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Regularity of Attractor for 3D Derivative Ginzburg-Landau Equation

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Abstract. In this paper, we are concerned with a three-dimensional derivative Ginzburg-Landau equation with a periodic initial value condition. The smoothing property of the solution is established by a uniform priori estimates. The existence of the global attractors, $\mathcal{A}_i \subset H_p^i(\Omega)$ $(i = 2, 3, \cdots)$, for the semi-group $\{S^{(i)}(t)\}_{t\geq 0}$ of the operators generated by the equation is proved by using the compactness principle. Finally, the regularity of the global attractors, namely, $\mathcal{A}_2 = \mathcal{A}_3 = \cdots = \mathcal{A}_m$, is proved by using the method of semi-group decomposition.

Contents

1. Introduction

Many physical and chemical phenomena can be described by nonlinear equations, such as the Korteweg-de Vries equation for propagated waves on shallow water surfaces [**1**] and the Ginzburg-Landau equation for a phase transition in superconductivity [**2**]. In applied mathematics and theoretical physics, the description of spatial pattern formation or chaotic dynamics in continuum systems, in particular biological systems or fluid dynamical systems, has been a challenging task. The

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mathematical theory behind these systems appears rich and interesting, and in the broad sense, is a topic which continuously attracts considerable attention from a variety of scientific fields. Due to the complexity of the corresponding nonlinear evolution equations, simpler model equations for which the mathematical issues can be solved with greater success, have been derived. The complex Ginzburg-Landau equation (GLE) is one of these models which takes the form

$$u_t - (1+\mathrm{i}\nu)\triangle u + (1+\mathrm{i}\mu)|u|^{2\sigma}u - \gamma u = 0. \tag{1.1}$$

This equation describes the evolution of the amplitude of perturbations to steady-state solutions at the onset of instability [**3, 4**]. In the past decades, this equation was widely studied for instability waves in hydrodynamics, such as the nonlinear growth of Rayleigh-Bénard convective rolls [**5**], the appearance of Taylor vortices in the Couette flow between counter-rotating cylinders [**6**], the development of Tollmien-Schlichting waves in plane Poiseuille flows [**7**], and the transition to turbulence in chemical reactions [**8**],

The derivative Ginzburg-Landau equation (DGLE)

$$u_t = \rho u + (1+\mathrm{i}\nu)\triangle u - (1+\mathrm{i}\mu)|u|^{2\sigma}u + \lambda_1 \cdot \nabla(|u|^2 u) + (\lambda_2 \cdot \nabla u)|u|^2 \tag{1.2}$$

arises as the envelope equation for a weakly subcritical to counter-propagating waves, and it is also important for a number of physical systems including the onset of oscillatory convection in binary fluid mixture [**9**]. In the case of one or two dimensions, finite dimensional global attractors and regularity of solutions were explored in [**10, 11**]. When $\nu = 0$, the equation (1.2) incorporates to the derivative nonlinear Schrödinger equation [**12**].

In the past decades, equations (1.1) and (1.2) have been extensively studied in the one or two spatial dimension. For example, Ghidaglia and Héron [**13**], Doering et al [**14**], Promislow [**15**], Bu [**16**] studied the finite-dimensional attractor and related dynamical properties for 1D or 2D GLE (1.1) with $\sigma = 1$ or 2. Lü [**17**] investigated the upper semi-continuity of approximate attractors of GLE (1.1) in one-dimensional space with $\sigma = 1$. Guo et al [**18, 19**] and Gao [**20, 21, 22**] dealt with the 1D and 2D DGLE (1.2) and explored the existence of the global solution and the finite-dimensional global attractor of DGLE (1.2) with periodic boundary conditions, Cauchy conditions or Dirichlet inhomogeneous boundary value conditions in the case of $\sigma = 2$.

Nevertheless, relevant theoretical results for the case of three spatial dimensions for the Ginzburg-Landau equation appear scarce. The main reason lies in the fact that the Sobolev interpolation inequalities used in one- or two-dimensional case become invalid for the three-dimensional case. Thus, it is necessary to make more subtle estimates for the nonlinear terms to overcome this difficulty. Doering et al [**23**] and Okazawa et al [**24**] considered the case of three-dimensional space for GLE (1.1) with periodic boundary conditions and initial boundary value conditions, respectively. They established the existence and uniqueness of global solution under certain parametric conditions. In [**25, 26, 27**], Lü et al also discussed the periodic initial-value problem of GLE (1.1) in three-dimensional space, and proved the existence and uniqueness of global solution under a weaker restriction of parametric conditions. Furthermore, the existence of global attractor and exponential attractor with finite dimensions as well as the upper semi-continuity of global attractor was also explored. Nader and Hatem [**28**] constructed a solution to DGLE (1.2) in N-dimensional space. Karachlios and Zographoppulos [**29**] investigated a degenerate case of DGLE (1.2) with the Dirichlet boundary value condition in N-dimensional space ($N \geq 2$) and proved the existence of the global attractor in L^2.

In this paper, we consider a periodic initial-value problem of a more general derivative 3D Ginzburg-Landau equation:

$$u_t = (1 + \mathrm{i}\nu)\triangle u - (1 + \mathrm{i}\mu)|u|^{2\sigma}u + \gamma u - \tag{1.3}$$

$$\lambda_1 \cdot |u|^2\nabla u - \lambda_2 \cdot u^2\nabla\bar{u}, \ (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \tag{1.4}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^3, \tag{1.5}$$

$$u(x + e_j, t) = u(x,t), \qquad j = 1,\, 2,\, 3, \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+, \tag{1.6}$$

where ν, μ and $\gamma > 0$ are real constants, λ_1 and λ_2 are complex constant vectors, and the over-line denotes the complex conjugate. We assume that the parameters μ and σ satisfy the condition

$$|\mu| < \frac{\sqrt{2\sigma + 1}}{\sigma}, \qquad \sigma > 2. \tag{1.7}$$

The existence of the unique solution, the finite-dimensional global attractor and the exponential attractor was investigated under the assumption (1.7) in [**30**] and the upper semi-continuity of global attractor was described in [**31**], but the regularity of global attractor was not discussed therein. Since regularity and global attractor are two of the most important qualities for dynamical systems and any possible regularity of the attractor is extremely helpful for a better understanding of the long term behavior of the semigroup, in this paper our main purpose is to present some regularity results on the global attractor for the problem (1.4)–(1.6).

On the assumption that the dissipative dynamical system associated with a partial differential equation possesses an global attractor $\mathcal{A}$ in the Sobolev space, say $H^2(\Omega)$, the regularity is to be understood here in the sense of the theory of partial differential equations. That is, if the data are sufficiently regular, then the global attractor lies in a set of more (spatially) regular functions, a Sobolev subspace $H^m(\Omega)$, for an appropriate m. In other words, the regularity of attractor means that even the system starts with a initial state $u_0(x)$ in a lower order Sobolev space $H^2(\Omega)$, its long-time state may be a more (spatially) regular function in a higher order Sobolev space $H^m(\Omega)$. The data mentioned here indicate the different functions and parameters appearing in the partial differential equation.

Throughout this paper we shall use the following notions: Let $\Omega = [0,1] \times [0,1] \times [0,1]$. We denote by $(\cdot,\cdot)$ the usual inner product of $L^2(\Omega)$, by $\|\cdot\|_m$ the norm of Sobolev spaces $H^m(\Omega)$, and $\|\cdot\| = \|\cdot\|_0$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$. Let

$$L^2_p(\Omega) = \{\phi \in L^2(\Omega)|\phi(x + \mathbf{e}_j) = \phi(x), \quad j = 1,2,3\}$$

with the norm defined as that of $L^2(\Omega)$. Let

$$H^m_p(\Omega) = \{\phi \in H^2_p(\Omega)|\phi(x + \mathbf{e}_j) = \phi(x), \quad j = 1,2,3\}$$

with the norm defined as that of $H^2(\Omega)$.

In our study, we need the following technical three Lemmas in the proofs of our main results.

Lemma 1.1 (Sobolev Interpolation Inequality [**32**]). *Let $u \in L^q(\Omega)$, $D^m u \in L^r(\Omega)$ and $\Omega \subset R^n$, where $1 \leq r \leq \infty$ and $0 \leq j \leq m$. Then there exists a constant $c = c(j, m, \Omega, p, q, r)$ independent of u such that*

$$\|D^j u\|_{L^p} \leq c\|u\|^a_{W^{m,r}(\Omega)}\|u\|^{1-a}_{L^q},$$

where

$$\frac{1}{p}=\frac{j}{n}+a\left(\frac{1}{r}-\frac{m}{n}\right)+(1-a)\frac{1}{q},\qquad \frac{j}{m}<a<1.$$

Lemma 1.2 (Gronwall's Inequality [**33**])**.** *Let $y(t),\ g(t)$ and $h(t)$ be three non-negative functions satisfying*

$$y'(t)\le g(t)y(t)+h(t),\qquad \forall\ t\ge t_0\ge 0.$$

and

$$\int_t^{t+r} g(s)\mathrm{d}s\le \alpha_1,\quad \int_t^{t+r} h(s)\mathrm{d}s\le \alpha_2,\quad \int_t^{t+r} y(s)\mathrm{d}s\le \alpha_3,\quad \forall\ t\ge t_0.$$

Then we have

$$y(t+r)\le\left(\frac{\alpha_3}{r}+\alpha_2\right)\mathrm{e}^{\alpha_1},\qquad \forall\ t\ge t_0.$$

Lemma 1.3 ([**34**])**.** *Let $\mathcal{E}$ be a Banach space. Suppose that $\{S(t)\}_{t\ge 0}$ is a semi-group of continuous operators, i.e. $S(t):\mathcal{E}\to\mathcal{E}$, with*

$$S(t)\cdot S(\tau)=S(t+\tau),\quad S(0)=I,$$

where I is the identical operator. Suppose that the operator $S(t)$ satisfies the following three conditions.

(i) The operator $S(t)$ is bounded, i.e. for any given $R>0$, if $\|u\|_{\mathcal{E}}\le R$, then there exists a constant $C(R)$ such that

$$\|S(t)u\|_{\mathcal{E}}\le C(R),\quad \text{for } t\in[0,+\infty).$$

(ii) There is a bounded absorbing set $\mathcal{B}_0\subset\mathcal{E}$, i.e., for any given bounded set $\mathcal{B}\subset\mathcal{E}$, there exists a constant $T=T(\mathcal{B})$ such that

$$S(T)\mathcal{B}\subset\mathcal{B}_0,\quad \text{for } t\ge T.$$

(iii) $S(t)$ is a completely continuous operator for the sufficiently large $t>0$.
Then the semi-group $\{S(t)\}_{t\ge 0}$ of operators has a compact global attractor $\mathcal{A}\subset\mathcal{E}$.

The rest of this paper is organized as follows. In Section 2, the smoothness of solutions is obtained by a priori estimates under a weaker restriction on the parameter σ. The existence of global attractors $\mathcal{A}_i\subset H_p^i(\Omega)$ $(i=3,4,\cdots m)$ for the semi-group of operators $\{S^{(i)}(t)\}_{t\ge 0}$ generated by the system (1.4)–(1.6) is proved. In Section 3, the solution operator $S^{(2)}(t)$ is decomposed as $S_1^{(2)}(t)+S_2^{(2)}(t)$, where $S_1^{(2)}(t)u_0$ is more regular than $S^{(2)}(t)u_0$, and $\|S_2^{(2)}u_0\|_2$ approaches zero as t tends to infinity uniformly for u_0 bounded in $H_p^2(\Omega)$. Section 4 is dedicated to the regularity of global attractors.

2. Existence of High-Order Attractors

In this section, we prove the existence of the global attractor $\mathcal{A}_m\subset H_p^m(\Omega)$ $(m=2,3,\cdots,)$.

Lemma 2.1 ([**30**])**.** *Suppose that the condition (1.7) holds and $u_0(x)\in H_p^2(\Omega)$. Then the problem (1.4)–(1.6) possesses a unique global solution*

$$u(x,t)\in L^\infty(R^+;H_p^2(\Omega))\cap L^2([0,T;H_p^3(\Omega)),$$

and for any $R > 0$ given, there exists $t_2 = t_2(R)$ such that

$$\|u\|_2 \le E_2, \quad \forall\ t \ge 0,$$

$$\|u\|_2 \le M_2, \quad \forall\ t \ge t_2,$$

if $\|u_0\|_2 \le R$, where the constant E_2 depends on the parameters σ, ν, μ, γ, λ_1, λ_2 and R; and M_2 only depends on the parameters σ, ν, μ, γ, λ_1 and λ_2.

Furthermore, the semi-group $\{S(t)\}_{t\ge 0}$ of operators generated by the problem (1.4)–(1.6) has a compact global attractor $\mathcal{A}_2 \overset{\triangle}{=} \mathcal{A} \subset H_p^2(\Omega)$, i.e. there exists a set $\mathcal{A} \subset H_p^2(\Omega)$ such that
(a) $S(t)\mathcal{A} = \mathcal{A}$ for all $t \ge 0$;
(b) $\operatorname{dist}(S(t)\mathcal{B}, \mathcal{A}) \to 0$ for any bounded set $\mathcal{B} \subset H_p^2(\Omega)$, where

$$\operatorname{dist}(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_2.$$

Next, we show the high-order smoothness of the global solution for the problem (1.4)–(1.6).

Proposition 2.1. *Under the conditions of Lemma 2.1, suppose that $m \ge 2$ is a positive integer, $\sigma \ge \frac{1}{2}\left[\frac{m}{2} - 1\right]$ or σ is a positive integer. Then there exists $t_m = t_m(R)$ such that*

$$\|u\|_m \le M_m, \qquad \forall\ t \ge t_m, \quad \|u_0\|_2 \le R, \tag{2.1}$$

and

$$\|u\|_m \le E_m, \qquad \forall\ t \ge 0, \quad \|u_0\|_m \le R, \tag{2.2}$$

where the constant E_m depends on the parameters σ, ν, μ, γ, λ_1, λ_2, m and R; and M_m only depends on the parameters σ, ν, μ, γ, λ_1, λ_2 and m.

Thus, the problem (1.4)–(1.6) possesses the global smooth solution

$$u \in C(R^+; H_p^m(\Omega)) \cap C^1(R^+; H_p^{m-2}(\Omega)),$$

and the closed ball

$$B_m = \{\varphi \in H_p^m(\Omega) \big| \ \|\varphi\|_m \le M_m\}$$

is the bounded absorbing set of the semi-group of operators $\{S^{(m)}(t)\}_{t\ge 0}$.

Proof. We prove that (2.1) and (2.2) by using the principle of mathematical induction.

When $m = 2$, (2.1) and (2.2) can be deduced by Lemma 2.1 directly.

Suppose that (2.1) and (2.2) hold for $m = 2, 3, \cdots, k-1$, i.e.

$$\|u\|_m \le M_m \text{ for all } t \ge t_m \text{ if } \|u_0\|_2 \le R, \quad m = 2, 3, \cdots, k-1; \tag{2.3}$$

$$\|u\|_m \le E_m \text{ for all } t \ge 0 \text{ if } \|u_0\|_m \le R, \quad m = 2, 3, \cdots, k-1.$$

By using the Sobolev interpolation inequality, there exist constants $E'_m = E'_m(R)$ and M'_m such that

$$\|u\|_{W^{m-2,\infty}} \le M'_m, \quad \forall\ t \ge t_m, \quad 3 \le m \le k-1, \tag{2.4}$$

and

$$\|u\|_{W^{m-2,\infty}} \le E'_m, \quad \forall\ t \ge 0, \quad 3 \le m \le k-1.$$

This implies that (2.1) and (2.2) also hold for $m = k$.

Setting $l = \left[\frac{k-1}{2}\right]$ and differentiating (1.4) for l times with respect to t, we have

$$u_{t^{l+1}} - (1+\mathrm{i}\nu)\triangle u_{t^l} + (1+\mathrm{i}\mu)(|u|^{2\sigma}u)_{t^l} - \gamma u_{t^l} + \lambda_1\cdot(|u|^2\nabla u)_{t^l} + \lambda_2\cdot(u^2\nabla\bar{u})_{t^l} = 0, \tag{2.5}$$

If $k = 2l+1$, by taking the real part of the L^2- inner product of (2.5) with $-\triangle u_{t^l}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \\
=\ & \gamma\|\nabla u_{t^l}\|^2 + \mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega (|u|^{2\sigma}u)_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x\right) \\
& -\mathrm{Re}\int_\Omega \lambda_1\cdot(|u|^2\nabla u)_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x - \mathrm{Re}\int_\Omega \lambda_2\cdot(u^2\nabla\bar{u})_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x.
\end{aligned} \tag{2.6}$$

For $t \geq t_{k-1} = t_{2l}$, $\sigma \geq \frac{1}{2}(l-1) = \frac{1}{2}\left(\left[\frac{k}{2}\right]-1\right)$ or σ is a positive integer, in view of Hölder's inequality, the Sobolev interpolation inequality and Young's inequality together with (2.3) and (2.4), the four terms on the right hand side of (2.6) can be estimated as

$$\begin{aligned}
\left|\mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega (|u|^{2\sigma}u)_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x\right)\right| &\leq \frac{1}{8}\|\triangle u_{t^l}\|^2 + c(M_{k-1}, M'_{k-1}), \\
\gamma\|\nabla u_{t^l}\|^2 &\leq \gamma\|\triangle u_{t^l}\|\,\|u_{t^l}\| \\
&\leq \frac{1}{8}\|\triangle u_{t^l}|^2 + 2\gamma^2\|u_{t^l}\|^2 \\
&\leq \frac{1}{8}\|\triangle u_{t^l}\|^2 + c(M_{k-1}, M'_{k-1}), \\
\left|\mathrm{Re}\int_\Omega \lambda_1\cdot(|u|^2\nabla u)_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x\right| &\leq \frac{1}{8}\|\triangle u_{t^l}\|^2 + c(M_{k-1}, M'_{k-1}),
\end{aligned}$$

and

$$\left|\mathrm{Re}\int_\Omega \lambda_2\cdot(u^2\nabla\bar{u})_{t^l}\triangle\bar{u}_{t^l}\mathrm{d}x\right| \leq \frac{1}{8}\|\triangle u_{t^l}\|^2 + c(M_{k-1},\ M'_{k-1}).$$

So (2.6) can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C_1(M'_{k-1}, M_{k-1}), \qquad \forall\ t \geq t_{k-1}. \tag{2.7}$$

By taking the L^2-inner product of (2.5) with u_{t^l} and using a similar way to the derivation of (2.7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{t^l}\|^2 + \|\nabla u_{t^l}\|^2 \leq C_2(M_{k-1}, M'_{k-1}), \qquad \forall\ t \geq t_{k-1}. \tag{2.8}$$

That is,

$$\begin{aligned}
\int_t^{t+1}\|\nabla u_{t^l}(s)\|^2\mathrm{d}s &\leq \|u_{t^l}(t)\|^2 + C_2(M'_{k-1}, M_{k-1}) \\
&\leq C(M_{k-1}) + C_2(M'_{k-1}, M_{k-1}) \\
&\triangleq \alpha_3, \qquad \forall\ t \geq t_{k-1}.
\end{aligned}$$

Set

$$\alpha_2 = C_1(M'_{k-1}, M_{k-1}).$$

Applying the Gronwall's inequality to (2.7), we have

$$\|\nabla u_{t^l}(t+1)\|^2 \leq \alpha_3 + \alpha_2, \quad \forall\ t \geq t_{k-1}, \tag{2.9}$$

which leads to

$$\|\nabla^k u\|^2 = \|\nabla^{2l+1} u\|^2 \leq C(\alpha_2, \alpha_3), \quad \forall\, t \geq t_{k-1} + 1.$$

Let $M_k = \sqrt{C(\alpha_2, \alpha_3) + M_{k-1}^2}$ and $t_k = t_{k-1} + 1$. So (2.1) holds for $m = k$.

If $u_0 \in H_p^k(\Omega)$, by using an analogous way to the derivation of (2.7), (2.6) can be re-expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C_3(E'_{k-1}, E_{k-1}), \quad \forall\, t \geq 0. \tag{2.10}$$

Multiplying (2.10) by e^t and integrating it with respect to t yields

$$\|\nabla u_{t^l}\|^2 \leq \|\nabla u_{t^l}(0)\|^2 + C_3 \leq C(R^2) + C_3 \stackrel{\triangle}{=} E''_k, \quad \forall\, t \geq 0.$$

Let $E_k = \sqrt{E''_k + E_{k-1}^2}$. So (2.2) holds for $m = k$ too.

When $k = 2l + 2$, by taking the L^2-inner products of (2.5) with $\triangle^2 u_{t^l}$ and $-\triangle u$, respectively, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u_{t^l}\|^2 + \|\nabla \triangle u_{t^l}\|^2 \leq C'_1(M_{k-1}, M'_{k-1}), \quad \forall\, t \geq t_{k-1},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C'_2(M_{k-1}, M'_{k-1}), \quad \forall\, t \geq t_{k-1}.$$

Using an analogue discussion to the derivation of (2.9), we have

$$\|\triangle u_{t^l}(t+1)\|^2 \leq \alpha'_3 + \alpha'_2, \quad \forall\, t \geq t_{k-1},$$

which leads to

$$\|\nabla^k u\|^2 = \|\nabla^{2l+2} u\|^2 \leq C'(\alpha'_2, \alpha'_3), \quad \forall\, t \geq t_{k-1} + 1.$$

Let $M_k = \sqrt{C(\alpha'_2, \alpha'_3) + M_{k-1}^2}$ and $t_k = t_{k-1} + 1$, so (2.1) holds for $m = k$.

If $u_0 \in H_p^k(\Omega)$, it is deduced that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C'_3(E_{k-1}, E'_{k-1}), \quad \forall\, t \geq 0. \tag{2.11}$$

Multiplying (2.11) by e^t and integrating it with respect to t yields

$$\|\triangle u_{t^l}\|^2 \leq \|\triangle u_{t^l}(0)\|^2 + C_1 \leq C(R^2) + C'_3 \stackrel{\triangle}{=} E''_k, \quad \forall\, t \geq 0.$$

Let $E_k = \sqrt{E''_k + E_{k-1}^2}$, thus (2.2) holds for $m = k$ too.

Consequently, we conclude that (2.1) and (2.2) hold for any positive integer $m \geq 2$. The proof of Proposition 2.1 is completed. □

In order to prove the existence of the high-order global attractor $\mathcal{A}_m$, we introduce another proposition as follows.

Proposition 2.2. *Suppose that the conditions of Proposition 2.1 hold with* $\sigma \geq \frac{1}{2}\left[\frac{m-1}{2}\right]$ *or* σ *is a positive integer. Then the semi-group of operations* $S^m(t)$ $(t \geq 0)$: $H_p^m(\Omega) \to H_p^m(\Omega)$ *is uniformly compact for the sufficiently large* $t > 0$.

Proof. If $m = 2l$, we consider the real parts of the inner product of (2.5) with $-\triangle u_{t^l}$ and u_{t^l}, respectively. It follows from Proposition 2.1 that there exist constants $C_1 = C_1(M'_m, M_m)$ and $C_2 = C_2(M'_m, M_m)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C_1(M'_m, M_m), \qquad \forall\, t \geq t_m, \tag{2.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_{t^l}\|^2 + \|\nabla u_{t^l}\|^2 \leq C_2(M_m, M'_m), \qquad \forall\, t \geq t_m. \tag{2.13}$$

Applying the Gronwall's inequality to (2.12) and using (2.13), we get

$$\|\nabla u_{t^l}\|^2 \leq C_3(M_m, M'_m), \qquad \forall\, t \geq t_{m+1} = t_m + 1,$$

which leads to

$$\begin{aligned} \|\nabla^{m+1}u\|^2 &= \|\nabla^{2l+1}u\|^2 \\ &\leq C(M_m, M'_m)\|\nabla u_{t^l}\|^2 \\ &\leq M_{m+1}, \qquad \forall\, t \geq t_{m+1} = t_m + 1. \end{aligned}$$

If $m = 2l+1$, using the same arguments as the above we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u_{t^l}\|^2 + \|\nabla\triangle u_{t^l}\|^2 \leq C'_1(M_m, M'_m), \qquad \forall\, t \geq t_m, \tag{2.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_{t^l}\|^2 + \|\triangle u_{t^l}\|^2 \leq C'_2(M_m, M'_m), \qquad \forall\, t \geq t_m. \tag{2.15}$$

Again, applying the Gronwall's inequality to (2.14) and using (2.15) gives

$$\|\nabla^{m+1}u\|^2 = \|\nabla^{2l+2}u\|^2 \leq C(M_m, M'_m)\|\triangle u_{t^l}\|^2 \leq M_{m+1}, \qquad \forall\, t \geq t_{m+1} = t_m + 1.$$

By virtue of the Sobolev compact imbedding theorem, we know that the semi-group of operators $S^{(m)}(t)$ is uniformly compact for $t \geq t_{m+1}$. So the proof of Proposition 2.2 is completed. □

On the other hand, if $\sigma \geq \frac{1}{2}\left[\frac{m}{2}\right]$ or $\sigma > 2$ is an integer, according to Propositions 2.1 and 2.2, one can see that $S^{(m)}(t)$ is strongly continuous. Making use of Lemma 1.3, we obtain the following theorem:

Theorem 2.3. *Suppose that all conditions of Proposition 2.2 hold. Then there exists a global attractor $\mathcal{A}_m \subset H^m_p(\Omega)$ of the semi-group $\{S^{(m)}(t)\}_{t\geq 0}$ of the operators generated by the problem (1.4)–(1.6).*

3. Decomposition of Semigroup

In order to prove the regularity of global attractor, it is necessary to decompose $S^{(2)}(t)$ appropriately. In this section, we decompose $S^{(2)}(t)$ as $S^{(2)}_1(t)+S^{(2)}_2(t)$; where $S^{(2)}_1(t)u_0$ is more regular than the solution $S^{(2)}(t)u_0$, and $\|S^{(2)}_2(t)u_0\|_2$ approaches zero as t tends to infinity uniformly for the bounded u_0 in $H^2_p(\Omega)$.

For any given positive integer N, let $S_N = \mathrm{Span}\{\mathrm{e}^{2\pi \mathrm{i}k\cdot x} : |k| \leq N\}$ and denote the orthogonal projection operator by $P_N : L^2_p(\Omega) \to S_N$ and $Q_N = I - P_N$ [**32**]. Then there have

Lemma 3.1 ([**35**]). *If $v \in H_p^m(\Omega)$, then there exists a constant c independent of v and N such that*

$$\|P_N v\|_m \le cN^{m-j}\|P_N v\|_j, \quad \forall\ 0 < j \le m,$$

$$\|Q_N v\|_j \le cN^{j-m}\|Q_N v\|_m, \quad \forall\ j = 0, 1, \cdots, m,$$

and

$$\|\nabla^j Q_N v\| \le cN^{j-m}\|\nabla^m Q_N v\|, \quad \forall\ j = 0, 1, \cdots, m.$$

We also decompose the solution $u(x,t)$. Let $u_0 \in H_p^2(\Omega)$ and $u = S(t)u_0$ be the solution of the problem (1.4)–(1.6), then it has

$$u = P_N u + Q_N u = p(t) + q(t),$$

where $p(t) = P_N u$ represents the low-frequency part of u and $q(t) = Q_N u$ represents the high-frequency part of u.

We split the high-frequency part $q(t)$ as

$$q(t) = y + z,$$

where $y,\ z \in Q_N L_p^2(\Omega)$ are the solutions of the following equations for $t \ge t_2$:

$$y_t - (1 + \mathrm{i}\nu)\triangle y - \gamma y$$

$$= -(1 + \mathrm{i}\mu)Q_N\left(|u|^{2\sigma}(p + y)\right) - Q_N\left(\lambda_1 \cdot |u|^2\nabla(p + y) + \lambda_2 \cdot u^2\nabla(\bar{p} + \bar{y})\right), \tag{3.1}$$

$$y(x,t) = y(x + \mathrm{e}_j, t), \quad j = 1,\ 2,\ 3; \quad y(x, t_2) = 0, \tag{3.2}$$

$$z_t - (1 + \mathrm{i}\nu)\triangle z = \gamma z - (1 + \mathrm{i}\mu)Q_N\left(|u|^{2\sigma} z\right) - Q_N\left(\lambda_1 \cdot |u|^2\nabla z + \lambda_2 \cdot u^2\nabla\bar{z}\right), \tag{3.3}$$

$$z(x,t) = z(x + \mathrm{e}_j, t), \quad j = 1,\ 2,\ 3; \quad z(x, t_2) = Q_N u(t_2), \tag{3.4}$$

respectively. For $t \le t_2$, $y(t) = 0$ and $z(t) = Q_N u(t)$, where t_2 is given by Proposition 2.1.

We first prove that y is smooth for $t \ge t_m$ and z converges toward zero in $H_p^2(\Omega)$ when t goes to infinity.

Theorem 3.1. *Under the condition (1.7), if $u_0 \in H_p^2(\Omega)$ satisfies $\|u_0\|_2 \le R$, then there exists a unique solution y of (3.1)–(3.2) and a unique solution z of (3.3)–(3.4) satisfying*

$$y,\ z \in C^1\left([0,\infty);\ L_p^2(\Omega)\right) \cap C\left([0,\infty);\ H_p^2(\Omega)\right).$$

Moreover, If σ is a positive integer or $\sigma \ge \frac{1}{2}\left[\frac{m}{2}\right]$, then there exists N_3 large enough, and constants $K_m = K_m(N)$ and $\delta = \delta(N) > 0$ such that for any given $N \ge N_3$, the following estimates hold:

$$\|y(t)\|_m \le K_m, \quad \forall\ t \ge t_m, \quad m = 2, 3, \cdots, \tag{3.5}$$

$$\|z(t)\|_2 \le \|u(t_2)\|_2 \mathrm{e}^{-\delta(t - t_2)} \le M_2 \mathrm{e}^{-\delta(t - t_2)}, \quad \forall\ t \ge t_2, \tag{3.6}$$

where M_2, t_m and R are given by Lemma 2.1 and Proposition 2.1, respectively.

Proof. The existence and uniqueness can be proved directly by using the Galerkin method [**30**]. We separate our proof for estimates (3.5) and (3.6) into three steps.

Step 1. Consider the estimates for y in $H_p^2(\Omega)$.

By taking the real part of the inner product of (3.1) with y, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2+\|\nabla y\|^2-\gamma\|y\|^2+\mathrm{Re}\Big((1+\mathrm{i}\mu)\int_\Omega Q_N\big(|u|^{2\sigma}(p+y)\big)\bar{y}\mathrm{d}x\Big)\\ &=-\mathrm{Re}\Big(\lambda_1\cdot\int_\Omega Q_N\big(|u|^2\nabla(p+y)\big)\bar{y}\mathrm{d}x\Big)-\mathrm{Re}\Big(\lambda_2\cdot\int_\Omega Q_N\big(u^2\nabla(\bar{p}+\bar{y})\big)\bar{y}\mathrm{d}x\Big).\end{aligned}\tag{3.7}$$

For the last term in the left-hand side of the equation (3.7), using the definition of Q_N gives

$$\mathrm{Re}\left((1+\mathrm{i}\mu)\left[Q_N\left(|u|^{2\sigma}(p+y),y\right)\right]\right)=\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x+\mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega|u|^{2\sigma}p\bar{y}\mathrm{d}x\right).$$

Furthermore, according to Proposition 2.1 and definitions of P_N and $p(t)$, and by using Hölder's inequality and the Sobolev interpolation inequality, we deduce that

$$\begin{aligned}\mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega|u|^{2\sigma}p\,\bar{y}\mathrm{d}x\right)&\le|1+\mathrm{i}\mu|\left(\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x\right)^{\frac12}\left(\int_\Omega|u|^{2\sigma}|p|^2\mathrm{d}x\right)^{\frac12}\\&\le\frac14\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x+|1+\mathrm{i}\mu|^2\int_\Omega|u|^{2\sigma}|p|^2\mathrm{d}x\\&\le\frac14\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x+|1+\mathrm{i}\mu|^2\|u\|_\infty^{2\sigma}\|u\|^2.\end{aligned}\tag{3.8}$$

Separate the first term in the right-hand side of (3.7) as

$$\begin{aligned}&\mathrm{Re}\left(\lambda_1\cdot\int_\Omega Q_N\left(|u|^2\nabla(p+y)\right)\bar{y}\mathrm{d}x\right)\\=\;&\mathrm{Re}\left(\lambda_1\cdot\int_\Omega|u|^2\nabla p\,\bar{y}\mathrm{d}x\right)+\mathrm{Re}\left(\lambda_1\cdot\int_\Omega|u|^2\nabla y\,\bar{y}\mathrm{d}x\right).\end{aligned}\tag{3.9}$$

Notice that

$$\begin{aligned}\mathrm{Re}\left(\lambda_1\cdot\int_\Omega|u|^2\nabla y\,\bar{y}\mathrm{d}x\right)&\le|\lambda_1|\left(\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x\right)^{\frac1\sigma}\|y\|^{\frac{\sigma-2}{\sigma}}\|\nabla y\|\\&\le\frac14\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x+\frac14\|\nabla y\|^2+4^{\frac{2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}}\|y\|^2,\end{aligned}$$

and

$$\mathrm{Re}\left(\lambda_1\cdot\int_\Omega|u|^2\nabla p\,\bar{y}\mathrm{d}x\right)\le|\lambda_1|\,\|u\|_\infty^2\|\nabla p\|\,\|y\|\le\frac{\gamma}{2}\|y\|^2+\frac{|\lambda_1|^2}{2\gamma}\|u\|_\infty^4\|\nabla u\|^2.$$

So (3.9) can be rewritten as

$$\begin{aligned}&\mathrm{Re}\left(\lambda_1\cdot\int_\Omega Q_N\left(|u|^2\nabla(p+y)\right)\bar{y}\mathrm{d}x\right)\\&\le\frac14\int_\Omega|u|^{2\sigma}|y|^2\mathrm{d}x+\frac14\|\nabla y\|^2+\Big(4^{\frac{2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}}\\&+\frac{\gamma}{2}\Big)\|y\|^2+\frac{|\lambda_1|^2}{2\gamma}\|u\|_\infty^4\|\nabla u\|^2.\end{aligned}\tag{3.10}$$

Similarly, from the second term in the right-hand side of the equation (3.7) we have

$$
\begin{aligned}
&\mathrm{Re}\left(\lambda_2 \cdot \int_\Omega Q_N\left(u^2\nabla(\bar{p}+\bar{y})\right)\bar{y}\mathrm{d}x\right)\\
&\le \frac{1}{4}\int_\Omega |u|^{2\sigma}|y|^2\mathrm{d}x + \frac{1}{4}\|\nabla y\|^2 + \Big(4^{\frac{2}{\sigma-2}}|\lambda_2|^{\frac{2\sigma}{\sigma-2}}\\
&+\frac{\gamma}{2}\Big)\|y\|^2 + \frac{|\lambda_1|^2}{2\gamma}\|u\|_\infty^4\|\nabla u\|^2. \qquad (3.11)
\end{aligned}
$$

Substituting (3.8)–(3.11) into (3.7), we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + \|\nabla y\|^2 - \big(4\gamma + 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}} + 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_2|^{\frac{2\sigma}{\sigma-2}}\big)\|y\|^2 \le C(M_2). \qquad (3.12)
$$

From Lemma 3.1, we know that

$$
\|\nabla y\|^2 \ge c_0 N^2 \| \ y\|^2.
$$

So (3.12) can be simplified as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2 + \big(c_0 N^2 - 4\gamma + 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}} + 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_2|^{\frac{2\sigma}{\sigma-2}}\big)\|y\|^2 \le C(M_2). \qquad (3.13)
$$

Let N_0 be large enough such that

$$
c_0 N_0^2 - 4\gamma - 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}} - 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_2|^{\frac{2\sigma}{\sigma-2}} > 0.
$$

Then for $N \ge N_0$, multiplying (3.13) by $\mathrm{e}^{\delta_0 t}$ and integrating it from t_2 to t with $y(t_2) = 0$, we have

$$
\|y\|^2 \le \frac{C(M_2)}{\delta_0} \triangleq K_0^2, \qquad \forall\ t \ge t_2, \qquad (3.14)
$$

where

$$
\delta_0 = \delta_0(N) = cN^2 - 4\gamma - 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_1|^{\frac{2\sigma}{\sigma-2}} - 2^{\frac{\sigma+2}{\sigma-2}}|\lambda_2|^{\frac{2\sigma}{\sigma-2}}.
$$

By taking the real part of the inner product of (3.1) with $-\triangle y$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y\|^2 + \|\triangle y\|^2\\
=\ &\gamma\|\nabla y\|^2 + \mathrm{Re}\Big((1+\mathrm{i}\mu)\int_\Omega Q_N\big(|u|^{2\sigma}(p+y)\big)\triangle\bar{y}\mathrm{d}x\Big)\\
&+\mathrm{Re}\Big(\lambda_1 \cdot \int_\Omega Q_N\big(|u|^2\nabla(p+y)\big)\triangle\bar{y}\mathrm{d}x\Big) +\\
&\mathrm{Re}\Big(\lambda_2 \cdot \int_\Omega Q_N\big(u^2\nabla(\bar{p}+\bar{y})\big)\triangle\bar{y}\mathrm{d}x\Big). \qquad (3.15)
\end{aligned}
$$

Using the Sobolev interpolation inequality, the four terms in the right hand side of (3.15) can be estimated as

$$
\left|(1+\mathrm{i}\mu)\int_\Omega Q_N\big(|u|^{2\sigma}(p+y)\big)\triangle\bar{y}\mathrm{d}x\right| \le \frac{1}{8}\|\triangle y\|^2 + C(K_0, M_2),
$$

$$
\gamma\|\nabla y\|^2 \le \frac{1}{8}\|\triangle y\|^2 + 2\gamma^2 K_0^2,
$$

$$
\left|\lambda_1 \cdot \int_\Omega Q_N\big(|u|^2\nabla(p+y)\big)\triangle\bar{y}\mathrm{d}x\right| \le \frac{1}{8}\|\triangle y\|^2 + C(K_0, M_2),
$$

and

$$\left|\lambda_2\cdot\int_\Omega Q_N\big(u^2\nabla(\bar p+\bar y)\big)\triangle\bar y\mathrm{d}x\right|\le\frac18\|\triangle y\|^2+C(K_0,M_2),$$

respectively. Thus, (3.15) can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y\|^2+\|\triangle y\|^2\le C(K_0,M_2).$$

Since $\|\triangle y\|^2\ge cN^2\|\nabla y\|^2$, we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y\|^2+cN^2\|\nabla y\|^2\le C(K_0,M_2).$$

Multiplying the above inequality by e^{cN^2t} and integrating it from t to t_2 with $y(t_2)=0$ yields

$$\|\nabla y\|^2\le\frac{C(M_0,K_0)}{cN^2}\triangleq K_1^2,\quad\forall\ t\ge t_2.\tag{3.16}$$

Using the same procedure, by considering the real part of the inner product of (3.1) with $\triangle^2 y$, we have

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle y\|^2+\|\nabla\triangle y\|^2\\ =\ &-\mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega Q_N\left(|u|^{2\sigma}(p+y)\right)\triangle^2\bar y\mathrm{d}x\right)\\ &-\mathrm{Re}\left(\lambda_1\cdot\int_\Omega Q_N\left(|u|^2\nabla(p+y)\right)\triangle^2\bar y\mathrm{d}x\right)\\ &+\gamma\|\triangle y\|^2-\mathrm{Re}\left(\lambda_2\cdot\int_\Omega Q_N\left(u^2\nabla(\bar p+\bar y)\right)\triangle^2\bar y\mathrm{d}x\right).\end{aligned}\tag{3.17}$$

We estimate each term of the right side in (3.17) for $\sigma\ge\frac12$. By using Hölder's inequality, the Sobolev interpolation inequality, as well as (3.14)–(3.16), we have

$$\left|(1+\mathrm{i}\mu)\int_\Omega Q_N\big(|u|^{2\sigma}(p+y)\big)\triangle^2\bar y\mathrm{d}x\right|\le\frac18\|\nabla\triangle y\|^2+C(K_0,K_1,M_2),$$

$$\gamma\|\triangle y\|^2\le\frac18\|\nabla\triangle y\|^2+2\gamma^2K_1^2,$$

$$\left|\lambda_1\cdot\int_\Omega Q_N\big(|u|^2\nabla(p+y)\big)\triangle^2\bar y\mathrm{d}x\right|\le\frac18\|\triangle y\|^2+C(K_0,K_1,M_2),$$

and

$$\left|\lambda_2\cdot\int_\Omega Q_N\big(u^2\nabla(\bar p+\bar y)\big)\triangle^2\bar y\mathrm{d}x\right|\le\frac18\|\triangle y\|^2+C(K_0,K_1,M_2).$$

So, (3.17) can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle y\|^2+\|\nabla\triangle y\|^2\le C(K_0,K_1,M_2).$$

Since $\|\nabla\triangle y\|^2\ge cN^2\|\triangle y\|^2$, we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle y\|^2+cN^2\|\triangle y\|^2\le C(K_0,K_1,M_2).\tag{3.18}$$

Multiplying (3.18) by e^{cN^2t} and integrating it from t from t_2 with $y(t_2)=0$ yields

$$\|\triangle y\|^2\le\frac{C(K_0,K_1,M_2)}{cN^2}\triangleq\widetilde K^2,\quad\forall\ t\ge t_2.$$

Set $K_2=\sqrt{K_0^2+K_1^2+\widetilde{K}_2^2}$. So it is easy to see that (3.5) holds for $m=2$.

Step 2. Estimates for y in $H_p^m(\Omega)$ $(m\geq 3)$.

We prove that (3.5) holds for any $m\geq 3$ by using the mathematical induction.

For $m=3$, differentiating (3.1) with respect to t and using the real part of the L^2-inner product with $-\triangle y_t$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_t\|^2+\|\triangle y_t\|^2\\
=\ &\mathrm{Re}\left(\lambda_1\cdot\int_\Omega Q_N\left(|u|^2\nabla(p+y)\right)_t\triangle\bar{y}_t\mathrm{d}x\right)\\
&+\mathrm{Re}\left(\lambda_2\cdot\int_\Omega Q_N\left(u^2\nabla(\bar{p}+\bar{y})\right)_t\triangle\bar{y}_t\mathrm{d}x\right)\\
&+\gamma\|\nabla y_t\|^2+\mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega Q_N\left(|u|^{2\sigma}(p+y)\right)_t,\triangle\bar{y}_t\mathrm{d}x\right).
\end{aligned}\tag{3.19}$$

For $t\geq t_2$, we estimate the four terms in the right hand side of (3.19). By using Hölder's inequality, the Sobolev interpolation inequality and Proposition 2.1, we deduce that

$$\left|(1+\mathrm{i}\mu)\int_\Omega Q_N\big(|u|^{2\sigma}(p+y)\big)_t,\triangle\bar{y}_t\mathrm{d}x\right|\leq\frac{1}{8}\|\triangle y_t\|^2+C(K_2,M_2),$$

$$\gamma\|\nabla y_t\|^2\leq\frac{1}{8}\|\triangle y_t\|^2+C(K_2,M_2),$$

$$\left|\lambda_1\cdot\int_\Omega Q_N\big(|u|^2\nabla(p+y)\big)_t\triangle\bar{y}_t\mathrm{d}x\right|\leq\frac{1}{8}\|\triangle y_t\|^2+C(K_2,M_2),$$

and

$$\left|\lambda_2\cdot\int_\Omega Q_N\big(u^2\nabla(\bar{p}+\bar{y})\big)_t\triangle\bar{y}_t\mathrm{d}x\right|\leq\frac{1}{8}\|\triangle y_t\|^2+C(K_2,M_2).$$

Thus, (3.19) can be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_t\|^2+\|\triangle y_t\|^2\leq C(K_2,M_2).$$

Since $\|\triangle y_t\|^2\geq cN^2\|\nabla y_t\|^2$, we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_t\|^2+cN^2\|\nabla y_t\|^2\leq C(K_2,M_2).\tag{3.20}$$

Multiplying (3.20) by e^{cN^2t}, integrating it from t from t_2 and using the inequality

$$\begin{aligned}
\|\nabla y(t_2)\|\ &\leq\ c\big(\|u(t_2)\|_2^{2\sigma+1}+\|u(t_2)\|_2^3\big)\\
&\leq\ C(M_2),
\end{aligned}$$

we have

$$\begin{aligned}
\|\nabla y_t\|^2\ &\leq\ \mathrm{e}^{-cN^2(t-t_2)}\|\nabla y_t(t_2)\|^2+\frac{C(K_2,M_2)}{cN^2}\\
&\leq\ \mathrm{e}^{-cN^2(t-t_2)}C(M_2)+\frac{C(K_2,M_2)}{cN^2}\\
&\triangleq\ \rho_3^2,\qquad\forall\ t\geq t_2.
\end{aligned}$$

That is,

$$\begin{aligned}\|\nabla\triangle y\| &\leq \|\nabla y_t\| + C_0(M_2, K_2)\\ &\leq \rho_3 + C(M_2, K_2)\\ &\triangleq \widetilde{K}_3, \qquad \forall\, t \geq t_3 = t_2 + 1.\end{aligned}$$

Set $K_3 = \sqrt{K_2^2 + \widetilde{K}_3^2}$. So (3.5) holds for $m = 3$.

Suppose that (3.5) holds for any $3 < m \leq k-1$ (k is a positive integer), namely, there exists a constant K_m such that

$$\|y\|_m \leq K_m, \qquad \forall\, t \geq t_m, \quad m \leq k-1. \tag{3.21}$$

It follows from the Sobolev interpolation inequality that

$$\|y\|_{W^{m-2,\infty}} \leq K'_m, \qquad \forall\, t \geq t_m, \quad m \leq k-1, \tag{3.22}$$

where the constant K'_m depends on K_m. From Proposition 2.1 and the definition of P_N we have

$$\|p\|_m \leq M_m, \qquad \|p\|_{W^{m-2,\infty}} \leq C(K'_m), \qquad \forall\, t \geq t_m, \quad m \leq k-1, \tag{3.23}$$

where t_m in (3.21)–(3.23) is the same as the one given in Proposition 2.1.

Now, we consider the case of $m = k$.

Let $l = \left[\frac{k-1}{2}\right]$. Differentiating (3.1) for l times with respect to t gives

$$\begin{aligned}&y_{t^{l+1}} - (1+\mathrm{i}\nu)\triangle y_{t^l} + (1+\mathrm{i}\mu)Q_N(|u|^{2\sigma}(p+y))_{t^l} - \gamma y_{t^l}\\ =\ &-Q_N\big(\lambda_1 \cdot |u|^2\nabla(p+y) + \lambda_2 \cdot u^2\nabla(\bar{p}+\bar{y})\big)_{t^l}.\end{aligned} \tag{3.24}$$

If $k = 2l+1$, by considering the real part of the inner product of (3.24) with $-\triangle y_{t^l}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_{t^l}\|^2 + \|\triangle y_{t^l}\|^2\\ =\ &\gamma\|\nabla y_{t^l}\|^2 + \mathrm{Re}\left((1+\mathrm{i}\mu)\int_\Omega Q_N(|u|^{2\sigma}(p+y))_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right)\\ &+\mathrm{Re}\left(\int_\Omega Q_N\left(\lambda_1\cdot|u|^2\nabla(p+y)\right)_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right)\\ &+\mathrm{Re}\left(\int_\Omega Q_N\big(\lambda_2\cdot u^2\nabla(\bar{p}+\bar{y})\big)_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right).\end{aligned} \tag{3.25}$$

We estimate each term in the right hand side of (3.25) by applying Hölder's inequality, the Sobolev interpolation inequality, Young's inequality, as well as (3.21)–(3.23). When $t \geq t_{k-1} = t_{2l}$ and σ is a positive integer or $\sigma \geq \frac{l}{2}$, we have

$$\left|(1+\mathrm{i}\mu)\int_\Omega(|u|^{2\sigma}(p+y))_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right| \leq \frac{1}{8}\|\triangle y_{t^l}\|^2 + C(K_{k-1}, K'_{k-1}, M_{k-1}, M'_{k-1}),$$

$$\gamma\|\nabla y_{t^l}\|^2 \leq \frac{1}{8}\|\triangle y_{t^l}\|^2 + C(K_{k-1}, K'_{k-1}, M_{k-1}, M'_{k-1}),$$

$$\left|\int_\Omega Q_N\big(\lambda_1\cdot|u|^2\nabla(p+y)\big)_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right| \leq \frac{1}{8}\|\triangle y_{t^l}\|^2 + C(K_{k-1}, K'_{k-1}, M_{k-1}, M'_{k-1}),$$

and

$$\left|\int_\Omega Q_N\big(\lambda_2\cdot u^2\nabla(\bar{p}+\bar{y})\big)_{t^l}\triangle\bar{y}_{t^l}\,\mathrm{d}x\right| \leq \frac{1}{8}\|\triangle y_{t^l}\|^2 + C(K_{k-1}, K'_{k-1}, M_{k-1}, M'_{k-1}).$$

Substituting the above four inequalities into (3.25) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_{t^l}\|^2 + \|\triangle y_{t^l}\|^2 \le C_1(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1}), \qquad \forall\ t \ge t_{k-1}. \tag{3.26}$$

Considering the inner product of (3.24) with y_{t^l} and using a similar argument as the derivation of (3.26), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_{t^l}\|^2 + \|\nabla y_{t^l}\|^2 \le C_2(K_{k-1}, K'_{k-1}, M_{k-1}, M'_{k-1}), \qquad \forall\ t \ge t_{k-1}. \tag{3.27}$$

When $t \ge t_{k-1}$, it follows from (3.27) that

$$\begin{aligned}\int_t^{t+1} \|\nabla y_{t^l}(s)\|^2 \mathrm{d}s &\le \|y_{t^l}(t)\|^2 + C_2(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1}) \\ &\le C(K_{k-1}) + C_2(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1}) \\ &\triangleq \alpha_3.\end{aligned}$$

On the other hand, setting $\alpha_2 = C_1(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1})$ and applying the Gronwall's inequality to (3.26), we have

$$\|\nabla y_{t^l}(t+1)\|^2 \le \alpha_3 + \alpha_2, \quad \forall\ t \ge t_{k-1}, \tag{3.28}$$

which leads to

$$\|\nabla^k y\|^2 = \|\nabla^{2l+1} y\|^2 \le C(\alpha_2, \alpha_3), \quad \forall\ t \ge t_{k-1} + 1.$$

Set $K_k = \sqrt{C(\alpha_2, \alpha_3) + K_{k-1}^2}$ and $t_k = t_{k-1} + 1$. So (3.5) holds for $m = k$.

Similarly, if $k = 2l+2$, considering the real parts of the inner products of (3.24) with $\triangle^2 y_{t^l}$ and $-\triangle y_{t^l}$, respectively, and using an analogous way to the derivation of (3.26) and (3.27), we derive that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle y_{t^l}\|^2 + \|\nabla\triangle y_{t^l}\|^2 \le C'_1(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1}), \qquad \forall\ t \ge t_{k-1},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla y_{t^l}\|^2 + \|\triangle y_{t^l}\|^2 \le C'_2(K_{k-1}, K'_{k-1}, M'_{k-1}, M_{k-1}), \qquad \forall\ t \ge t_{k-1}.$$

Following the derivation of (3.28), we get

$$\|\triangle y_{t^l}(t+1)\|^2 \le \alpha'_3 + \alpha'_2, \quad \forall\ t \ge t_{k-1},$$

which implies

$$\|\nabla^k y\|^2 = \|\nabla^{2l+2} y\|^2 \le C'(\alpha_2, \alpha_3), \quad \forall\ t \ge t_{k-1} + 1.$$

By setting $K_k = \sqrt{C'(\alpha'_2, \alpha'_3) + K_{k-1}^2}$ and $t_k = t_{k-1} + 1$, we see that (3.5) holds for $m = k$.

Consequently, the proof of (3.5) is completed.

Step 3. Estimates for z in $H_p^2(\Omega)$.

By considering the real part of the inner product of the equation (3.3) with z, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + \|\nabla z\|^2 + \int_\Omega |u|^{2\sigma}|z|^2 \mathrm{d}x - \gamma\|z\|^2 = -\mathrm{Re}\int_\Omega \big(\lambda_1 \cdot |u|^2\nabla z + \lambda_2 \cdot u^2\nabla\bar{z}\big)\bar{z}\mathrm{d}x. \tag{3.29}$$

We estimate the right hand side of (3.29) by applying Hölder's inequality and Young's inequality:

$$\begin{aligned}\operatorname{Re}\int_\Omega \lambda_1\cdot|u|^2\nabla z\bar{z}\mathrm{d}x &\le |\lambda_1|\,\|\nabla z\|\left(\int_\Omega |u|^{2\sigma}|z|^2\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\int_\Omega |z|^2\mathrm{d}x\right)^{\frac{\sigma-2}{2\sigma}}\\ &\le \frac{1}{4}\|\nabla z\|^2+\frac{1}{2}\int_\Omega |u|^{2\sigma}|z|^2\mathrm{d}x+(\sigma-2)|\lambda_1|^{\frac{2\sigma}{\sigma-2}}2^{\frac{4-\sigma}{\sigma-2}}\|z\|^2,\end{aligned}$$

and

$$\operatorname{Re}\int_\Omega \lambda_2\cdot u^2\nabla\bar{z}\bar{z}\mathrm{d}x \le \frac{1}{4}\|\nabla z\|^2+\frac{1}{2}\int_\Omega |u|^{2\sigma}|z|^2\mathrm{d}x+(\sigma-2)|\lambda_2|^{\frac{2\sigma}{\sigma-2}}2^{\frac{4-\sigma}{\sigma-2}}\|z\|^2.$$

By Lemma 3.1, we know that

$$\|\nabla z\|^2\ge c_0N^2\|z\|^2.$$

Thus, (3.29) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2+\left(c_0N^2-2\gamma-2^{\frac{4-\sigma}{\sigma-2}}(\sigma-2)(|\lambda_1|^{\frac{2\sigma}{\sigma-2}}+|\lambda_2|^{\frac{2\sigma}{\sigma-2}})\right)\|z\|^2\le 0. \tag{3.30}$$

Choose $N_1\ge N_0$ sufficient large such that

$$\begin{aligned}\delta_1(N_1) &= c_0N_1^2-2\gamma-2^{\frac{4-\sigma}{\sigma-2}}(\sigma-2)\left(|\lambda_1|^{\frac{2\sigma}{\sigma-2}}+|\lambda_2|^{\frac{2\sigma}{\sigma-2}}\right)\\ &> 0.\end{aligned}$$

For $N\ge N_1$, multiplying (3.30) with $e^{\delta_1 t}$ and integrating it for from t_2 to t yields

$$\|z(t)\|^2\le\|z(t_2)\|^2\mathrm{e}^{-\delta_1(t-t_2)},\qquad \forall\ t\ge t_2,$$

where

$$\delta_1=\delta_1(N)=c_0N^2-2\gamma-2^{\frac{4-\sigma}{\sigma-2}}(\sigma-2)\left(|\lambda_1|^{\frac{2\sigma}{\sigma-2}}+|\lambda_2|^{\frac{2\sigma}{\sigma-2}}\right).$$

Then, we consider the real part of the inner product of (3.3) with $-\triangle z$ and obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z\|^2+\|\triangle z\|^2-\gamma\|\nabla z\|^2\\ &=\operatorname{Re}\left((1+\mathrm{i}\mu)\int_\Omega |u|^{2\sigma}z\triangle\bar{z}\mathrm{d}x\right)+\operatorname{Re}\int_\Omega\left(\lambda_1\cdot|u|^2\nabla z+\lambda_2\cdot u^2\nabla\bar{z}\right)\triangle\bar{z}\mathrm{d}x.\end{aligned}\tag{3.31}$$

For the right hand side of (3.31), using Hölder's inequality and Young's inequality again yields

$$\operatorname{Re}\left((1+\mathrm{i}\mu)\int_\Omega Q_N(|u|^{2\sigma}z)\triangle\bar{z}\mathrm{d}x\right)\le\frac{1}{4}\|\triangle z\|^2+\|u\|_\infty^{4\sigma}\|z\|^2, \tag{3.32}$$

and

$$\begin{aligned}&\left|\int_\Omega\left(\lambda_1\cdot|u|^2\nabla z+\lambda_2\cdot u^2\nabla\bar{z}\right)\triangle\bar{z}\mathrm{d}x\right|\\ &\le\frac{1}{4}\|\triangle z\|^2+54(|\lambda_1|^4+|\lambda_2|^4)\|u\|_\infty^8\|z\|^2.\end{aligned}\tag{3.33}$$

Substituting (3.32) and (3.33) into (3.31), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z\|^2+\|\triangle z\|^2-2\gamma\|\nabla z\|^2-C(M_2))\|z\|^2\le 0. \tag{3.34}$$

Thanks to Lemma 3.1, we know that

$$\|\triangle z\|^2\ge c_1N^2\|\nabla z\|^2,\qquad \|z\|^2\le c_2N^{-2}\|\nabla z\|^2.$$

Substituting the above expressions into (3.34), we further have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z\|^2 + (c_1 N^2 - (2\gamma^2 + c_2 C(M_2))N^{-2})\|\nabla z\|^2 \leq 0. \tag{3.35}$$

Choose $N_2 \geq N_1$ large enough such that

$$\delta_2(N_2) = c_1 N_2^2 - (2\gamma^2 + c_2 C(M_2))N_2^{-2} > 0.$$

For $N \geq N_2$, multiplying (3.35) by $\mathrm{e}^{\delta_2 t}$ and integrating it from t_2 to t yields

$$\|\nabla z\|^2 \leq \|\nabla z(t_2)\|^2 \mathrm{e}^{-\delta_2(t-t_2)}, \quad \forall\, t \geq t_2.$$

where

$$\delta_2 = \delta_2(N) = c_1 N^2 - (2\gamma^2 + c_2 C(M_2))N^{-2} > 0.$$

By considering the real part of the inner product of equation (3.1) with $\triangle^2 z$ and using the same discussion as the derivation of (3.34), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle z\|^2 + \|\nabla\triangle z\|^2 - 2\gamma\|\triangle z\|^2 - C(M_2)\|z\|_1^2 \leq 0. \tag{3.36}$$

From Lemma 3.1, we know that

$$\|\nabla\triangle z\|^2 \geq c_3 N^2 \|\triangle z\|^2, \quad \|z\|_1^2 \leq c_4 N^{-2}\|\triangle z\|^2.$$

Thus, (3.36) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle z\|^2 + (c_3 N^2 - 2\gamma - c_4 C(M_2)N^{-2})\|\triangle z\|^2 \leq 0. \tag{3.37}$$

Similarly, take $N_3 \geq N_2$ large enough such that

$$\delta_3(N_3) = c_3 N_3^2 - (2\gamma + c_4 C(M_2))N_3^{-2} > 0.$$

For $N \geq N_3$, multiplying (3.37) with $e^{\delta_3 t}$ and integrating it from t_2 to t leads to

$$\|\triangle z\|^2 \leq \|\triangle z(t_2)\|^2 \mathrm{e}^{-\delta_3(t-t_2)}, \quad \forall\, t \geq t_2,$$

where

$$\delta_3 = \delta_3(N) = c_3 N^2 - (2\gamma + c_4 C(M_2))N^{-2}.$$

Let $\delta = \dfrac{1}{2}\min(\delta_1,\, \delta_2,\, \delta_3)$. It is easy to see that (3.6) holds. Consequently, the proof of Theorem 3.1 is completed. □

In virtue of Theorem 3.1, the solution operator $S(t) = S^{(2)}(t) : \ H_p^2(\Omega) \to H_p^2(\Omega)$ generated by the problem (1.4)–(1.6) can be decomposed as

$$S^{(2)}(t) = S_1^{(2)}(t) + S_2^{(2)}(t), \qquad \forall\, t \geq 0,$$

where $S_1^{(2)}(t)$ and $S_2^{(2)}(t)$ are defined by

$$S_1^{(2)}(t)u_0 = \begin{cases} P_N u(t) + y(t) = p(t) + y(t), & \geq t_2, \\ P_N u(t) = p(t), & t \leq t_2, \end{cases} \tag{3.38}$$

and

$$S_2^{(2)}(t)u_0 = \begin{cases} z(t), & t \geq t_2, \\ Q_N u(t) = q(t), & t \leq t_2. \end{cases} \tag{3.39}$$

Here $u(t) = S^{(2)}u_0$ for $t \geq t_2$, and $y(t)$ and $z(t)$ are solutions of systems (3.1)-(3.2) and (3.3)-(3.4), respectively. Hence, for every $u \in H_p^2(\Omega)$, we have

$$S^{(2)}(t)u = S_1^{(2)}(t)u + S_2^{(2)}(t)u. \tag{3.40}$$

4. Regularity of Attractor

Theorem 4.1. *Suppose that the condition (1.7) holds and σ is a positive integer or $\sigma \geq \frac{1}{2}([\frac{m}{2}])$ for any positive integer $m \geq 2$. Let $\mathcal{A}_m$ be the global attractors of the semi-group of operators, and $\{S^{(m)}(t)\}_{t\geq 0}$ generated by the problem (1.4)–(1.6). Then we have*

(i) For any $m \geq 3$, $\mathcal{A}_2$ is a bounded and closed set in $H_p^m(\Omega)$.

(ii) $\mathcal{A}_2 = \mathcal{A}_m$ for $m \geq 3$.

Proof. (i) Suppose that $u \in \mathcal{A}_2$. We shall prove $u \in H_p^m(\Omega)$ for any $m \geq 3$.

Owing to the well-known characterization of the ω-limit set [**29**], there exists a sequence of elements u_n in B_2 and a sequence of positive real numbers t_n' which approaches infinity as n tends to infinity such that

$$S^{(2)}(t_n')u_n \to u \text{ in } H_p^2(\Omega), \quad \text{as } n \to +\infty. \tag{4.1}$$

From (3.40) it holds

$$S^{(2)}(t_n')u_n = S_1^{(2)}(t_n')u_n + S_2^{(2)}(t_n')u_n, \quad \forall\, n \in \mathbf{N}. \tag{4.2}$$

Based on the definitions of $S_1^{(2)}(t)$ and $S_2^{(2)}(t)$ (i.e. (3.38) and (3.39)) and using Theorem 3.1, if N is large enough, then we have

$$\left\|S_1^{(2)}(t_n')u_n\right\|_m \leq C(N,m), \quad \forall\, n \in \mathbf{N}, \tag{4.3}$$

and

$$\left\|S_2^{(2)}(t_n')u_n\right\|_2 \leq c\mathrm{e}^{-\delta t_n'}\|u_n\|_2, \quad \forall\, n \in \mathbf{N}. \tag{4.4}$$

From (4.3), there exists a subsequences $\{t_{n'}'\}_{n'>0}$ and $w \in H_p^m(\Omega)$ such that

$$S_1^{(2)}(t_{n'}')u_{n'} \rightharpoonup w \quad \text{weakly in } H_p^m(\Omega), \quad \text{as } n' \to \infty, \tag{4.5}$$

and

$$\|w\|_m \leq \lim_{n'\to\infty} \inf \left\|S_1^{(2)}(t_{n'}')u_{n'}\right\|_m \leq C(N,m). \tag{4.6}$$

Taking account of $\varphi \in L_p^2(\Omega)$, from (4.2) we get

$$\big(S^{(2)}(t_{n'}')(u_{n'}),\varphi\big) = \big(S_1^{(2)}(t_{n'}')(u_{n'}),\varphi\big) + \big(S_2^{(2)}(t_{n'}')(u_{n'}),\varphi\big).$$

Letting n' tends to $+\infty$ in the above expression and using (4.1), (4.4) and (4.5), we find

$$(u,\varphi) = (w,\varphi), \quad \forall\, \varphi \in L_p^2(\Omega).$$

Setting $\varphi = (-\triangle)^m u$ and using (4.6), we have

$$\|\nabla^m u\| \leq \|w\|_m \leq C(N,m),$$

which shows $u \in H_p^m(\Omega)$. In other words, $\mathcal{A}_2$ is a bounded set in $H_p^m(\Omega)$.

(ii) We show that $\mathcal{A}_2 \subset \mathcal{A}_m$.

Since $\mathcal{A}_m$ attracts all bounded sets in $H_p^m(\Omega)$ and $\mathcal{A}_2$ is bounded in $H_p^m(\Omega)$, we get

$$\mathrm{dist}_{H_p^m(\Omega)}(S^{(2)}(t)\mathcal{A}_2,\mathcal{A}_m) = \mathrm{dist}_{H_p^m(\Omega)}(S^{(m)}(t)\mathcal{A}_2,\mathcal{A}_m) \to 0, \quad \text{as } t \to \infty.$$

In addition, $\mathcal{A}_2$ is an invariant set of $S^{(2)}(t)$, namely, $S^{(2)}(t)\mathcal{A}_2 = \mathcal{A}_2$, which leads to

$$\mathrm{dist}_{H_p^m(\Omega)}(\mathcal{A}_2,\mathcal{A}_m) = 0.$$

Hence, in view of $\mathcal{A}_m$ being closed in $H_p^m(\Omega)$, we have

$$\mathcal{A}_2 \subset \mathcal{A}_m.$$

Next, we show that $\mathcal{A}_m \subset \mathcal{A}_2$. Since $\mathcal{A}_2$ attracts the bounded set $\mathcal{A}_m$ in $H_p^2(\Omega)$, it implies that

$$\text{dist}_{H_p^2(\Omega)}(S^{(m)}(t)\mathcal{A}_m, \mathcal{A}_2) = \text{dist}_{H_p^2(\Omega)}(S^{(2)}(t)\mathcal{A}_m, \mathcal{A}_2) \to 0, \quad \text{as } t \to \infty.$$

It follows from $S^{(m)}(t)\mathcal{A}_m = \mathcal{A}_m$ and the Sobolev embedding theorem that

$$\text{dist}_{H_p^2(\Omega)}(\mathcal{A}_m, \mathcal{A}_2) = 0,$$

that is,

$$\mathcal{A}_m \subset \mathcal{A}_2.$$

Consequently, the proof of Theorem 4.1 is completed. □

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