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Controllability of Sobolev Type Fractional Evolution Systems

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Abstract. The main purpose of this paper is to investigate a class of Sobolev type semilinear fractional evolution systems in a separable Banach space. Applying a suitable fixed point theorem as well as condensing mapping, controllability results for two class of control sets are established by means of the theory of propagation family and technique of measure of noncompactness. An application involving a partial differential equation with a Caupto fractional derivative is considered.

Contents

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1. Introduction

In this paper, we consider the following Sobolev type fractional evolution system in a separable Banach space X:

$$\begin{cases} {}_0^C D_t^q (Ex(t)) = Ax(t) + Ef(t, x(t)) + EBu(t), \ t \in J := [0, a], \\ Ex(0) = Ex_0, \ x_0 \in D(E), \end{cases} \tag{1}$$

where ${}_0^C D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$ with the lower limit zero (see Definition 1.3), $A : D(A) \subset X \to X$ and $E : D(E) \subset X \to X$ are two closed linear operators and the pair (A, B) generates an exponentially bounded propagation family $\{W(t), t \geq 0\}$ of $D(E)$ to X (see Definition 2.11, Liang and Xiao [**1**]). The state $x(\cdot)$ takes values in X and the control function $u(\cdot)$ is given in $\mathscr{U}$, the Banach space of admissible control functions, where

$$\mathscr{U} := \begin{cases} L^p(J, U), \ \text{for } q \in (\frac{1}{p}, 1) \text{ with } 1 < p < \infty, \\ L^\infty(J, U), \ \text{for } q \in (0, 1), \end{cases}$$

and U is a Banach space. B is a bounded linear operator from U into $D(E)$ and $f : J \times X \to D(E) \subset X$ will be specified later.

Sobolev type evolution equations often arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids. Meanwhile, fractional calculus was planted over three hundred years ago and provided an excellent tool for the description of memory and hereditary properties of various materials and processes. In particular, the subject of fractional differential equations is gaining much importance and attention. So-called fractional differential equations are specified by generalizing the standard integer order derivative to arbitrary order. Due to the effective memory function of fractional derivative, fractional differential equations have been widely used to describe many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more interesting theory results and scientific applications of fractional differential equations, we cite the monographs [**2, 3, 4, 5, 6**], the research papers [**7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31**] and the references therein.

In the past decade, many researchers have studied the existence and controllability of the mild solutions for Cauchy problem of all kinds of Sobolev type evolution equations under the various of conditions on the pair (A, E). After reviewing these interesting results, the reader can find that $D(A) \subset D(E)$, boundedness or compactness of E^{-1} are posed (see [**10, 32, 33**]). In particular, Li et al. [**34**] obtained new existence results for Sobolev type fractional evolution equations by virtue of the theory of propagation family which generated by the pair (A, E) via the techniques of the measure of noncompactness and the condensing maps. The restrict conditions on the $D(A)$, $D(E)$ and E^{-1} are removed.

However, to the best of our knowledge, controllability results of Sobolev type fractional evolution systems via the theory of propagation family have not been explored. Thus, we offer to study the controllability of the system (1) via the theory of propagation family $\{W(t), t \geq 0\}$ generating by the pair (A, E). Our aim in this paper is to present sufficient conditions for the controllability results corresponding to two class of the possible admissible control sets. To simplify the process, we construct $\{\mathscr{T}_{(A,E)}(t), t \geq 0\}$ and $\{\mathscr{S}_{(A,E)}(t), t \geq 0\}$ associated with the pair (A, E) and give their boundedness and norm continuity in the sense of uniform operator topology. Here, we mixed and modify the conditions and techniques used in [**26, 32, 34, 35**] to prove the controllability results.

To end this section, we recall the following known definitions of fractional calculus (see, e.g., [**3, 4, 5**]).

DEFINITION 1.1. *The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$ {}_0I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0, $$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

DEFINITION 1.2. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$ {}_0^L D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n. $$

DEFINITION 1.3. *The Caputo derivative of order γ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$$ {}_0^C D_t^{\gamma} f(t) = {}_0^L D_t^{\gamma} \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \ t > 0, \ n-1 < \gamma < n. $$

REMARK 1.4. *(i) If $f(t) \in C^1[0, \infty)$, then ${}_0^C D_t^{\gamma} f(t) = {}_0I_t^{1-\gamma} f'(t)$, $t > 0$, $0 < \gamma < 1$. (ii) The Caputo derivative of a constant is equal to zero. (iii) If f is an abstract function with values in X, then integrals which appear in Definitions 1.1 and 1.2 are taken in Bochner's sense.*

2. Preliminaries

Let's recall some definitions and properties of measure of noncompactness and condensing maps (see, e.g., [**36, 37, 38**]).

DEFINITION 2.1. *Let Y^+ be the positive cone of an order Banach space $(Y, \leq)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in Y^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subset X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.*

The *MNC* Φ is said to be:

(i) Monotone: if for all bounded subsets Ω_1, Ω_2 of X, $\Omega_1 \subseteq \Omega_2$ implies $\Phi(\Omega_1) \leq \Phi(\Omega_2)$;

(ii) Nonsingular: if $\Phi(\{\theta\} \cup \Omega) = \Phi(\Omega)$ for every $\theta \in X$ and every nonempty subset $\Omega \subseteq X$;

(iii) Regular: if $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.

One of the most important examples of *MNC* is the noncompactness measure of Hausdorff χ defined on each bounded subset Ω of X by

$$ \chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net in } X\}. $$

It is well known that Hausdorff *MNC* χ enjoys the above properties (i)-(iii) and other properties (see [**37, 38**]).

(iv) $\chi(\Omega_1 + \Omega_2) \leq \chi(\Omega_1) + \chi(\Omega_2)$, where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, \ y \in \Omega_2\}$;

(v) $\chi(\Omega_1 \cup \Omega_2) \leq \max\{\chi(\Omega_1), \chi(\Omega_2)\}$;

(vi) $\chi(\lambda\Omega) \leq |\lambda|\chi(\Omega)$ for any $\lambda \in \mathbb{R}$;

(vii) If the map $Q : D(Q) \subseteq X \to Z$ is Lipschitz continuous with constant k, then $\chi_Z(Q\Omega) \leq k\chi(\Omega)$ for any bounded subset $\Omega \subseteq D(\Omega)$, where Z is a Banach space.

Now, let $G : J \to 2^X$ be a multifunction where 2^X denotes the class of all nonempty subsets of X. It is called:

(i) Integrable: if it admits a Bochner integrable selection $g : J \to X$, $g(t) \in G(t)$ for a.e. $t \in J$.

(ii) Integrably bounded: if there exists a function $\kappa(\cdot) \in L^1(J, \mathbb{R}^+)$ such that $\|G(t)\| := \sup\{\|g\| : g \in G(t)\} \leq \kappa(t)$ for a.e. $t \in J$.

Lemma 2.2. *(see Theorem 4.2.3,* [**38**]*) For an integrable, integrably bounded multifunction $G : J \to 2^X$ where X is a separable Banach space, let $\chi(G(t)) \leq g(t)$ for a.e. $t \in J$, where $g \in L^1(J, \mathbb{R}^+)$. Then $\chi(\int_0^t G(s)ds) \leq \int_0^t g(s)ds$ for all $t \in J$.*

We also recall definition of condensing maps and fixed point theorems via condensing maps (see, e.g., [**36, 38**]).

Definition 2.3. *Let β be a monotone nonsingular MNC in Banach space Y. A continuous map $\mathscr{P} : Y \subseteq Y \to Y$ is called condensing with respect to a MNC β (or β-condensing) if for every bounded set $\Omega \subseteq Y$ which which is not relatively compact, we have $\beta(\mathscr{P}(\Omega)) \ngeq \beta(\Omega)$.*

Theorem 2.4. *Let $\mathfrak{B}$ be a bounded convex closed subset of Y and $\mathscr{P} : \mathfrak{B} \to \mathfrak{B}$ a β-condensing map. Then $Fix\mathscr{P} = \{x : x = \mathscr{P}(x)\}$ is nonempty.*

Theorem 2.5. *Let $\mathcal{V} \subset Y$ be a bounded open neighborhood of zero and $\mathscr{H} : \overline{\mathcal{V}} \to Y$ a β-condensing map satisfying the boundary condition $x \neq \hat{\lambda}\mathscr{H}(x)$ for all $x \in \partial V$ and $\hat{\lambda} \in (0, 1]$. Then $Fix\mathscr{H} = \{x : x = \mathscr{H}(x)\}$ is a nonempty compact set.*

Next, we recall the concept of exponentially bounded propagation family (see Definition 1.4, [**1**]).

Definition 2.6. *A strongly continuous operator family $\{W(t), t \geq 0\}$ of $D(E)$ to a Banach space X satisfying that $\{W(t), t \geq 0\}$ is exponentially bounded, which means that there exist $\omega > 0$ and $M > 0$ such that $\|W(t)x\| \leq Me^{\omega t}\|x\|$ for any $x \in D(E)$ and $t \geq 0$, is called an exponentially bounded propagation family for the following abstract degenerate Cauchy problem*

$$\begin{cases} (Ex(t))' = Ax(t), \ t \in J, \\ Ex(0) = Ex_0, \ x_0 \in D(E), \end{cases} \tag{2}$$

if for $\lambda > \omega$,

$$(\lambda E - A)^{-1}Ex = \int_0^\infty e^{-\lambda t}W(t)xdt, \ x \in D(E). \tag{3}$$

In this case, we say that the problem (2) has an exponentially bounded propagation family $\{W(t), t \geq 0\}$.

Moreover, if (3) holds, we also say that the pair (A, E) generates an exponentially bounded propagation family $\{W(t), t \geq 0\}$.

Remark 2.7. *Since $D(E) \subset X$ is dense, $W(t)$ can be uniquely extended on X as a linear bounded mapping so that $\|W(t)x\| \leq Me^{\omega t}\|x\|$ for any $x \in X$ and $t \geq 0$. From now on, we consider such $W(t)$ on X directly.*

Denote

$$(4)\quad \mathscr{T}_{(A,E)}(t) = \int_0^\infty \xi_q(\theta)W(t^q\theta)d\theta,\ \mathscr{S}_{(A,E)}(t) = q\int_0^\infty \theta\xi_q(\theta)W(t^q\theta)d\theta,$$

where

$$(5)\quad \begin{aligned} \xi_q(\theta) &= \frac{1}{q}\theta^{-(1+\frac{1}{q})}\varpi_q(\theta^{-\frac{1}{q}}) \geq 0, \\ \varpi_q(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q). \end{aligned}$$

And, ξ_q is a probability density function defined on $(0,\infty)$, that is

$$(6)\ \xi_q(\theta) \geq 0,\ \theta \in (0,\infty),\ \int_0^\infty \xi_q(\theta)d\theta = 1,\ \text{and}\ \int_0^\infty \theta\xi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)}.$$

Using the similar method in [**12, 13, 34**], we can introduce the following definition of mild solution for the system (1).

Definition 2.8. *For each $u \in \mathscr{U}$ and $x_0 \in D(E)$, a mild solution of the system (1) we mean a function $x \in C(J,X)$ which satisfies*

$$\begin{aligned} x(t) &= \mathscr{T}_{(A,E)}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{S}_{(A,E)}(t-s)f(s,x(s))\,ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathscr{S}_{(A,E)}(t-s)Bu(s)ds,\ t \in J. \end{aligned}$$

The following results of $\mathscr{T}_{(A,E)}(\cdot)$ and $\mathscr{S}_{(A,E)}(\cdot)$ will be used throughout this paper.

Lemma 2.9. *Suppose the pair (A,E) generates an exponentially bounded propagation family $\{W(t), t \geq 0\}$. If $\{W(t), t \geq 0\}$ is a norm continuous family for $t > 0$ and $\|W(t)\| \leq M_1$ for $t \geq 0$, then the following two properties hold:*

(i) For any fixed $t \geq 0$, $\mathscr{T}_{(A,E)}(t)$ and $\mathscr{S}_{(A,E)}(t)$ are bounded operators on X, i.e., for any $x \in X$,

$$\|\mathscr{T}_{(A,E)}(t)x\| \leq M_1\|x\| \ \text{and}\ \|\mathscr{S}_{(A,E)}(t)x\| \leq \frac{M_1}{\Gamma(q)}\|x\|.$$

(ii) $\{\mathscr{T}_{(A,E)}(t), t \geq 0\}$ and $\{\mathscr{S}_{(A,E)}(t), t \geq 0\}$ are norm continuous family for $t > 0$ in the sense of uniform operator topology.

Proof. The first assertion has been proved (see Remark 2.1.3, [**34**]). Next, we verify the second assertion. We only need to prove that $\|\mathscr{T}_{(A,E)}(t_1) - \mathscr{T}_{(A,E)}(t_2)\|$ and $\|\mathscr{S}_{(A,E)}(t_1) - \mathscr{S}_{(A,E)}(t_2)\|$ tend to zero as $t_1 \to t_2$ respectively in the sense of uniform operator topology.

For $0 < t_1 < t_2 < \infty$, a simple computation implies

$$(7)\quad \|\mathscr{T}_{(A,E)}(t_1) - \mathscr{T}_{(A,E)}(t_2)\| \leq \int_0^\infty \xi_q(\theta)\|W(t_1^q\theta) - W(t_2^q\theta)\|d\theta,$$

$$(8)\quad \|\mathscr{S}_{(A,E)}(t_1) - \mathscr{S}_{(A,E)}(t_2)\| \leq q\int_0^\infty \theta\xi_q(\theta)\|W(t_1^q\theta) - W(t_2^q\theta)\|d\theta.$$

Note that $\|W(t_1^q\theta) - W(t_2^q\theta)\| \to 0$ as $t_1 \to t_2$ in the sense of uniform operator topology for any fixed $\theta > 0$. Linking (6) and (7), (8), one can obtain the second assertion immediately. The proof is complete. □

3. Main results

In this section, we study the controllability of the system (1) by utilizing the theory of propagation family and techniques of measure of noncompactness.

DEFINITION 3.1. *The system (1) is said to be controllable on the interval J if for every $x_0 \in D(E)$ and every $x_1 \in D(E)$ there exists a control $u \in \mathscr{U}$ such that the mild solution x of system (1) satisfies $x(a) = x_1$.*

We pose the following assumptions:

$[H_1]$: The pair (A, B) generates an exponentially bounded propagation family $\{W(t), t \geq 0\}$ of $D(E)$ to X.

$[H_2]$: $\{W(t), t \geq 0\}$ is norm continuous family for $t > 0$ and $\|W(t)\| \leq M_1$ for $t \geq 0$.

$[H_3]$: The control function $u(\cdot)$ takes from $\mathscr{U}$, the Banach space of admissible control functions, either $\mathscr{U} := L^p(J, U)$ for $q \in (\frac{1}{p}, 1)$ with $1 < p < \infty$ or $\mathscr{U} := L^\infty(J, U)$ for $q \in (0, 1)$ where U is a Banach space.

$[H_4]$: $B : U \to D(E)$ is a bounded linear operator and a linear operator $\mathbb{W} : \mathscr{U} \to X$ defined by

$$\mathbb{W}u = \int_0^a (a-s)^{q-1} \mathscr{S}_{(A,E)}(a-s) Bu(s) ds$$

has a bounded right inverse operator $\mathbb{W}^{-1} : X \to \mathscr{U}$.

It is easy to see that $\mathbb{W}u \in X$ and $\mathbb{W}$ is well defined due to the following fact:

$$\begin{aligned}
\|\mathbb{W}u\| &= \left\| \int_0^a (a-s)^{q-1} \mathscr{S}_{(A,E)}(a-s) Bu(s) ds \right\| \\
&\leq \frac{M_1 \|B\|}{\Gamma(q)} \int_0^a (a-s)^{q-1} \|u(s)\| ds \\
&\leq \begin{cases} \frac{M_1\|B\|}{\Gamma(q)} \left(\frac{p-1}{qp-1} a^{\frac{qp-1}{p-1}} \right)^{\frac{p-1}{p}} \|u\|_{\mathscr{U}}, \text{ if } q \in (\frac{1}{p}, 1), \\ u \in \mathscr{U} = L^p(J, U),\ 1 < p < \infty, \\ \frac{M_1\|B\| a^q}{\Gamma(q+1)} \|u\|_{\mathscr{U}}, \text{ if } q \in (0,1),\ u \in \mathscr{U} = L^\infty(J, U). \end{cases}
\end{aligned}$$

Meanwhile,

$$\int_0^t (t-s)^{q-1} \|u(s)\| ds \leq K_q \|u\|_{\mathscr{U}}, \tag{9}$$

where

$$K_q := \begin{cases} \left(\frac{p-1}{qp-1} a^{\frac{qp-1}{p-1}} \right)^{\frac{p-1}{p}} \|u\|_{\mathscr{U}}, \text{ if } q \in (\frac{1}{p}, 1),\ u \in \mathscr{U} = L^p(J, U),\ 1 < p < \infty, \\ \frac{a^q}{q} \|u\|_{\mathscr{U}}, \text{ if } q \in (0,1),\ u \in \mathscr{U} = L^\infty(J, U), \end{cases}$$

for any $t \in J$.

Next we assume:

$[H_5]$: f satisfies the following two conditions:

(i) For each $x \in X$ the function $f(\cdot, x) : J \to D(E) \subset X$ is strongly measurable and for each $t \in J$, the function $f(t, \cdot) : X \to D(E) \subset X$ is continuous.

(ii) For each $k > 0$, there is a measurable function g_k such that

$$\sup_{\|x\| \leq k} \|f(t,x)\| \leq g_k(t), \text{ with } \|g_k\|_\infty := \sup_{s \in J} g_k(s) < \infty,$$

$$\sup_{t\in J}\int_0^t (t-s)^{q-1} g_k(s)ds \le \gamma k,$$

for any $k>0$ sufficiently large and some γ.

(iii) There exists a positive constant $L>0$ such that

$$\chi(f(t,D)) \le L\chi(D),$$

for any bounded set $D\subset X$ and a.e. $t\in J$.

The first step in studying the controllability problem is to determine if an objective can be reached by some suitable control. A standard approach is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. For the sake of simplicity, we present the standard framework to deal with controllability problems here.

Based on our assumptions, for an arbitrary function $x(\cdot)$, it is suitable to define the following control formula

$$u(t) = \mathbb{W}^{-1}\left[x_1 - \mathscr{T}_{(A,E)}(a)x_0 - \int_0^a (a-s)^{q-1}\mathscr{S}_{(A,E)}(a-s)f(s,x(s))ds\right]. \tag{10}$$

In what follows, it is necessary to show that when using the control u in (10), the operator $\mathcal{P}$ defined by

$$\begin{aligned}(\mathcal{P}x)(t) &= \mathscr{T}_{(A,E)}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{S}_{(A,E)}(t-s)f(s,x(s))ds \\ &\quad + \int_0^t (t-s)^{q-1}\mathscr{S}_{(A,E)}(t-s)Bu(s)ds,\ \text{for } t\in J,\end{aligned} \tag{11}$$

from $C(J,X)$ into $C(J,X)$, has a fixed point. Clearly, this fixed point is just a mild solution of system (1). Further, one can check

$$\begin{aligned}(\mathcal{P}x)(a) &= \mathscr{T}_{(A,E)}(a)x_0 + \int_0^a (a-s)^{q-1}\mathscr{S}_{(A,E)}(a-s)f(s,x(s))ds \\ &\quad + \int_0^a (a-s)^{q-1}\mathscr{S}_{(A,E)}(a-s) \\ &\quad \times B\mathbb{W}^{-1}\Big[x_1 - \mathscr{T}_{(A,E)}(a)x_0 \\ &\quad - \int_0^a (a-\tau)^{q-1}\mathscr{S}_{(A,E)}(a-\tau)f(\tau,x(\tau))d\tau\Big]ds \\ &= x_1,\end{aligned}$$

which means that u steers the fractional system (1) from x_0 to x_1 in finite time a. Consequently, we can claim the system (1) is controllable on J.

For each number $k>0$, define

$$\mathcal{B}_k = \{x\in C(J,X):\ \|x(t)\|\le k,\ t\in J\}.$$

Of course, $\mathcal{B}_k$ is clearly a bounded, closed, convex subset in $C(J,X)$.

Under the assumptions $[H_1]-[H_5]$, we will establish some important results as follows.

Lemma 3.2. *Assuming*

$$\rho := \begin{cases} \frac{\gamma M_1}{\Gamma(q)}\left(1+\frac{\sqrt{a}M_1\|B\|K_q\|\mathbb{W}^{-1}\|}{\Gamma(q)}\right) < 1,\ \textit{if } \mathscr{U} = L^2(J,U), \\ \frac{\gamma M_1}{\Gamma(q)}\left(1+\frac{M_1\|B\|K_q\|\mathbb{W}^{-1}\|}{\Gamma(q)}\right) < 1,\ \textit{if } \mathscr{U} = L^\infty(J,U), \end{cases} \tag{12}$$

there exists a constant $K \geq \frac{M^*}{1-\rho}$ *such that* $\mathcal{P}\mathcal{B}_K \subset \mathcal{B}_K$*, where*

$$M^* := \begin{cases} M_1\|x_0\| + \frac{\sqrt{a}M_1\|B\|}{\Gamma(q)} K_q \|\mathbb{W}^{-1}\| \Big(\|x_1\| + M_1\|x_0\| \Big), & \text{if } \mathscr{U} = L^2(J,U), \\ M_1\|x_0\| + \frac{M_1\|B\|}{\Gamma(q)} K_q \|\mathbb{W}^{-1}\| \Big(\|x_1\| + M_1\|x_0\| \Big), & \text{if } \mathscr{U} = L^\infty(J,U). \end{cases}$$

Proof. Let $x \in \mathcal{B}_K$. For $t \in J$, using our assumptions and Lemma 2.9(i), we obtain

$$\begin{aligned} \|(\mathcal{P}x)(t)\| &\leq M_1\|x_0\| + \frac{M_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g_K(s) ds + \\ &\quad \frac{M_1\|B\|}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|u(s)\| ds \\ &\leq M_1\|x_0\| + \frac{M_1\gamma K}{\Gamma(q)} + \frac{M_1\|B\|}{\Gamma(q)} K_q \|u\|_{\mathscr{U}} \\ &= \rho K + M^* \\ &\leq K, \end{aligned}$$

where we note that the control u defined in (10) satisfies

$$\begin{aligned} \|u(t)\| &\leq \|\mathbb{W}^{-1}\| \left\| x_1 - \mathscr{T}_{(A,E)}(a)x_0 - \int_0^a (a-s)^{q-1} \mathscr{S}_{(A,E)}(a-s) f(s,x(s)) ds \right\| \\ &\leq \|\mathbb{W}^{-1}\| \left(\|x_1\| + M_1\|x_0\| + \frac{M_1}{\Gamma(q)} \gamma K \right), \end{aligned}$$

which implies that

$$\|u\|_{\mathscr{U}} \leq \begin{cases} \sqrt{a}\|\mathbb{W}^{-1}\| \left(\|x_1\| + M_1\|x_0\| + \frac{M_1}{\Gamma(q)}\gamma K \right), & \text{if } \mathscr{U} = L^2(J,U), \\ \|\mathbb{W}^{-1}\| \left(\|x_1\| + M_1\|x_0\| + \frac{M_1}{\Gamma(q)}\gamma K \right), & \text{if } \mathscr{U} = L^\infty(J,U). \end{cases} \tag{13}$$

Hence, $\mathcal{P}\mathcal{B}_K \subset \mathcal{B}_K$ for any $K \geq \frac{M^*}{1-\rho}$ sufficiently large. The proof is complete. □

LEMMA 3.3. *The operator* $\mathcal{P}$ *defined by (11) is continuous.*

Proof. Let $\{x_m\}_{m\in N} \subseteq \mathcal{B}_K$ be a sequence such that $x_m \to x$ as $m \to \infty$. Note that $(t-s)^{q-1} f(s,x_m(s)) \to (t-s)^{q-1} f(s,x(s))$ as $m \to \infty$ for very $t \in J$ and almost each $s \in [0,t]$ and

$$(t-s)^{q-1}\|f(s,x_m(s)) - f(s,x(s))\| \leq 2(t-s)^{q-1} g_K(s).$$

Since $\int_0^t (t-s)^{q-1} g_K(s) \le \frac{\|g_K\|_\infty}{q}$, by the Lebesgue's Dominated Convergence Theorem, we get

$$\begin{aligned}
& \|(\mathcal{P}x_m)(t) - (\mathcal{P}x)(t)\| \\
\le\ & \frac{M_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Bigg[\|f(s, x_m(s)) - f(s, x(s))\| \\
+\ & \|B\|\|\mathbb{W}^{-1}\| \int_0^a (a-z)^{q-1} \|f(z, x_m(z)) - f(z, x(z))\| dz \Bigg] ds \\
=\ & \frac{M_1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x_m(s)) - f(s, x(s))\| ds \\
+\ & \frac{M_1 \|B\|\|\mathbb{W}^{-1}\| a^q}{\Gamma(q+1)} \left\| \int_0^a (a-s)^{q-1} \|f(s, x_m(s)) - f(z, x(s)) \right\| ds \\
\to\ & 0, \text{ as } m \to \infty,
\end{aligned}$$

for $t \in J$. This yields that $\mathcal{P}$ is continuous. The proof is complete. □

Let χ be a Hausdorff *MNC* in X. Consider the measure of noncompactness ν in the space $C(J, X)$ with values in the cone $\mathbb{R}^2$ of the way: for every bounded subset $\Omega \subset C(J, X)$,

$$\nu := (\psi(\Omega), mod_c(\Omega))$$

where $\psi(\Omega) := \sup_{t \in J} \chi(\Omega(t))$ and $mod_c(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \le \delta} \|x(t_1) - x(t_2)\|$.

Lemma 3.4. *Assume*

$$\frac{a^q M_1}{\Gamma(q+1)} L \left[1 + \frac{a^q M_1}{\Gamma(q+1)} \|B\|\|\mathbb{W}^{-1}\| \right] < 1. \tag{14}$$

If $\nu(\mathcal{P}(\mathcal{B}_K)) \ge \nu(\mathcal{B}_K)$ *then* $\psi(\mathcal{B}_K) = 0$.

Proof. Clearly $\mathcal{B}_K \subset C(J, X)$ is nonempty and bounded. For any $t \in J$, we set

$$\Theta(\mathcal{B}_K(t)) = \int_0^t G(s) ds,$$

where a function $s \in [0, t] \multimap G(s)$ is defined as:

$$\begin{aligned}
G(s) \quad = \quad & \Big\{ (t-s)^{q-1} \mathscr{S}_{(A,E)}(t-s) f(s, x(s)) \\
& + (t-s)^{q-1} \mathscr{S}_{(A,E)}(t-s) B u(s) : x \in \mathcal{B}_K \Big\}
\end{aligned}$$

and $u(t)$ is given by (10). It is obvious that G is integrable and integrably bounded. Moreover, a simple computation implies that

$$\begin{aligned}
\chi(G(s)) &\leq \frac{M_1}{\Gamma(q)}(t-s)^{q-1}\chi\Bigg(\Bigg\{f(s,x(s)) \\
&\quad +B\mathbb{W}^{-1}\Bigg[x_1-\mathscr{T}_{(A,E)}(a)x_0 \\
&\quad -\int_0^a (a-s)^{q-1}\mathscr{S}_{(A,E)}(a-s)f(s,x(s))ds\Bigg] : x\in\mathcal{B}_K\Bigg\}\Bigg) \\
&\leq \frac{M_1}{\Gamma(q)}(t-s)^{q-1}\Bigg[\chi\Bigg(\Bigg\{f(s,\mathcal{B}_K(s))\Bigg\}\Bigg) \\
&\quad +\chi\Bigg(\Bigg\{B\mathbb{W}^{-1}\Bigg[x_1-\mathscr{T}_{(A,E)}(a)x_0 \\
&\quad -\int_0^a (a-s)^{q-1}\mathscr{S}_{(A,E)}(a-s)f(s,\mathcal{B}_K(s))ds\Bigg]\Bigg\}\Bigg)\Bigg] \\
&\leq \frac{M_1}{\Gamma(q)}(t-s)^{q-1}\Bigg[L\chi(\mathcal{B}_K(s)) \\
&\quad +\frac{M_1}{\Gamma(q)}\|B\|\|\mathbb{W}^{-1}\|\Bigg(\int_0^a (a-s)^{q-1}L\chi(\mathcal{B}_K(s))ds\Bigg)\Bigg] \\
&\leq \frac{M_1}{\Gamma(q)}(t-s)^{q-1}L\Bigg[1+\frac{a^qM_1}{\Gamma(q+1)}\|B\|\|\mathbb{W}^{-1}\|\Bigg]\psi(\mathcal{B}_K) := \kappa(s).
\end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned}
\chi(\Theta(\mathcal{B}_K(t))) &\leq \int_0^t \kappa(s)ds \\
&\leq \frac{t^qM_1}{\Gamma(q+1)}L\Bigg[1+\frac{a^qM_1}{\Gamma(q+1)}\|B\|\|\mathbb{W}^{-1}\|\Bigg]\psi(\mathcal{B}_K).
\end{aligned}$$

Thus,

$$\begin{aligned}
\psi(\mathcal{P}(\mathcal{B}_K(t))) &\leq \chi(\Theta(\mathcal{B}_K(t))) \\
&\leq \frac{a^qM_1}{\Gamma(q+1)}L\Bigg[1+\frac{a^qM_1}{\Gamma(q+1)}\|B\|\|\mathbb{W}^{-1}\|\Bigg]\psi(\mathcal{B}_K),
\end{aligned}$$

which implies $\psi(\mathcal{B}_K)=0$ due to the condition (14) and $\nu(\mathcal{P}(\mathcal{B}_K))\geq\nu(\mathcal{B}_K)$. The proof is complete. □

LEMMA 3.5. *If* $\nu(\mathcal{P}(\mathcal{B}_K))\geq\nu(\mathcal{B}_K)$ *then* $mod_c(\mathcal{B}_K)=0$.

Proof. To achieve our aim, we need to prove $\mathcal{P}(\mathcal{B}_K)$ is equicontinuous. Let $x \in \mathcal{B}_K$ and $t', t'' \in J$ such that $0 < t' < t''$, then

$$\begin{aligned}
& \|(\mathcal{P}x)(t'') - (\mathcal{P}x)(t')\| \\
\leq\ & \|\mathscr{T}_{(A,E)}(t'')x_0 - \mathscr{T}_{(A,E)}(t')x_0\| \\
& + \bigg\| \int_0^{t''} (t''-s)^{q-1}\mathscr{S}_{(A,E)}(t''-s) f(s, x(s))\, ds \\
& - \int_0^{t'} (t'-s)^{q-1}\mathscr{S}_{(A,E)}(t'-s) f(s, x(s))\, ds \bigg\| \\
& + \bigg\| \int_0^{t''} (t''-s)^{q-1}\mathscr{S}_{(A,E)}(t''-s) Bu(s) ds \\
& - \int_0^{t'} (t'-s)^{q-1}\mathscr{S}_{(A,E)}(t'-s) Bu(s) ds \bigg\|
\end{aligned}$$

$$\begin{aligned}
\leq\ & \|\mathscr{T}_{(A,E)}(t'') - \mathscr{T}_{(A,E)}(t')\| \|x_0\| \\
& + \int_0^{t''} |(t''-s)^{q-1} - (t'-s)^{q-1}| \left\| \mathscr{S}_{(A,E)}(t''-s) f(s, x(s)) \right\| ds \\
& + \int_0^{t'} (t'-s)^{q-1} \left\| [\mathscr{S}_{(A,E)}(t''-s) - \mathscr{S}_{(A,E)}(t'-s)] f(s, x(s)) \right\| ds \\
& + \int_0^{t''} |(t''-s)^{q-1} - (t'-s)^{q-1}| \left\| \mathscr{S}_{(A,E)}(t''-s) Bu(s) \right\| ds \\
& + \int_0^{t'} (t'-s)^{q-1} \left\| [\mathscr{S}_{(A,E)}(t''-s) - \mathscr{S}_{(A,E)}(t'-s)] Bu(s) \right\| ds \\
& + \int_{t'}^{t''} (t'-s)^{q-1} \left\| \mathscr{S}_{(A,E)}(t''-s) f(s, x(s)) \right\| ds \\
& + \int_{t'}^{t''} (t'-s)^{q-1} \left\| \mathscr{S}_{(A,E)}(t''-s) Bu(s) \right\| ds \\
\leq\ & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \|\mathscr{T}_{(A,E)}(t'') - \mathscr{T}_{(A,E)}(t')\| \|x_0\|, \\
I_2 &= \frac{M_1}{\Gamma(q)} \int_0^{t''} [(t'-s)^{q-1} - (t''-s)^{q-1}] g_K(s) ds, \\
I_3 &= \sup_{s \in [0,t']} \|\mathscr{S}_{(A,E)}(t''-s) - \mathscr{S}_{(A,E)}(t'-s)\| \int_0^{t'} (t'-s)^{q-1} g_K(s) ds,
\end{aligned}$$

$$
\begin{aligned}
I_4 &= \frac{M_1\|B\|}{\Gamma(q)}\int_0^{t''}[(t'-s)^{q-1}-(t''-s)^{q-1}]\|u(s)\|ds,\\
I_5 &= \sup_{s\in[0,t']}\|\mathscr{S}_{(A,E)}(t''-s)-\mathscr{S}_{(A,E)}(t'-s)\|\|B\|\int_0^{t'}(t'-s)^{q-1}\|u(s)\|ds\\
I_6 &:= \frac{M_1}{\Gamma(q)}\int_{t'}^{t''}(t'-s)^{q-1}g_K(s)ds,\\
I_7 &:= \frac{M_1\|B\|}{\Gamma(q)}\int_{t'}^{t''}(t'-s)^{q-1}\|u(s)\|ds.
\end{aligned}
$$

Note that Lemma 2.9(ii), $\mathscr{T}_{(A,E)}(t)$ and $\mathscr{S}_{(A,E)}(t)$ are continuous in the uniform operator topology for $t\geq 0$, $\sup_{s\in J}|g_K(s)|<\infty$ and $u(\cdot)$ is bounded by (13). We can obtain the terms $I_1, I_3, I_5, I_6, I_7 \to 0$ as $t''\to t'$. Moreover, applying

$$\int_0^{t''}[(t'-s)^{q-1}-(t''-s)^{q-1}]ds=\frac{t'^q-t''^q+(t''-t')^q}{q}$$

one can check the terms $I_2, I_4\to 0$ as $t''\to t'$. Thus, $\mathcal{P}(\mathcal{B}_K)$ is equicontinuous.

Hence, $mod_c(\mathcal{P}(\mathcal{B}_K))=0$. This implies that $mod_c(\mathcal{B}_K)=0$ from $\nu(\mathcal{P}(\mathcal{B}_K))\geq\nu(\mathcal{B}_K)$. The proof is complete. □

LEMMA 3.6. *The operator $\mathcal{P}$ defined by (11) is ν-condensing on $\mathcal{B}_K$.*

Proof. It follows from Lemmas 3.4 and 3.5 that $\nu(\mathcal{B}_K)=(0,0)$. The regularity property of v implies the relative compactness of $\mathcal{B}_K$. It follows from Definition 2.3 that $\mathcal{P}$ is ν-condensing on $\mathcal{B}_K$. □

For $\hat{\lambda}\in(0,1]$, consider a one-parameter family of maps $\mathcal{H}:[0,1]\times C(J,X)\to C(J,X)$ given by

$$(\hat{\lambda},x)\to\mathcal{H}(\hat{\lambda},x)=\hat{\lambda}\mathcal{P}(x).$$

LEMMA 3.7. *The fixed point set of the family of maps $\mathcal{H}$:*

$$Fix\mathcal{H}=\{x\in\mathcal{H}(\hat{\lambda},x)\ \textit{for some}\ \hat{\lambda}\in(0,1]\}$$

has a priori bounded.

Proof. The result can be derived by Lemma 3.2 immediately. We omit it here. □

Now we are ready to state the main results in this paper.

THEOREM 3.8. *Assume $[H_1]-[H_5]$ are satisfied. Then the system (1) is controllable on J provided that the conditions (12) and (14) hold.*

Proof. To obtain our conclusion, we need to prove $\mathcal{P}$ has a fixed point in $\mathcal{B}_K$. In fact, it follows from Lemmas 3.2 and 3.6 that $\mathcal{P}:\mathcal{B}_K\to\mathcal{B}_K$ is ν-condensing map. By Theorem 2.5, $\mathcal{P}$ has a fixed point in $\mathcal{B}_K$. This implies that any fixed point of $\mathcal{P}$ is just a mild solution of the system (1) on J which satisfying $(\mathcal{P}x)(a)=x_1$ with $u(t)$ given by (10). Therefore, the system (1) is controllable on J. □

COROLLARY 3.9. *Let the assumptions of Theorem 3.8 be satisfied. The set of mild solutions of the system (1) is a nonempty and compact subset of $C(J,X)$ with $u(t)$ given by (10).*

Proof. It follows from Lemma 3.7 that we can take a closed ball $\mathcal{B}_K$ to contain the set $Fix\mathcal{H}$ inside itself. Moreover, $\mathcal{P}$ maps $\mathcal{B}_K$ into $C(J,X)$ and is ν-condensing map. By Theorem 2.5, we have the conclusion. □

4. Example

Take $X = U = L^2[0,\pi]$. We consider the following fractional partial differential equation with control

$$(15)\quad \begin{cases} {}_0^C D_t^{\frac{4}{5}} (x(t,y) - x_{yy}(t,y)) = x_{yy}(t,y) + \mu t^2 \left(\sin \frac{x(t,y)}{t} - \sin_{yy} \frac{x(t,y)}{t}\right) + Bu(t), \\ y \in [0,\pi],\ t \in J_1 = [0,1],\ 0 < \mu < \infty, \\ x(t,0) = x(t,\pi) = 0,\ t \geq 0, \\ x(0,y) - x_{yy}(0,y) = x_0(y),\ 0 \leq y \leq \pi. \end{cases}$$

Define $A : D(A) \subset X \to X$ by $Ax = x_{yy}$ and $E : D(E) \subset X \to X$ by $Ex = x - x_{yy}$ respectively, where each domain $D(A)$, $D(E)$ is given by $\{x \in X : x, x_y$ are absolutely continuous, $x_{yy} \in X,\ x(0) = x(\pi) = 0\}$.

It follows from Theorem 2.2 in [**1**] that the pair (A,E) can generate a propagation family $\{W(t), t \geq 0\}$ of uniformly bounded and $\{W(t), t \geq 0\}$ is norm continuous for $t > 0$ and $\|W(t)\| \leq 1$. Meanwhile, it follows from [**39**] that A and E can be written as $Ax = -\sum_{n=1}^{\infty} n^2 \langle x, x_n\rangle,\ x \in D(A)$ and $Ex = \sum_{n=1}^{\infty}(1+n^2)\langle x, x_n\rangle x_n,\ x \in D(E)$, respectively, where $x_n(y) = \sqrt{\frac{2}{\pi}} \sin ny,\ n = 1, 2, \cdots$ is the orthonormal set of eigenfunctions of A. Hence for any $x \in D(E)$, $\lambda > 0$ we obtain

$$(\lambda E - A)^{-1} Ex = \sum_{n=1}^{\infty} \frac{1+n^2}{\lambda(1+n^2)+n^2} \langle x, x_n\rangle x_n = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\lambda t} e^{-\frac{n^2}{1+n^2}t} dt \langle x, x_n\rangle x_n.$$

Therefore, $\{W(t), t \geq 0\}$ can be generated by $-AE^{-1}$ and written as

$$W(t)x := \sum_{n=1}^{\infty} e^{-\frac{n^2}{1+n^2}t} \langle x, x_n\rangle x_n.$$

Then, $\mathscr{T}_{(A,E)}(\cdot)$ and $\mathscr{S}_{(A,E)}(\cdot)$ can be written as

$$\begin{aligned} \mathscr{T}_{(A,E)}(t)x &= \int_0^{\infty} \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} e^{-\frac{n^2}{1+n^2} t^{\frac{4}{5}}\theta} \langle x, x_n\rangle x_n d\theta, \\ \mathscr{S}_{(A,E)}(t)x &= \frac{4}{5} \int_0^{\infty} \theta \xi_{\frac{4}{5}}(\theta) \sum_{n=1}^{\infty} e^{-\frac{n^2}{1+n^2} t^{\frac{4}{5}}\theta} \langle x, x_n\rangle x_n d\theta. \end{aligned}$$

Clearly, $\|\mathscr{T}_{(A,E)}(t)\| \leq 1$ and $\|\mathscr{S}_{(A,E)}(t)\| \leq \frac{1}{\Gamma(\frac{4}{5})}$ for $t \geq 0$.

Next, $B : U \to D(E)$ is defined by $B = bI$, $b > 0$ and defined by

$$\mathbb{W}u = b \int_0^1 (1-s)^{-\frac{1}{5}} \mathscr{S}_{(A,E)}(1-s) u(s,y) ds.$$

Since $q=\frac{4}{5}>\frac{1}{2}$, we can take $p=2$ and $\mathscr{U}=L^2(J_1,U)$ and so $K_{\frac{4}{5}}=\sqrt{\frac{5}{3}}$. It is easy to show that $\mathbb{W}$ is surjective. Indeed, if $u(s,y):=x(y)\in\mathscr{U}$. Then

$$\begin{aligned}\mathbb{W}u &= b\int_0^1(1-s)^{-\frac{1}{5}}\frac{4}{5}\int_0^\infty\theta\xi_{\frac{4}{5}}(\theta)\sum_{n=1}^\infty e^{-\frac{n^2}{1+n^2}(1-s)^{\frac{4}{5}}\theta}\langle x,x_n\rangle x_n d\theta ds\\ &= b\int_0^\infty\xi_{\frac{4}{5}}(\theta)\sum_{n=1}^\infty\int_0^1\frac{4}{5}\theta(1-s)^{-\frac{1}{5}}e^{-\frac{n^2}{1+n^2}(1-s)^{\frac{4}{5}}\theta}ds\langle x,x_n\rangle x_n d\theta\\ &= b\int_0^\infty\xi_{\frac{4}{5}}(\theta)\sum_{n=1}^\infty\int_0^1\frac{1+n^2}{n^2}\frac{d}{ds}\left[e^{-\frac{n^2}{1+n^2}(1-s)^{\frac{4}{5}}\theta}\right]ds\langle x,x_n\rangle x_n d\theta\\ &= b\int_0^\infty\xi_{\frac{4}{5}}(\theta)\sum_{n=1}^\infty\frac{1+n^2}{n^2}\left[1-e^{-\frac{n^2}{1+n^2}\theta}\right]\langle x,x_n\rangle x_n d\theta\\ &= b\sum_{n=1}^\infty\frac{1+n^2}{n^2}\left[1-\mathbb{E}_{\frac{4}{5}}\left(-\frac{n^2}{1+n^2}\right)\right]\langle x,x_n\rangle x_n,\end{aligned}$$

where $\mathbb{E}_{\frac{4}{5}}$ is a Mittag-Leffler function [**40, 41**]. So we can define a right inverse $\mathbb{W}^{-1}:X\to\mathscr{U}$ by

$$(\mathbb{W}^{-1}x)(t,y):=\frac{1}{b}\sum_{n=1}^\infty\frac{n^2}{1+n^2}\frac{\langle x,x_n\rangle x_n}{1-\mathbb{E}_{\frac{4}{5}}\left(-\frac{n^2}{1+n^2}\right)},\ \text{for } x=\sum_{n=1}^\infty\langle x,x_n\rangle x_n,$$

with

$$\|\mathbb{W}^{-1}\|=\frac{1}{b\left(1-\mathbb{E}_{\frac{4}{5}}\left(-\frac{n^2}{1+n^2}\right)\right)}\le\frac{1}{b\left(1-\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)\right)}.$$

Now $f:J_1\times\mathbb{R}\to\mathbb{R}$ is defined by $f(t,x(t,y))=\mu t^2\sin\frac{x(t,y)}{t}$. It is easy to see that f is measurable for the first variable and $f(t,x)$ is continuous for the second variable. Moreover, clearly $\limsup_{k\to\infty}\frac{1}{k}\sup_{t\in J_1,|x|\le k}|f(t,x)|=0$ and $\chi(f(t,D_1))\le\mu t\chi(D_1)\le\mu\chi(D_1)$ for any bounded set $D_1\subset X$ and $t\in J_1$. Hence $\gamma=0$ and $L=\mu$.

Define $F:J_1\times C(J,X)\to D(E)$ by $F(t,z)(y)=f(t,z(y))$. Now, the system (15) can be abstracted as

$$\begin{cases} {}_0^C D_t^{\frac{4}{5}}(Ex(t))=-Ax(t)+EF(t,x(t))+EBu(t),\ t\in J_1,\\ Ex(0)=Ex_0.\end{cases}$$

From the above discussion, all the assumptions in Theorem 3.8 are satisfied, since by $\gamma=0$, (12) holds, while (14) holds when:

$$\frac{\mu}{\Gamma(\frac{9}{5})}\left[1+\frac{1}{\Gamma(\frac{9}{5})\left(1-\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)\right)}\right]<1.$$

Then the system (15) is controllable on J_1.

Finally, one can numerically find that $\mu<0.229071$. It is key to compute $\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)$.

We only provide a possible way to compute $\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)$. In fact, we can use the definition

$$\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k\Gamma(1+\frac{5i}{4})}=\sum_{k=0}^{25}\frac{(-1)^k}{2^k\Gamma(1+\frac{5i}{4})}+\sum_{k=26}^{\infty}\frac{(-1)^k}{2^k\Gamma(1+\frac{5i}{4})}.$$

Using Mathematica we get

$$\sum_{k=0}^{25}\frac{(-1)^k}{2^k\Gamma(1+\frac{5i}{4})}\doteq 0.626879.$$

On the other hand, it holds

$$\sum_{k=26}^{\infty}\frac{(-1)^k}{2^k\Gamma(1+\frac{5i}{4})}\le\sum_{k=26}^{\infty}\frac{1}{2^k}=\frac{1}{2^{25}}\doteq 2.98023\times 10^{-8}.$$

Hence $\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)\doteq 0.626879$. The rest computations to estimate μ is given again by Mathematica, since Γ is built in Mathematica.

We also remark that one can compute $\mathbb{E}_{\frac{4}{5}}\left(-\frac{1}{2}\right)$ by using the formula (21) in [**41**].

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