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Dynamics of stochastic modified Boussinesq approximation equation driven by fractional Brownian motion

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ABSTRACT. The current paper is devoted to stochastic modified Boussinesq approximation equation driven by fractional Brownian motion with $H \in (\frac{1}{4}, \frac{1}{2})$. Based on the different diffusion operators $P\Delta^2$ and $-\Delta$ in stochastic systems, we combine two types operators $\Phi_1 = I$ and a Hilbert-Schmidt operator $\Phi_2 = \Phi$ to guarantee the convergence of the corresponding Wiener-type stochastic integrals, and show the existence and regularity of the stochastic convolution corresponding to the stochastic modified Boussinesq approximation equation. By the Banach modified fixed point theorem in the selected intersection space, the existence and uniqueness of global mild solution are obtained. Finally, the existence of a random attractor for the random dynamical system generated by the mild solution for the modified Boussinesq approximation equation is also established.

CONTENTS

1. Introduction

The Navier-Stokes equations are often coupled with other equations, especially, with the scalar transport equations for fluid density, salinity, or temperature. These

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coupled equations (often with the Boussinesq approximation) model a variety of phenomena arising in environmental, geophysical, and climate systems ([**7, 8, 26**]). For important models such as the Navier-Stokes equation, KDV equation, Burgers equation and the Schrödinger equation, one can consult [**3, 6, 13, 14, 21**] for results on the existence, uniqueness of solution, and existence of attractors. The modified Boussinesq approximation equation is a reasonable model to describe the essential phenomena of the highly viscous incompressible fluid in the Earth's mantle, we refer to Hills and Roberts [**19**], Padula [**27**] for a derivation of the following Boussinesq approximation equation

$$\begin{cases} u_t + u \cdot \nabla u - \nabla \cdot \tau(e(u)) = -\nabla \pi + f(x) + e_2\theta, \\ \theta_t + (u \cdot \nabla)\theta - \triangle\theta = g(x), \end{cases} \tag{1.1}$$

where the vector function u represents the velocity of the fluid, θ is the scalar temperature, function $f(x)$ and $g(x)$ are periodic external forces with respect to space variable x, the vector $e_2 = (0, 1)$ is a unit vector in R^2, and the scalar function π is the pressure, $\tau_{ij}(e(u))$ is a symmetric stress tensor with the form

$$\tau_{ij}(e(u)) = 2\mu_0(\epsilon + |e|^2)^{\frac{p-2}{2}} e_{ij} - 2\mu_1 \triangle e_{ij}, \epsilon > 0, i, j = 1, 2,$$
$$e_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \quad |e(u)|^2 = \sum_{i,j=1}^{2} |e_{ij}(u)|^2,$$

where μ_0 and μ_1 are positive constants. In the shear thinning case, $1 < p < 2 (n = 2, 3)$ and the shear thickening case, $1 < p \leq 2 + \frac{2}{n+2}, (n = 2, 3)$. There are many contributions to investigate the existence and uniqueness of the solution, attractors and manifold for the modified Boussinesq approximation equation, we refer to [**5, 17, 11, 30**] for deterministic non-Newtonian flow (with the absence of θ), and refer to [**18**] and the monograph [**17**] for the well-posedness and long-time behavior of modified Boussinesq approximation equation.

Recently, C. Guo [**16**] showed the existence of random attractor for the stochastic Boussinesq approximation equations driven by Gaussian white noise in domain $D = [0, L] \times [0, L]$

$$\begin{aligned} & du + (u \cdot \nabla u - \nabla \cdot \tau(e(u)) + \nabla\pi)dt \\ & = (f(x) + e_2\theta)dt + \Phi_1(t)dW(t), x \in D, t > 0, \\ & d\theta_t + ((u \cdot \nabla)\theta - \triangle\theta)dt = g(x)dt + \Phi_2(t)dW(t), x \in D, t > 0, \\ & \nabla \cdot u(x, t) = 0, \quad x \in D, t > 0, \\ & u(x, 0) = u_0(x), \theta(x, 0) = \theta_0(x), x \in D, \\ & u_i(x, t) = u_i(x + L\chi_j, t), \quad \theta(x, t) = \theta(x + L\chi_j, t) \quad i = 1, 2. \end{aligned} \tag{1.2}$$

where $\{\chi_j\}_{j=1}^2$ is the natural basis of R^2,$W(t) = \sum_i \beta_i(t)h_i$ is the cylindrical Wiener process for white noise, $\beta_i(t)$ is a family of mutually independent real-valued standard Wiener process, $\Phi_i(t), i = 1, 2$ are Hilbert-Schmidt operators.

The fractional Brownian motion(FBM) is a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0, 1)$. For $H \neq \frac{1}{2}$, the FBM is not a semimartingale and the increments of the process are not independent. These properties can be used in modeling "cluster" phenomena (systems with memory and persistence) such as hydrology [**20**],et al. There are many papers and monographs on fractional Brownian motion and its stochastic integral, and stochastic partial differential equation(SPDE) driven by FBM, we refer to [**4**], [**29**], [**22, 23, 24**] and so on. Recently, The authors in [**11**] and [**12**] studied the regularity for stochastic convolution, and showed the existence of the mild solution for stochastic non-Newtonian fluid driven by space-time fractional Brownian motion. Futhermore, they also established the existence of random attractor.

Since fractional Brownian motion is neither markov nor martingale, the classical Itô stochastic integral fails to apply for one of FBM. It is interesting to study the well-posedness and long time behavior of stochastic modified Boussinesq approximation driven by FBM. It is necessary to mention the current work in [**10**], in which they studied the two dimensional stochastic Navier-Stokes equation driven by FBM. Motivated by the ideas in [**11**],[**10**] and [**16**]. In the present paper, we consider the following stochastic modified Boussinesq equation driven by fractional Brownian motion

$$\begin{cases} du(t) + (u\cdot\nabla u - \nabla\cdot\tau(e(u)) + \nabla\pi)dt = (f(x) + e_2\theta)dt + dB^H(t), \quad x\in\mathcal{O},\ t>0,\\ d\theta(t) + ((u\cdot\nabla)\theta - \triangle\theta)dt = g(x)dt + \Phi(t)dB^H(t), \quad x\in\mathcal{O}, t>0,\\ \nabla\cdot u(x,t) = 0, \quad x\in\mathcal{O}, t>0,\\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x), x\in\mathcal{O}, \end{cases} \tag{1.3}$$

where $\mathcal{O}\subset R^2$ be a bounded domain with smooth boundary $\partial\mathcal{O}$.

Due to the regularity of the stochastic convolution depends on the value of Hurst parameter H, stochastic Wiener-type integral is different for $H\in(1/2,1)$ and $H\in(0,1/2)$ respectively. For $H\in(0,1/2)$, it needs the fractional Riemann-Liouville integral to transfer the fractional Brownian motion to be represented in terms of standard Brownia motion, and the computation for the regularity in the case of Hurst parameter $H\in(0,\frac{1}{2})$ is more complicated than one for $H\in(\frac{1}{2},1)$. More details are present in section 3. Based on the different diffusion operators $P\Delta^2$ and $-\Delta$, we combine two types operators $\Phi_1 = I$ and a Hilbert-Schmidt operator $\Phi_2 = \Phi$ to guarantee the convergence of the corresponding to Wiener-type stochastic integrals, and show the existence and regularity of the stochastic convolution corresponding to the stochastic modified Boussinesq approximation equation. By the modified Banach fixed point theorem in the selected intersection space, the existence and uniqueness of mild solution are obtained. Finally, the existence of a random attractor for the random dynamical system generated by the mild solution for the modified Boussinesq approximation equation is also presented.

The rest of the paper is organized as follows. Section 2 is devote to some functional setting and operators, some definitions and criteria for random dynamical systems. In section 3, we introduce the definition of the infinite dimensional fractional Brownian motion and its stochastic integral, and present some properties of stationary solutions z_1 and z_2 for some kinds of different Hilbert-Schmidt operators. The section 4 is devoted to the existence of mild solution. The existence of random attractor of random dynamical system generated by the mild solution of stochastic equation (1.3) is shown in section 5.

2. Preliminaries

In this section, we will present some notations for operators and working spaces, and then represent the stochastic modified Boussinesq approximation equation as an stochastic evolution equation in product space.

In what follows, we introduce some notations as follows

$$H_1 = \{u \in \{L^2(\mathcal{O})\}^2 : \nabla \cdot u = 0,\, u \cdot \boldsymbol{n}\big|_{\partial\mathcal{O}} = 0\}, \quad H_2 = L^2(\mathcal{O}).$$

Denote $H = H_1 \times H_2$ endowed with the norm

$$|\phi|_H^2 := |u|_{H_1}^2 + |\theta|_{H_2}^2 = |u|_{L^2(\mathcal{O})^2}^2 + |\theta|_{L^2(\mathcal{O})}^2$$

for any $\phi = (u, \theta) \in H$, where $u \in H_1$ and $\theta \in H_2$. For simplicity, we use the notation $|\cdot|$ to represent the norm for space H_1, H_2 and H respectively. It is easy to verify that H_1, H_2 and H are Hilbert spaces with the inner product denoted by $(\cdot\,,\cdot)$ for each of the spaces.

Followed the same notation in [**11**], denote

$$V_1 = \{u \in \{H_0^2(\mathcal{O})\}^2 : \nabla \cdot u = 0\}, \quad V_2 = H_0^1(\mathcal{O}), \quad V = V_1 \times V_2.$$

Then V_1 is a Hilbert space with the norm

$$|u|_{V_1}^2 = \frac{1}{2}|\triangle u|^2 = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \left| \frac{\partial e_{ij}(u)}{\partial x_k} \right|^2 dx.$$

Define bilinear operator $a_1(\cdot\,,\cdot) : V_1 \times V_1 \to \mathbb{R}$ and $a_2(\cdot\,,\cdot) : V_2 \times V_2 \to \mathbb{R}$ by

$$a_1(u, v) = (u, v)_{V_1}, \qquad a_2(\theta, \xi) = (\theta, \xi)_{V_2}.$$

By Lax-Milgram lemma, we can use the bilinear operators $a_1(\cdot\,,\cdot)$ and $a_2(\cdot\,,\cdot)$ to define the following linear operators $A_1 \in \mathscr{L}(V_1, V_1')$ and $A_2 \in \mathscr{L}(V_2, V_2')$:

$$< A_1 u, v >= a_1(u, v), \qquad < A_2\theta, \xi >= a_2(\theta, \xi).$$

Similar to the argument in Proposition 2.3 in [**2**] and [**11**], the operators A_i is an isometry from V_i to V_i' for $i = 1, 2$.

Denote

$$D(A_1) = V_1 \cap \{H^4(\mathcal{O})\}^2, \qquad D(A_2) = V_2 \cap H^2(\mathcal{O}),$$

then $A_i \in \mathscr{L}(D(A_i), H_i)$ is an isometry from $D(A_i)$ to H_i, and A_i is a self-adjoint positive operator with compact inverse A_i^{-1}, where $i = 1, 2$.

It follows from the Hilbert-Schmidt theorem that there exist eigenvalues $\{\lambda_j\}_{j=1}^\infty$, $\{\hat{\lambda_j}\}_{j=1}^\infty$ and the corresponding eigenvectors $\{e_j\}_{j=1}^\infty \subset D(A_1)$, $\{\hat{e}_j\}_{j=1}^\infty \subset D(A_2)$ such that

$$A_1 e_j = \lambda_j e_j, \quad j = 1, 2, \ldots, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \quad \lambda_j \to \infty \ (j \to \infty),$$
$$A_2 \hat{e}_j = \hat{\lambda}_j \hat{e}_j, \quad j = 1, 2, \ldots, \quad 0 < \hat{\lambda}_1 \le \hat{\lambda}_2 \le \cdots \le \hat{\lambda}_j \le \cdots, \quad \hat{\lambda}_j \to \infty (j \to \infty).$$

Moreover, $\{e_j\}_{j=1}^\infty$ and $\{\hat{e}_j\}_{j=1}^\infty$ are the orthonormal basis for H_1 and H_2 respectively.

Since operator $A_i (i = 1, 2)$ is the densely-defined, self-adjoint, positive operator in Hilbert space $H_i (i = 1, 2)$, then $A_i (i = 1, 2)$ is a sectional operator, and $S_i(t) \in \mathscr{L}(H_i)$ is an analytic semigroup generated by $A_i, (i = 1, 2)$.

$$S_i(t) := e^{-tA_i} = \int_0^\infty e^{-t\lambda} dE_{i,\lambda}, \quad i = 1, 2,$$

where $\{E_{i,\lambda}\}$ are the projections to the eigenspace determined by $A_i, i = 1, 2$.

For any $\phi = (u, \theta) \in V$, denote

$$A\phi = \begin{pmatrix} 2\mu_1 A_1 u \\ A_2\theta \end{pmatrix}, \qquad S(t)\phi = \begin{pmatrix} S_1(t)u \\ S_2(t)\theta \end{pmatrix},$$

and define the trilinear operator $b(\cdot,\cdot,\cdot)$ by

$$b(\phi_1,\phi_2,\phi_3)=b_1(u_1,u_2,u_3)+b_2(u_1,\theta_2,\theta_3),\quad \forall\phi_i=(u_i,\theta_i)\in V,$$

where

$$b_1(y,v,w)=\sum_{i,j=1}^{2}\int_{\mathcal{O}} y_i\frac{\partial v_j}{\partial x_i}w_j dx,\quad \forall y,v,w\in\left\{H^1(\mathcal{O})\right\}^2,$$

$$b_2(y,\theta,\xi)=\sum_{i=1}^{2}\int_{\mathcal{O}} y_i\frac{\partial \theta}{\partial x_i}\xi dx,\quad \forall y\in\left\{H^1(\mathcal{O})\right\}^2,\theta,\xi\in H^1(\mathcal{O}).$$

For any $\phi_i=(u_i,\theta_i)$, we define the continuous bilinear functionals $B(\phi_1,\phi_2)\in V'$, $B_1(u_1,u_2)\in V_1'$ and $B_2(u_1,\theta_2)\in V_2'$ by

$$\begin{aligned}<B(\phi_1,\phi_2),\phi_3> &= b(\phi_1,\phi_2,\phi_3),\\ <B_1(u_1,u_2),u_3> &= b_1(u_1,u_2,u_3),\\ <B_2(u_1,\theta_2),\theta_3> &= b_2(u_1,\theta_2,\theta_3).\end{aligned}$$

In what follows, we abbreviate $B(\phi,\phi)$ as $B(\phi)$ for any $\phi\in V$.

Define the functional $N(u)\in V_1'$ by

$$<N(u),v>=\int_{\mathcal{O}}\mu(u)e_{ij}(u)e_{ij}(v)dx,\quad \forall v\in V_1,$$

where $\mu(u)=(\epsilon+|e(u)|^2)^{\frac{p-2}{2}}$, and

$$\widetilde{N}(\phi)=\begin{pmatrix} N(u)\\ 0\end{pmatrix},\quad \forall\phi=(u,\theta)\in V.$$

We also denote $\widetilde{N}$ as N without any confusion, and

$$R\phi=\begin{pmatrix} -\theta\\ 0\end{pmatrix},\quad \Phi dB^H(t)=\begin{pmatrix} dB_1^H(t)\\ \Phi_2 dB_2^H(t)\end{pmatrix}.$$

With the above notations, the stochastic modified Boussinesq approximation equation (1.3) can be rewritten as the following abstract stochastic evolution equation

$$\begin{cases} d\phi(t)+\big(A\phi(t)+B(\phi(t))+N(\phi)(t)+R(\phi(t))\big)dt=\Phi dB^H(t),\\ \phi(0)=(u_0,\theta_0).\end{cases}\tag{2.1}$$

Finally, we introduce the definitions of random dynamical system, random attractor which are taken from [**6**]. Let $(\mathbb{H},d)$ be a complete separable metric space, $(\Omega,\mathbb{F},P)$ be a probability space. The following definition is from [**12**].

Definition 2.1. *$(\Omega,\mathbb{F},P,(\theta_t)_{t\in R})$ is called a metric dynamical system if $\theta: R\times\Omega\to\Omega$ is $(B(R)\times\mathbb{F},\mathbb{F})$ measurable, $\theta_0=I,\theta_{s+t}=\theta_s\circ\theta_t$ for all $t,s\in R$, and $\theta_tP=P$ for all $t\in R$.*

Definition 2.2. *A random dynamical system (RDS) with time T on a metric, complete and separable space $(\mathbb{H},d)$ with Borel $\sigma-$algebra $\mathcal{B}$ over $\{\theta_t\}$ on $(\Omega,\mathbb{F},P)$ is a measurable map*

$$S:\quad T\times\mathbb{H}\times\Omega\longmapsto\mathbb{H},\quad (t,x,\omega)\longmapsto S(t,\omega)x$$

such that

(i) $S(0,\omega)=Id$ *(identity on $\mathbb{H}$),*

(ii) (**Cocycle property**) $S(t+s,\omega)=S(t,\theta_s\omega)\circ S(s,\omega)$ *for all* $s,t\in T$ *and* $\omega\in\Omega$.

Definition 2.3. *An RDS is said to be continuous or differentiable if* $S(t,\omega):\mathbb{H}\longmapsto\mathbb{H}$ *is continuous or differentiable respectively for all* $t\in T$. *A set* $B\subset\Omega$ *is called invariant with respect to* $(\theta_t)_{t\in R}$ *if for all* $t\in R$, $\theta_t^{-1}B=B$.

Definition 2.4. *A random set* $K(\omega)$ *is said to be* $S(t,\omega)$ *forward invariant if*

$$S(t,\omega)K(\omega)=K(\theta_t\omega).$$

Definition 2.5. *A random set* $A(\omega)$ *is said to attract another random set* $B(\omega)$ *if* P*-almost surely,*

$$d(S(t,\theta_{-t}\omega)B(\theta_{-t}\omega),A(\omega))\to 0, \text{as}\, t\to\infty.$$

Definition 2.6. *If* $K(\omega)$ *and* $B(\omega)$ *are random sets such that for* P*-almost all* $\omega\in\Omega$, *there exists a time* $t_B(\omega)$ *such that for all* $t\geq t_B(\omega)$

$$S(t,\theta_{-t}\omega)B(\theta_{-t}\omega)\subset K(\omega),$$

then $K(\omega)$ *is called as an absorbing set with respect to* $B(\omega)$, *and* $t_B(\omega)$ *is called the absorption time.*

Definition 2.7. *Suppose* $S(t,\omega)$ *is an RDS such that there exists a random compact set* $\omega\to\mathcal{A}(\omega)$ *which satisfies the following conditions:*

(i) $S(t,\omega)\mathcal{A}(\omega)=\mathcal{A}(\theta_t\omega)$ *for all* $t>0$, *and*

(ii) $\mathcal{A}(\omega)$ *attracts every bounded deterministic set* $B\subset H$.

Then $\mathcal{A}(\omega)$ *is called global random attractor.*

Definition 2.8. *The random Omega limit set of a bounded set* $B\subset X$ *at time* t *is defined as*

$$A(B,t,\omega)=\bigcap_{T<t}\overline{\bigcup S(t,s,\omega)B},$$

and

$$\Omega_B=\overline{\bigcup_{B\subset X}A(B,t,\omega)}.$$

Theorem 2.1. *(*[**6**]*) Suppose* $\{S(t,\omega)\}_{t\geq s,\omega\in\Omega}$ *be a random dynamical systems on a Polish space* $\mathbb{H}$, *and suppose that there exists a compact set* $\omega\mapsto K(\omega)$ *absorbing every bounded nonrandom set* $B\subset H$. *Then the set*

$$\mathcal{A}(\omega)=\overline{\bigcup_{B\subset H}\Omega_B(\omega)}$$

is a global random attractor for $S(t,\omega)$. *Furthermore,* $\mathcal{A}$ *is measurable with respect to* $\mathbb{F}$ *is* T *is discrete, and it is measurable with respect to the completion of* $\mathbb{F}$, *where* $\Omega_B(\omega)$ *is the Omega limit set of the set* B.

3. The Wiener-type stochastic integral with respect to FBM

In this section, we introduce the definition of the infinite dimensional fractional Brownian motion only with $H\in(0,\frac{1}{2})$ and its Wiener-type stochastic integral, and present some properties of stationary solutions z_1 and z_2 for a class of different Hilbert-Schmidt operators.

First, we introduce the definition of the standard cylindrical fractional Brownian motion, which is taken from [**9**], and the definition of the wiener-type stochastic integral are taken from [**11**].

Definition 3.1. *([**9**]) Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. A cylindrical process $< B^H, \cdot >: \Omega \times \mathbb{R}_+ \times V \to \mathbb{R}$ on $(\Omega, \mathscr{F}, P)$ is called a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0,1)$ if*

(1) *for each $x \in V \backslash \{0\}$, $\frac{1}{||x||} < B^H(\cdot), x >$ is a standard scalar FBM with Hurst parameter H;*
(2) *for $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$,*

$$< B^H(t), \alpha x + \beta y >= \alpha < B^H(t), x > + \beta < B^H(t), y > \quad P\text{-a.s.} \tag{3.1}$$

Remark 3.1. *For $H = \frac{1}{2}$, the definition 3.1 reduces to the usual one for a standard cylindrical Wiener process. The fractional Brownian motion has the differential regularity for $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$ respectively. In what follows, we just focus on the case $H \in (0, \frac{1}{2})$ with lower regularity, we need the fractional Riemann Liouville integrals to define the Wiener-type stochastic integral in terms of standard Brownian motion.*

Let f be a deterministic Banach-space valued function that belongs to $L^1(0,T;V)$. The fractional Riemann Liouville integrals of order $\alpha > 0$ are determined at almost every $t \in [0,T]$ are defined by

(1) Left-sided integral:

$$(I^{\alpha}_{0+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

(2) Right-sided integral:

$$(I^{\alpha}_{T-} f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

For $\alpha \in (0,1)$, we denote by $I^{\alpha}_{0+}(L^2(0,T;V))$ (respectively, $I^{\alpha}_{T-}(L^2(0,T;V))$) the class of functions f in $L^2(0,T;V)$ which can be represented as an I^{α}_{0+}-integral (respectively, I^{α}_{T-}-integral) of some function $g \in L^2(0,T;V)$. If $f \in I^{\alpha}_{T-}(L^2(0,T;V))$, then the function g such that $f = I^{\alpha}_{T-} g$ is unique in L^2 and it agrees with the right-sided Riemann-Liouville derivative of f of order α defined by

$$D^{\alpha}_{T-} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{f(s)}{(s-t)^{\alpha}} ds.$$

This derivative has the Weyl representation

$$D^{\alpha}_{T-} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(T-t)^{\alpha}} - \alpha \int_t^T \frac{f(s)-f(t)}{(s-t)^{\alpha+1}} ds \right),$$

where the convergence of the integrals at the singularity $t = s$ holds in L^2-sense. It follows that the space $I^{\alpha}_{T-}(L^2(0,T;V))$ is Hilbert space with product

$$< f, g >_{\alpha} = < f, g >_{L^2(0,T;V)} + < D^{\alpha}_{T-} f, D^{\alpha}_{T-} g >_{L^2(0,T;V)} .$$

Now, we introduce the Wiener-type stochastic integral with respect to FBM. Fix an interval $[0,T]$ and let $\beta^H(t)$ be a one-dimensional fBm of Hurst index $H \in$

$(0, \frac{1}{2})$ on the probability space $(\Omega, \mathscr{F}, P)$. By definition β^H is a centered Gaussian process with covariance

$$R(t,s) = \mathbb{E}(\beta^H(t)\beta^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Fractional Brownian motion β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s),$$

where β is a Wiener process, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = b_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + (\frac{1}{2} - H) s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du \right]. \tag{3.2}$$

The constant b_H is defined by

$$b_H = \left[\frac{2H}{(1-2H)\cdot\beta(1-2H, H+\frac{1}{2})} \right]^{\frac{1}{2}},$$

where $\beta(\cdot)$ is the Euler Beta function. It follows from (3.2) that

$$\frac{\partial K_H}{\partial t}(t,s) = b_H(H - \frac{1}{2})(t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}.$$

Denote by $\mathscr{E}$ the linear space of V-valued step functions of the form

$$\phi(t) = \sum_{i=1}^n a_i 1_{(t_i, t_{i+1}]}(t)$$

where $a_i \in V$ and $0 = t_1 < t_2 < \cdots < t_{n+1} = T$. The Wiener integral with respect to FBM for $\phi \in \mathscr{E}$ is defined by

$$\int_0^T \phi(s) d\beta^H(s) = \sum_{i=1}^n a_i(\beta^H_{t_{i+1}} - \beta^H_{t_i}).$$

Define the linear operator $K_H^* := \mathscr{E} \mapsto L^2(0,T;V)$ induced from the kernel K_H by

$$(K_H^*\phi)(t) = \phi(t)K_H(T,t) + \int_s^T (\phi(s) - \phi(t)) \frac{\partial K_H}{\partial s}(s,t)ds.$$

It follows that

$$\mathbb{E}\left| \int_0^T \phi(s)d\beta^H(s) \right|_V^2 = |K_H^*\phi|_{L^2(0,T;V)}. \tag{3.3}$$

Let $\mathscr{H}$ be the Hilbert space obtained by the completion of the pre-Hilbert space $\mathscr{E}$ with the inner product

$$< \phi, \psi >_{\mathscr{H}} := < K_H^*\phi, K_H^*\psi >_{L^2(0,T;V)}$$

for $\phi, \psi \in \mathscr{E}$. We refer to [**1**] for the proof of the fact that K_H^* is an isometry between the space $\mathscr{E}$ and $L^2(0,T)$ that can be extended to the Hilbert space $\mathscr{H}$ and $L^2(0,T)$. Thus the stochastic integral is extended to $\mathscr{H}$ by isometry (3.3) and the image on an element $\Psi \in \mathscr{H}$ by this isometry is called the Wiener integral of Ψ with respect to β^H.

For $0 < H < \frac{1}{2}$, the reproducing kernel Hilbert space $\mathscr{H}$ can be represented by the fractional integral space. Namely,

$$\mathscr{H} = (K_H^*)^{-1}(L^2(0,T;V)) = I_{T^-}^{\frac{1}{2}-H}(L^2(0,T;V)),$$

As a consequence, we have the following relationship between the Wiener integral with respect to FBM and the Wiener integral with respect to the Wiener process:

$$\int_0^t \varphi(s)d\beta^H(s) = \int_0^t (K_H^*\varphi)(s)dW(s) \tag{3.4}$$

for every $t \leq T$ and $\varphi \in \mathscr{H}$ if and only if $K_H^*\varphi \in L^2(0,T;V)$.

Denote $\widetilde{e}_n = \lambda_n^{-\frac{1}{2}} e_n$, $n \in \mathbb{N}$. It follows that $\{\widetilde{e}_n\}_{n\in\mathbb{N}}$ forms a standard orthonormal basis of V. Letting $\beta_n^H(t) =< B^H(t), \widetilde{e}_n >$ for $n \in \mathbb{N}$, the sequence of scalar processes $\{\beta_n^H\}_{n\in\mathbb{N}}$ is independent and B^H can be represented by the formal series

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t)\widetilde{e}_n \tag{3.5}$$

that does not converge a.s. in H. Although for any fixed t the series (3.5) is not convergent in $L^2(\Omega \times H)$, we can always consider a Hilbert space U_1 such that $H \subset U_1$ such that this inclusion is a Hilbert-Schmidt operator. In this way, B^H given by (3.5) is a well-defined U_1-valued Gaussian stochastic process.

Let $\{\Phi(s), 0 \leq s \leq T\}$ be a deterministic $\mathscr{L}_2(V)$-valued function, where $\mathscr{L}_2(V)$ be the space of Hilbert-Schmidt operators on V. The stochastic integral of Φ with respect to B^H can be defined by

$$\int_0^t \Phi(s)dB^H(s) = \sum_{n=1}^{\infty}\int_0^t \Phi(s)\widetilde{e}_n d\beta_n^H(s) = \sum_{n=1}^{\infty}\int_0^t (K_H^*(\Phi\widetilde{e}_n))(s)d\beta_n(s), \tag{3.6}$$

where β_n is the standard Brownian motion.

However, the stochastic linear additive equation in its mild form admits a solution even if $\int_0^t \Phi(t)dB^H(s)$ is not properly defined as a V-valued process. Here we use notations: $\mathscr{L}_2(H)$ be the space of all Hilbert-Schmidt operators on H.

- **(Hyper-1)** $Q \equiv id_H$, $\Phi \in \mathscr{L}_2(H)$**;**

These conditions are proposed by Maslowski, Schmalfuss in [**24**] and Duncan, Maslowski, Pasik-Duncan in [**9**] and Tindel, Tudor, Veins in [**29**] respectively. We mention that the Hilbert-Schmidt operators (elements of $\mathscr{L}_2(H)$) is compact. Indeed, the key feature of these conditions is the compactness which guarantees that we can handle the infinite-dimensional problem in a finite-dimensional manner.

Consider the stochastic linear differential equation

$$\begin{cases} dz_2 = A_2 z dt + \Phi dB^H, \\ z_2(0) = 0 \in V. \end{cases} \tag{3.7}$$

LEMMA 3.1. [**24**] *If $H \in (0, \frac{1}{2})$ and Φ satisfies the condition of (**Hyper-1**), then there is a version of the stochastic convolution ($z_2(t) = \int_0^t S_2(t-s)\Phi dB^H(s), t \in [0,T]$) with $C([0,T];V)$ sample paths.*

Consider the stochastic linear differential equation

$$\begin{cases} dz_1 = A_1 z dt + dB^H, \\ z_1(0) = 0 \in V. \end{cases} \tag{3.8}$$

As noted in [**29**] and [**9**], the stochastic integral $\int_0^t I_d dB^H(s)$ is not well-defined as a V-valued random variable since the identity operator $I_d \notin \mathcal{L}_2(V)$. We then consider the mild form of the equation, whose unique solution, if it exists, can be written in the evolution form

$$z_1(t) = \int_0^t S_1(t-s)dB^H(s). \tag{3.9}$$

LEMMA 3.2. *([**12**]) If $H \in (\frac{1}{4}, \frac{1}{2})$, then the stochastic convolution $z_1(t) := \int_0^t S(t-s)dB^H(s)$ is well defined and the process $(z_1(t), t \in [0,T])$ has a V-continuous modification.*

Noticing that the sample orbit of fractional Brownian motion is not differentiable almost everywhere in the classical sense, so we consider the stochastic convolution in the product space H:

$$\begin{aligned} z(t) = \int_0^t S(t-s)\Phi dB^H(t) &:= \begin{pmatrix} \int_0^t S_1(t-s)dB_1^H(t) \\ \int_0^t S_2(t-s)\Phi_2 dB_2^H(t) \end{pmatrix} \\ &\triangleq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \end{aligned} \tag{3.10}$$

4. Existence and uniqueness of the mild solutions

In this section, we will apply the modified Banach Fixed Point theorem to show the existence and uniqueness of the mild solution for stochastic modified Boussinesq approximation equation (2.1) in the space $E = C([0,T]; H) \cap L^2(0,T; V)$. We first give the definition of mild solution for equation (2.1).

DEFINITION 4.1. *A H-values random process $(\phi(t), t \geq 0)$ on a fixed probability space $(\Omega, \mathscr{F}, P)$ with a given infinite-dimensional fractional Brownian motion is called a mild solution of stochastic equation* (2.1) *if $(\phi(t), t \geq 0)$ satisfies the following equation*

$$\begin{aligned} \phi(t) =& S(t)\phi_0 - \int_0^t S(t-s)B(\phi(s))ds - \int_0^t S(t-s)N(\phi(s))ds \\ & - \int_0^t S(t-s)R(\phi(s))ds + \int_0^t S(t-s)\Phi dB^H(s). \end{aligned} \tag{4.1}$$

where the first three terms are operator-valued Bochner integrals, and the last one is the Wiener-type stochastic integral defined by (3.10).

Denote

$$E_1 = C([0,T]; H_1) \cap L^2(0,T; V_1), \quad E_2 = C([0,T]; H_2) \cap L^2(0,T; V_2).$$

It is easy to verify that space E, E_1 and E_2 are Banach spaces. In order to apply the modified Banach Fixed point theorem, it is necessary to estimate each term of the integral equation (4.1) in space E.

For any $\phi \in E$, denote

$$J_1(\phi) := -\int_0^{\cdot} S(\cdot - s)B(\phi(s))ds, \tag{4.2}$$

$$J_2(\phi) := -\int_0^{\cdot} S(\cdot - s)N(\phi(s))ds. \tag{4.3}$$

$$J_3(\phi) := -\int_0^{\cdot} S(\cdot - s)R(\phi(s))ds \tag{4.4}$$

then operators J_1, J_2 and J_3 possess the following properties:

LEMMA 4.1. *$J_1 : E \to E$, and for any $\phi, \psi \in E$, it follows*

$$\begin{aligned}
&|J_1(\phi)|_E^2 \leq c_1 |\phi|_E^4, \\
&|J_1(\phi) - J_1(\psi)|_E^2 \leq c_2 \Big(|\phi|^2_{C([0,T];H)} \cdot |\phi|^2_{L^2(0,T;V)} \\
&+ |\psi|^2_{C([0,T];H)} \cdot |\psi|^2_{L^2(0,T;V)} \Big)^{\frac{1}{2}} \cdot |\phi - \psi|_E^2.
\end{aligned}$$

PROOF. It follows from Lemma 2.6 in [**2**] that

$$B(\phi) \in L^2(0,T;V'), \quad \forall \phi \in E.$$

So $J_1(\phi)$ is the weak solution of the following linear differential equations

$$\begin{cases} \frac{dJ(t)}{dt} + AJ(t) + B(\phi(t)) = 0, \quad t \in [0,T], \\ J(0) = 0, \end{cases} \tag{4.5}$$

and $J_1 \in C([0,T];H) \cap L^2(0,T;V) = E$, that is, J_1 maps E onto E.

Taking inner product with the first equation in equation(4.5) by J_1, we obtain

$$\frac{1}{2}\frac{d|J_1(t)|^2}{dt} + |J_1(t)|_V^2 = - < B(\phi(t)), J_1(t) > \leq \frac{1}{2}|B(\phi(t))|^2_{V'} + \frac{1}{2}|J_1(t)|_V^2, \tag{4.6}$$

where the sharp bracket $<,>$ denotes dual pairing.

Integrating equation (4.6) over $[0,t]$,

$$|J_1(t)|^2 + \int_0^t |J_1(s)|_V^2 ds \leq \int_0^t |B(\phi(s))|^2_{V'} ds. \tag{4.7}$$

we notice that

$$\begin{aligned}
&\int_0^T |B_1(u,u)|^2_{V_1'} dt \\
&\leq c_1 \int_0^T |u(t)|^2 \cdot |u(t)|^2_{V_1} dt \leq c_1 \cdot |u|^2_{C([0,T];H_1)} \cdot \int_0^T |u(t)|^2_{V_1} dt \\
&\leq \frac{c_1}{2} \left(|u|^4_{C([0,T];H_1)} + |u|^4_{L^2(0,T;V_1)} \right) \leq \frac{c_1}{2} |u|^4_{E_1},
\end{aligned} \tag{4.8}$$

and

$$
\begin{aligned}
&\int_0^T |B_2(u,\theta)|^2_{V_2'}dt \\
(4.9)\qquad &\le c_2\int_0^T |u(t)|\cdot|u(t)|_{V_1}|\theta(t)|\cdot|\theta(t)|_{V_2}dt \\
&\le c_2\cdot|u(t)||\theta(t)|_{C([0,T];H)}\cdot\int_0^T |u(t)|_V|\theta(t)|_{V_2}dt \\
&\le \frac{c_2}{4}\left(|u|^4_{C([0,T];H_1)}+|\theta|^4_{C([0,T];H_2)}+|u|^4_{L^2(0,T;V_1)}+|\theta|^4_{L^2(0,T;V_2)}\right) \\
(4.10)\qquad &\le \frac{c_2}{4}(|u|^4_{E_1}+|\theta|^4_{E_2}),
\end{aligned}
$$

Combining estimation (4.8) with (4.9), we have

$$
\begin{aligned}
\int_0^t |B(\phi(s))|^2_{V'}ds = \int_0^T |B_1(u,u)|^2_{V_1'}dt + \int_0^T |B_2(u,\theta)|^2_{V_2'}dt
&\le \frac{c_2}{4}(|u|^4_{E_1}+|\theta|^4_{E_2})+\frac{c_1}{2}|u|^4_{E_1} \\
&\le c_1(|u|^4_{E_1}+|\theta|^4_{E_2}) \\
&\le c_1(|u|^4_{E_1}+|\theta|^4_{E_2}+(|u|_{E_1}|\theta|_{E_2})^2) \\
&\le c_1|\phi|^4_E.
\end{aligned}
$$

Thus

$$
(4.11)\qquad |J_1|^2_E \le 2\left(|J_1|^2_{C([0,T];H)}+|J_1|^2_{L^2(0,T;V)}\right)\le c_1|\phi|^4_E.
$$

Next, we will prove the second estimation in lemma 4.1. For any $\phi,\psi\in E$, let $w=J_1(\phi)-J_1(\psi)$, then w is the weak solution of the following linear equation

$$
\begin{cases}
\frac{dw(t)}{dt}+Aw(t)+B(\phi(t))-B(\psi(t))=0, \\
w(0)=0.
\end{cases}
$$

Hence,

$$
(4.12)\qquad |w(t)|^2+\int_0^t |w(s)|^2_V ds \le \int_0^t |B(\phi(s))-B(\psi(s))|^2_{V'}ds.
$$

It follows that for any $\varphi\in V$,

(4.13)

$$
\begin{aligned}
&|<B_1(u)-B_1(v),\varphi>| \\
&=|b(u,u,\varphi)-b(v,v,\varphi)| \\
&\le |b(u-v,u,\varphi)|+|b(v,u-v,\varphi)| \\
&\le C\left(|u-v|^{\frac12}|u-v|^{\frac12}_{V_1}\cdot|\varphi|_{V_1}\cdot|u|^{\frac12}|u|^{\frac12}_{V_1}+|v|^{\frac12}|v|^{\frac12}_{V_1}\cdot|\phi|_{V_1}\cdot|u-v|^{\frac12}|u-v|^{\frac12}_{V_1}\right) \\
&=C\left(|u|^{\frac12}|u|^{\frac12}_{V_1}+|v|^{\frac12}|v|^{\frac12}_{V_1}\right)|u-v|^{\frac12}|u-v|^{\frac12}_{V_1}|\varphi|_{V_1},
\end{aligned}
$$

which implies that

$$
(4.14)\qquad |B_1(u)-B_1(v)|_{V_1'}\le c_2\left(|u|^{\frac12}|u|^{\frac12}_{V_1}+|v|^{\frac12}|v|^{\frac12}_{V_1}\right)|u-v|^{\frac12}|u-v|^{\frac12}_{V_1}.
$$

A similar argument applied to operator B_2, we obtain

(4.15)
$$\begin{aligned}
&| < B_2(u,\theta) - B_2(v,\eta), \varphi > | \\
&= |b(u,\theta,\varphi) - b(v,\eta,\varphi)| \\
&\leq |b(u-v,\theta,\varphi)| + |b(v,\theta-\eta,\varphi)| + |b(u-v,\eta,\varphi)| + |b(u,\theta-\eta,\varphi)| \\
&\leq C\left(|u-v|^{\frac{1}{2}}|u-v|_{V_1}^{\frac{1}{2}} \cdot |\varphi|_{V_2} \cdot |\theta|^{\frac{1}{2}}|\theta|_{V_2}^{\frac{1}{2}} + |\eta|^{\frac{1}{2}}|\eta|_{V_2}^{\frac{1}{2}} \cdot |\varphi|_{V_2} \cdot |u-v|^{\frac{1}{2}}|u-v|_{V_1}^{\frac{1}{2}}\right)
\end{aligned}$$

(4.16)
$$\begin{aligned}
&+ \left(|\theta-\eta|^{\frac{1}{2}}|\theta-\eta|_{V_2}^{\frac{1}{2}} \cdot |\varphi|_{V_2} \cdot |u|^{\frac{1}{2}}|u|_{V_1}^{\frac{1}{2}} + |v|^{\frac{1}{2}}|v|_{V_1}^{\frac{1}{2}} \cdot |\varphi|_{V_2} \cdot |\theta-\eta|^{\frac{1}{2}}|\theta-\eta|_{V_2}^{\frac{1}{2}}\right) \\
&= C\Bigg(\left(|u|^{\frac{1}{2}}|u|_{V_1}^{\frac{1}{2}} + |v|^{\frac{1}{2}}|v|_{V_1}^{\frac{1}{2}}\right)|\theta-\eta|^{\frac{1}{2}}|\theta-\eta|_{V_2}^{\frac{1}{2}}|\varphi|_{V_2} \\
&\quad + \left(|\theta|^{\frac{1}{2}}|\theta|_{V_2}^{\frac{1}{2}} + |\eta|^{\frac{1}{2}}|\eta|_{V_2}^{\frac{1}{2}}\right)|u-v|^{\frac{1}{2}}|u-v|_{V_1}^{\frac{1}{2}}|\varphi|_{V_2}\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
|B_2(u,\theta) - B_2(v,\eta)|_{V_2'} \leq C\Bigg(&\left(|u|^{\frac{1}{2}}|u|_{V_1}^{\frac{1}{2}} + |v|^{\frac{1}{2}}|v|_{V_1}^{\frac{1}{2}}\right)|\theta-\eta|^{\frac{1}{2}}|\theta-\eta|_{V_2}^{\frac{1}{2}} \\
&+ \left(|\theta|^{\frac{1}{2}}|\theta|_{V_2}^{\frac{1}{2}} + |\eta|^{\frac{1}{2}}|\eta|_{V_2}^{\frac{1}{2}}\right)|u-v|^{\frac{1}{2}}|u-v|_{V_1}^{\frac{1}{2}}\Bigg).
\end{aligned} \tag{4.17}$$

Finally, direct calculations yields

$$\begin{aligned}
&\int_0^t |B_1(u(s)) - B_1(v(s))|_{V_1'}^2 ds \\
\leq& 2C\int_0^T \Bigg(|u(s)|^{\frac{1}{2}}|u(s)|_{V_1}^{\frac{1}{2}} \\
&+ |v(s)|^{\frac{1}{2}}|v(s)|_{V_1}^{\frac{1}{2}}\Bigg)^2 |u(s)-v(s)| \cdot |u(s)-v(s)|_{V_1} ds \\
\leq& C\Bigg(\int_0^T \left(|u(s)|^{\frac{1}{2}}|u(s)|_{V_1}^{\frac{1}{2}} + |v(s)|^{\frac{1}{2}}|v(s)|_{V_1}^{\frac{1}{2}}\right)^4
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&|u(s)-v(s)|^2 ds + \int_0^T |u(s)-v(s)|_{V_1}^2 ds\Bigg) \\
\leq& C\Bigg(|u-v|_{C([0,T];H_1)}^2 \int_0^T \left(|u(s)|^{\frac{1}{2}}|u(s)|_{V_1}^{\frac{1}{2}} + |v(s)|^{\frac{1}{2}}|v(s)|_{V_1}^{\frac{1}{2}}\right)^4 ds
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&+ |u-v|_{L^2(0,T;V_1)}^2\Bigg) \\
\leq& C\Bigg(4|u-v|_{C([0,T];H_1)}^2 \int_0^T \left(|u(s)|^2|u(s)|_{V_1}^2 + |v(s)|^2|v(s)|_{V_1}^2\right) ds
\end{aligned} \tag{4.20}$$

$$
\begin{aligned}
&+|u-v|^2_{L^2(0,T;V_1)}\Big) \\
\le & C\Big(4|u-v|^2_{C([0,T];H_1)}\left(|u|^2_{C([0,T];H_1)}|u|^2_{L^2(0,T;V_1)}+|v|^2_{C([0,T];H_1)}|v|^2_{L^2(0,T;V_1)}\right) \\
&+|u-v|^2_{L^2(0,T;V_1)}\Big) \\
\le & 2C\Big(|u|^2_{C([0,T];H_1)}|u|^2_{L^2(0,T;V_1)}+|v|^2_{C([0,T];H_1)}|v|^2_{L^2(0,T;V_1)}\Big)^{\frac{1}{2}} \\
&\left(|u-v|^2_{C([0,T];H_1)}+|u-v|^2_{L^2(0,T;V_1)}\right) \\
\le & C\Big(|u|^2_{C([0,T];H_1)}|u|^2_{L^2(0,T;V_1)} \\
&+|v|^2_{C([0,T];H_1)}|v|^2_{L^2(0,T;V_1)}\Big)^{\frac{1}{2}}|u-v|^2_{E_1},
\end{aligned}
\tag{4.21–4.23}
$$

By a similar argument, we have

$$
\begin{aligned}
&\int_0^t |B_2(u(s),\theta(s))-B_2(v(s),\eta(s))|^2_{V_2'}ds \\
\le & C\Bigg(\left(|u|^2_{C([0,T];H_1)}|u|^2_{L^2(0,T;V_1)}+|v|^2_{C([0,T];H_1)}|v|^2_{L^2(0,T;V_1)}\right)^{\frac{1}{2}}|\theta-\eta|^2_{E_2} \\
&+\left(|\theta|^2_{C([0,T];H_2)}|\theta|^2_{L^2(0,T;V_2)}+|\eta|^2_{C([0,T];H_2)}|\eta|^2_{L^2(0,T;V_2)}\right)^{\frac{1}{2}}|u-v|^2_{E_1}\Bigg) \\
\le & C\left(|\phi|^2_{C([0,T];H)}|\phi|^2_{L^2(0,T;V)}+|\psi|^2_{C([0,T];H)}|\psi|^2_{L^2(0,T;V)}\right)^{\frac{1}{2}}(|u-v|^2_{E_1}+|\theta-\eta|^2_{E_2}).
\end{aligned}
\tag{4.24}
$$

Combining the estimation (4.11)-(4.17), we conclude that

$$
\begin{aligned}
&|J_1(\phi)-J_1(\psi)|^2_E \\
\le & 2\sup_{t\in[0,T]}|w(t)|^2+2\int_0^T|w(s)|^2_V ds \\
\le & \int_0^t |B(\phi(s))-B(\psi(s))|^2_{V'}ds \\
\le & \int_0^t |B_1(u(s))-B_1(v(s))|^2_{V_1'}ds+\int_0^t |B_2(u(s),\theta(s))-B_2(v(s),\eta(s))|^2_{V_2'}ds \\
\le & C\left(|\phi|^2_{C([0,T];H)}|\phi|^2_{L^2(0,T;V)}+|\psi|^2_{C([0,T];H)}|\psi|^2_{L^2(0,T;V)}\right)^{\frac{1}{2}}(|u-v|^2_{E_1}+|\theta-\eta|^2_{E_2}) \\
\le & c_2\left(|\phi|^2_{C([0,T];H)}|\phi|^2_{L^2(0,T;V)}+|\psi|^2_{C([0,T];H)}|\psi|^2_{L^2(0,T;V)}\right)^{\frac{1}{2}}|\phi-\psi|^2_E.
\end{aligned}
$$

Thus, the proof is completed. □

LEMMA 4.2. *$J_2 : E \to E$, and for any $\phi, \psi \in E$, it follows*

$$|J_2(\phi)|_E^2 \leq c_3 |\phi|^2_{L^2(0,T;V)}, \tag{4.25}$$

$$|J_2(\phi) - J_2(\phi)|_E^2 \leq c_4 T |\phi - \psi|_E^2. \tag{4.26}$$

PROOF. It follows from Lemma 2.6 in [**2**] that for any $\phi \in E$, $N(\phi) \in L^2(0,T;V)$. Similar to the proof in lemma 4.1, J_2 maps E onto E, and $J_2(\phi)$ is the weak solution of the linear differential equations

$$\begin{cases} \frac{dJ(t)}{dt} + AJ(t) + N(\phi(t)) = 0, \quad t \in [0,T], \\ J(0) = 0, \end{cases}$$

and satisfies

$$|J_2(t)|^2 + \int_0^t |J_2(s)|_V^2 ds \leq \int_0^t |N(\phi(s))|_{V'}^2 ds.$$

Noticing that $|N(\phi)|_{V'} = |N(u)|_{V_1'} \leq C|u|_{V_1}$, then we get

$$|J_2(\phi)|_E^2 \leq 2\int_0^t |N(\phi(s))|_{V'}^2 ds \leq 2C \int_0^t |u(s)|_{V_1}^2 ds = c_3 |u|^2_{L^2(0,T;V_1)} \leq c_3 |\phi|^2_{L^2(0,T;V)}.$$

Next, we will show the inequality (4.26) holds.

For any $\phi, \psi \in E$, denote $w = J_2(\phi) - J_2(\psi)$, then w is the weak solution for the linear differential equation

$$\begin{cases} \frac{dw(t)}{dt} + Aw(t) + N(\phi(t)) - N(\psi(t)) = 0, \\ w(0) = 0. \end{cases}$$

and satisfies

$$|w(t)|^2 + \int_0^t |w(s)|_V^2 ds \leq \int_0^t |N(u(s)) - N(v(s))|_{V_1'}^2 ds.$$

Followed the similar arguments in Lemma 3.1 in [**30**] and the Sobolev interpolation theorem, we have

$$\begin{aligned} < N(u) - N(v), \varphi > \quad &\leq \quad C|e(u-v)| \cdot |\nabla \varphi| \leq C|u-v|_1 \cdot |\varphi|_1 \\ &\leq \quad C|u-v|^{1/2} \cdot |u-v|_2^{1/2} \cdot |\varphi|_1, \end{aligned}$$

Hence, we get

$$\begin{aligned} |J_2(\phi) - J_2(\psi)|_E^2 &\leq 2\int_0^T |N(u(s)) - N(v(s))|_{V_1'}^2 ds \\ &\leq C \int_0^T |u(s) - v(s)| \cdot |u(s) - v(s)|_2 ds \\ &\leq C|u-v|_{C([0,T];H_1)} \cdot |u-v|_{L^1(0,T;V_1)} \\ &\leq CT|u-v|_{C([0,T];H_1)} \cdot |u-v|_{L^2(0,T;V_1)} \\ &\leq c_4 T |u-v|_{E_1}^2 \leq c_4 T |\phi - \psi|_E^2. \end{aligned}$$

Thus, the proof is completed. □

LEMMA 4.3. *$J_3 : E \to E$, and for any $\phi, \psi \in E$, it follows*

$$|J_3(\phi)|_E^2 \leq |\phi|^2_{L^2(0,T;V)}, \tag{4.27}$$

$$|J_3(\phi) - J_3(\psi)|_E^2 \leq T|\phi - \psi|_E^2. \tag{4.28}$$

PROOF. Repeating the similar argument in lemma 4.1, we can prove that J_3 maps E onto E, and $J_3(\phi)$ is the weak solution of the linear differential equation

$$\begin{cases} \frac{dJ(t)}{dt} + AJ(t) + R(\phi(t)) = 0, \quad t \in [0,T], \\ J(0) = 0, \end{cases}$$

and

$$\begin{aligned} |J_3(t)|^2 + \int_0^t |J_3(s)|_V^2 ds &\le \int_0^t |R(\phi(s))|_{V'}^2 ds \\ &\le \int_0^t |\theta|_{V_2}^2 ds = |\theta|^2_{L^2(0,T,V_2)} \le |\phi|^2_{L^2(0,T,V)}. \end{aligned}$$

For any $\phi, \psi \in E$, let $w = J_3(\phi) - J_3(\psi)$, then w is the weak solution for the following differential equation

$$\begin{cases} \frac{dw(t)}{dt} + Aw(t) + R(\phi(t)) - R(\psi(t)) = 0, \\ w(0) = 0, \end{cases}$$

and satisfies

$$\begin{aligned} |w(t)|^2 + \int_0^t |w(s)|_V^2 ds &\le \int_0^t |R(\phi(s)) - R(\psi(s))|^2 ds \\ &\le \int_0^t |\theta - \eta|^2 ds \le T|\theta - \eta|^2_{C([0,T],H_1)} \le T|\phi - \psi|_E^2. \end{aligned}$$

Hence, we have

$$|J_3(\phi) - J_3(\psi)|_E^2 \le T|\phi - \psi|_E^2.$$

Thus, the proof is completed. □

Since the process $z(t), t \in [0,T]$ has a V-valued continuous modification, then, we obtain the existence and uniqueness of the mild solution for stochastic equation (1.3):

THEOREM 4.1. *If $H \in (\frac{1}{4}, \frac{1}{2})$ and* **(Hyper-1)** *holds, then for any initial value $\phi_0 \in H$ and for any $T > 0$, stochastic modified Boussinesq approximation equation* (1.3) *has a unique mild solution in space $C([0,T]; H) \cap L^2(0,T;V)$.*

PROOF. It follows from Lemma 3.1 and 3.2 that $S(\cdot)\phi_0 \in C([0,T];V) \subset E$, and $z \in E$. Hence,

$$|S(\cdot)\phi_0 + z|_E \le |S(\cdot)\phi_0|_E + |z|_E \le 2|\phi_0| + |z|_E. \tag{4.29}$$

Consider the transformation $\mathscr{T} : E \to E$:

$$\mathscr{T}(\phi) = J_1(\phi) + J_2(\phi) + J_3(\phi). \tag{4.30}$$

Then for any $\phi, \psi \in E$, it follows from lemma 4.1, lemma 4.2 and lemma 4.3 that

$$\begin{aligned} &|\mathscr{T}(\phi) - \mathscr{T}(\psi)|_E \\ &\le |J_1(\phi) - J_1(\psi)|_E + |J_2(\phi) - J_2(\psi)|_E + |J_3(\phi) - J_3(\psi)|_E \\ &\le c_2^{\frac{1}{2}} \left(|\phi|^2_{C([0,T];H)} \cdot |\phi|^2_{L^2(0,T;V)} + |\psi|^2_{C([0,T];H)} \cdot |\psi|^2_{L^2(0,T;V)} \right)^{\frac{1}{4}} |\phi - \psi|_E \\ &\quad + (c_4 T)^{\frac{1}{2}} |\phi - \psi|_E + T^{\frac{1}{2}} |\phi - \psi|_E \\ &\le (c_2 M)^{\frac{1}{2}} \left(|\phi|^2_{L^2(0,T;V)} + |\psi|^2_{L^2(0,T;V)} \right)^{\frac{1}{4}} |\phi - \psi|_E + (c_4^{\frac{1}{2}} + 1) T^{\frac{1}{2}} |\phi - \psi|_E, \end{aligned} \tag{4.31}$$

where $M = 4|\phi_0| + 2|z|_E$.

By the absolute continuity of the Bochner integral, we can choose $\tau \in (0,1]$ such that

$$\left(|\phi|^2_{L^2(0,\tau;V)} + |\psi|^2_{L^2(0,\tau;V)}\right)^{\frac{1}{4}} \leq (2Mc_2)^{-\frac{1}{2}}. \tag{4.32}$$

Denote $T_0 = \min\{\tau, 1, 16^{-1}(c_4^{1/2}+1)^2\}$ and $E_{T_0} := C([0,T_0];H) \cap L^2(0,T_0;V)$, then

$$|\mathscr{T}(\phi) - \mathscr{T}(\psi)|_{E_{T_0}} \leq (\frac{1}{4} + \frac{1}{4})|\phi - \psi|_{E_{T_0}} = \frac{1}{2}|\phi - \psi|_{E_{T_0}}. \tag{4.33}$$

Applying the modified Banach Fixed point theorem (Lemma 15.2.6 in [**15**]), the equation

$$\begin{aligned}\phi(t) &= S(t)\phi_0 + z(t) + \mathscr{T}(\phi) \\ &\equiv S(t)\phi_0 + \int_0^t S(t-s)\Phi dB^H(s) - \int_0^t S(t-s)B(\phi(s))ds \\ &\quad - \int_0^t S(t-s)N(\phi(s))ds - \int_0^t S(t-s)R(\phi(s))ds\end{aligned}$$

admits a unique mild solution $u(t)$ in space $C([0,T_0];H) \cap L^2(0,T_0;V)$. Moreover, the solution satisfies the estimate $|\phi|_{E_{T_0}} \leq M$. Thus, the proof is completed. □

Next, we will show the existence of global mild solution for stochastic equation (1.3).

Let ϕ be the local mild solution of stochastic equation (1.3) on $[0,T_0]$, and denote $\psi(t) = \phi(t) - z(t)$, then $\psi(t)$ is the mild solution of the following equation

(4.34)
$$\begin{aligned}\psi(t) =& S(t)\phi_0 - \int_0^t S(t-s)B(\psi(s)+z(s))ds - \int_0^t S(t-s)N(\psi(s)+z(s))ds \\ &- \int_0^t S(t-s)R(\psi(s)+z(s))ds\end{aligned}$$

It is easy to see that $\psi(t)$ is also the weak solution of the following evolution equation with random coefficients:

$$\begin{aligned}&\frac{d}{dt}\psi(t) + A\psi(t) + B(\psi(t)+z(t)) \\ &+N(\psi(t)+z(t)) + R(\psi(s)+z(s)) = 0 \\ &\psi(0) = \phi_0.\end{aligned} \tag{4.35}$$

Followed by the arguments in section 15.3 ([**15**]), we can get a upper boundedness for ψ in the space E.

LEMMA 4.4. *Let ψ be the local solution of the stochastic evolution equation (4.34) on $[0,T]$, then*

$$\sup_{t\in[0,T]} |\psi(t)|^2 \leq e^{c_5\int_0^T |z(s)|_1^2 ds}|\phi_0|^2 + \int_0^T e^{c_5\int_s^T |z(r)|_1^2 dr} h_1(s)ds, \tag{4.36}$$

$$\int_0^T |\psi(t)|_V^2 dt \leq c_6|\phi_0|^2 + c_5c_6 \sup_{t\in[0,T]} |\psi(t)|^2 \int_0^T (|z(s)|_1^2)ds + c_6\int_0^T h_1(s)ds \tag{4.37}$$

where c_5 and c_6 are positive constants which depends on the domain $\mathcal{O}$, integral functions λ_1 and h_1 depend on z.

PROOF. Integrating both sides of equation (4.35) with $\psi(t)$ over $\mathcal{O}$, and applying the facts $< N(\psi), \psi > \geq 0$, $< R(\psi), \psi > \geq 0$ and the orthogonality of the trilinear term b, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d|\psi(t)|^2}{dt} + |\psi(t)|_V^2 \\
&= -b(\psi(t)+z(t), \psi(t)+z(t), \psi(t)) - < N(\psi(t)+z(t)), \psi(t) > \\
&\quad - < R(\psi(t)+z(t)), \psi(t) > \\
&\leq |b(\psi+z(t), z(t), \psi+z(t))| - < N(z(t)), \psi(t) > - < R(z(t)), \psi(t) > .
\end{aligned} \tag{4.38}$$

It follows that for any $r_1 > 0$,

$$\begin{aligned}
&b_1(v+z_1, z_1, v+z_1) \\
&\leq C_1|v+z_1| \cdot |z_1|_1 \cdot |v+z_1|_1 \\
&\leq \frac{C_1}{2C_2}|z_1|_1^2 \cdot |v+z_1|^2 + \frac{C_1C_2}{2}|v+z_1|_1^2 \\
&\leq \frac{C_1}{C_2}|z_1|_1^2 \cdot |v|^2 + C_1C_2|v|_1^2 + \frac{C_1}{C_2}|z_1|^2 \cdot |z_1|_1^2 + C_1C_2|z_1|_1^2,
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
&b_2(v+z_1, z_2, \eta+z_2) \\
&\leq C_1|v+z_1| \cdot |z_2|_1 \cdot |\eta+z_2|_1 \\
&\leq \frac{C_1}{2C_2}|z_2|_1^2 \cdot |v+z_1|^2 + \frac{C_1C_2}{2}|\eta+z_2|_1^2 \\
&\leq \frac{C_1}{C_2}|z_2|_1^2 \cdot |v|^2 + C_1C_2|\eta|_1^2 + \frac{C_1}{C_2}|z_1|^2 \cdot |z_2|_1^2 + C_1C_2|z_2|_1^2,
\end{aligned} \tag{4.40}$$

Hence, combining (4.39) and (4.40), we get

$$\begin{aligned}
&b(\psi(t)+z(t), \psi(t)+z(t), \psi(t)) \\
&\leq b_1(v+z_1, z_1, v+z_1) + b_2(v+z_1, z_2, \eta+z_2) \\
&\leq \frac{C_1}{C_2}|z|_1^2 \cdot |v|^2 + C_1C_2|\psi|_1^2 + \frac{C_1}{C_2}|z_1|^2 \cdot |z|_1^2 + C_1C_2|z|_1^2 \\
&\leq \frac{C_1}{C_2}|z|_1^2 \cdot |\psi|^2 + C_1C_2|\psi|_1^2 + \frac{C_1}{C_2}|z|^2 \cdot |z|_1^2 + C_1C_2|z|_1^2,
\end{aligned} \tag{4.41}$$

Similarly, direct calculations show that

$$- < N(z), \psi > = - < N(z_1), v > \leq \mu_0\epsilon^{-\alpha/2}|z_1|_1 \cdot |v|_1 \leq r_1|\psi|_1^2 + \frac{\mu_0^2}{4r_1\epsilon^\alpha}|z|_1^2, \tag{4.42}$$

and

$$- < R(z), \psi > \leq |(e_2z_2, v)| \leq \frac{\lambda_1^{\frac{1}{2}}}{8}|v|_1^2 + \frac{32}{\lambda_1^{\frac{1}{2}}}|z_2|^2 \leq \frac{\lambda_1^{\frac{1}{2}}}{8}|\psi|_1^2 + \frac{32}{\lambda_1^{\frac{1}{2}}}|z|^2. \tag{4.43}$$

Combining (4.41), (4.42) with (4.43), we obtain

(4.44)
$$\frac{1}{2}\frac{d}{dt}|\psi|^2+\frac{\lambda_1}{2}|\psi|^2+\frac{1}{2}|\psi|_V^2$$
$$\le\frac{C_1}{C_2}|z|_1^2|\psi|^2+(C_1C_2+r_1+\frac{\lambda_1^{\frac{1}{2}}}{4})|\psi|_1^2+(\frac{C_1}{C_2}|z|_1^2+\frac{32}{\lambda_1^{\frac{1}{2}}})|z|^2+C_1C_2|z|_1^2+\frac{\mu_0^2}{4r_1\epsilon^\alpha}|z|_1^2,$$

where C_2 be some positive constant which determined later.

Let
$$h_1=(2\frac{C_1}{C_2}|z|_1^2+\frac{64}{\lambda_1^{\frac{1}{2}}})|z|^2+2C_1C_2|z|_1^2+\frac{\mu_0^2}{2r_1\epsilon^\alpha}|z|_1^2,$$

then
$$\frac{d}{dt}|\psi|^2+\left(\frac{1}{2}-2(\frac{C_1C_2+r_1}{\lambda_1^{\frac{1}{2}}})\right)|\psi|_V^2+\left(\lambda_1-2\frac{C_1|z|_1^2}{C_2}\right)|\psi|^2\le h_1. \tag{4.45}$$

Choosing $C_2<\frac{\lambda_1^{\frac{1}{4}}}{4C_1}$ and let r_1 be small enough such that $C_1C_2+r_1<\frac{\lambda_1^{\frac{1}{2}}}{4}$, then we deduce
$$\frac{d}{dt}|\psi|^2+\left(\lambda_1-2\frac{C_1|z|_1^2}{C_2}\right)|\psi|^2\le h_1. \tag{4.46}$$

Applying Gronwall lemma, we have
$$|\psi(t)|^2\le|\phi(0)|^2e^{-\int_0^t\left(\lambda_1-2\frac{C_1|z(s)|_1^2}{C_2}\right)ds}+\int_0^t h_1(s_1)e^{-\int_{s_1}^t\left(\lambda_1-2\frac{C_1|z(s_2)|_1^2}{C_2}\right)ds_2}ds_1,$$

which implies that
$$\sup_{t\in[0,T]}|\psi(t)|^2\le e^{2\frac{C_1}{C_2}\int_0^T|z(s)|_1^2ds}|\phi_0|^2+\int_0^T e^{2\frac{C_1}{C_2}\int_s^T|z(r)|_1^2dr}h_1(s)ds.$$

Let $c_5=2C_1/C_2$, then the inequality (4.36) holds.

Integrating both sides of equation (4.45) over $(0,T)$, we have

(4.47)
$$|\psi(T)|^2-|\psi(0)|^2+\left(\frac{1}{2}-2\frac{C_1C_2+r_1}{\lambda_1^{\frac{1}{2}}}\right)\int_0^T|\psi(s)|_V^2ds$$
$$\le\int_0^T 2\frac{C_1}{C_2}|z(s)|_1^2|\Phi(s)|^2ds+\int_0^T h_1(s)ds.$$

Let $c_6=\left(\frac{1}{2}-2(C_1C_2+r_1)\lambda_1^{-1/2}\right)^{-1}$, then the inequality (4.37) holds. Thus, we complete the proof of lemma 4.4. □

Based on theorem 4.1 for the existence of local mild solution and lemma 4.4 for the extension of local mild solution, we state the existence of global mild solution for stochastic equation (1.3). In fact, define a stopping time
$$\tau_n=T\wedge inf\{t\in[t_0,T]:\ |\psi(t)|\ge n\}$$
then for some given $\omega\in\Omega$, $\psi(t)$ is bounded on $[t_0,T]$, and $|\psi(t)|<n$ for large enough n, and $\tau_n=T$, which implies that $\tau_n\to T, t\wedge\tau_n\to t$ as $n\to\infty$ and $t\in[t_0,T]$, we replace t in the argument of lemma 4.4 by $t\wedge\tau_n$, we can obtain the following existence of the global mild solution

THEOREM 4.2. *For* $H \in (\frac{1}{4}, \frac{1}{2})$, *and assume the condition* **(Hyper-1)** *hold, then for any* $\phi_0 \in H$ *and* $T > 0$, *then stochastic modified Boussinesq approximate equation* (1.3) *has a unique global mild solution in space* $C([0,T]; H) \cap L^2(0,T;V)$.

5. Existence of random attractor

In this section, we will show the existence of random attractor for random dynamical systems generalized by the mild solution of stochastic equation (1.3). To the end, it suffices to prove the absorbing set in the space $\dot{H}^1$, where

$$\dot{H}^1 = \text{the closure of } \mathscr{V} \text{ in space } \left(H^1(\mathcal{O})\right)^2.$$

Consider the following fractional Ornstein-Uhlenback stationary process:

$$Z(t,\omega) = Z(\theta_t\omega) = \int_{-\infty}^t S(t-r)dB^H(r,\omega).$$

Then Z is a stationary solution of the following linear stochastic evolution equation

$$dZ(t) = AZ(t) + dB^H(t), \quad t \in \mathbb{R}.$$

By lemma 3.1 and 3.2, it suffices to verify the existence of $Z(0)$ in $L^2(\Omega; H^1)$.

For $H \in (\frac{1}{4}, \frac{1}{2})$, direct computation gives

$$\begin{aligned}
&\mathbb{E}\,|Z(0)|_1^2 \\
&= \mathbb{E}\left|\lim_{t\to\infty}\sum_{i=1}^\infty \int_{-t}^0 S(-s)dB^H(s)\right|_1^2 \\
&= \limsup_{t\to\infty}\sum_{i=1}^\infty |S_1(\cdot)\widetilde{e}_i|^2_{I_{T-}^{\frac{1}{2}-H}(L^2(0,t;H^1))} + |S_2(\cdot)\Phi_2\bar{e}_i|^2_{I_{T-}^{\frac{1}{2}-H}(L^2(0,t;H^1))} \\
&= \limsup_{t\to\infty}\sum_{i=1}^\infty \Bigg(|S_1(\cdot)\widetilde{e}_i|^2_{L^2(0,t;H^1)} + \left|D_{T-}^{\frac{1}{2}-H} S_1(\cdot)\widetilde{e}_i\right|^2_{L^2(0,t;H^1)} \\
&\qquad + |S_2(\cdot)\Phi\bar{e}_i|^2_{L^2(0,t;H^1)} + \left|D_{T-}^{\frac{1}{2}-H} S_2(\cdot)\Phi\bar{e}_i\right|^2_{L^2(0,t;H^1)} \Bigg) \\
&\quad = \limsup_{t\to\infty}\sum_{i=1}^\infty \int_0^t \left(\lambda_i^{-\frac{1}{2}} + \lambda_i^{\frac{1}{2}-2H}\right)\cdot e^{-2\lambda_i t}dt \\
&\quad + |\Phi_2\bar{e}_i|^2_{V_2}\int_0^t \left(1 + \hat{\lambda}_1^{1-2H}\right)\cdot e^{-2\hat{\lambda}_1 t}dt \\
&\quad = \frac{1}{2}\limsup_{t\to\infty}\sum_{i=1}^\infty \left(\lambda_i^{-\frac{3}{2}} + \lambda_i^{-\frac{1}{2}-2H}\right)\cdot\left(1 - e^{-2\lambda_i t}\right) \\
&\quad + |\Phi_2|^2_{\mathscr{L}_2(V_2)}\left(\hat{\lambda}_1^{-1} + \hat{\lambda}_1^{-2H}\right)\cdot\left(1 - e^{-2\hat{\lambda}_1 t}\right) \\
&\quad \le \frac{1}{2}\sum_{i=1}^\infty \left(\lambda_i^{-\frac{3}{2}} + \lambda_i^{-\frac{1}{2}-2H}\right) + |\Phi_2|^2_{\mathscr{L}_2(V_2)}(2^{-1} + 2^{-2H}) \\
&\quad \le \beta_D(4H+1)\zeta(4H+1) + \beta_D(3)\zeta(3) - \zeta(8H+2) - \zeta(6) \\
&\quad + |\Phi_2|^2_{\mathscr{L}_2(V_2)}(2^{-1} + 2^{-2H}) \\
&\quad < \infty,
\end{aligned}$$

where $\zeta(\cdot)$ is the Riemann-zeta function.

For the real-valued continuous function $|Z(\theta.\omega)|_1^2$, since $(\Omega, \mathscr{F}, \{\theta(t)\}_{t\in\mathbb{R}})$ is the metric dynamical systems, then the Birkhoff-Chintchin Ergodic theorem implies that

$$\lim_{n\to\pm\infty} \frac{1}{n}\int_0^n |Z(\theta_t\omega)|_1^2 dt = \mathbb{E}|Z(\omega)|_1^2 < \infty. \tag{5.1}$$

Next, we will show that the mild solution of stochastic equation (1.3) can generate a random dynamical systems.

It follows from theorem 4.1 that for any $t_0 \in \mathbb{R}$, $\phi(t,\omega;t_0,\phi_0)$ is the unique mild solution

$$\begin{aligned}\phi(t;t_0) =& S(t)\phi_0 - \int_{t_0}^t S(t-s)B(\phi(s))ds - \int_{t_0}^t S(t-s)N(\phi(s))ds \\ &- \int_{t_0}^t S(t-s)R(\phi(s))ds + \int_{t_0}^t S(t-s)\Phi dB^H(s).\end{aligned}$$

Taking the change of variable $\phi(t,\omega;t_0) = \psi(t,\omega;t_0) + Z(t,\omega)$, we can claim that $\psi(t,\omega;t_0,\phi_0 - Z(\theta_{t_0}\omega))$ is the unique solution of the integral equation

$$\begin{aligned}\psi(t) =& S(t)(\phi_0 - Z(\theta_{t_0}\omega)) - \int_0^t S(t-s)B(\psi(s)+Z(s))ds \\ &- \int_0^t S(t-s)N(\psi(s)+Z(s))ds - \int_0^t S(t-s)R(\psi(s)+Z(s))ds.\end{aligned} \tag{5.2}$$

Thus, ψ is the weak solution of the following evolution equation with random coefficients

$$\begin{cases} \frac{d\psi}{dt} + A(\psi) + B(\psi+Z) + N(\psi+Z) + R(\psi+Z) = 0, \\ \psi(t_0) = \phi_0 - Z(\theta_{t_0}\omega). \end{cases} \tag{5.3}$$

Define a continuous map:

$$\varphi(t,\omega,\phi_0) = \psi(t,\omega;0,\phi_0 - Z(\omega)) + Z(\theta_t\omega), \quad \forall (t,\omega,\phi_0) \in \mathbb{R}\times\Omega\times H. \tag{5.4}$$

It can be verified that the measurability of φ follows from the continuous dependence of the solution with respect to initial value, and the cocycle property follows from the uniqueness of the solution. The solution φ generates a random dynamical systems associate with stochastic equation (1.3).

Next, we will show two important lemmas, which given the existence of the absorbing sets of ϕ and ψ in the space H and H^1 respectively.

For simplicity, we give the notation

- **(Hyper-2)** $\beta_D(4H+1)\zeta(4H+1) + \beta_D(3)\zeta(3) - \zeta(8H+2) - \zeta(6) + |\Phi_2|^2_{\mathscr{L}_2(V_2)}(2^{-1} + 2^{-2H}) < \frac{\lambda_1^{3/2}}{4C_1^2}$.

LEMMA 5.1. *Assume the conditions* **(Hyper-1)** *and* **(Hyper-2)** *are satisfied, then for* $H \in (\frac{1}{4}, \frac{1}{2})$*, there exists a random radii* $\rho_H(\omega)$ *and* $\rho_1(\omega)$ *such that for any* $M > 0$*, there exists* $t_2(\omega) < -1$ *such that for any* $t_0 < t_2$ *and* $|\phi_0| < M$*, the* *following inequalities hold:*

$$|\psi(t,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega))|^2\le\rho_H(\omega),\quad \forall\ t\in[-1,0], \tag{5.5}$$

$$|\phi(t,\omega;t_0,\phi_0)|^2\le\rho_H(\omega),\quad \forall\ t\in[-1,0], \tag{5.6}$$

$$\int_{-1}^0|\psi(t)|_V^2dt\le\rho_1(\omega),\quad \int_{-1}^0|\psi(t)+Z(t)|_V^2dt\le\rho_1(\omega). \tag{5.7}$$

Proof. Firstly, we will prove both $|\phi(t)|^2$ and $|\psi(t)|^2$ are bounded in space H. Similar to the argument technique in lemma 4.4, we have

$$\frac{d}{dt}|\psi|^2+\left(\frac{1}{2}-2(\frac{C_1C_2+r_1}{\lambda_1^{\frac{1}{2}}})\right)|\psi|_V^2+\left(\lambda_1-2\frac{C_1|Z|_1^2}{C_2}\right)|\psi|^2\le h_2, \tag{5.8}$$

where $h_2=(2\frac{C_1}{C_2}|Z|_1^2+\frac{64}{\sqrt{\lambda_1}})|Z|^2+2C_1C_2|Z|_1^2+\frac{\mu_0^2}{2r_1\epsilon^\alpha}|Z|_1^2$.

Choosing

$$C_2\in\big(2C_1(\beta_D(4H+1)\zeta(4H+1)+\beta_D(3)\zeta(3)-\zeta(8H+2)-\zeta(6)$$
$$+|\Phi_2|^2_{\mathscr{L}_2(V_2)}(2^{-1}+2^{-2H}))\lambda_1^{-1},\frac{\sqrt{\lambda_1}}{2C_1}\big),$$

and let r_1 be small enough such that the following inequality holds:

$$\frac{d}{dt}|\psi|^2+\left(\lambda_1-2\frac{C_1|Z|_1^2}{C_2}\right)|\psi|^2\le h_2.$$

By the Gronwall lemma, it follows that for any $t\in[-1,0]$ and $t_0<-1$,

$$\begin{aligned}|\psi(t)|^2\le&|\psi(t_0)|^2e^{-\int_{t_0}^t\left(\lambda_1-2\frac{C_1|Z(s)|_1^2}{C_2}\right)ds}+\int_{t_0}^t h_2(s_1)e^{-\int_{s_1}^t\left(\lambda_1-2\frac{C_1|Z(s_2)|_1^2}{C_2}\right)ds_2}ds_1\\ \le&|\psi(t_0)|^2e^{-\int_{t_0}^0\left(\lambda_1-2\frac{C_1|Z(s)|_1^2}{C_2}\right)ds}+\int_{t_0}^0 h_2(s_1)e^{-\int_{s_1}^0\left(\lambda_1-2\frac{C_1|Z(s_2)|_1^2}{C_2}\right)ds_2}ds_1.\end{aligned}$$

Applying the Ergodic theorem, we have

$$\lim_{t_0\to-\infty}\frac{1}{-t_0}\int_{t_0}^0|Z(s)|_1^2ds=\mathbb{E}|Z(\omega)|_1. \tag{5.9}$$

Let r_2 be small enough such that

$$\begin{aligned}&\frac{C_1}{C_2}\Big[\beta_D(4H+1)\zeta(4H+1)+\beta_D(3)\zeta(3)-\zeta(8H+2)-\zeta(6)\\ &+|\Phi_2|^2_{\mathscr{L}_2(V_2)}(2^{-1}+2^{-2H})\Big]\\ &<\frac{\lambda_1}{2}-\frac{r_2}{2}.\end{aligned} \tag{5.10}$$

Then, there exists $t_1(\omega)<-1$ such that for any $t_0<t_1$ and $t\in[-1,0]$,

$$|\psi(t)|^2\le e^{(1+t_0)r_2}|\phi_0|^2+\int_{t_0}^0e^{(1+t_0)r_2}h_2(s)ds. \tag{5.11}$$

Noticing that h_2 has at most polynomial growth as $t_0\to-\infty$ for P-a.s. $\omega\in\Omega$, we derive,

$$\int_{t_0}^0h_2(s)e^{(1+s)r_2}ds\le\int_{-\infty}^0h_2(s)e^{(1+s)r_2}ds\le\infty,\quad \text{P-a.s..} \tag{5.12}$$

Let $\rho_H = 4\int_{-\infty}^{0} h_2(s)e^{(1+s)r_2}ds + 2\sup_{t\in[-1,0]}|Z(t)|^2$, there exists $t_2(\omega) < t_1(\omega) < -1$ such that for all $|\phi_0| \le M$, $t_0 < t_2$ and $t \in [-1,0]$,

$$|\psi(-1,\omega;t_0,\phi_0 - Z(\theta_{t_0}\omega))|^2 \le 2\int_{-\infty}^{0} h_2(s)e^{(1+s)r_2}ds,$$

and

$$|\phi(-1,\omega;t_0,\phi_0)|^2 \le 2|\psi(-1,\omega;t_0,\phi_0 - Z(\theta_{t_0}\omega))|^2 + 2\sup_{t\in[-1,0]}|Z(t)|^2 \le \rho_H(\omega).$$

Next, we will prove $\int_{-1}^{0}|\psi(t)|_V^2 dt$ and $\int_{-1}^{0}|\phi(t)|_V^2 dt$ are bounded. Integrating equation (5.8) over the interval $[-1,0]$, we obtain

(5.13)

$$|\psi(0)|^2 - |\psi(-1)|^2 + c_4^{-1}\int_{-1}^{0}|\psi(t)|_V^2 dt \le \int_{-1}^{0} h_2(t)dt + \int_{-1}^{0} 2\frac{C_1}{C_2}|Z(t)|_1^2\cdot|\psi(t)|^2 dt.$$

It follows that for $t_0 < t_2$,

(5.14)

$$\int_{-1}^{0}|\psi(t)|_V^2 dt \le c_4\left(\int_{-1}^{0} h_2(t)dt + \frac{C_1\rho_H}{C_2}\int_{-1}^{0}|Z(t)|_1^2 dt + |\psi(-1)|^2\right) \triangleq C(\omega).$$

Similarly,

$$\begin{aligned}\int_{-1}^{0}|\psi(t)+Z(t)|_V^2 dt \le 2c_4\Big(&\int_{-1}^{0} h_2(t)dt + \frac{C_1\rho_H}{C_2}\\ &\int_{-1}^{0}|Z(t)|_1^2 dt + |\psi(-1)|^2\Big) + 2\int_{-1}^{0}|Z(t)|_V^2 dt \triangleq \widetilde{C}(\omega).\end{aligned}$$

Denote $\rho_1(\omega) = \max\{C(\omega), \widetilde{C}(\omega)\}$, thus, the proof is completed. □

LEMMA 5.2. *Assume the conditions* **(Hyper-1)** *and* **(Hyper-2)** *are satisfied, then for $H \in (\frac{1}{4}, \frac{1}{2})$, there exists a random radius $\rho_2(\omega)$ such that for all $M > 0$, $|\phi_0| < M$, $t_0 < t_2$ and $t \in [-\frac{1}{2}, 0]$, such that $t_2(\omega) < -1$ the following inequalities hold in Probability 1*

$$|\psi(t,\omega;t_0,\phi_0 - Z(\theta_{t_0}\omega))|_1^2 \le \rho_2(\omega), \qquad |\phi(t,\omega;t_0,\phi_0)|_1^2 \le \rho_2(\omega). \tag{5.15}$$

PROOF. Multiplying both sides of the first equation in systems (5.3) with $-\triangle v$, and integrating over $\mathcal{O}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi|_1^2 + |\psi|_3^2 \le& |b(\psi+Z,\psi+Z,\triangle\psi)| - < N(\psi+Z), -\triangle\psi >\\ &- < R(\psi(t)+z(t)), -\triangle\psi > .\end{aligned}\tag{5.16}$$

By using Gagliardo-Nirenberg's inequality and Young's inequality, we have

$$\begin{aligned}&|b_1(v+Z_1, v+Z_1, \triangle v)|\\ \le& C|v+Z_1|^{1/2}\cdot|v+Z_1|_2^{1/2}\cdot|v+Z_1|_1\cdot|v|_2\\ \le& C|v+Z_1|^{1/2}\cdot|v+Z_1|_1\cdot|v|_2^{3/2} + C|v+Z_1|^{1/2}\cdot|v+Z_1|_1\cdot|Z_1|_2^{1/2}\cdot|v|_2\\ \le& \frac{\lambda_1}{8}|v|_2^2 + \frac{54C^4}{\lambda_1^3}|v+Z_1|^2\cdot|v+Z_1|_1^4 + \frac{\lambda_1}{8}|v|_2^2 + \frac{2C^2}{\lambda_1}|v+Z_1|\cdot|v+Z_1|_1^2\cdot|Z_1|_2,\end{aligned}$$

and

$$\begin{aligned}
&|b_2(v+Z_1,\eta+Z_2,\triangle\eta)|\\
\leq&C|v+Z_1|^{1/2}\cdot|\eta+Z_2|_2^{1/2}\cdot|\eta+Z_2|_1\cdot|\eta|_2\\
\leq&C|v+Z_1|^{1/2}\cdot|\eta+Z_2|_1\cdot|\eta|_2^{3/2}+C|v+Z_1|^{1/2}\cdot|\eta+Z_2|_1\cdot|Z_2|_2^{1/2}\cdot|\eta|_2\\
\leq&\frac{\lambda_1}{8}|\eta|_2^2+\frac{54C^4}{\lambda_1^3}|v+Z_1|^2\cdot|\eta+Z_2|_1^4+\frac{\lambda_1}{8}|\eta|_2^2+\frac{2C^2}{\lambda_1}|v+Z_1|\cdot|\eta+Z_2|_1^2\cdot|Z_2|_2,
\end{aligned}$$

Hence,

$$\begin{aligned}
&|b(\psi+Z,\psi+Z,\triangle\psi)|\\
\leq&|b_1(v+Z_1,v+Z_1,\triangle v)|+|b_2(v+Z_1,\eta+Z_2,\triangle\eta)|\\
\leq&\frac{\lambda_1}{8}|\psi|_2^2+\frac{54C^4}{\lambda_1^3}|\psi+Z|^2\cdot|\psi+Z|_1^4+\frac{\lambda_1}{8}|\psi|_2^2+\frac{2C^2}{\lambda_1}|\psi+Z|\cdot|\psi+Z|_1^2\cdot|Z|_2,
\end{aligned}$$

Finally, we estimate the following two terms in (5.16)

$$\begin{aligned}
&-<N(\psi+Z),\triangle\psi>=\\
&-<N(v+Z_1),\triangle v>\leq\mu_0\epsilon^{-\frac{\alpha}{2}}\int_D|e_{ij}(v+Z_1)e_{ij}(\triangle v)|dx\\
\leq&\mu_0\epsilon^{-\frac{\alpha}{2}}|v+Z_1|_1\cdot|v|_3\leq\frac{1}{4}|v|_3^2+\frac{\mu_0^2}{\epsilon^\alpha}|v+Z_1|_1^2\leq\frac{1}{4}|\psi|_3^2+\frac{\mu_0^2}{\epsilon^\alpha}|\psi+Z|_1^2,
\end{aligned}\tag{5.17}$$

and

$$\begin{aligned}
&-<R(\psi+Z),\triangle\psi>=-<e_2(\eta+Z_2),\triangle v>\\
&\leq\frac{\lambda_1}{8}|v|_2^2+\frac{2}{\lambda_1}|\eta+Z_2|^2\leq\frac{\lambda_1}{8}|\psi|_2^2+\frac{2}{\lambda_1}|\psi+Z|^2.
\end{aligned}\tag{5.18}$$

Denote

$$h_3(t)=2\bigg(\frac{54C}{\lambda_1^3}|\psi+Z|^2\cdot|Z|_1^4+\frac{2C^2}{\lambda_1}|\psi+Z|\cdot|\psi+Z|_1^2\cdot|Z|_2+\frac{\mu_0^2}{\epsilon^\alpha}|\psi+Z|_1^2+\frac{4}{\lambda_1}|\psi+Z|^2,$$

and

$$h_4(t)=2\left(\frac{54C}{\lambda_1^3}|\psi+Z|^2\cdot|\psi|_1^2\right).$$

Then, the inequality (5.16) can be rewritten as the following inequality

$$\frac{d}{dt}|\psi(t)|_1^2+\frac{\lambda_1}{4}|\psi(t)|_2^2\leq h_3(t)+h_4(t)|\psi(t)|_1^2.$$

Thus,

$$\frac{d}{dt}|\psi(t)|_1^2\leq h_3(t)+h_4(t)|\psi(t)|_1^2.\tag{5.19}$$

By the variation of constant formula, it follows from inequality (5.19) that for any $-1\leq s\leq t\leq 0$,

$$\begin{aligned}
|\psi(t)|_1^2&\leq|\psi(s)|_1^2\cdot e^{\int_s^t h_4(s_1)ds_1}+e^{\int_s^t h_4(s_1)ds_1}\cdot\int_s^t h_3(s_2)e^{-\int_s^{s_2}h_4(s_1)ds_1}ds_2\\
&\leq\left(|\psi(s)|_1^2+\int_{-1}^0 h_3(s_2)ds_2\right)\cdot e^{\int_{-1}^0 h_4(s_1)ds_1}.
\end{aligned}\tag{5.20}$$

Integrating inequality (5.20) with respect to s over the interval $[-1, t]$, we obtain

$$(1+t)|\psi(t)|_1^2 \leq \left(\int_{-1}^{0} |\psi(s)|_1^2 ds + \int_{-1}^{0} h_3(s)ds\right) \cdot e^{\int_{-1}^{0} h_4(s)ds}. \tag{5.21}$$

Recalling that for $\int_{-1}^{0} h_3(s)ds$, $\int_{-1}^{0} h_4(s)ds$ and $\int_{-1}^{0} |\psi(s)|_1^2 ds$ are all bounded $t_0 \to -\infty$. Therefore, for any $t_0 < t_2$ and $t \in [-\frac{1}{2}, 0]$, we have

$$|\psi(t)|_1^2 \leq C(\omega). \tag{5.22}$$

Finally, we will prove the second inequality (5.15) holds. There exists a random radius $\rho_2(\omega)$ such that

$$\begin{aligned} |\phi(t,\omega;t_0,\phi_0)|_1^2 &\leq 2|\psi(t,\omega;t_0,\phi_0) - Z(\theta_{t_0}\omega))|_1^2 + \sup_{t\in[-\frac{1}{2},0]} |Z(t)|_1^2 \\ &\leq \rho_2(\omega), \quad \forall t_0 < t_2, t \in [-\frac{1}{2}, 0]. \end{aligned} \tag{5.23}$$

Especially for $t = 0$, it follows

$$|\phi(0,\omega;t_0,\phi_0)|_1^2 \leq \rho_2(\omega), \quad \forall t_0 < t_2.$$

Thus, the proof is completed. □

Since $\dot{H}^1$ is compactly embedded in H, then it follows from lemma 5.1 and lemma 5.2 that there exists a compact absorbing set in space H. Hence, we can apply theorem 2.1 to obtain the existence of the random attractor for stochastic equation (1.3).

Theorem 5.1. *Assume the conditions* **(Hyper-1)** *and* **(Hyper-2)** *are satisfied, then for* $H \in (\frac{1}{4}, \frac{1}{2})$*, the stochastic modified Boussinesq approximation equation* (1.3) *posses a random attractor.*

Remark 5.1. *If the temperature variable* $\theta = 0$*, then stochastic modified Boussinesq approximation equation reduces to stochastic non-Newtonian fluid driven by infinite dimensional fractional Brownian motion, the authors in* [**11**] *and* [**12**] *studied the regularity of the stochastic convolution, and showed the existence of random attractor for stochastic non-Newtonian fluid, both* $H \in (\frac{1}{4}, \frac{1}{2})$ *and* $(\frac{1}{2}, 1)$ *respectively. By the computation for the eigenvalue of* A_1*, they verified the* **(hyper-2)** *is valid when the stochastic modified Boussinesq approximation equation reduces to stochastic non-Newtonian fluid.*

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References

[1] E. Alòs, O. Mazet, and D. Nualart. Stochastic calculus with respect to Gaussian processes. Ann. Probab., 29(2):766-801, 2001

[2] F. Bloom and W. Hao. Regularization of a non-Newtonian system in an unbounded channel: existence and uniqueness of solutions. Nonlinear Anal,44(3):281-309, 2001.
[3] A. Bouard, A. Debussche. On the stochastic korteweg-de vries equation. Functional Analysis, 154: 215-251,1998.
[4] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang. Stochastic calculus for fractional Brownian motion and applications. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2008.
[5] F. Bloom, Attractors of non-Newtonian fluids. J. Dyn. Diff. Eqs., 7(1):109-140,1995.
[6] H. Crauel, A. Debussche, and F. Flandoli. Random attractors. J. Dyn.Diff. Eqs., 9(2):307-341, 1997.
[7] H. Dijkstra, Nonlinear Physical Oceanography, Kluwer Academic Publishers, Boston, 2000.
[8] J. Duan, H. Gao and B. Schmalfuss, Stochastic Dynamics of a Coupled Atmosphere-Ocean Model, Stochastics and Dynamics 2 , 357-380,2002.
[9] T. Duncan, B. Maslowski, and B. Pasik-Duncan. Semilinear stochastic equations in a Hilbert space with a fractional Brownian motion. SIAM J. Math. Anal., 40(6):2286-2315, 2009.
[10] L. Fang, Two-dimensional Stochastic Navier-Stokes Equations with Fractional Brownian Motions, Random operators and Stochastic Equations, 21, 135-158,2013.
[11] J. Li and J. Huang. Dynamics of stochastic non-Newtonian fluids driven by fractional Brownian motion with Hurst parameter $H \in (1/4, 1/2)$. Appl. Math. Mech,Ed., 34(2),189-208, 2013.
[12] J. Li and J. Huang, Dynamics of 2D Stochastic non-Newtonian fluid driven by fractional Brownian motion. Discrete and Continuous Dynamical Systems-B,17(7),2483-2508,2012.
[13] G. Da Prato and J. Zabczyk. Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge, 1992.
[14] G.Da Prato, A. Debussche, R. Temam, et al., Stochastic Burgers equation. NoDEA, 1: 389-402,1994.
[15] G. Da Prato and J. Zabczyk. Ergodicity for infinite-dimensional systems, Cambridge University Press, Cambridge, 1996.
[16] C. Guo, Random attractors of stochastic modified Boussinesq approximation, Advances in Mathematics, 40(6), 766-768,2011.
[17] B. Guo, G. Lin and Y. Shang, Non-Newtonian Fluids Dynamical Systems(In Chinese). National Defense Industry Press, Beijing, 2006.
[18] B. Guo and Y. Shang, The periodic initial value problem and initial value problem for modified Boussinesq approximation. J.PDE., 15(2):57-71,2002.
[19] R. Hills and H. Roberts, On the motion of a fluid that is incompressible in a generalized sense and it relationship to the Boussinesq approximation. *Stability and Analysis of Cotinuous Media*,1,205-212,1991.
[20] H. Hurst, Long Term Storage Capacity of Reservoirs, Trans. Amer. Soc. Civ. Eng,116, 770-799,1951.
[21] N. Krylov and B. Rozovskii, Stochastic evolution equations. J. Soviet Math.(Russian), 1979, 71-147. Transl. 16: 1233-1277,1981.
[22] M.Garrido-Atenza, Keling Lu and B. Schmalfuss, Unstable invariant manifolds for stochastic PDEs driven by a FBM, J of Differential Equations,248, 1637-1667, 2010.
[23] M.Garrido-Atenza, Keling Lu and B. Schmalfuss, Random dynamical systems for stochastic partial diffusion equations driven by driven by a FBM, Discrete and Continuous Dynamical Systems-B, 14(2),473-493, 2010.
[24] B. Maslowski, and B. Schmalfuss, Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion. *Stochastic Anal. Appl.*, **22**(6), 1577-1607,2004.
[25] D. Nualart, Stochastic integration with respect to fractional Brownian motion and applications, Stochastic Models (Mexico City 2002),Contemp. Math. 336, American Mathematical Society, Providence,339, 2003.
[26] T. Ozgokmen, T. Iliescu, P. Fischer, A. Srinivasan and J. Duan. Large eddy simulation of stratified mixing in two-dimensional dam-break problem in a rectangular enclosed domain. Ocean Modeling 16, 106-140,2007.
[27] M. Padula, Mathematical properties of motion of viscous compressible fluids, in Progress in Theoretical Computational Fluid Mechanics, G.Galidi, J. Malek and J.Necas eds., Pitman Research Notes Series 308, Longman Scientific Technical, Essex,128-173,1994.

[28] V. Pipiras and M. Taqqu. Are classes of deterministic integrands for fractional Brownian motion on an interval complete. Bernoulli, 7(6):873-897,2001.

[29] S. Tindel, C. A. Tudor and F.Viens, Stochastic evolution equations with fractional Brownian motion. Probab. Theory Related Fields, **127**(2), 186-204, 2003.

[30] C. Zhao and S. Zhou. Pullback attractors for a non-autonomous incompressible non-Newtonian fluid. J. Differential Equations, 238(2):394-425, 2007.

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