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Homothetic variant of fractional Sobolev space with application to Navier-Stokes system revisited

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Abstract. This note provides a deeper understanding of the main results obtained in the author's 2007 DPDE paper [**25**].

Contents

1. Introduction

This note is devoted to a further understanding of the results on the so-called Q-spaces on $\mathbb{R}^n$ and the incompressible Navier-Stokes equations on $\mathbb{R}^{1+n}_+ = (0,\infty)\times\mathbb{R}^n$ established in the author's 2007 DPDE paper [**25**].

For $\alpha\in(-\infty,\infty)$, the space Q_α on $\mathbb{R}^n$ is defined as the class of all measurable complex-valued functions f on $\mathbb{R}^n$ with

$$|||f|||_{Q_\alpha} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(r^{2\alpha-n}\iint_{B(x,r)\times B(x,r)}\frac{|f(y)-f(z)|^2}{|y-z|^{n+2\alpha}}\,dydz\right)^{\frac{1}{2}}<\infty. \tag{1.1}$$

Here and henceforth, $B(x,r)\subseteq\mathbb{R}^n$ stands for the open ball centered at x with radius r.

This space exists as a homothetic variant of the fractional Sobolev space $\dot{L}^2_\alpha$ on $\mathbb{R}^n$, where

$$f\in\dot{L}^2_\alpha \Longleftrightarrow \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(y)-f(z)|^2}{|y-z|^{n+2\alpha}}\,dydz<\infty.$$

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According to [**6, 25**], $(Q_\alpha/\mathbb{C}, |||f|||_{Q_\alpha})$ is not only a Banach space, but also affine invariant: if $(\lambda, x_0) \in \mathbb{R}^{1+n}_+$ then

$$\phi(x) = \lambda x + x_0 \Rightarrow |||f \circ \phi|||_{Q_\alpha} = |||f|||_{Q_\alpha}.$$

Interestingly, one has the following structure:

$$Q_\alpha = \begin{cases} BMO \text{ as } \alpha \in (-\infty, 0); \\ (-\Delta)^{-\frac{\alpha}{2}} \mathcal{L}_{2,n-2\alpha} \text{ between } W^{1,n} \textit{ and } BMO \text{ as } \alpha \in (0,1); \\ \mathbb{C} \text{ as } \alpha \in [1, \infty), \end{cases}$$

where $(-\Delta)^{-\alpha/2}$ stands for the $-\alpha/2$-th power of the Laplacian operator, and

$$\begin{cases} f \in \mathcal{L}_{2,n-2\alpha} \iff |||f|||^2_{\mathcal{L}_{2,n-2\alpha}} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{2(\alpha-n)} \iint_{B(x,r)\times B(x,r)} |f(y)-f(z)|^2\, dydz < \infty; \\ f \in W^{1,n} \iff |||f|||^n_{W^{1,n}} = \int_{\mathbb{R}^n} |\nabla f(x)|^n\, dx < \infty; \\ f \in BMO \iff |||f|||^2_{BMO} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{-2n} \iint_{B(x,r)\times B(x,r)} |f(y)-f(z)|^2\, dydz < \infty. \end{cases}$$

As showed in [**25**], the importance of the structure lies in an application of Q_α to treating the existence and uniqueness of the so-called mild solution $u = u(t,x) = (u_1(t,x), ..., u_n(t,x))$ of the normalized incompressible Navier-Stokes system with the pressure function $p = p(t,x)$ and the initial data $a = a(x) = (a_1(x), ..., a_n(x))$ below

$$\text{(1.2)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \text{ on } \mathbb{R}^{1+n}_+; \\ \nabla \cdot u = 0 \text{ on } \mathbb{R}^n; \\ u(0, \cdot) = a(\cdot) \text{ on } \mathbb{R}^n, \end{cases}$$

namely, u solves the integral equation

$$\text{(1.3)} \qquad u(t,x) = e^{t\Delta} a(x) - \int_0^t e^{(t-s)\Delta} P\nabla \cdot (u \otimes u) ds,$$

where

$$\begin{cases} e^{t\Delta} a(x) = (e^{t\Delta} a_1(x), ..., e^{t\Delta} a_n(x)); \\ P = \{P_{jk}\}_{j,k=1,...,n} = \{\delta_{jk} + R_j R_k\}_{j,k=1,...,n}; \\ \delta_{jk} = \text{Kronecker symbol}; \\ R_j = \partial_j (-\Delta)^{-\frac{1}{2}} = \text{Riesz transform}. \end{cases}$$

Even more interestingly, several relevant advances were made in [**21, 19, 12, 8, 18, 20, 15, 16, 17**]. The principal results in these papers have strongly inspired the author to revisit and optimize the main results in [**25**]. The present article is divided into the following two sections between this Introduction and the References at the end:

2. $\{Q^{-1}_\alpha\}_{0\le\alpha<1}$ and its Navier-Stokes equations;
3. $\lim_{\alpha\to 1} Q^{-1}_\alpha$ and its Navier-Stokes equations.

Notation. $U \lesssim V$ or $V \gtrsim U$ stands for $U \le CV$ for a constant $C > 0$ independent of U and V; $U \approx V$ is used for both $U \lesssim V$ and $V \lesssim U$.

2. $\{Q^{-1}_\alpha\}_{0\le\alpha<1}$ and its Navier-Stokes equations

2.1. $\{(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha}\}_{0\le\alpha<1}$ & $\{Q^{-1}_\alpha\}_{0\le\alpha<1}$. As an extension of the John-Nirenberg's BMO-space [**13**], the Q-space Q_α was studied first in [**6**], and then in [**4, 5**]. Among several characterizations of Q_α, the following, as a variant of [**4**, Theorem 3.3] (expanding Fefferman-Stein's basic result for $BMO = (-\Delta)^{-0}\mathcal{L}_{2,n}$ in [**7**]), is of independent interest: given $\alpha \in [0,1)$ and a C^∞ function ψ on $\mathbb{R}^n$ with

$$
(2.1)\qquad \begin{cases} \psi \in L^1; \\ |\psi(x)| \lesssim (1+|x|)^{-(n+1)} \ \text{ for } \ x\in\mathbb{R}^n; \\ \int_{\mathbb{R}^n}\psi(x)dx = 0; \\ \psi_t(x) = t^{-n}\psi(\frac{x}{t}) \ \text{ for } \ (t,x)\in\mathbb{R}^{1+n}_+, \end{cases}
$$

one has:

$$
(2.2)\qquad f\in(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \Longleftrightarrow \sup_{(r,x\in\mathbb{R}^{1+n}_+} r^{2\alpha-n}\int_0^r\Big(\int_{B(x,r)}|f*\psi_t(y)|^2\,dy\Big)t^{-1-2\alpha}\,dt<\infty.
$$

Obviously, $*$ stands for the convolution operating on the space variable and

$$
(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} = \begin{cases} BMO \text{ for } \alpha=0; \\ Q_\alpha \text{ for } \alpha\in(0,1). \end{cases}
$$

Upon choosing four ψ-functions in (2.1)-(2.2), we can get four descriptions of

$$
(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha}
$$

involving the Poisson and heat semi-groups. To see this, denote by $e^{-t\sqrt{-\Delta}}(\cdot,\cdot)$ and $e^{t\Delta}(\cdot,\cdot)$ the Poisson and heat kernels respectively:

$$
\begin{cases} e^{-t\sqrt{-\Delta}}(x,y) = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}t(|x-y|^2+t^2)^{-\frac{n+1}{2}}; \\ e^{t\Delta}(x,y) = (4\pi t)^{-\frac{n}{2}}\exp\big(-\frac{|x-y|^2}{4t}\big). \end{cases}
$$

And, for $\beta\in(-\infty,\infty)$ the notation $(-\Delta)^{\frac{\beta}{2}}f$, determined by the Fourier transform $\widehat{(\cdot)}$: $\widehat{(-\Delta)^{\frac{\beta}{2}}f}(x) = (2\pi|x|)^\beta\hat{f}(x)$, represents the $\beta/2$-th power of the Laplacian

$$
-\Delta f = -\Delta_x f = -\sum_{j=1}^n \partial_j^2 f = -\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}.
$$

Choice 1: If

$$
\begin{cases} \psi_{1,0}(x) = \Big(1+|x|^2-(n+1)\Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}\Big)(1+|x|^2)^{-\frac{n+3}{2}}; \\ (\psi_{1,0})_t(x) = t\partial_t e^{-t\sqrt{-\Delta}}(x,0), \end{cases}
$$

then

$$
f\in(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \Longleftrightarrow \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{2\alpha-n}\int_0^r\Big(\int_{B(x,r)}|\partial_t e^{-t\sqrt{-\Delta}}f(y)|^2\,dy\Big)t^{1-2\alpha}dt<\infty.
$$

Choice 2: If

$$
\begin{cases} \psi_{1,j}(x) = -(n+1)\Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}(1+|x|^2)^{-\frac{n+3}{2}}; \\ (\psi_{1,j})_t(x) = t\partial_j e^{-t\sqrt{-\Delta}}(x,0), \end{cases}
$$

then

$$
f\in(-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \Longleftrightarrow \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{2\alpha-n}\int_0^r\Big(\int_{B(x,r)}|\nabla_y e^{-t\sqrt{-\Delta}}f(y)|^2\,dy\Big)t^{1-2\alpha}dt<\infty,
$$

where ∇_y is the gradient with respect to the space variable $y=(y_1,...,y_n)\in\mathbb{R}^n$.

Choice 3: If

$$
\begin{cases} \psi_{2,0}(x) = -(4\pi)^{-\frac{n}{2}}\Big(n-\frac{|x|^2}{2}\Big)\exp\Big(-\frac{|x|^2}{4}\Big); \\ (\psi_{2,0})_t(x) = t\partial_t e^{t^2\Delta}(x,0), \end{cases}
$$

then

$$f \in (-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \Longleftrightarrow \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{2\alpha-n}\int_0^r \Big(\int_{B(x,r)} |\partial_t e^{t^2\Delta} f(y)|^2\,dy\Big)t^{1-2\alpha}dt < \infty.$$

Choice 4: If

$$\begin{cases}\psi_{2,j}(x) = -(4\pi)^{-\frac{n}{2}}\Big(\frac{x_j}{2}\Big)\exp\Big(-\frac{|x|^2}{4}\Big);\\ (\psi_{2,j})_t(x) = t\partial_j e^{t^2\Delta}(x,0),\end{cases}$$

then

$$f \in (-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \Longleftrightarrow \sup_{(r,x)\in\mathbb{R}^{1+n}_+} r^{2\alpha-n}\int_0^r \Big(\int_{B(x,r)} |\nabla_y e^{t^2\Delta} f(y)|^2\,dy\Big)t^{1-2\alpha}dt < \infty.$$

The previous characterizations lead to the following assertion uniting [**25**, Theorem 1.2 (iii)] and the corresponding result on BMO^{-1} in [**11**].

Theorem 2.1. *For $\alpha \in [0,1)$ let $Q^{-1}_\alpha = ((-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha})^{-1}$ be the class of all functions f on $\mathbb{R}^n$ with*

$$\|f\|_{Q^{-1}_\alpha} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\left(r^{2\alpha-n}\int_0^{r^2}\Big(\int_{B(x,r)} |e^{t\Delta} f(y)|^2\,dy\Big)\,t^{-\alpha}dt\right)^{\frac{1}{2}} < \infty, \tag{2.3}$$

then

$$\nabla\cdot(Q_\alpha)^n = div(Q_\alpha)^n = Q^{-1}_\alpha. \tag{2.4}$$

Consequently,

$$0 \le \alpha_1 < \alpha_2 < 1 \Longrightarrow Q^{-1}_{\alpha_2} \subseteq Q^{-1}_{\alpha_1}. \tag{2.5}$$

Proof. The argument below, taken essentially from the proofs of [**25**, Lemma 2.2 and Theorem 1.2 (ii)], is valid for all $\alpha \in [0,1)$.

Step 1. We prove

$$f_{j,k} = \partial_j\partial_k(-\Delta)^{-1}f \ \ \&\ \ f \in Q^{-1}_\alpha \Longrightarrow f_{j,k} \in Q^{-1}_\alpha \ \text{ for } \ j,k = 1,2,...n.$$

Taking a C^∞_0 function ϕ with

$$\begin{cases}\text{supp}\phi \subset B(0,1);\\ \int_{\mathbb{R}^n}\phi(x)dx = 1;\\ \phi_r(x) = r^{-n}\phi(x/r);\\ g_r(t,x) = \phi_r * \partial_j\partial_k(-\Delta)^{-1}e^{t\Delta}f(x),\end{cases}$$

we get

$$e^{t\Delta}f_{j,k}(x) = \partial_j\partial_k(-\Delta)^{-1}e^{t\Delta}f(x) = f_r(t,x) + g_r(t,x).$$

Upon denoting by $\dot{B}^{1,1}_1$ the predual of the homogeneous Besov space $\dot{B}^{-1,\infty}_\infty$ (consisting of all functions f on $\mathbb{R}^n$ with $\|e^{t\Delta}f\|_{L^\infty} \lesssim t^{-1/2}$), we find (cf. [**14**, p. 160, Lemma 16.1])

$$\begin{aligned}&f \in Q^{-1}_\alpha \Longrightarrow f \in BMO^{-1} \subseteq \dot{B}^{-1,\infty}_\infty\\ &\Longrightarrow \|g_r(t,\cdot)\|_{L^\infty} \le \left\|\partial_j\partial_k(-\Delta)^{-1}e^{t\Delta}f\right\|_{\dot{B}^{-1,\infty}_\infty}\|\phi_r\|_{\dot{B}^{1,1}_1} \lesssim r^{-1}\|f\|_{\dot{B}^{-1,\infty}_\infty},\end{aligned}$$

thereby reaching

$$\int_0^{r^2}\Big(\int_{B(x,r)} |g_r(t,y)|^2\,dy\Big)t^{-\alpha}dt \lesssim r^{n-2\alpha}\|f\|^2_{\dot{B}^{-1,\infty}_\infty} \lesssim r^{n-2\alpha}\|f\|^2_{Q^{-1}_\alpha}. \tag{2.6}$$

Next, taking another C_0^∞ function ψ with $\psi = 1$ on $B(0, 10)$, writing

$$\begin{cases} \psi_{r,x} = \psi(\frac{y-x}{r}); \\ f_r = F_{r,x} + G_{r,x}; \\ G_{r,x} = \partial_j\partial_k(-\Delta)^{-1}\psi_{r,x}e^{t\Delta}f - \phi_r * \partial_j\partial_k(-\Delta)^{-1}\psi_{r,x}e^{t\Delta}f, \end{cases}$$

and employing the Plancherel formula for the space variable, we find out

$$\begin{aligned} \int_0^{r^2} \left\|\partial_j\partial_k(-\Delta)^{-1}\psi_{r,x}e^{t\Delta}f\right\|_{L^2}^2 t^{-\alpha}dt &\lesssim \int_0^{r^2} \Big(\int_{\mathbb{R}^n} \left|y_jy_k|y|^{-2}(\widehat{\psi_{r,x}e^{t\Delta}}f)(y)\right|^2 dy\Big)t^{-\alpha}dt \\ &\lesssim \int_0^{r^2} \left\|\psi_{r,x}e^{t\Delta}f\right\|_{L^2}^2 t^{-\alpha}dt. \end{aligned}$$

At the same time, using Minkowski's inequality (for ϕ_r) and the Plancherel formula once again, we read off

$$\int_0^{r^2} \left\|\phi_r * \partial_j\partial_k(-\Delta)^{-1}\psi_{r,x}e^{t\Delta}f\right\|_{L^2}^2 t^{-\alpha}dt \lesssim \int_0^{r^2} \left\|\psi_{r,x}e^{t\Delta}f\right\|_{L^2}^2 t^{-\alpha}dt.$$

Consequently

$$\int_0^{r^2} \left\|G_{r,x}(t,\cdot)\right\|_{L^2}^2 t^{-\alpha}dt \lesssim \int_0^{r^2} \left\|\psi_{r,x}e^{t\Delta}f\right\|_{L^2}^2 t^{-\alpha}dt.$$

To handle $F_{r,x}$, we apply the following inequality (cf. [**14**, p. 161])

$$\int_{B(x,r)} |F_{r,x}(t,y)|^2\,dy \lesssim r^{n+1} \int_{\mathbb{R}^n\setminus B(x,10r)} |e^{t\Delta}f(w)|^2|x-w|^{-(n+1)}\,dw$$

to obtain

$$\begin{aligned} &\int_0^{r^2} \Big(\int_{B(x,r)} |F_{r,x}(t,y)|^2 dy\Big)t^{-\alpha}dt \\ &\lesssim \sum_{l=1}^{\infty} \int_{B(x,r10^{1+l})\setminus B(x,r10^l)} \frac{\Big(\int_0^{r^2} |e^{t\Delta}f(w)|^2 t^{-\alpha}dt\Big)}{(|w-x|r^{-1})^{n+1}}\,dw \lesssim \|f\|_{Q_\alpha^{-1}}^2 r^{2\alpha-n}. \end{aligned}$$

A combination of the above estimates for $F_{r,x}$ and $G_{r,x}$ yields

$$\int_0^{r^2}\int_{B(x,r)} |f_r(t,y)|^2 t^{-\alpha}dydt \lesssim r^{n-2\alpha}\|f\|_{Q_\alpha^{-1}}^2. \tag{2.7}$$

Of course, both (2.6) and (2.7) produce $f_{j,k} \in Q_\alpha^{-1}$, as desired.

Step 2. We check $\nabla\cdot(Q_\alpha)^n = Q_\alpha^{-1}$.

If $f \in \nabla\cdot(Q_\alpha)^n$, then there exist $f_1, ..., f_n \in Q_\alpha$ such that $f = \sum_{j=1}^n \partial_j f_j$. Thus, an application of the Minkowski inequality derives

$$\|f\|_{Q_\alpha^{-1}} \le \sum_{j=1}^n \left\|\partial_j f_j\right\|_{Q_\alpha^{-1}} \lesssim \sum_{j=1}^n \|f_j\|_{Q_\alpha} < \infty.$$

Conversely, if $f \in Q_\alpha^{-1}$, then an application of *Step 1* derives $f_{j,k} = \partial_j\partial_k(-\Delta)^{-1}f \in Q_\alpha^{-1}$, whence giving $f_k = -\partial_k(-\Delta)^{-1}f \in Q_\alpha$. So,

$$\widehat{\sum_{k=1}^n \partial_k f_k} = -\sum_{k=1}^n \widehat{f_{k,k}} = \hat{f} \Longrightarrow f \in \nabla\cdot(Q_\alpha)^n.$$

Step 3. (2.5) follows immediately from (2.4). □

2.2. Navier-Stokes system initiated in $\{(Q_\alpha^{-1})^n\}_{0\le\alpha<1}$. Classically, the Cauchy problem for (1.2) is to establish the existence of a solution (velocity)

$$u = u(t,x) = \big(u_1(t,x), ..., u_n(t,x)\big)$$

with a pressure $p = p(t,x)$ of the fluid at time $t \in (0,\infty)$ and position $x \in \mathbb{R}^n$ assuming the initial data/velocity $a = a(x) = (a_1(x), ..., a_n(x))$. Of particularly important is the invariance of (1.2) under the scaling transform:

$$\begin{cases} u(t,x) \mapsto u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x); \\ p(t,x) \mapsto p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x); \\ a(x) \mapsto a_\lambda(x) = \lambda a(\lambda x). \end{cases}$$

Namely, if $(u(t,x), p(t,x), a(x))$ solves (1.2) then $(u_\lambda(t,x), p_\lambda(t,x), a_\lambda(x))$ also solves (1.2) for any $\lambda > 0$. This suggests a consideration of (1.2) with an initial data being of the scaling invariance. Through the scale invariance

$$\|a_\lambda\|_{(L^n)^n} = \sum_{j=1}^n \|(a_j)_\lambda\|_{L^n} = \|a\|_{(L^n)^n},$$

Kato proved in [**9**] that (1.2) has mild solutions locally in time if $a \in (L^n)^n$ and globally if $\|a\|_{(L^n)^n}$ is small enough (for some generalizations of Kato's result, see e.g. [**24**] and [**26**]). Note that $\|\cdot\|_{Q_\alpha^{-1}}$ is invariant under the scale transform $a(x) \mapsto \lambda a(\lambda x)$. So it is a natural thing to extend the Kato's results to $\{Q_\alpha^{-1}\}_{0\le\alpha<1}$. To do this, we introduce the following concept whose case with $\alpha = 0$ coincides with the space triple $(BMO_T^{-1}, \overline{VMO}^{-1}, X_T)$ in [**11**].

Definition 2.2. Let $(\alpha, T) \in [0,1) \times (0,\infty]$.
(i) A distribution f on $\mathbb{R}^n$ is said to be in $Q_{\alpha;T}^{-1}$ provided

$$\|f\|_{Q_{\alpha;T}^{-1}} = \sup_{(r,x)\in(0,T)\times\mathbb{R}^n} \left(r^{2\alpha-n} \int_0^{r^2} \int_{B(x,r)} |e^{t\Delta} f(y)|^2 t^{-\alpha}\, dydt\right)^{\frac{1}{2}} < \infty.$$

(ii) A distribution f on $\mathbb{R}^n$ is said to be in $\overline{VQ_\alpha}^{-1}$ provided $\lim_{T\to 0} \|f\|_{Q_{\alpha;T}^{-1}} = 0$.
(iii) A function g on $\mathbb{R}^{1+n}_+$ is said to be in $X_{\alpha;T}$ provided

$$\|g\|_{X_{\alpha,T}} = \sup_{t\in(0,T)} t^{\frac{1}{2}} \|g(t,\cdot)\|_{L^\infty} + \sup_{(r,x)\in(0,T)\times\mathbb{R}^n} \left(r^{2\alpha-n} \int_0^{r^2} \int_{B(x,r)} |g(t,y)|^2 t^{-\alpha} dydt\right)^{\frac{1}{2}} < \infty.$$

Clearly, if $0 \le \alpha_1 \le \alpha_2 < 1$ then $X_{\alpha_2;T} \subseteq X_{\alpha_1;T}$. Moreover, one has:

$$\begin{cases} f_\lambda(x) = \lambda f(\lambda x); \\ g_\lambda(t,x) = \lambda g(\lambda^2 t, \lambda x); \\ (\lambda, t, x) \in (0,\infty)\times(0,\infty)\times\mathbb{R}^n, \end{cases} \Longrightarrow \begin{cases} \|f_\lambda\|_{Q_{\alpha;\infty}^{-1}} = \|f\|_{Q_{\alpha;\infty}^{-1}}; \\ \|g_\lambda\|_{X_{\alpha;\infty}} = \|g\|_{X_{\alpha;\infty}}. \end{cases}$$

Also, recalling (cf. [**3**])

$$f \in \dot{B}_{p,\infty}^{-1+\frac{n}{p}} \text{ under } p > n \Longleftrightarrow \|e^{t\Delta} f\|_{L^p} \lesssim t^{\frac{n-p}{2p}} \text{ for all } t > 0,$$

one has

$$p > n > \alpha p \Longrightarrow L^n \subseteq \dot{B}_{p,\infty}^{-1+\frac{n}{p}} \subseteq Q_{\alpha;\infty}^{-1} = Q_\alpha^{-1},$$

which follows from Hölder's inequality based calculation for $r \in (0,1)$:

$$\int_0^{r^2} \int_{B(x,r)} |e^{t\Delta} f(y)|^2 t^{-\alpha} dydt \lesssim r^{n(1-\frac{2}{p})} \int_0^{r^2} \|e^{t\Delta} f\|_{L^p}^2 t^{-\alpha} dt \lesssim r^{n-2\alpha}.$$

In order to establish the existence and uniqueness of a mild solution of (1.2) with an initial data in $(Q^{-1}_\alpha)^n$, we need two lemmas.

Lemma 2.3. *Given $(\alpha, T) \in [0,1) \times (0,\infty]$ and a function $f(\cdot,\cdot)$ on $\mathbb{R}^{1+n}_+$, let*

$$\mathsf{I}(f,t,x) = \int_0^t e^{(t-s)\Delta}\Delta f(s,x)ds \quad \forall \quad (t,x) \in \mathbb{R}^{1+n}_+.$$

Then

$$\int_0^T \left\|\mathsf{I}(f,t,\cdot)\right\|^2_{L^2} t^{-\alpha}dt \lesssim \int_0^T \left\|f(t,\cdot)\right\|^2_{L^2} t^{-\alpha}dt. \tag{2.8}$$

Proof. This lemma and its proof are basically the same as [**25**, Lemma 3.1] and its argument under $\alpha \in (0,1)$.

It is enough to verify (2.8) for $T = \infty$ thanks to three facts: (i) $\mathsf{I}(f,\cdot,\cdot)$ counts only on the values of f on $(0,t) \times \mathbb{R}^n$; (ii) if $T < \infty$ then one can extend f by letting $f = 0$ on (T,∞); (iii) we can define $f(\cdot) = 0 = \mathsf{I}(f,t,\cdot)$ for $t \in (-\infty, 0)$.

Through defining

$$\kappa(t,x) = \begin{cases} \Delta e^{t\Delta}(x,0) & \text{for } t > 0; \\ 0 & \text{for } t \le 0, \end{cases}$$

we get

$$\mathsf{I}(f,t,x) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} \kappa(t-s, x-y) f(s,y)dyds,$$

whence finding that $\mathsf{I}(f,t,x)$ is actually a convolution operator over $\mathbb{R}^{1+n}$. Due to

$$\widehat{\kappa(t,\cdot)}(\zeta) = \int_{\mathbb{R}^n} \kappa(t,x)\exp(-2\pi i x\cdot\zeta)dx = -(2\pi)^2|\zeta|^2 \exp\big(-(2\pi)^2 t|\zeta|^2\big),$$

we have

$$\begin{aligned}\widehat{\mathsf{I}(f,t,\cdot)}(\zeta) &= \int_{\mathbb{R}^{1+n}} f(s,y)\left(\int_{\mathbb{R}^n}\kappa(t-s,v)\exp(-2\pi i(v+y)\cdot\zeta)dv\right)dyds \\ &= -(2\pi)^2\int_0^t |\zeta|^2 \exp\big(-(2\pi)^2(t-s)|\zeta|^2\big)\widehat{f(s,\cdot)}(\zeta)ds.\end{aligned}$$

This last formula, along with the Fubini theorem and the Plancherel formula, derives

$$\begin{aligned}\int_0^\infty \left\|\mathsf{I}(f,t,\cdot)\right\|^2_{L^2} t^{-\alpha}dt &\le \int_0^\infty \left(\int_{\mathbb{R}^n}\Big(\int_0^t \frac{|\zeta|^2|\widehat{f(s,\cdot)}(\zeta)|}{\exp\left((2\pi)^2(t-s)|\zeta|^2\right)}ds\Big)^2 d\zeta\right)t^{-\alpha}dt \\ &\approx \int_{\mathbb{R}^n}\left(\int_0^\infty \Big(\int_0^\infty \frac{(1_{\{0\le s\le t\}})|\zeta|^2|\widehat{f(s,\cdot)}(\zeta)|}{\exp\left((2\pi)^2(t-s)|\zeta|^2\right)}ds\Big)^2 t^{-\alpha}dt\right)d\zeta.\end{aligned}$$

This indicates that if one can verify

$$\int_0^\infty\left(\int_0^\infty \Big(1_{\{0\le s\le t\}}\Big)\frac{|\zeta|^2|\widehat{f(s,\cdot)}(\zeta)|}{\exp\left((t-s)|\zeta|^2\right)}ds\right)^2 t^{-\alpha}dt \lesssim \int_0^\infty |\widehat{f(t,\cdot)}(\zeta)|^2 t^{-\alpha}dt, \tag{2.9}$$

then the Plancherel formula can be used once again to produce

$$\int_0^\infty \left\|\mathsf{I}(f,t,\cdot)\right\|^2_{L^2} t^{-\alpha}dt \lesssim \int_0^\infty \|f(t,\cdot)\|^2_{L^2} t^{-\alpha}dt,$$

as required.

To prove (2.9), let us rewrite its left side as $\int_0^\infty \left(\int_0^\infty K(s,t)F(s,\zeta)ds\right)^2 dt$, where

$$\begin{cases} F(s,\zeta) = s^{-\frac{\alpha}{2}}|\widehat{f(s,\cdot)}(\zeta)|; \\ K(s,t) = (1_{\{0\le s\le t\}})(\frac{s}{t})^{\frac{\alpha}{2}}|\zeta|^2\exp\big(-(t-s)|\zeta|^2\big). \end{cases}$$

A simple calculation shows

$$\begin{cases} \int_0^\infty K(s,t)ds = |\zeta|^2\int_0^t (\frac{s}{t})^{\frac{\alpha}{2}}\exp(-(t-s)|\zeta|^2)ds \lesssim 1; \\ \int_0^\infty K(s,t)dt = |\zeta|^2\int_s^\infty (\frac{s}{t})^{\frac{\alpha}{2}}\exp(-(t-s)|\zeta|^2)dt \lesssim 1, \end{cases}$$

and then an application of the Schur lemma gives

$$\int_0^\infty \left(\int_0^\infty K(s,t)F(s,\zeta)ds\right)^2 dt \lesssim \int_0^\infty (F(t,\zeta))^2 dt,$$

as desired. □

Lemma 2.4. *Given $\alpha\in[0,1)$ and a function f on $(0,1)\times\mathbb{R}^n$, let*

$$\mathsf{J}(f;\alpha) = \sup_{(r,x)\in(0,1)\times\mathbb{R}^n} r^{2\alpha-n}\int_0^{r^2}\int_{B(x,r)}|f(t,y)|t^{-\alpha}dtdy.$$

Then

$$\int_0^1 \left\|\sqrt{-\Delta}e^{t\Delta}\int_0^t f(s,\cdot)ds\right\|_{L^2}^2 t^{-\alpha}dt \lesssim \mathsf{J}(f;\alpha)\int_0^1 \left\|f(t,\cdot)\right\|_{L^1}t^{-\alpha}dt. \tag{2.10}$$

Proof. This lemma and its argument follow from [**25**, Lemma 3.2] and its proof.

To be short, let $\langle\cdot,\cdot\rangle$ be the inner product in L^2 with respect to the space variable $x\in\mathbb{R}^n$. Then

$$\begin{aligned} \|\cdots\|_{L^2}^2 &= \int_{\mathbb{R}^n}\left|\sqrt{-\Delta}e^{t\Delta}\int_0^t f(s,y)ds\right|^2 dy \\ &= \int_0^t\int_0^t \left\langle \sqrt{-\Delta}e^{t\Delta}f(s,\cdot),\ \sqrt{-\Delta}e^{t\Delta}f(h,\cdot)\right\rangle dsdh. \end{aligned}$$

Consequently

$$\begin{aligned} \int_0^1 \|\cdots\|_{L^2}^2 t^{-\alpha}dt &\lesssim \iint_{0<h<s<1}\left\langle |f(s,\cdot)|, (e^{2\Delta}-e^{2s\Delta})|f(h,\cdot)|\right\rangle s^{-\alpha}dsdh \\ &\lesssim \left(\int_0^1 \|f(s,\cdot)\|_{L^1}\, s^{-\alpha}ds\right)\sup_{s\in(0,1]}\left\|\int_0^s e^{2s\Delta}|f(h,\cdot)|dh\right\|_{L^\infty}. \end{aligned}$$

From [**14**, p. 163] it follows that

$$\sup_{(s,z)\in(0,1]\times\mathbb{R}^n}\int_0^s e^{2s\Delta}|f(h,z)|dh \lesssim \sup_{(r,x)\in(0,1)\times\mathbb{R}^n} r^{-n}\int_0^{r^2}\int_{B(x,r)}|f(s,y)|dyds.$$

and so that

$$\sup_{(s,z)\in(0,1]\times\mathbb{R}^n}\int_0^s e^{2s\Delta}|f(h,z)|dh \lesssim \sup_{(r,x)\in(0,1)\times\mathbb{R}^n} r^{2\alpha-n}\int_0^{r^2}\int_{B(x,r)}|f(s,y)|\, s^{-\alpha}dsdy.$$

This in turn implies

$$\int_0^1 \|\cdots\|_{L^2}^2 t^{-\alpha}dt \lesssim \mathsf{J}(f;\alpha)\int_0^1 \|f(s,\cdot)\|_{L^1}\, s^{-\alpha}ds,$$

whence giving (2.10). □

Below is the so-called existence and uniqueness result for a mild solution to (1.2) established in [**11, 25**].

Theorem 2.5. *Let* $\alpha \in [0,1)$. *Then*
(i) *(1.2) has a unique small global mild solution* u *in* $(X_\alpha)^n$ *for all initial data* a *with* $\nabla\cdot a = 0$ *and* $\|a\|_{(Q_\alpha^{-1})^n}$ *being small.*
(ii) *For any* $T \in (0,\infty)$ *there is an* $\epsilon > 0$ *such that (1.2) has a unique small mild solution* u *in* $(X_{\alpha;T})^n$ *on* $(0,T)\times\mathbb{R}^n$ *when the initial data* a *satisfies* $\nabla\cdot a = 0$ *and* $\|a\|_{(Q_{\alpha;T}^{-1})^n} \le \epsilon$. *Consequently, for all* $a \in (\overline{VQ_\alpha^{-1}})^n$ *with* $\nabla\cdot a = 0$ *there exists a unique small local mild solution* u *in* $(X_{\alpha;T})^n$ *on* $(0,T)\times\mathbb{R}^n$.

Proof. For completeness, we give a proof based on a slight improvement of the argument for [**25**, Theorem 1.4 (i)-(ii)].

Notice that the following estimate for a distribution f on $\mathbb{R}^n$ (cf. [**14**, Lemma 16.1]):

$$\|e^{t\Delta}f\|_{L^\infty} \lesssim t^{-\frac{1+n}{2}} \sup_{x\in\mathbb{R}^n}\int_0^t\int_{B(x,t)} |e^{s\Delta}f(y)|^2\,dyds \quad \forall \quad t\in(0,\infty)$$

implies

$$t^{\frac12}\|e^{t\Delta}f\|_{L^\infty} \lesssim \|f\|_{Q_{0;T}^{-1}} \lesssim \|f\|_{Q_{\alpha;T}^{-1}} \quad \text{for} \quad 0<t<T\le\infty.$$

So, according to the Picard contraction principle (see e.g. [**14**, p. 145, Theorem 15.1]), we know that verifying Theorem 2.5 via the integral equation (1.3) is equivalent to showing that the bilinear operator

$$\mathsf{B}(u,v;t) = \int_0^t e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)\,ds$$

is bounded from $(X_{\alpha;T})^n\times(X_{\alpha;T})^n$ to $(X_{\alpha;T})^n$. Of course, $u\in(X_{\alpha;T})^n$ and $a\in(Q_{\alpha;T}^{-1})^n$ are respectively equipped with the norms:

$$\begin{cases} \|u\|_{(X_{\alpha;T})^n} = \sum_{j=1}^n \|u_j\|_{X_{\alpha;T}}; \\ \|a\|_{(Q_{\alpha;T}^{-1})^n} = \sum_{j=1}^n \|a_j\|_{Q_{\alpha;T}^{-1}}. \end{cases}$$

Step 1. We are about to show L^∞-bound:

$$|\mathsf{B}(u,v;t)| \lesssim t^{-\frac12}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n} \;\;\forall\;\; t\in(0,T). \tag{2.11}$$

Indeed, if $\frac{t}{2}\le s<t$ then

$$\|e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)\|_{L^\infty} \lesssim (t-s)^{-\frac12}\|u\|_{L^\infty}\|v\|_{L^\infty} \lesssim \big(s(t-s)^{\frac12}\big)^{-1}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n}.$$

Meanwhile, if $0<s<\frac{t}{2}$ then

$$\begin{aligned} |e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)| &\lesssim \int_{\mathbb{R}^n} \frac{|u(s,y)||v(s,y)|}{(t^{\frac12}+|x-y|)^{n+1}}dy \\ &\lesssim \sum_{k\in\mathbb{Z}^n}(t^{\frac12}(1+|k|)^{-(n+1)}\int_{x-yt^{\frac12}(k+[0,1]^n)} \frac{|u(s,y)|}{|v(s,y)|^{-1}}dy. \end{aligned}$$

The Cauchy-Schwarz inequality is applied to imply

$$\int_0^t\int_{x-y\in t^{\frac12}(k+[0,1]^n)} |u(s,y)||v(s,y)|dyds \lesssim t^{\frac{n}{2}}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n}.$$

These inequalities in turn derive

$$\begin{aligned}|\mathsf{B}(u,v;t)| &\lesssim \int_0^{\frac{t}{2}} |e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)|ds + \int_{\frac{t}{2}}^{t} |e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)|ds\\ &\lesssim \left(t^{-\frac12} + \int_{\frac{t}{2}}^{t} s^{-1}(t-s)^{-\frac12}ds\right)\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n}\\ &\lesssim t^{-\frac12}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n},\end{aligned}$$

producing (2.11).

Step 2. We are about to prove L^2-bound:

$$r^{2\alpha-n}\int_0^{r^2}\int_{B(x,r)} |\mathsf{B}(u,v;t)|^2 s^{-\alpha}dyds \lesssim \|u\|^2_{(X_{\alpha;T})^n}\|v\|^2_{(X_{\alpha;T})^n}, \tag{2.12}$$

$\forall\ (r^2,x)\in(0,T)\times\mathbb{R}^n$. In fact, if

$$\begin{cases} 1_{r,x} = 1_{B(x,10r)};\\ \mathsf{B}(u,v;t) = \mathsf{B}_1(u,v;t) - \mathsf{B}_2(u,v;t) - \mathsf{B}_3(u,v;t);\\ \mathsf{B}_1(u,v;t) = \int_0^s e^{(s-h)\Delta}P\nabla\cdot((1-1_{r,x})u\otimes v)dh;\\ \mathsf{B}_2(u,v;t) = (-\Delta)^{-\frac12}P\nabla\cdot\int_0^s e^{(s-h)\Delta}\Delta\Big((-\Delta)^{-\frac12}(I-e^{h\Delta})(1_{r,x})u\otimes v\Big)dh;\\ \mathsf{B}_3(u,v;t) = (-\Delta)^{-\frac12}P\nabla\cdot(-\Delta)^{\frac12}e^{s\Delta}\Big(\int_0^s(1_{r,x})u\otimes v)dh\Big);\\ I = \text{the identity operator,}\end{cases}$$

then one has the following consideration under $0<s<r^2$ and $|y-x|<r$.

First, we utilize the Cauchy-Schwarz inequality to get

$$\begin{aligned}|\mathsf{B}_1(u,v;t)| &\lesssim \int_0^s\int_{\mathbb{R}^n\setminus B(x,10r)} \frac{|u(h,z)||v(h,z)|}{((s-h)^{\frac12}+|y-z|)^{n+1}}dzdh\\ &\lesssim \int_0^{r^2}\int_{\mathbb{R}^n\setminus B(x,10r)} |u(h,z)||v(h,z)||x-z|^{-(n+1)}dzdh\\ &\lesssim \frac{\left(\int_0^{r^2}\int_{\mathbb{R}^n\setminus B(x,10r)}|u(h,z)|^2|x-z|^{-(n+1)}dzdh\right)^{\frac12}}{\left(\int_0^{r^2}\int_{\mathbb{R}^n\setminus B(x,10r)}|v(h,z)|^2|x-z|^{-(n+1)}dzdh\right)^{-\frac12}}\\ &\lesssim r^{-1}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n},\end{aligned}$$

whence obtaining

$$\int_0^{r^2}\int_{B(x,r)} |\mathsf{B}_1(u,v;t)|^2 t^{-\alpha}dydt \lesssim r^{n-2\alpha}\|u\|^2_{(X_{\alpha;T})^n}\|v\|^2_{(X_{\alpha;T})^n}.$$

Next, for $\mathsf{B}_2(u,v;t)$ set

$$\mathsf{M}(h,y) = 1_{r,x}(u\otimes v) = 1_{r,x}(y)\big(u(h,y)\otimes v(h,y)\big).$$

From the L^2-boundedness of the Riesz transform and Lemma 2.3 it follows that

$$\int_0^{r^2}\left\|\mathsf{B}_2(u,v;t)\right\|^2_{L^2} t^{-\alpha}dt \lesssim \int_0^{r^2}\left\|\Big((-\Delta)^{-\frac12}(I-e^{s\Delta})\mathsf{M}(s,\cdot)\Big)\right\|^2_{L^2} s^{-\alpha}ds.$$

Note that $\sup_{s\in(0,\infty)} s^{-1}(1-\exp(-s^2)) < \infty$. So, $(-\Delta)^{-\frac{1}{2}}(I - e^{s\Delta})$ is bounded on L^2 with operator norm $\lesssim s^{\frac{1}{2}}$. This fact, along with the Cauchy-Schwarz inequality, implies

$$\int_0^{r^2} \left\|\mathsf{B}_2(u,v;t)\right\|_{L^2}^2 t^{-\alpha}dt \lesssim r^{n-2\alpha}\|u\|^2_{(X_{\alpha;T})^n}\|v\|^2_{(X_{\alpha;T})^n}.$$

In a similar manner, we establish the following estimate for $\mathsf{B}_3(u,v;t)$:

$$\int_0^{r^2} \left\|\mathsf{B}_3(u,v;t)\right\|_{L^2}^2 t^{-\alpha}dt \lesssim r^{4+n-2\alpha}\int_0^1 \left\|(-\Delta)^{\frac{1}{2}}e^{\tau\Delta}\int_0^\tau |\mathsf{M}(r^2\theta, r\cdot)|d\theta\right\|_{L^2}^2 \tau^{-\alpha}d\tau.$$

Note that Lemma 2.4 ensures that if

$$\mathsf{K}(M;\alpha) = \sup_{\rho\in(0,1)} \rho^{-n}\int_0^{\rho^2}\int_{B(x,\rho)} |\mathsf{M}(r^2\theta, rw)|\tau^{-\alpha}dwd\tau$$

then

$$\int_0^1 \left\|(-\Delta)^{\frac{1}{2}}e^{\tau\Delta}\int_0^\tau |\mathsf{M}(r^2\theta, r\cdot)|d\theta\right\|_{L^2}^2 \tau^{-\alpha}d\tau \lesssim \mathsf{K}(M;\alpha)\int_0^1 \left\|\mathsf{M}(r^2\theta, r\cdot)\right\|_{L^1}\theta^{-\alpha}d\theta.$$

So, the easily-verified estimates

$$\begin{cases} \mathsf{K}(M;\alpha) \lesssim r^{-2}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n}; \\ \int_0^1 \|\mathsf{M}(r^2\theta, r\cdot)\|_{L^1}\,\theta^{-\alpha}d\theta \lesssim r^{-2}\|u\|_{(X_{\alpha;T})^n}\|v\|_{(X_{\alpha;T})^n}; \end{cases}$$

derive

$$\int_0^{r^2} \|\mathsf{B}_3(u,v;t)\|_{L^2}^2 t^{-\alpha}dt \lesssim r^{n-2\alpha}\|u\|^2_{(X_{\alpha;T})^n}\|v\|^2_{(X_{\alpha;T})^n}.$$

Putting the estimates for $\{\mathsf{B}_j(u,v)\}_{j=1}^3$ together, we reach (2.12).

Finally, the boundedness of $\mathsf{B}(\cdot,\cdot;t) : (X_{\alpha;T})^n \times (X_{\alpha;T})^n \mapsto (X_{\alpha;T})^n$ follows from both (2.11) and (2.12). Of course, $T=\infty$ and $T\in(0,\infty)$ assure (i) and (ii) respectively. □

3. $\lim_{\alpha\to 1} Q_\alpha^{-1}$ and its Navier-Stokes equations

3.1. $(-\Delta)^{-1/2}L_{2,n-2}$ & $\lim_{\alpha\to1} Q_\alpha^{-1}$**.** A careful observation of the analysis carried out in Section 2 reveals that one cannot take $\alpha = 1$ in those lemmas and theorems. But, upon recalling

$$Q_\alpha = (-\Delta)^{-\alpha/2}\mathcal{L}_{2,n-2\alpha} \quad \forall \quad \alpha\in(0,1),$$

for which the proof given in the first group of estimates on [**25**, p. 234] unfortunately contains five typos and the correct formulation reads as:

$$\begin{aligned} |(\psi_0)_t * f_2(y)| &\lesssim \int_{\mathbb{R}^n\setminus 2B} \frac{t|f(z)-f_{2B}|}{(t+|x-z|)^{n+1}}\,dz \\ &\lesssim \int_{\mathbb{R}^n\setminus 2B} \frac{t|f(z)-f_{2B}|}{|x-z|^{n+1}}\,dz \\ &\lesssim t\sum_{k=1}^\infty \int_{B_k} \frac{|f(z)-f_{2B}|}{|x-z|^{n+1}}\,dz \\ &\lesssim t\sum_{k=1}^\infty (2^k r)^{-(n+1)}\int_{B_k} |f-f_{2B}|\,dz \\ &\lesssim tr^{-(1+\alpha)}\|f\|_{\mathcal{L}_{2,n-2\alpha}}, \end{aligned}$$

and considering the limiting process of (2.3) as $\alpha \to 1$ via the fact that $(1-\alpha)t^{-\alpha}dt$ converges weak-$*$ as $\alpha \to 1$ to the point-mass at 0 but also $\int_{B(x,r)} |e^{t\Delta}f(y)|^2dy$ approaches $\int_{B(x,r)} |f(y)|^2dy$ as $t \to 0$, in Theorem 2.1, (2.3) and Definition 2.2 we can naturally define the limiting space $\lim_{\alpha\to 1} Q_\alpha^{-1}$ as the square Morrey space $L_{2,n-2}$ (cf. [**21**]) - the class of all L^2_{loc}-functions f with

$$\|f\|_{L_{2,n-2}} = \sup_{(r,x)\in\mathbb{R}^{1+n}_+} \left(r^{2-n}\int_{B(x,r)} |f(y)|^2\,dy\right)^{\frac{1}{2}} < \infty. \tag{3.1}$$

In the light of (3.1) and a result on the Riesz operator $(-\Delta)^{-1/2}$ acting on the square Morrey space in [**1**], we have

$$\begin{cases} (-\Delta)^{-1/2}L_{2,n-2} \subseteq BMO; \\ L_{2,n-2} \subseteq BMO^{-1}; \\ f_\lambda(x) = \lambda f(\lambda x) \ \forall \ (\lambda, x) \in \mathbb{R}^{1+n}_+; \\ \|f_\lambda\|_{L_{2,n-2}} = \|f\|_{L_{2,n-2}} \ \forall \ \lambda \in (0,\infty). \end{cases}$$

Here it is worth pointing out that $(-\Delta)^{-1/2}L_{2,n-2}$ is also affine invariant under the norm

$$\|f\|_{(-\Delta)^{-1/2}L_{2,n-2}} = \|(-\Delta)^{\frac{1}{2}}f\|_{L_{2,n-2}}.$$

To see this, note that

$$f \in (-\Delta)^{-1/2}L_{2,n-2} \Longleftrightarrow f(x) = \int_{\mathbb{R}^n} g(y)|y-x|^{1-n}\,dy \ \text{ for some } \ g \in L_{2,n-2}.$$

So, a simple computation gives

$$\begin{cases} f(\lambda x + x_0) = \int_{\mathbb{R}^n} G_\lambda(y)|y-x|^{1-n}\,dy \ \text{ with} \\ G_\lambda(x) = \lambda g(\lambda x + x_0)\ \&\ \|G_\lambda\|_{L_{2,n-2}} = \|g\|_{L_{2,n-2}}. \end{cases}$$

The following assertion supports the above limiting process.

Theorem 3.1. $(-\Delta)^{-\frac{1}{2}}L_{2,n-2} \subseteq \cap_{\alpha\in(0,1)}Q_\alpha$ & $L_{2,n-2} \subseteq \cap_{\alpha\in(0,1)}Q_\alpha^{-1}$.

Proof. Given $\alpha \in (0,1)$. For $f \in (-\Delta)^{-\frac{1}{2}}L_{2,n-2} \subseteq BMO$, $j \in \mathbb{Z}$ and a Schwartz function ψ, let

$$\begin{cases} f = (-\Delta)^{-\frac{1}{2}}g; \\ \psi_j(x) = 2^{jn}\psi(2^jx); \\ \Delta_j(f)(x) = \psi_j * f(x); \\ \widehat{\Delta'_j(f)}(x) = |2^jx|^\alpha\hat{\psi}(2^{-j}x)\hat{f}(x); \\ \mathrm{supp}\hat{\psi} \subset \{y \in \mathbb{R}^n :\ 2^{-1} \le |y| \le 2\}; \\ \sum_j \hat{\psi}_j \equiv 1. \end{cases}$$

A simple computation gives that for any cube I (whose edges are parallel to the coordinate axes) in $\mathbb{R}^n$ with side length $\ell(I)$,

$$\ell(I)^{2\alpha-n}\iint_{I\times I} |f(x)-f(y)|^2|x-y|^{-(n+2\alpha)}\,dxdy \lesssim T_1(I) + T_2(I), \tag{3.2}$$

where

$$\begin{cases} T_1(I) = \ell(I)^{2\alpha-n}\iint_{I\times I} |\sum_{j<-\log_2\ell(I)}\Delta_j(f)(x) - \sum_{j<-\log_2\ell(I)}\Delta_j(f)(y)|^2\||x-y|^{-(n+2\alpha)}\,dxdy; \\ T_2(I) = \ell(I)^{2\alpha-n}\iint_{I\times I} |\sum_{j\ge-\log_2\ell(I)}\Delta_j(f)(x) - \sum_{j\ge-\log_2\ell(I)}\Delta_j(f)(y)|^2\||x-y|^{-(n+2\alpha)}\,dxdy. \end{cases}$$

According to [**19**, (3.2)] and the last estimate for $\mathbb{IV}$ in [**19**] as well as [**2**, (22)], we get

$$
\begin{cases} \sup_I T_1(I) \lesssim |||f|||_{BMO}^2 \sup_I \ell(I)^{2\alpha-n-2} \int_I \int_I |x-y|^{2-2\alpha-n}\, dxdy \lesssim \|g\|^2_{L_{2,n-2}}; \\ \sup_I T_2(I) \lesssim \sup_I \ell(I)^{2\alpha-n} \sum_{j\ge -\log_2 \ell(I)} 2^{2\alpha j} \|(-\Delta)^{\frac12}\Delta'_j g\|^2_{L^2(I)} \lesssim \|g\|^2_{L_{2,n-2}}. \end{cases} \tag{3.3}
$$

Each $\sup_I$ in (3.3) ranges over all cubes I with edges being parallel to the coordinate axes. Thus, $f \in Q_\alpha$ follows from (3.2) and (3.3) as well as (1.1). This shows the first inclusion of Theorem 3.1.

Next, suppose $f \in L_{2,n-2}$. Then the easily-verified uniform boundedness of the map $f \mapsto e^{t\Delta} f$ on $L_{2,n-2}$, i.e.,

$$
\sup_{t\in(0,\infty)} \|e^{t\Delta} f\|_{L_{2,n-2}} \lesssim \|f\|_{L_{2,n-2}},
$$

yields

$$
r^{2\alpha-n} \int_0^{r^2} \Big(\int_{B(x,r)} |e^{s\Delta} f|^2\, dy \Big)\, s^{-\alpha} ds \lesssim r^{2(\alpha-1)} \int_0^{r^2} \|e^{s\Delta} f\|^2_{L_{2,n-2}}\, s^{-\alpha} ds \lesssim \|f\|^2_{L_{2,n-2}},
$$

whence giving $f \in Q_\alpha^{-1}$ and verifying the second inclusion of Theorem 3.1.

□

3.2. Navier-Stokes equations initiated in $(\lim_{\alpha\to1} Q_\alpha^{-1})^n$**.** When applying ∇ to

$$
((-\Delta)^{-1/2} L_{2,n-2})^n
$$

or

$$
((-\Delta)^{-1/2} \lim_{\alpha\to1} Q_\alpha^{-1})^n
$$

(cf. Theorem 2.1), we are suggested to consider $L_{2,n-2}$ in a further study of (1.2). To see this clearly, let us introduce the following definition.

Definition 3.2.
(i) A function $f \in L_{2,n-2}$ is said to be in $VL_{2,n-2}$ provided that for any $\epsilon > 0$ there is a C_0^∞ function h such that $\|f-h\|_{L_{2,n-2}} < \epsilon$, namely, $VL_{2,n-2}$ is the closure of C_0^∞ in $L_{2,n-2}$.
(ii) Given $T \in (0,\infty]$, a function $g \in L^2_{loc}((0,T)\times\mathbb{R}^n)$ is said to be in $X_{2,n-2;T}$ provided

$$
\|g\|_{X_{2,n-2;T}} = \sup_{t\in(0,T)} t^{\frac12} \|g(t,\cdot)\|_{L^\infty} + \sup_{t\in(0,T)} \|g(t,\cdot)\|_{L_{2,n-2}} < \infty.
$$

Related to Theorem 3.1 is the following inclusion $X_{2,n-2;T} \subseteq \cap_{\alpha\in(0,1)} X_{\alpha;T}$ which follows from

$$
\int_0^{r^2} \int_{B(x,r)} |g(t,y)|^2\, t^{-\alpha} dydt \lesssim r^{n-2} \|g(t,\cdot)\|^2_{L_{2,n-2}} \int_0^{r^2} t^{-\alpha}\, dt \lesssim r^{n-2\alpha}.
$$

As a limiting case $\alpha \to 1$ of Theorem 2.5, we have the following generalization of the 3D result [**15**, Theorem 1 (A)-(B)] (cf. [**10, 16**]) on the existence of a mild solution to (1.2) under $a = (a_1, ..., a_n) \in (L_{2,n-2})^n$ and $\|a\|_{(L_{2,n-2})^n} = \sum_{j=1}^n \|a_j\|_{L_{2,n-2}}$.

Theorem 3.3.
(i) *(1.2) has a small global mild solution u in $(X_{2,n-2;\infty})^n$ for all initial data $a = (a_1, ..., a_n)$ with $\nabla \cdot a = 0$ and $\|a\|_{(L_{2,n-2})^n}$ being small.*
(ii) *For any $a = (a_1, ..., a_n) \in (VL_{2,n-2})^n$ with $\nabla \cdot a = 0$ there exists a $T > 0$ depending on a such that (1.2) has a small local mild solution u in $C([0,T], (L_{2,n-2})^n)$.*

Proof. To prove this assertion, for $T \in (0, \infty]$ let us introduce the following middle space $X_{4,2;T}$ of all functions g on $\mathbb{R}^{1+n}_{+}$ with

$$\|g\|_{X_{4,2;T}} = \sup_{t\in(0,T)} t^{\frac{1}{2}}\|g(t,\cdot)\|_{L^\infty} + \sup_{t\in(0,T)} t^{\frac{1}{4}}\|g(t,\cdot)\|_{L_{4,n-2}} < \infty,$$

where

$$\|g(t,\cdot)\|_{L_{4,n-2}} = \left(\sup_{(r,x)\in\mathbb{R}^{1+n}_{+}} r^{2-n}\int_{B(x,r)} |g(t,y)|^4\,dy\right)^{\frac{1}{4}}.$$

Note that the following estimate for $f \in L_{2,n-2}$ (cf. [**14**, Theorem 18.1]):

$$|e^{t\Delta}f(x)| \lesssim \sum_{k\in\mathbb{Z}^n} \sup_{z\in k+[0,1]^n} \exp\Big(-\frac{|z|^2}{4}\Big)\int_{k+[0,1]^n} |f(x - t^{\frac{1}{2}}y)|\,dy \quad \forall \quad (t,x) \in \mathbb{R}^{1+n}_{+}, \tag{3.4}$$

along with the Cauchy-Schwarz inequality, deduces $t^{\frac{1}{2}}\|e^{t\Delta}f\|_{L^\infty} \lesssim \|f\|_{L_{2,n-2}}$. So, (3.4), plus the uniform boundedness of the map $f \mapsto e^{t\Delta}f$ on $L_{2,n-2}$, gives

$$\|e^{t\Delta}f\|_{L_{4,n-2}} \lesssim \|e^{t\Delta}f\|^{\frac{1}{2}}_{L^\infty}\|e^{t\Delta}f\|^{\frac{1}{2}}_{L_{2,n-2}} \lesssim t^{-\frac{1}{4}}\|f\|_{L_{2,n-2}}.$$

Thus

$$\begin{cases} \|e^{t\Delta}f(x)\|_{X_{4,2;T}} \lesssim \|f\|_{L_{2,n-2}}; \\ \lim_{T\to 0}\|e^{t\Delta}f(x)\|_{X_{4,2;T}} = 0 \quad \text{as} \quad f \in VL_{2,n-2}. \end{cases}$$

Keeping the previous preparation and the Picard contraction principle in mind, we find that showing Theorem 2.1, via the integral equation (1.3) and the iteration process

$$\begin{cases} u^{(0)}(t,\cdot) = e^{t\Delta}a(\cdot); \\ u^{(j+1)}(t,\cdot) = u^{(0)}(t,\cdot) - \mathsf{B}(u^{(j)}(t,\cdot), u^{(j)}(t,\cdot), t); \\ j = 0, 1, 2, 3, ..., \end{cases}$$

amounts to proving the boundedness of the bilinear operator $\mathsf{B}(\cdot,\cdot,t) : (X_{4,2;T})^n \times (X_{4,2;T})^n \mapsto (X_{4,2;T})^n$. However, this boundedness follows directly from the following estimates (cf. [**15**, (25)-(24)]) for $0 < s < t < T$:

$$\begin{cases} \frac{\|e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)\|_{(L^\infty)^n}}{(t-s)^{-\frac{1}{2}}} \lesssim \min\Big\{\big(s(t-s)\big)^{\frac{1}{2}} s^{\frac{1}{4}}\|u\|_{(L_{4,n-2})^n} s^{\frac{1}{4}}\|v\|_{(L_{4,n-2})^n}\big(s(t-s)\big)^{\frac{1}{2}}, \\ s^{-1}s^{\frac{1}{2}}\|u\|_{(L^\infty)^n} s^{\frac{1}{2}}\|v\|_{(L^\infty)^n}\Big\}; \\ \frac{\|e^{(t-s)\Delta}P\nabla\cdot(u\otimes v)\|_{(L_{4,n-2})^n}}{(t-s)^{-\frac{1}{2}}} \lesssim s^{-\frac{3}{4}}\big(s^{\frac{1}{4}}\|u\|_{(L_{4,n-2})^n}\big)\big(s^{\frac{1}{2}}\|v\|_{(L^\infty)^n}\big). \end{cases}$$

□

Remark 3.4. *Though Theorem 2.5 can be used to derive that if* $\|a\|_{(L_{2,n-2})^n}$ *is sufficiently small then there is a unique solution* u *of (1.2) in* $(X_{\alpha;\infty})^n$*, Theorem 2.5 cannot guarantee* $u \in (X_{2,n-2,\infty})^n$ *due to* $X_{2,n-2,\infty} \subseteq \cap_{0<\alpha<1} X_{\alpha;\infty}$*. In any event, we always have* $\sup_{t\in(0,\infty)} t^{\frac{1}{2}}\|u(t,\cdot)\|_{L^\infty} < \infty$ *and even more general estimate (cf.* [**15**, (49) & Lemma 3]*):* $\sup_{t\in(0,\infty)} t^{\frac{1}{2}}\|u^{(j+1)}(t,\cdot) - u^{(j)}(t,\cdot)\|_{L^\infty} \lesssim (j+1)^{-2}$.

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