \epsilon,$$

for some m-Lipschitz function $\Phi : U \to U^\perp$, i.e.

$$\big\|\Phi(u) - \Phi(v)\big\| \le m\|u - v\| \quad \text{for all} \quad u, v \in U,$$

where $U^\perp$ is orthogonal complement of U in H and $\mathbf{G}_U[\Phi]$ is the graph of Φ over U:

$$\mathbf{G}_U[\Phi] = \{ u + \Phi(u) : \ u \in U \}.$$

*The m-*Lipschitz deviation *of X, $\mathrm{dev}_m(X)$, is given by*

$$\mathrm{dev}_m(X) = \limsup_{\epsilon\to 0} \frac{\log \delta_m(X,\epsilon)}{-\log \epsilon}.$$

Since this quantity is bounded and non-increasing in m, the limit as m tends to infinity exists and is equal to the infimum. It is therefore natural to define (Pinto de Moura and Robinson [**32**]) the *Lipschitz deviation* of X, $\mathrm{dev}(X)$, as

$$\mathrm{dev}(X) = \lim_{m\to\infty} \mathrm{dev}_m(X).$$

Just as in [**32**], we show that the existence of a family of approximating Lispchitz manifolds, such as that provided by Proposition 4.2, implies that the associated global attractor have zero Lipschitz deviation.

Corollary 4.4. *Suppose that, for some constants $\gamma > 0$ and $C > 0$,*

$$\big\|A(u-v)\big\| \le C\|u-v\| \log\big(M_1^2/\|u-v\|^2\big)^\gamma, \quad \text{for all} \quad u,v\in\mathcal{A}, \tag{13}$$

where $M_1 \ge 4\sup_{u\in\mathcal{A}}\|u\|$, and that

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0. \tag{14}$$

Then $\mathrm{dev}(\mathcal{A}) = 0$.

Proof. Let $\epsilon_n = 2M_1^2 \mathrm{e}^{-\lambda_{n+1}/\sqrt{2}C}$. It follows from Proposition 4.2 that the global attractor $\mathcal{A}$ is contained in an ϵ_n-neighbourhood of a finite-dimensional Lipschitz manifold $\mathcal{M}_n$, defined as a graph of $\Phi : P_nH \to Q_nH$, with

$$\big\|\Phi(p_1) - \Phi(p_2)\big\| \le \|p_1 - p_2\| \quad \text{for all} \quad p_1, p_2 \in P_nH.$$

Hence, $\delta_1(\mathcal{A},\epsilon_n) = n$ and

$$\limsup_{n\to\infty} \frac{\log \delta_1(\mathcal{A},\epsilon_n)}{-\log \epsilon_n} = \limsup_{n\to\infty} \frac{\log n}{\sigma\lambda_{n+1} - \log c_0} = 0.$$

Therefore, the global attractor $\mathcal{A}$ for a dynamical system generated by a partial differential equation of the form (2), such that A satisfies (13), has $\mathrm{dev}_1(\mathcal{A}) = 0$. □

Having Lipschitz deviation zero is interesting, since one can then apply the following abstract embedding result due to Olson and Robinson [**30**] (see also [**32**]); this is essentially the embedding result of Hunt & Kaloshin [**20**], with their 'thickness exponent' replaced by the Lipschitz deviation.

Theorem 4.5 ([**30**]). *Let $\mathcal{A}$ be a compact subset of a real Hilbert space H with box-counting dimension d and Lipschitz deviation τ. Let $N > 2d$ be an integer and let θ be a real number with*

$$0 < \theta < \frac{N-2d}{N(1+\tau/2)}. \tag{15}$$

Then for a prevalent set of linear maps $L : H \to \mathbb{R}^N$ there exists a $C > 0$ such that

$$C|L(x) - L(y)|^\theta \ge \|x-y\| \quad \text{for all} \quad x,y \in \mathcal{A}.$$

In particular, these maps are injective on $\mathcal{A}$.

Note that the Lipschitz deviation is used to bound explicitly the Hölder exponent of the inverse of a linear map L restricted to the image of $\mathcal{A}$, and that if $\tau = 0$ it follows that θ can be chosen to satisfy

$$0 < \theta < 1 - \frac{2d}{N}.$$

Coupled with the result of Corollary 4.4 this yields a weakened version of Proposition 4.1 (if A is Lipschitz continuous on $\mathcal{A}$ then there exists a bi-Lipschitz embedding of the attractor into a Euclidean space of sufficiently high dimension): if A is log-Lipschitz continuous on $\mathcal{A}$ then there are Hölder embeddings of $\mathcal{A}$ into $\mathbb{R}^N$ with exponent arbitrarily close to one (for N large enough).

We will show in the following section that A is 1-log-Lipschitz on $\mathcal{A}$ for a wide class of parabolic equations, from which it follows that they have zero Lipschitz deviation and hence can be 'nicely embedded' in the above sense.

Note that while we have already shown in [**32**] that many attractors have zero Lipschitz deviation, using the dynamical 'squeezing property' [**9**] and ideas from the theory of approximate inertial manifolds (see [**12**]), the analysis here shows that this can also be seen as a consequence of regularity properties of the attractor in a way that is independent of the dynamics.

5. Log-Lipschitz continuity of the vector field

In general, the regularity of the vector field $\mathcal{G}$ is determined by the regularity of the linear term A, which can be related to the smoothness of functions on the attractor $\mathcal{A}$. For example, it follows from the standard interpolation inequality

$$\|Au - Av\| \leq \|u - v\|^{1-(1/r)} \|A^r(u - v)\|^{1/r}, \quad \text{for } u, v \in \mathcal{A}, \tag{16}$$

that, if $\mathcal{A}$ is bounded in $D_H(A^r)$, then A is Hölder continuous on $\mathcal{A}$. In this way, the continuity of F on $\mathcal{A}$ can be deduced from the regularity of solutions on the attractor.

As an example of how one can develop this approach, suppose that $\mathcal{A}$ is bounded in the Gevrey class $D_H(\mathrm{e}^{\tau A^{1/2}})$, i.e. that

$$\|\mathrm{e}^{\tau A^{1/2}} u\| \leq M \quad \text{for all} \quad u \in \mathcal{A}, \tag{17}$$

where

$$\mathrm{e}^{\tau A^{1/2}} u = \sum_{k=0}^{\infty} \frac{A^{k/2} u}{k!}$$

(see Foias & Temam [**16**]). If (17) holds then, using the expansion of A^k in terms of a basis of eigenfunctions, elementary manipulations lead to the identity

$$\|\mathrm{e}^{\tau A^{1/2} u}\|^2 = \sum_{k=0}^{\infty} \frac{(2\tau)^k}{k!} \|A^{k/4} u\|^2,$$

whence it follows that

$$\|A^{k/4} u\|^2 \leq \frac{M^2 k!}{(2\tau)^k}.$$

Inequality (16) therefore implies that

$$\|Au - Av\| \leq \left(\frac{(2M)^2 k!}{(2\tau)^k}\right)^{2/k} \|u - v\|^{1-(4/k)} = \frac{1}{4\tau^2}(k!)^{2/k} Q^{4/k}\|u\|$$
$$\leq \frac{Q^{4/k}k^2}{4\tau^2}\|u - v\|,$$

where $Q = 2M/\|u - v\|$. Minimising with respect to k yields

$$\|Au - Av\| \leq \left(\frac{\mathrm{e}}{\tau}\right)^2 \|u - v\| \left[\log \frac{2M}{\|u - v\|}\right]^2,$$

i.e. $A : \mathcal{A} \to H$ is 2-log-Lipschitz (cf. [**35**]).

Such a result relies only on the smoothness of solutions. But one can do much better by making use of the underlying equation. Indeed, Kukavica [**23**] used the structure of the differential equation (2) and far less restrictive conditions on $\mathcal{A}$ than above to show that $A^{1/2} : \mathcal{A} \to H$ is 1/2-log-Lipschitz. We briefly outline his argument, which was primarily developed to study the problem of backwards uniqueness for nonlinear equations with rough coefficients, and then show that it can be used to prove that $A : \mathcal{A} \to H$ is 1-log-Lipschitz.

In what follows we will consider the same equation as in Section 2

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = F(u). \tag{18}$$

In order to simplify the presentation we will assume here that $\alpha = 1/2$, i.e. that the nonlinear term F is locally Lipschitz from $D_H(A^{1/2})$ into H, although the following argument works for $0 \leq \alpha \leq 1/2$. Moreover, we assume that the maximal invariant set $\mathcal{A}$ is bounded in $D_H(A^{1/2})$. The argument that follows is simple – the key observation is that the result is sufficiently abstract that one can make a variety of choices of H $\big($e.g. we will take $H = L^2$ and then $H = D_{L^2}(A^{1/2})\big)$.

Let $u(t)$ and $v(t)$ be solutions of (18). The equation for the evolution of the difference $w(t) := u(t) - v(t)$ can be expressed as

$$\frac{\mathrm{d}w}{\mathrm{d}t} + Aw = f, \tag{19}$$

where $f(t) := F\big(u(t)\big) - F\big(v(t)\big)$. Our assumptions imply that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw, w) = (w_t, Aw) = -(Aw, Aw) + (f, Aw) \tag{20}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw, Aw) = (w_t, A^2 w) = -(Aw, A^2 w) + (f, A^2 w). \tag{21}$$

Moreover, it follows from (4) with $\alpha = 1/2$ that

$$\|f\| \leq \|F(u) - F(v)\| \leq K(\|A^{1/2}u\|, \|A^{1/2}v\|)\|A^{1/2}w\| \leq K_1\|A^{1/2}w\| \tag{22}$$

and, consequently,

$$(f, w) \geq -K_2\|w\|\|A^{1/2}w\| \tag{23}$$

for some $K_1, K_2 \geq 0$.

Under these mild regularity assumptions, Kukavica [**23**] proved the following backward uniqueness property: if $w : [T_0, 0] \to H$ is a solution of (19), then $w(0) = 0$

implies that $w(t) = 0$ for all $t \in [T_0, 0]$. His approach consists in establishing upper bounds for the log-Dirichlet quotient

$$\widetilde{Q}(t) = \frac{(Aw(t), w(t))}{\|w(t)\|^2 \Big(\log \frac{M^2}{\|w(t)\|^2} \Big)},$$

where M is a sufficiently large constant. This quantity is a variation of the standard Dirichlet quotient $Q(t) = \|A^{1/2}w\|^2/\|w\|^2$ (see [**28**], [**3**] for details). Kukavica showed that, for equations of the form of (19), the log-Dirichlet quotient is bounded for all $t \geq 0$ and, as an application of this result, stated the following theorem in the particular case of the two-dimensional Navier–Stokes equations.

THEOREM 5.1 (After Kukavica [**23**]). *Suppose that $F : D_H(A^{1/2}) \to H$ and $\mathcal{A}$ is a bounded subset of $D_H(A^{1/2})$, invariant for the flow generated by*

$$u_t = -Au + F(u)$$

and such that

$$\|F(u) - F(v)\| \leq K_1 \|A^{1/2}(u - v)\| \quad \text{for all} \quad u, v \in \mathcal{A}.$$

Then there exists a constant $C > 0$ such that

$$\|A^{1/2}(u - v)\|^2 \leq C\|u - v\|^2 \log(M^2/\|u - v\|^2), \quad \text{for all} \quad u, v \in \mathcal{A},\ u \neq v,$$

where $M = 4 \sup_{u \in \mathcal{A}} \|u\|$.

We give a quick summary of Kukavica's proof, filling in some details in the closing part of the argument. An expanded version of this proof can be found in Robinson [**37**].

PROOF. Let

$$L(\|w\|) = \log \frac{M^2}{\|w\|^2},$$

where M is any constant such that

$$M \geq 4 \sup_{u_0 \in \mathcal{A}} \|u_0\|.$$

Note that $L(\|w(t)\|) \geq 1$ for all $t \in [0, T_0]$. For $t \in [0, T_0]$, denote $\widetilde{L}(t) = L\big(\|w(t)\|\big)$. Define the log-Dirichlet quotient as

$$\widetilde{Q}(t) = \frac{Q(t)}{L\big(\|w\|\big)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 L\big(\|w\|\big)} = \frac{\|A^{1/2}w\|^2}{\|w\|^2 \widetilde{L}(t)}$$

where $Q(t) = \|A^{1/2}w\|^2/\|w\|^2$.

Using (20) and (21), Kukavica [**23**] showed in the proof of his Theorem 2.1 that

$$\widetilde{Q}'(t) + K_3 \widetilde{Q}(t)^2 \leq K_4, \tag{24}$$

with $K_3 = 1/2$ and $K_4 = 2K_1^4$. Applying a variant of Gronwall's inequality[4] proved in Temam [**44**, Lemma 5.1] to (24), we obtain that there exists T such that

$$\widetilde{Q}(t) \le C(K_3, K_4), \quad \text{for all} \quad t \ge T,$$

where $C(K_3, K_4)$ and T are constants independent of $\widetilde{Q}(0)$.

Now, consider $u_0, v_0 \in \mathcal{A}$. Since solutions in the attractor exist for all time, we know there exists $t \ge T$ such that $u_0 = S(t)u(-t)$ and $v_0 = S(t)v(-t)$ with $u_0 \ne v_0$. So, $u(-t) \ne v(-t)$. Moreover, $\widetilde{Q}(-t) < \infty$ implies that $\widetilde{Q}(0) \le C(K_3, K_4)$. Hence,

$$\sup_{u_0, v_0 \in \mathcal{A},\ u_0 \ne v_0} \widetilde{Q}(t) \le C(K_3, K_4).$$

□

We now show how this result can be used to obtain the 1-log-Lipschitz continuity of $A : \mathcal{A} \to H$.

Corollary 5.2. *Suppose that $\mathcal{A}$ is a bounded subset of $D_H(A)$, invariant for the flow generated by*

$$u_t = -Au + F(u)$$

and such that for all $u, v \in \mathcal{A}$

$$\|F(u) - F(v)\| \le K_1 \|A^{1/2}(u - v)\| \quad \text{and} \quad \|A^{1/2}(F(u) - F(v)\| \le K_2 \|A(u - v)\|.$$

Then there exists a constant $K > 0$ such that

$$\|A(u - v)\| \le K\|u - v\| \log(M^2/\|u - v\|^2), \quad \text{for all} \quad u, v \in \mathcal{A},\ u \ne v,$$

for some $M \ge 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u\|$.

Proof. Write $w = u - v$. Since (22) (and hence (23)) holds with $H = L^2$, then there exits a constant $C_0 > 0$ such that

$$\|A^{1/2}w\|_{L^2}^2 \le C_0 \|w\|_{L^2}^2 \log\left(M_0^2/\|w\|_{L^2}^2\right), \tag{27}$$

where

$$M_0 \ge 4 \sup_{u \in \mathcal{A}} \|u\|_{L^2}.$$

Now we use the fact that $\mathcal{A}$ is bounded in $D_H(A)$. Since (22) also holds with $H = D_{L^2}(A^{1/2})$, there exits a constant $C_1 > 0$ such that

$$\|Aw\|_{L^2}^2 \le C_1 \|A^{1/2}w\|_{L^2}^2 \log\left(M_1^2/\|A^{1/2}w\|_{L^2}^2\right) \tag{28}$$

where

$$M_1 \ge 4 \sup_{u \in \mathcal{A}} \|A^{1/2}u_0\|_{L^2}.$$

So,

$$\|Aw\|_{L^2}^2 \le C_0 C_1 \|w\|_{L^2}^2 \log\left(M_0^2/\|w\|_{L^2}^2\right) \log\left(M_1^2/\|A^{1/2}w\|_{L^2}^2\right).$$

[4] Lemma 5.1 (p167 in Temam [**44**]): *Let y be a positive absolutely continuous function on $(0, \infty)$, which satisfies*

$$y' + \gamma y^p \le \delta \tag{25}$$

with $p > 1$, $\gamma > 0$, $\delta \ge 0$. Then, for $t > 0$

$$y(t) \le \left(\frac{\delta}{\gamma}\right)^{1/p} + \left(\gamma(p-1)t\right)^{-1/(p-1)}. \tag{26}$$

Since $\|w\|_{L^2} \leq c\|A^{1/2}w\|$,

$$\|Aw\|_{L^2}^2 \leq C_0 C_1 \|w\|_{L^2}^2 \log\left(M_0^2/\|w\|_{L^2}^2\right) \log\left(c^2 M_1^2/\|w\|_{L^2}^2\right).$$

One can choose M_0, M_1, and M such that $M_0 \leq cM_1 \leq M$. Hence,

$$\|Aw\|_{L^2} \leq K\|w\|_{L^2} \log\left(M^2/\|w\|_{L^2}^2\right), \tag{29}$$

where $K = \sqrt{C_0 C_1}$. □

Using either Theorem 5.1 or Corollary 5.2, it follows from Proposition 4.2 to show that there exists a family of approximating Lipschitz manifolds $\mathcal{M}_N$, such that the global attractor $\mathcal{A}$ associated with equation (18) lies within an exponentially small neighbourhood of $\mathcal{M}_N$ and hence has zero Lipschitz deviation. It is interesting to note that our earlier proof of this [**32**] used the 'squeezing property', which in fact follows from an appropriate form of control over the classical Dirichlet quotient, see [**9**] for details. As remarked above, however, it is less our purpose here to reprove the fact that 'many attractors have zero Lipschitz deviation' than to investigate the relationship between regularity of A on $\mathcal{A}$ and other properties of the attractor.

To illustrate Corollary 5.2, we consider the incompressible Navier-Stokes equations

$$\partial_t u - \nu\triangle u + u\cdot\nabla u + \nabla p = F,$$
$$\nabla\cdot u = 0,$$

with periodic boundary conditions on $\Omega = [0, 2\pi]^2$ and initial condition $u(x,0) = u_0(t)$. Here $u(x,t)$ is the velocity vector field, $p(x,t)$ the pressure scalar function, ν the kinematic viscosity and $F(x,t)$ represents the volume forces that are applied to the fluid. We restrict ourselves to the space-periodic case for simplicity. Let $\mathcal{H}$ be the space of all the C^∞ periodic divergence-free functions that have zero average on Ω. Let H be the closure of $\mathcal{H}$ with scalar product $(\cdot,\cdot)_{L^2}$ and norm $\|\cdot\|_{L^2}$, and let V be similarly the closure of $\mathcal{H}$ with scalar product $(\cdot,\cdot)_{H^1}$ and norm $\|\cdot\|_{H^1}$. Let A be the Stokes operator defined by

$$Au = -\triangle u,$$

for all u in the domain $D(A)$ of A in H. Now consider the Navier-Stokes equations written in its functional form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu Au + B(u,u) = F, \tag{30}$$

using the operator A and the bilinear operator B from $V\times V$ into V' defined by

$$(B(u,v),w) = b(u,v,w), \quad \text{for all} \quad u,v,w\in V.$$

If $F\in H$ is independent of time, then the equation (30) possesses a global attractor

$$\mathcal{A} = \left\{u_0\in H : S(t)u_0 \text{ exists for all } t\in\mathbb{R},\ \sup_{t\in\mathbb{R}}\|S(t)u_0\|_{L^2_{\mathrm{per}}(\Omega)} < \infty\right\},$$

where $S(t)u_0$ denotes a solution starting at u_0 on its maximal interval of existence (cf. Constantin and Foias [**6**]). Under these assumptions, the difference of solutions $w = u - v$ will satisfy

$$\frac{\mathrm{d}w}{\mathrm{d}t} + \nu Aw = -\left[B(w,u) + B(v,w)\right].$$

So, in this case we use Kukavica's Theorem with $f = -\big[B(w,u) + B(v,w)\big]$. Note that

$$\|f\|_{H^1} \leq K_1 \|A^{1/2} w\|_{H^1},$$

and consequently

$$(f, Aw) \geq -K_2 \|w\|_{H^1} \|A^{1/2} w\|_{H^1}.$$

Therefore, one can apply Proposition 4.2 to the two dimensional Navier-Stokes equation with forcing $F \in L^2$ to show the existence of a family of approximate inertial manifolds of exponential order.

6. Conclusion

We first discussed in this paper the concept of finite-dimensionality of a flow. We then studied the consequences of the regularity of the vector field on the global attractor associated with certain parabolic equations. Namely, if the linear term A is Lipschitz continuous, then the global attractor $\mathcal{A}$ is a subset of a Lipschitz manifold given as a graph over a finite-dimensional eigenspace of A. If A is only log-Lipschitz continuous, then $\mathcal{A}$ lies within a small neighbourhood of a finite-dimensional Lipschitz manifold. Nevertheless, we are able to obtain in this case linear embeddings of the attractor into $\mathbb{R}^N$, whose inverse is Hölder continuous with exponent arbitrarily close to one by choosing N sufficiently large. Finally, we prove that the linear term A of the 2D Navier-Stokes equations is actually 1-Log-Lipschitz continuous when restricted to the attractor.

All the results presented in this paper were motivated by the problem of constructing a system of ordinary differential equation whose asymptotic behaviour reproduces the dynamics on an arbitrary finite-dimensional global attractor. We have shown in [**33**] that if the global attractor has finite Assouad dimension and the vector field restricted to the attractor is log-Lipschitz with exponent $\gamma < 1/2$, then there exists a system of ODEs with unique solutions that reproduces the dynamics on the global attractor.

Since this paper was written, Eden et al. [**10**] have constructed an abstract evolution equation of the form $u_t + Au = F(u)$ whose attractor cannot be embedded into any Euclidean space in a log-Lipschitz way; and Zelik [**46**] adapts the proof of Theorem 5.1 to show that when $\|F(u) - F(v)\| \leq K_1 \|u - v\|$ then A is 1/2-log-Lipschitz on the attractor, but that this exponent is optimal.

While these results are grave obstacles in applying our programme to construct such a system of ODEs, this remains a significant and important open problem. As Eden et al. remark, these counterexamples '*do not imply* at least in a straightforward way that the idea with Lipschitz and Log-Lipschitz Mane projections will not work for the concrete classes of equations of mathematical physics, like reaction-diffusion systems, or 2D Navier-Stokes equations on a torus.'

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References

[1] P. Assouad, 'Plongements Lipschitziens dans $\mathbb{R}^n$', *Bull. Soc. Math. France* 111 (1983), 429–448.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations* (North-Holland Publishing Co., Amsterdam, 1992).

[3] C. Bardos and L. Tartar 'Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines' *Arch. Rational Mech. Anal.* 50 (1973), 10–25.

[4] A. Ben-Artzi, A. Eden, C. Foias and B. Nicolaenko 'Hölder continuity for the inverse of Mañé's projection' *J. Math. Anal. Appl.* 178, no.1 (1993), 22–29.

[5] G. Bouligand 'Ensembles impropres et ordre dimensionnel' *Bull. Sci. Math.* 52 (1928), 320–344 and 361–376.

[6] P. Constantin and C. Foias *Navier-Stokes Equations* (University of Chicago Press, Chicago, 1988).

[7] P. Constantin, C. Foias, Nicolaenko and R. Temam 'Inertial manifolds for nonlinear evolutionary equations' *J. Diff. Eq.* 73 (1988), 309–353.

[8] A. Debussche and R. Temam 'Inertial manifolds and the slow manifolds in meteorology' *Diff. and Int. Equ.* 4, no.5 (September, 1991), 897–931.

[9] A. Eden, C. Foias, B. Nicolaenko, and R. Temam *Exponential Attractors for Dissipative Evolution Equations* Research in applied Mathematics Series (Wiley, New York, 1994).

[10] A. Eden, S.V. Zelik, and V.K. Kalantarov 'Counterexamples to regularity of Mañé projections in the theory of attractors' *Russian Math. Surveys* 68 (2013), 199-226.

[11] C. Foias and E. Olson 'Finite fractal dimension and Hölder-Lipschitz parametrization' *Indiana Univ. Math. J.* 45, no.3 (1996), 603–616.

[12] C. Foias, O. Manley and R. Temam 'Modelling of the interaction of small and large eddies in two-dimensional turbulent flows' *RAIRO Modél. Math. Anal. Numér.* 22, no.1 (1988), 93–118.

[13] C. Foias, G. Sell and R. Temam 'Variétés inertielles des équations différentielles dissipatives' *C. R. Acad. Sci. Paris I* 301 (1985), 139–141.

[14] C. Foias, G. R. Sell and R. Temam 'Inertial Manifolds for nonlinear evolutionary equations' *J. Diff. Eq.* 73 (1988), 309–353.

[15] C. Foias and R. Temam 'Some analytic and geometric properties of solutions of the Navier-Stokes equations' *J. Math. Pures Appl.* 58 (1979), 339–365.

[16] C. Foias and R. Temam 'Gevrey class regularity for the solutions of the Navier-Stokes equations' *J. Funct. Anal.* 87 (1989), 359–369.

[17] J.K. Hale *Asymptotic Behavior of Dissipative Systems* Mathematical Surveys and Monographs Number 25 (American Mathematical Society, Providence, RI, 1988).

[18] P. Hartman (1982) *Ordinary Differential Equations* (2nd edition, Birkhäuser, Boston, Mass.)

[19] D. Henry *Geometric Theory of Semilinear Parabolic Equations* Lecture Notes in Mathematics vol. 840 (Springer-Verlag, 1981).

[20] B. R. Hunt and V. Y. Kaloshin 'Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces' *Nonlinearity* 12 (1999), 1263–1275.

[21] I. Kukavica 'Fourier Parametrization of Attrators for Dissipative Equations in one space dimension' *J. of Dyn. Diff. Eqns.* 15, no.2/3 (July 2003), 473–484.

[22] I. Kukavica 'On Fourier Parametrization of Global Attrators for Equations in one space dimension' *Discrete Contin. Dyn. Syst.* 13, no.3 (2005), 553–560.

[23] I. Kukavica 'Log-log convexity and backward uniqueness' *Proc. Amer. Math. Soc.* 135, no.8 (2007), 2415–2421.

[24] J. Luukkainen 'Assouad dimension: Antifractal metrization, porous sets, and homogeneous measures' *J. Korean Math. Soc.* 35 no.1 (1998), 23–76.

[25] R. Mané 'On the dimension of the compact invariant sets of certain nonlinear maps' *Springer Lectures Notes in Math.* 898 (1981), 230–242.

[26] E. J. McShane 'Extension of the range of functions' *Bull. Am. Math. Soc.* 40 (1934), 837–842.

[27] H. Movahedi-Lankarani 'On the inverse of Mañé projection' *Proc. Amer. Math. Soc.* 116, no.2 (1992), 555–560.

[28] H. Ogawa 'Lower bounds for solutions of differential inequalities in Hilbert space' *Proc. Amer. Math. Soc.* 16 (1965), 1241–1243.

[29] E. Olson 'Bouligand dimension and almost Lipschitz embeddings' *Pacific J. Math.* 202 (2002), 459–474.

[30] E. J. Olson and J. C. Robinson 'Almost bi-Lipschitz embeddings and almost homogeneous sets' *Trans. Amer. Math. Soc.* 362, no.1 (2010), 145–168.
[31] E. Pinto de Moura and J. C. Robinson 'Orthogonal sequences and regularity of embeddings into finite-dimensional spaces' *J. Math. Anal. Appl.* 368, (2010a), 254–262.
[32] E. Pinto de Moura and J. C. Robinson 'Lipschitz deviation and embeddings of global attractors' *Nonlinearity* 23, (2010b), 1695–1708.
[33] E. Pinto de Moura, J. C. Robinson and J. J. Sanches-Gabites 'Embedding of global attractors and their dynamics' *Proc. Amer. Math. Soc.* 139 (2011), no. 10, 3497–3512
[34] J.C. Robinson *Infinite-dimensional dynamical systems.* Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2001).
[35] J. C. Robinson 'Attractors and finite-dimensional behaviour in the 2D Navier-Stokes equations'. *ISRN Math. Anal.* 2013, Art. ID 291823.
[36] J. C. Robinson 'Linear embeddings of finite-dimensional subsets of Banach spaces into Euclidean spaces' *Nonlinearity* 22 (2009), 711–728.
[37] J. C. Robinson *Dimensions, Embeddings, and Attractors.* Cambridge University Press, Cambridge, UK, 2012. 2010.
[38] J. C. Robinson Log-Lipschitz embeddings of homogeneous sets with sharp logarithmic exponents and slicing products of balls. *Proc. Amer. Math. Soc.* 142 (2014), 1275–1288.
[39] A. V. Romanov 'Finite-dimensional limiting dynamics for dissipative parabolic equations' *Sbornik: Mathematics* 191, no. 3 (2000), 415–429.
[40] A. V. Romanov (2001) Finite-dimensional dynamics on attractors of non-linear parabolic equations. *Izv. Math.* **65**, 977–1001.
[41] A. V. Romanov (2006) Effective parametrization in phase spaces of parabolic equations. *Izv. Math.* **70**, 1015–1029.
[42] R. Rosa 'Approximate inertial manifolds of exponential order' *Discrete and continuous dynamical systems* 1, no. 3 (1995), 421–448.
[43] G. R. Sell and Y. You *Dynamics of evolutionary equations* Applied Mathematical Sciences vol. 143 (Springer-Verlag, New York, 2002).
[44] R. Temam *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* Springer Applied Mathematical Sciences vol 68, 2nd edition (Springer-Verlag, Berlin, 1997).
[45] J. H. Wells and L. R. Williams *Embeddings and extensions in analysis* (Springer-Verlag, Berlin, 1975).
[46] S. Zelik (2013) Inertial manifolds and finite-dimensional reduction for dissipative PDEs. arXiv: 1303.4457.

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