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Regularity of solutions of a phase field model

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Abstract. Phase field models are widely-used for modelling phase transition processes such as solidification, freezing or CO_2 sequestration. In this paper, a phase field model proposed by G. Caginalp is considered. The existence and uniqueness of solutions are proved in the case of nonsmooth initial data. Continuity of solutions with respect to time is established. In particular, it is shown that the governing initial boundary value problem can be considered as a dynamical system.

Contents

1. Introduction 353
2. The Model 354
3. Statement of the problem and main result 355
4. Construction of Approximations 356
5. Existence of weak solutions 359
6. Uniqueness and stability 360
7. Conclusions 364
Acknowledgements 364
References 364

1. Introduction

Nowadays, phase field techniques for modeling of solidification and freezing processes become very popular (see e.g. [**1**], [**6**], [**2**], [**5**], [**8**], and [**17**]). They are based on the consideration of the Gibbs free energy which depends on an order

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parameter that assumes values from -1 (solid) to 1 (liquid) and changes sharply but smoothly over the solidification front so that the sharp liquid/solid interface becomes smoothed. The rate of smoothing is controlled by a small parameter, which enables to reach arbitrary approximation of the sharp interface.

Phase field models are also appropriate for the description of phase transitions when modeling CO_2 sequestration. The supercritical carbon dioxide, CO_2 that has been pressurized to a phase between gas and liquid, may be injected into a saline aquifer where it may either dissolve in the brine, react with the dissolved minerals or the surrounding rock, or become trapped in the pore space of the aquifer.

In this paper, we consider a phase field model proposed by G. Caginalp in [**1**]. The aim of the investigation is to prove the existence and uniqueness of solutions to this model for very general initial data (comp. with [**13**], [**9**], [**4**], [**3**], [**16**], and [**14**]). Moreover, it will be proved that the solutions are continuous in time, and their values at each time instant lie in the same space as the initial data. Thus, the model can be considered as a dynamical system.

The paper is structured as follows. Section 2 introduces the phase field model. The precise formulation of the main results of the paper are given in Section 3. Approximate solutions are constructed in Section 4, and the existence of weak solutions is shown in Section 5. Uniqueness, stability and continuity in time of solutions are proved in Section 6.

2. The Model

The phase field model derived by G. Caginalp (see [**1**]) is given by system (2.1). The temperature u and the phase-function ϕ are defined on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$. The evolution of these functions is given by the initial boundary value problem

$$
\begin{aligned}
u_t + \frac{l}{2}\,\phi_t - \kappa\,\Delta u &= 0 && \text{in } \Omega \times (0,T),\\
\tau\,\phi_t - 2u + \frac{1}{2}\,\left(\phi^3 - \phi\right) - \xi^2 \Delta\phi &= 0 && \text{in } \Omega \times (0,T),\\
-\kappa\,\partial_{\boldsymbol{\nu}} u = \lambda(u-g), \qquad -\partial_{\boldsymbol{\nu}}\phi &= 0 && \text{on } \partial\Omega \times (0,T),\\
u(\boldsymbol{x},0) = u^0(\boldsymbol{x}), \qquad \phi(\boldsymbol{x}) &= \phi^0(\boldsymbol{x}) && \text{for } t=0 \text{ on } \Omega.
\end{aligned}
\tag{2.1}
$$

The constants κ and l appearing in (2.1) denote the heat conductivity and the latent heat, respectively. The boundary temperature regime is defined by a given function g. The region Ω is assumed to be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, i.e. $\partial\Omega$ is of class $\mathcal{C}^{0,1}$. The evolution of u and ϕ is considered on the time interval $[0,T]$ where the finial time instant T is an arbitrary positive and finite real number.

In order to analyze the regularity of solutions of system (2.1), we rewrite the equations in terms of different unknowns. Instead of considering the unknown temperature u with an initial value u^0, introduce the functions

$$
v := u + \frac{l}{2}\phi, \qquad v^0(\boldsymbol{x}) := u^0(\boldsymbol{x}) + \frac{l}{2}\,\phi^0(\boldsymbol{x}).
\tag{2.2}
$$

If the functions u and ϕ satisfy system (2.1), the unknowns v and ϕ solve the following initial-boundary value problem:

$$
\begin{aligned}
v_t - \kappa\,\Delta v + \frac{\kappa\, l}{2}\Delta\phi &= 0 && \text{in } \Omega,\\
\tau\phi_t - 2v + \left(l - \frac{1}{2}\right)\phi + \frac{1}{2}\phi^3 - \xi^2\Delta\phi &= 0 && \text{in } \Omega,\\
-\kappa\,\partial_{\boldsymbol{\nu}} v = \lambda\left(v - \frac{l}{2}\phi - g\right), \qquad -\partial_{\boldsymbol{\nu}}\phi &= 0 && \text{on } \partial\Omega,\\
v(\boldsymbol{x}) = v^0(\boldsymbol{x}), \qquad \phi(\boldsymbol{x}) &= \phi^0(\boldsymbol{x}) && \text{for } t = 0 \text{ on } \Omega.
\end{aligned}
\tag{2.3}
$$

3. Statement of the problem and main result

In this section, we first introduce some notations that will be used throughout the paper and then formulate the main results, see Th. 3.2 and Cor. 3.3 below.

If not stated differently, the following notation is used throughout the paper:

$$
\begin{aligned}
&\Omega_T := \Omega \times (0,T), \qquad \partial\Omega_T := \partial\Omega \times (0,T),\\
&\mathcal{H} := H^1(\Omega_T), \quad \mathcal{H}_T := \{f \in \mathcal{H} \,:\, f(\boldsymbol{x}, T) = 0 \quad \text{for a. a. } \boldsymbol{x} \in \Omega\},\\
&\mathcal{C}^1_T([0,T]) := \left\{f \in \mathcal{C}^1([0,T]) \,:\, f(T) = 0\right\},\\
&Y := L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),\\
&\|y\|_Y = \|y\|_{L^2(0,T;H^1(\Omega))} + \|y\|_{L^\infty(0,T;L^2(\Omega))},\\
&X := L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T),\\
&\|x\|_X = \|x\|_{L^2(0,T;H^1(\Omega))} + \|x\|_{L^4(\Omega_T)},\\
&X' := L^2(0,T;(H^1(\Omega))') + L^{4/3}(\Omega_T),\\
&\|f\|_{X'} = \inf_{\substack{f_1 \in L^2(0,T;(H^1(\Omega))')\\ f_2 \in L^{4/3}(\Omega_T)\\ f_1 + f_2 = f}} \max\left\{\|f_1\|_{L^2(0,T;(H^1(\Omega))')}, \|f_2\|_{L^{4/3}(\Omega_T)}\right\},\\
&W := \{u \in X \,:\, u_t \in X'\}, \quad \|u\|_W := \|u\|_X + \|u_t\|_{X'}.
\end{aligned}
\tag{3.1}
$$

Indeed, $(X', \|\cdot\|_{X'})$ is the normed dual of the separable and reflexive Banach space $(X, \|\cdot\|_X)$, see [**7**, Chap. IV].

The next definition states the sense of weak solutions.

Definition 3.1. A pair of functions (v, ϕ) with $v \in L^2(0,T;H^1(\Omega))$ and $\phi \in X$ is called weak solution of problem (2.3), if the following equation

$$
\begin{aligned}
0 = &-\int_\Omega \left[v^0\,\psi^0 + \tau\phi^0\,\eta^0\right]\,\mathrm{d}\boldsymbol{x} - \int_0^T\int_\Omega \left[v\,\psi_t + \tau\phi\,\eta_t\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&+ \int_0^T\int_\Omega \left[\kappa\,\nabla v - \frac{\kappa\, l}{2}\nabla\phi\right]\nabla\psi\,\mathrm{d}\boldsymbol{x}\mathrm{d}t + \lambda\int_0^T\int_{\partial\Omega}\left[v - \frac{l}{2}\phi - g\right]\psi\,\mathrm{d}s\mathrm{d}t\\
&+ \int_0^T\int_\Omega \left[\left\{-2v + \left(l - \frac{1}{2}\right)\phi + \frac{1}{2}\phi^3\right\}\eta + \xi^2\nabla\phi\cdot\nabla\eta\right]\,\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned}
\tag{3.2}
$$

holds true for all ψ, $\eta \in \mathcal{H}$ with $\eta \in L^4(\Omega_T)$.

The main result on problem (2.3) is stated in the following theorem.

THEOREM 3.2. *Let v^0, $\phi^0 \in L^2(\Omega)$, $g \in L^2(\partial\Omega_T)$ be arbitrary functions. Then system* (2.3) *has a unique weak solution* (v, ϕ) *in the sense of Definition 3.1. The components v and ϕ of the solution have regularity: v, $\phi \in \mathcal{C}\left([0,T]; L^2(\Omega)\right)$. Moreover, if v_i^0, $\phi_i^0 \in L^2(\Omega)$, and $g_i \in L^2(\partial\Omega_T)$, $i = 1, 2$, then*

$$\|\bar{v}\|_Y^2\,,\ \|\bar{\phi}\|_Y^2 \le C\left(1 + T\,e^T\right)\left[\|\bar{v}^0\|^2_{L^2(\Omega)} + \|\bar{\phi}^0\|^2_{L^2(\Omega)} + \|\bar{g}\|^2_{L^2(\partial\Omega_T)}\right], \tag{3.3}$$

where $\bar{g} = g_1 - g_2$, $\bar{v} = v_1 - v_2$, $\bar{\phi} = \phi_1 - \phi_2$, $\bar{v}^0 = v_1^0 - v_2^0$, and $\bar{\phi}^0 = \phi_1^0 - \phi_2^0$. The constant C is independent of v_i, ϕ_i, v_i^0, ϕ_i^0, and g_i, $i = 1, 2$.

The proof of Th. 3.2 is given in Sections 5 and 6: the existence of weak solutions is stated in Lemma 5.1, the continuity in time follows from the technical Lemma 6.1, and the stability estimate (3.3) is given in Lemma 6.2.

The following corollary is a consequence of the continuity in time of the solution components v and ϕ asserted in Th. 3.2.

COROLLARY 3.3. *Assume the hypotheses of Th. 3.2. Model* (2.3) *can be considered as a dynamical system with state space $(L^2(\Omega))^2$ and evolution function*

$$\Psi\ :\ [0,T] \times (L^2(\Omega))^2 \to (L^2(\Omega))^2, \quad (t,\, v^0,\, \phi^0) \mapsto (v(t),\, \phi(t)).$$

To show Cor. 3.3, notice that Ψ is a well-defined mapping under the assumptions of Th. 3.2. More precisely, weak solutions of problem (2.3) are unique by estimate (3.3), and the mapping $\Psi(t, v^0, \phi^0) = (v(t), \phi(t))$ is defined for each time instant, $t \in [0, T]$, since v and ϕ are continuous in time with values in $L^2(\Omega)$.

4. Construction of Approximations

In this section, we construct approximate solutions to problem (2.3) that will be used in Section 5 to establish the existence of weak solutions in the sense of Definition 3.1.

Let $\{\omega_i\}_{i=1}^{\infty}$ be an orthogonal basis in $H^1(\Omega)$ and $L^2(\Omega)$ simultaneously (such a basis really exists). Consider Galerkin-Approximations of the form

$$v^m(\boldsymbol{x}, t) = \sum_{i=0}^{m} a_i^m(t)\omega_i(\boldsymbol{x})\,, \quad \phi^m(\boldsymbol{x}, t) = \sum_{i=0}^{m} b_i^m(t)\,\omega_i(\boldsymbol{x}), \tag{4.1}$$

where $a_i^m(t)$ and $b_i^m(t)$ are unknown functions.

To determine the coefficients a_i^m and b_i^m, we derive a system of ordinary differential equations. Extend g by zero for $t > T$, and (similar to [**10**, Chap. II, §4]) let g^m be the Steklov average of g

$$g^m(\boldsymbol{x}, t) = m \int_t^{t+1/m} g(\boldsymbol{x}, \tau)\,\mathrm{d}\tau.$$

Then $g^m \in \mathcal{C}(\mathbb{R}; L^2(\partial\Omega)) \cap L^2(\partial\Omega_{\mathbb{R}})$, and $g^m \to g$ strongly in $L^2(\partial\Omega_T)$ as $m \to \infty$.

Substitute the approximations (4.1) into (3.2), cancel the integration over time, and replace the couples of test functions $(\psi,\, \eta)$ first by $(\omega_j,\, 0)$ and second by $(0,\, \omega_k)$,

$j,\, k = 1, \ldots, m$. This yields the ordinary differential equations

$$
(4.2)\quad
\begin{aligned}
0 &= \sum_{i=1}^{m} \dot{a}_i^m(t) \int_\Omega \omega_i\, \omega_j \,\mathrm{d}\boldsymbol{x} - \lambda \int_{\partial\Omega} g^m(t)\, \omega_j \,\mathrm{d}s \\
&\quad + \sum_{i=1}^{m} \left[\kappa\, a_i^m(t) + \frac{\kappa\, l}{2}\, b_i^m(t)\right] \int_\Omega \nabla\omega_i \cdot \nabla\omega_j \,\mathrm{d}\boldsymbol{x} \\
&\quad + \lambda \sum_{i=1}^{m} \int_{\partial\Omega} \left[a_i^m(t) - \frac{l}{2}\, b_i^m(t)\right] \omega_i\, \omega_j \,\mathrm{d}s \qquad (j = 1, \ldots, m), \\
0 &= \sum_{i=1}^{m} \tau\, \dot{b}_i^m(t) \int_\Omega \omega_i\, \omega_k \,\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{m} \xi^2\, b_i^m(t) \int_\Omega \nabla\omega_i \cdot \nabla\omega_k \,\mathrm{d}\boldsymbol{x} \\
&\quad + \frac{1}{2} \int_\Omega \left(\sum_{i=1}^{m} b_i^m(t)\, \omega_i\right)^3 \omega_k \,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{i=1}^{m} \int_\Omega \left[\left\{-2\, a_i^m(t) + \left(l - \frac{1}{2}\right) b_i^m(t)\right\} \omega_i\, \omega_k \,\mathrm{d}\boldsymbol{x}\right] \qquad (k = 1, \ldots, m).
\end{aligned}
$$

Assuming that $\{\omega_i\}$ is also orthonormal in $L^2(\Omega)$, equation (4.2) can be written as a system of ordinary differential equations determining the coefficients $a_i^m(t)$ and $b_i^m(t)$ as follows:

$$
\left.\begin{aligned}
\dot{a}_j^m(t) + A_j^m(t,\, a_1^m, \ldots,\, a_m^m,\, b_1^m, \ldots,\, b_m^m) = 0 \\
\tau\, \dot{b}_j^m(t) + B_j^m(t,\, a_1^m, \ldots,\, a_m^m,\, b_1^m, \ldots,\, b_m^m) = 0
\end{aligned}\right\} \qquad \text{for } j = 1, \ldots, m,
$$

where the functions A_j^m and B_j^m depend analytically on their variables. This system can be rewritten as

$$
(4.3)\qquad \begin{bmatrix} \mathbb{I}_m & 0 \\ 0 & \tau\, \mathbb{I}_m \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{a}}^m \\ \dot{\boldsymbol{b}}^m \end{bmatrix} + \begin{bmatrix} \boldsymbol{A}^m \\ \boldsymbol{B}^m \end{bmatrix} = \boldsymbol{0},
$$

where $\mathbb{I}_m$ denotes the $m \times m$ identity-matrix. The initial conditions for system (4.3) are given by

$$
(4.4)\qquad a_j^m(0) = \int_\Omega \omega_j\, v^0 \,\mathrm{d}\boldsymbol{x}, \qquad b_j^m(0) = \int_\Omega \omega_j\, \phi^0 \,\mathrm{d}\boldsymbol{x}, \qquad \text{for } j = 1, \ldots, m.
$$

Since the functions $\boldsymbol{A}^m$ and $\boldsymbol{B}^m$ are smooth with respect to all their variables, the theory of ordinary differential equations shows that, for each fixed m, there exists a non-empty time interval $[0, T_m]$ on which (4.3) is solvable.

Next, we show that T_m can be chosen independently of m. To this end, note that if a_i^m and b_i^m are the solution of (4.3), they are continuously differentiable. Multiply the first equations in (4.2) by $a_j^m(t)$ and sum up over $j = 1, \ldots, m$. Multiply the second equations in (4.2) by $\alpha\, b_k^m(t)$ (the constant α will be specified later) and sum up over $k = 1, \dots, m$. By the ansatz (4.1) this yields

$$
\begin{aligned}
0 = &\int_\Omega \Big[v^m\, v^m_t + \kappa\,|\nabla v^m|^2 - \frac{\kappa\, l}{2}\, \nabla \phi^m \cdot \nabla v^m \Big]\, \mathrm{d}\boldsymbol{x} \\
&+ \lambda \int_{\partial\Omega} \Big[|v^m|^2 - \frac{l}{2}\, \phi^m\, v^m - g^m\, v^m \Big]\, \mathrm{d}s \\
&+ \alpha \int_\Omega \Big[\tau\, \phi^m\, \phi^m_t + \Big(l - \frac{1}{2}\Big) |\phi^m|^2 + \frac{1}{2}\, |\phi^m|^4 + \xi^2\, |\nabla \phi^m|^2 \Big]\, \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Integrating over a time interval $(0, t)$, $t \in (0, T_m]$, using the product rule for the time derivatives and applying Young's inequality yield (with $\epsilon > 0$)

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \big[v^m(t)^2 + \alpha\, \tau\, \phi^m(t)^2 \big]\, \mathrm{d}\boldsymbol{x} + \lambda \int_0^t \int_{\partial\Omega} |v^m|^2\, \mathrm{d}s\mathrm{d}\tau \\
&\qquad + \int_0^t \int_\Omega \Big[\kappa\, |\nabla v^m|^2 + \frac{\alpha}{2}\, |\phi^m|^4 + \alpha\, \xi^2\, |\nabla \phi^m|^2 \Big]\, \mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\qquad \le \frac{1}{2} \int_\Omega \big[|v^0|^2 + \alpha\, \tau\, |\phi^0|^2 \big]\, \mathrm{d}\boldsymbol{x} \\
&\qquad + \lambda \int_0^t \int_{\partial\Omega} \Big[\epsilon |v^m|^2 + \frac{|g^m|^2}{2\epsilon} + \frac{l^2}{8\epsilon}\, |\phi^m|^2 \Big]\, \mathrm{d}s\mathrm{d}\tau \\
&\qquad + \int_0^t \int_\Omega \Big[\alpha \Big(l - \frac{1}{2}\Big) |\phi^m|^2 + \frac{\kappa}{2} |\nabla v^m| \; + \frac{l^2\, \kappa}{2} |\nabla \phi^m|^2 \Big]\, \mathrm{d}\boldsymbol{x}\mathrm{d}\tau.
\end{aligned} \tag{4.5}
$$

Choose $\epsilon = {}^1/_2$ and use the embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ to obtain

$$
\frac{l^2\, \lambda}{4} \int_{\partial\Omega} |\phi^m|^2\, \mathrm{d}s \le C \int_\Omega \big[|\phi^m|^2 + |\nabla \phi^m|^2 \big]\, \mathrm{d}\boldsymbol{x}, \tag{4.6}
$$

where C is a constant that is independent of ϕ^m. Now, choose α such that

$$
\beta := \alpha\, \xi^2 - \frac{l^2\, \kappa}{2} - C > 0. \tag{4.7}
$$

Reinserting (4.6) and (4.7) into (4.5) yields

$$
\begin{aligned}
&\frac{1}{2} \int_\Omega \big[v^m(t)^2 + \alpha\, \tau\, \phi^m(t)^2 \big]\, \mathrm{d}\boldsymbol{x} + \frac{\lambda}{2} \int_0^t \int_{\partial\Omega} |v^m|^2\, \mathrm{d}s\mathrm{d}\tau \\
&\qquad + \int_0^t \int_\Omega \Big[\frac{\kappa}{2}\, |\nabla v^m|^2 + \frac{\alpha}{2}\, |\phi^m|^4 + \beta\, \xi^2\, |\nabla \phi^m|^2 \Big]\, \mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\qquad \le \frac{1}{2} \int_\Omega \big[|v^0|^2 + \alpha\, \tau\, |\phi^0|^2 \big]\, \mathrm{d}\boldsymbol{x} + \lambda \int_0^t \int_{\partial\Omega} |g^m|^2 \mathrm{d}s\mathrm{d}\tau \\
&\qquad + \int_0^t \int_\Omega \Big(C + l - \frac{1}{2} \Big) |\phi^m|^2\, \mathrm{d}\boldsymbol{x}\mathrm{d}\tau.
\end{aligned} \tag{4.8}
$$

By Gronwall's inequality we obtain that

$$
\begin{aligned}
&v^m,\, \phi^m \text{ bounded in } L^\infty(0, T_m; L^2(\Omega)) \cap L^2(0, T_m; H^1(\Omega)), \\
&\qquad \phi^m \text{ bounded in } L^4(0, T_m; L^4(\Omega))
\end{aligned} \tag{4.9}
$$

and moreover the bounds are independent of m and t. Due to the choice of $\{\omega_j\}_{j\in\mathbb{N}}$ it holds

$$\|v^m(t)\|^2_{L^2(\Omega)} = \sum_{i=1}^{m} |a_i^m(t)|^2 \qquad \text{and} \qquad \|\phi^m(t)\|^2_{L^2(\Omega)} = \sum_{i=1}^{m} |b_i^m(t)|^2.$$

Therefore (4.9) shows that a_i^m and b_i^m $(i = 1,\dots,m)$ are bounded on $[0,T_m]$ and can be continued beyond T_m. Consequently, there exists no maximal T_m, and a_i^m and b_i^m can be defined on $[0,\infty)$ for each m.

5. Existence of weak solutions

In this section, we show that weak solutions to problem (2.3) can be extracted from the sequences $\{v^m\}$ and $\{\phi^m\}$ constructed in Section 4. The next lemma establishes the existence of solutions to problem (2.3) and a regularity result for the time derivatives of the solution components. This regularity is needed in Section 6 to show the uniqueness of the solution.

Lemma 5.1. *Problem* (2.3) *has at least one weak solution* $(v,\,\phi)$ *in the sense of Definition 3.1. Each of the weak solutions satisfies:* $v_t \in L^2(0,T;(H^1(\Omega))')$ *and* $\phi_t \in L^2(0,T;(H^1(\Omega))') + L^{4/3}(\Omega_T)$.

Proof. Due to (4.9), there exist functions v, ϕ and ζ such that (up to subsequences)

$$\begin{aligned} v^m &\overset{*}{\rightharpoonup} v && \text{in } Y,\\ \phi^m &\overset{*}{\rightharpoonup} \phi && \text{in } Y\cap X,\\ (\phi^m)^3 &\overset{*}{\rightharpoonup} \zeta && \text{in } L^{4/3}(\Omega_T), \end{aligned} \tag{5.1}$$

see definitions (3.1). By the construction of the approximate solutions in (4.1), (4.2), and (4.4) the functions v^m and ϕ^m satisfy the equation

$$\begin{aligned} 0 = &-\int_\Omega \left[v^{0,m}\,\psi^0 + \phi^{0,m}\,\eta^0\right] \mathrm{d}\boldsymbol{x}\\ &+\int_0^T\int_\Omega \left[-v^m\,\psi_t + \left(\kappa\,\nabla v^m - \frac{\kappa\, l}{2}\nabla\phi^m\right)\nabla\psi\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ &+\lambda\int_0^T\int_{\partial\Omega}\left(v^m - \frac{l}{2}\,\phi^m - g^m\right)\psi\,\mathrm{d}s\mathrm{d}t\\ &+\int_0^T\int_\Omega\left[-\tau\phi^m\,\eta_t + \left\{-2v^m + \left(l-\frac{1}{2}\right)\phi^m + \frac{1}{2}(\phi^m)^3\right\}\eta\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\\ &+\int_0^T\int_\Omega \xi^2\,\nabla\phi^m\nabla\eta\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \end{aligned} \tag{5.2}$$

for all ψ and η which are linear combinations of functions $c_j(t)\,\omega_j(\boldsymbol{x})$, $c_j(t) \in \mathcal{C}^1_T([0,T])$, $j = 1,\dots,m$. Consider the limit as $m\to\infty$. Due to (4.4), the initial functions $v^{0,m}$ and $\phi^{0,m}$ can be replaced by v^0 and ϕ^0 in the first integral on the right-hand side. By the properties of Steklov averages, g^m can be replaced by g and, by (5.1), $(v^m,\phi^m,(\phi^m)^3)$ can be replaced by the limiting functions (v,ϕ,ζ) in (5.2). Notice that linear combinations of functions $c_j(t)\,\omega_j(\boldsymbol{x}), c_j \in \mathcal{C}^1_T[0,T]$ lie dense in the spaces $\mathcal{H}_T$ and $\mathcal{H}_T \cap L^4(\Omega_T)$. Therefore v, ϕ, and ζ satisfy the equation

$$
\begin{aligned}
0 = &- \int_\Omega \left[v^0\, \psi^0 + \phi^0\, \eta^0\right] \mathrm{d}\boldsymbol{x} \\
&+ \int_0^T \int_\Omega \left[-v\, \psi_t - \tau \phi\, \eta_t + \left(\kappa\, \nabla v - \frac{\kappa\, l}{2} \nabla \phi\right) \nabla \psi\right] \mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&+ \lambda \int_0^T \int_{\partial\Omega} \left(v - \frac{l}{2}\, \phi\, g\right) \psi\, \mathrm{d}s\mathrm{d}t \\
&+ \int_0^T \int_\Omega \left[\left\{-2v + \left(l - \frac{1}{2}\right) \phi + \frac{1}{2}\, \zeta\right\} \eta + \xi^2\, \nabla\phi \cdot \nabla \eta\right] \mathrm{d}\boldsymbol{x}\mathrm{d}t
\end{aligned} \tag{5.3}
$$

for all $\psi \in \mathcal{H}_T$ and $\eta \in \mathcal{H}_T \cap L^4(\Omega_T)$.

It remains to show that $\zeta = \phi^3$. To this end, show first that $\phi^m \to \phi$ in $L^2(0,T;H^{1-\epsilon}(\Omega))$ strongly ($\epsilon > 0$) by applying [**18**, Section 8, Corollary 4]. To apply this corollary, ϕ_t has to be estimated. Choose $\psi = 0$ in (5.3), use the boundedness of Ω_T and the embedding $H^1(\Omega_T) \hookrightarrow L^4(\Omega_T)$ and apply Hölder's inequality to derive the estimate

$$
\begin{aligned}
|\langle \phi_t; \eta\rangle| &= \left| - \int_0^T \int_\Omega \phi\, \eta_t\, \mathrm{d}\boldsymbol{x}\mathrm{d}t \right| \\
&= \left| \frac{1}{\tau} \int_0^T \int_\Omega \left[\left\{-2v + \left(l - \frac{1}{2}\right) \phi + \frac{1}{2}\zeta\right\} \eta + \xi^2 \nabla\phi\nabla\eta\right] \mathrm{d}\boldsymbol{x}\mathrm{d}t \right| \\
&\le C \left[\|v\|_{L^{3/4}(\Omega_T)} + \|\phi\|_{L^{3/4}(\Omega_T)} + \|\zeta\|_{L^{3/4}(\Omega_T)}\right] \cdot \|\eta\|_{L^4(\Omega_T)} \\
&\quad + C\, \|\phi\|_{L^2(0,T;H^1(\Omega))} \cdot \|\eta\|_{L^2(0,T;H^1(\Omega))}
\end{aligned} \tag{5.4}
$$

for all η of the form $\eta(\boldsymbol{x},t) = c(t)\, \omega(\boldsymbol{x})$, $c \in \mathcal{D}(0,T)$ and $\omega \in H^1(\Omega)$. Inequality (5.4) and the bounds (4.9) show that ϕ_t is a continuous functional on $L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)$. Therefore (see also the proof of Lemma 6.1) it holds

$$
\begin{aligned}
\phi_t &\in \big(L^2(0,T;H^1(\Omega)) \cap L^4(\Omega_T)\big)' \\
&= L^2(0,T;(H^1(\Omega))') + L^{4/3}(\Omega_T) \subset L^1\big(0,T;(H^1(\Omega))'\big),
\end{aligned}
$$

because T is finite. Applying [**18**, Section 8, Corollary 4] with $X = H^1(\Omega)$, $B = H^{1-\epsilon}(\Omega)$, $Y = \big(H^{-1}(\Omega)\big)'$ and $p = 2$ we obtain

$$
\phi^m \to \phi \qquad \text{strongly in } L^2(0,T;H^{1-\epsilon}(\Omega))
$$

for a subsequence still denoted by $\{\phi^m\}$. This implies the convergence $\phi^m(\boldsymbol{x},t)^3 \to \phi(\boldsymbol{x},t)^3$ almost everywhere in Ω_T. Now $\zeta = \phi^3$ follows from $(\phi^m)^3 \rightharpoonup \zeta$ in $L^{4/3}(\Omega_T)$ (see (5.1)) and [**12**, Chap. 1, Lemma 1.3]. □

6. Uniqueness and stability

In order to show the uniqueness of the solution (v, ϕ) obtained in Sections 4 and 5, we need a certain regularity of the phase function ϕ. Using methods presented in book [**7**], the following lemma that provides a formula for the integration by parts can be proved.

Lemma 6.1. *Let H be a Hilbert space and V_1 and V_2 be reflexive and separable Banach spaces which are continuously embedded in H. Assume that $V := V_1 \cap V_2$ is* *dense in H. For given $1 < p_1,\, p_2 \le \infty$, denote by p_1' and p_2' the Lebesgue conjugate exponents. Define the following normed spaces*

$$X := L^{p_1}(0,T;V_1) \cap L^{p_2}(0,T;V_2), \qquad \|x\|_X := \|x\|_{L^{p_1}(0,T;V_1)} + \|x\|_{L^{p_2}(0,T;V_2)}\,,$$
$$W := \{u \in X \;:\; u_t \in X'\}\,, \qquad \|u\|_W := \|u\|_X + \|u_t\|_{X'}\,.$$

Then $W \subset \mathcal{C}([0,T];H)$ with the continuous embedding (because T is finite). The formula

$$\begin{aligned}\langle u(t); v(t)\rangle_{H\times H} &- \langle u(s); v(s)\rangle_{H\times H}\\ &= \int_s^t \left[\langle u_t(\tau); v(\tau)\rangle_{V'\times V} + \langle v_t(\tau); u(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\end{aligned}$$

(integration by parts) holds for arbitrary $u,\, v \in W$ and $s,\, t \in [0,T]$.

Proof. Denote the norms in V_i by $\|\cdot\|_i$, $i = 1,2$. The spaces $V_1 \cap V_2$ and $V_1 + V_2$ are Banach spaces when they are endowed with the norms

$$\|v\|_{V_1\cap V_2} := \|v\|_{V_1} + \|v\|_{V_2}\,, \qquad v \in V_1 \cap V_2.$$
$$\|z\|_{V_1+V_2} := \inf_{\substack{v_1\in V_1,\\ v_2\in V_2,\\ v_1+v_2=z.}} \max\left\{\|v_1\|_{V_1}\,,\, \|v_2\|_{V_2}\right\}.$$

Since $V_i \hookrightarrow H$ and H is locally convex, [**7**, Chap. I, Th. 5.13] yields the relations

$$V_1' + V_2' = (V_1 \cap V_2)'\,, \qquad (V_1 + V_2)' = V_1' \cap V_2'. \tag{6.1}$$

Set $V := V_1 \cap V_2$, then (6.1) implies that $V' = V_1' + V_2'$. The normed dual X' of X (defined in the lemma) is given by

$$X' := L^{p_1'}(0,T;V_1') + L^{p_2'}(0,T;V_2'),$$
$$\|f\|_{X'} := \inf_{\substack{f_1\in L^{p_1'}(0,T;V_1'),\\ f_2\in L^{p_2'}(0,T;V_2'),\\ f_1+f_2=f.}} \max\left\{\|f_1'\|_{L^{p_1}(0,T;V_1)}\,,\, \|f_2\|_{L^{p_2'}(0,T;V_2')}\right\},$$

see [**7**, Chap. I, Th. 5.13 and Chap. IV, Th. 1.14] which also imply that X is reflexive.

The space W defined in the lemma is a Banach space, see [**7**, Chap. IV, Th. 1.16]. Using the method of the proof of [**7**, Chap. IV, Lemma 1.11], we obtain the continuity of the embedding $W \hookrightarrow \mathcal{C}([0,T]; V_1' + V_2')$ because T is assumed to be finite. Adopting the proof of [**7**, Chap. IV, Lemma 1.12], we deduce that $\mathcal{C}^1([0,T];V) \cap W$ is dense in W. Moreover, the proof of [**7**, Chap. IV, Th. 1.17] shows that the embedding $W \hookrightarrow \mathcal{C}([0,T];H)$ is continuous because T is finite.

It remains to show the formula for integration by parts.
Let $u,\, v \in W$ and $u^n,\, v^n \in \mathcal{C}^1([0,T];V) \cap W$, $n \in \mathbb{N}$, be such that

$$\|u - u^n\|_W\,,\, \|v - v^n\|_W \le 1/n.$$

Then u^n and v^n satisfy

$$\begin{aligned}\langle u^n(t); v^n(t)\rangle_{H\times H} &- \langle u^n(s); v^n(s)\rangle_{H\times H}\\ &= \int_s^t \left[\langle u_t^n(\tau); v^n(\tau)\rangle_{V'\times V} + \langle v_t^n(\tau); u^n(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\end{aligned} \tag{6.2}$$

for arbitrary $s,\, t \in [0,T]$ and for all $n \in \mathbb{N}$.

Consider the limit as $n \to \infty$ in the first summand on the left-hand side of (6.2). The embedding $W \hookrightarrow \mathcal{C}([0,T];H)$ implies that

$$\begin{aligned}&\left|\langle u^n(t); v^n(t)\rangle_{H\times H} - \langle u(t); v(t)\rangle_{H\times H}\right|\\ &\quad\le \left|\langle u^n(t); v^n(t) - v(t)\rangle_{H\times H} + \langle u^n(t) - u^n(t); v(t)\rangle_{H\times H}\right|\\ &\quad\le \frac{C_W}{n}\left[\frac{C_W}{n} + \|u\|_{\mathcal{C}([0,T];H)} + \|v\|_{\mathcal{C}([0,T];H)}\right],\end{aligned}$$

where C_W is the norm of the identity map from $W \to \mathcal{C}([0,T];H)$. A similar argument with t replaced by s shows that the left-hand side of (6.2) satisfies

$$\begin{aligned}&\langle u^n(t); v^n(t)\rangle_{H\times H} - \langle u^n(s); v^n(s)\rangle_{H\times H}\\ &\quad\longrightarrow \langle u(t); v(t)\rangle_{H\times H} - \langle u(s); v(s)\rangle_{H\times H}\end{aligned}\tag{6.3}$$

as $n \to \infty$. Consider the limit as $n \to \infty$ of the first summand on the right-hand side of (6.2) to obtain:

$$\begin{aligned}&\left|\int_s^t \left[\langle u_t^n(\tau); v^n(\tau)\rangle_{V'\times V} - \langle u_t(\tau); v(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\right|\\ &\quad\le \left|\int_s^t \left[\langle u_t^n(\tau); v^n(\tau) - v(\tau)\rangle_{V'\times V} + \langle u_t^n(\tau) - u_t(\tau); v(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\right|\\ &\quad\le \frac{1}{n}\left[\frac{1}{n} + \|u_t\|_{X'} + \|v\|_X\right].\end{aligned}$$

Interchanging the roles of u and v shows that the right-hand side of (6.2) satisfies

$$\begin{aligned}&\int_s^t \left[\langle u_t^n(\tau); v^n(\tau)\rangle_{V'\times V} - \langle v_t^n(\tau); u^n(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\\ &\quad\longrightarrow \int_s^t \left[\langle u_t(\tau); v(\tau)\rangle_{V'\times V} - \langle v_t(\tau); u(\tau)\rangle_{V'\times V}\right] \mathrm{d}\tau\end{aligned}\tag{6.4}$$

as $n \to \infty$. Substituting (6.3) and (6.4) into (6.2) yields the claimed formula. This completes the proof of the lemma. □

Lemma 6.1 is applied with $V_1 = H^1(\Omega)$, $p_1 = 2$, $V_2 = L^4(\Omega)$, $p_2 = 4$ and $H = L^2(\Omega)$. Using the notation introduced in (3.1), we obtain from (5.4) that $\phi_t \in X'$. Thus, Lemma 6.1 yields $\phi \in W \hookrightarrow \mathcal{C}([0,T];L^2(\Omega))$. The following lemma establishes the uniqueness of weak solutions to problem (2.3) and a stability result in space Y.

LEMMA 6.2. *Let v_i^0, $\phi_i^0 \in L^2(\Omega)$, and $g_i \in L^2(\partial\Omega_T)$, $i = 1, 2$ be given functions. Let (v_i, ϕ_i) be weak solutions of problem (2.3) in the sense of Definition 3.1. Then*

$$\|\bar{v}\|_Y^2,\ \|\bar{\phi}\|_Y^2 \le C\left(1 + T\,e^T\right)\left[\|\bar{v}^0\|_{L^2(\Omega)}^2 + \|\bar{\phi}^0\|_{L^2(\Omega)}^2 + \|\bar{g}\|_{L^2(\partial\Omega_T)}^2\right],$$

where $\bar{g} = g_1 - g_2$, $\bar{v} = v_1 - v_2$, and $\bar{\phi} = \phi_1 - \phi_2$. The constant C is independent of v_i, ϕ_i, v_i^0, ϕ_i^0 and g_i, $i = 1, 2$.

PROOF. Use the notations from (3.1). Let $\bar{g}$, $\bar{v}$, and $\bar{\phi}$ be the functions defined in the lemma. Set $\zeta = \phi_1^2 + \phi_1\,\phi_2 + \phi_2^2 \geq 0$. Then $\bar{v}$ and $\bar{\phi}$ satisfy the equation

$$
\begin{aligned}
0 &= \langle \bar{v}_t; \psi\rangle + \int_0^T \int_\Omega \left(\kappa\,\nabla\bar{v} - \frac{\kappa\,l}{2}\nabla\bar{\phi}\right)\nabla\psi\,\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\quad + \lambda\int_0^T\int_{\partial\Omega}\left(\bar{v} - \frac{l}{2}\bar{\phi} - g\right)\psi\,\mathrm{d}s\mathrm{d}t + \tau\left\langle\bar{\phi}_t;\eta\right\rangle_{X'\times X} \\
&\quad + \int_0^T\int_\Omega\left[\left\{-2\bar{v} + \left(l - \frac{1}{2} + \frac{1}{2}\zeta\right)\bar{\phi}\right\}\eta + \xi^2\nabla\bar{\phi}\nabla\eta\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t,
\end{aligned}
\tag{6.5}
$$

where $\langle\cdot\,;\cdot\rangle$ denotes the duality product between $L^2(0,T,(H^1(\Omega))')$ and $L^2(0,T;H^1(\Omega))$. The right-hand side of (6.5) is a continuous linear functional defined on functions $\psi \in L^2(0,T;H^1(\Omega))$ and $\eta \in X$. Therefore, $(\psi,\eta) = (\bar{v},\bar{\phi})$ is an admissible choice and $\bar{\phi} \in W$.

Denote by $\chi_{(0,t)}$ the characteristic function of the interval $(0,t)$ for fixed $t \in (0,T]$. Let the constants α and β satisfy (4.7) and set $(\psi,\eta) = (\chi_{(0,t)}\,\bar{v}, \alpha\,\chi_{(0,t)}\,\bar{\phi})$ in (6.5) to obtain

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\left[|\bar{v}(t)|^2 + \alpha\tau|\bar{\phi}(t)|^2\right]\mathrm{d}\boldsymbol{x} + \lambda\int_0^t\int_{\partial\Omega}|\bar{v}|^2\,\mathrm{d}s\mathrm{d}\tau \\
&\quad + \int_0^t\int_\Omega\left[\kappa\,|\nabla\bar{v}|^2 + \alpha\,\xi^2|\nabla\bar{\phi}|^2 + \frac{\alpha}{2}\zeta|\bar{\phi}|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau \\
&\leq \frac{1}{2}\int_\Omega\left[|\bar{v}^0|^2 + \alpha\tau|\bar{\phi}^0|^2\right]\mathrm{d}\boldsymbol{x} + \lambda\int_0^t\int_{\partial\Omega}\left[|\bar{v}|^2 + \frac{l^2}{4}|\bar{\phi}|^2 + \frac{1}{2}|\bar{g}|^2\right]\mathrm{d}s\mathrm{d}\tau \\
&\quad + \int_0^t\int_\Omega\left[\alpha|\bar{v}|^2 + \alpha\left(l + \frac{1}{2}\right)|\bar{\phi}|^2 + \frac{l^2\,\kappa}{8}|\nabla\bar{\phi}|^2 + \frac{\kappa}{2}|\nabla\bar{v}|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}\tau.
\end{aligned}
\tag{6.6}
$$

Note that Lemma 6.1 is used here to evaluate $\langle\bar{\phi}_t;\bar{\phi}\rangle_{X'\times X}$. Due to (4.7) and the imbedding $H^1(\Omega)\big|_{\partial\Omega} \subset L^2(\partial\Omega)$, inequality (6.6) yields the estimate

$$
\begin{aligned}
&\int_\Omega\left[|\bar{v}(t)|^2 + |\bar{\phi}(t)|^2\right]\mathrm{d}\boldsymbol{x} + \int_0^T\int_\Omega\left[|\nabla\bar{v}|^2 + |\nabla\bar{\phi}|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t \\
&\quad\leq C\left\{\int_\Omega\left[|\bar{v}^0|^2 + |\bar{\phi}^0|^2\right]\mathrm{d}\boldsymbol{x} + \int_0^T\int_{\partial\Omega}|\bar{g}|^2\,\mathrm{d}s\mathrm{d}t + \int_0^T\int_\Omega\left[|\bar{v}|^2 + |\bar{\phi}|^2\right]\mathrm{d}\boldsymbol{x}\mathrm{d}t\right\}
\end{aligned}
$$

with a constant C that is independent of $\bar{v}$ and $\bar{\phi}$. By Lemma 6.1 it holds $\bar{v}$, $\bar{\phi} \in \mathcal{C}([0,T];L^2(\Omega))$ so that Gronwall's inequality yields

$$
\|\bar{v}\|_Y^2\,,\ \|\bar{\phi}\|_Y^2 \leq C\left(1 + T\,e^T\right)\left[\left\|\bar{v}^0\right\|_{L^2(\Omega)}^2 + \left\|\bar{\phi}^0\right\|_{L^2(\Omega)}^2 + \|\bar{g}\|_{L^2(\partial\Omega_T)}^2\right].
$$

In particular, for $\bar{v}^0 = \bar{\phi}^0 = \bar{g} \equiv 0$, we obtain the uniqueness of the solution. This proves the lemma. $\square$

REMARK 6.3. It should be noticed that model (2.3) is considered in a domain whose boundary is of class $C^{0,1}$. This assumption is done to allow the consideration of problems related to cryopreservation where regions of freezing may have boundaries with sharp and concave kinks. In the case of smooth domains, say of class C^2, the regularity of solutions can be improved, see [**10, 15, 11**]. The following scheme shows that $(v,\,\phi)$ is a strong solution of problem (2.3) provided the data have additional regularity $v^0,\ \phi^0 \in W^{\frac{1}{2}}_{\frac{4}{3}}(\Omega)$, and $g \in W^{\frac{1}{4},\,\frac{1}{8}}_{\frac{4}{3}}(\partial\Omega_T)$, see [**10, 15**] for a definition of this space. The embedding $Y \subset L^{\frac{4}{3}}(\Omega_T)$ and relations (5.1) imply that $-v + (l - \frac{1}{2})\phi + \frac{1}{2}\phi^3 \in L^{\frac{4}{3}}(\Omega_T)$. Therefore, application of the L^p-estimate technique to the second parabolic equation in (2.3) yields the regularity $\phi \in W^{2,\,1}_{\frac{4}{3}}(\Omega_T) = \{f \in L^{\frac{4}{3}}(0,T;\, W^2_{\frac{4}{3}}(\Omega)) \ :\ f_t \in L^{\frac{4}{3}}(\Omega_T)\}$. Using the first equation in (2.3), observing that $\Delta\phi \in L^{\frac{4}{3}}(\Omega_T)$, and applying the L^p-estimate technique yield the regularity $v \in W^{2,\,1}_{\frac{4}{3}}(\Omega_T)$. Notice again that the estimation of the L^p-norm of the time derivatives and second spatial derivatives is only possible in the case of smooth domains and regular data.

Remark 6.4. Finally, we remark that model (2.3) can be considered as a dynamical system with state space $(L^2(\Omega))^2$ and evolution function $\Psi(t,\, v^0,\, \phi^0) := (v(t),\, \phi(t))$. To see this, notice that Ψ is a well-defined mapping from $[0,T] \times (L^2(\Omega))^2 \to (L^2(\Omega))^2$ since weak solutions of problem (2.3) are unique by Lemma 6.2 and satisfy $v,\, \phi \in \mathcal{C}([0,T], L^2(\Omega))$ by Lemma 6.1.

7. Conclusions

A phase field model proposed by G. Caginalp is analyzed in bounded Lipschitz domains. The well-posedness of the problem is established for square integrable initial data. The continuity in time properties of the temperature and phase function show that the model can be considered as a dynamical system.

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