

## Trace formula for fourth order operators on the circle

Andrey Badanin and Evgeny Korotyaev

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ABSTRACT. We determine the trace formula for the fourth order operator on the circle. This formula is similar to the famous trace formula for the Hill operator obtained by Dubrovin, Its-Matveev and McKean-van Moerbeke.

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### 1. Introduction and main results

In the present paper we determine the trace formula for the fourth order operator on the circle. Before we recall the famous trace formula for the second order operator [6], [7], [11]. Consider the differential equation

$$(1.1) \quad -y'' + qy = \lambda y, \quad \lambda \in \mathbb{C},$$

where  $q$  is the 1-periodic potential and  $\mathbb{C}$  is a complex plane. Denote by  $\alpha_0^+ < \alpha_1^- \leq \alpha_1^+ < \alpha_2^- \leq \dots$  eigenvalues of the equation (1.1) with the 2-periodic boundary conditions.

Consider the equation (1.1) with the Dirichlet boundary conditions for the shifted potential  $q(\cdot + t)$  by any fix parameter  $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ . Denote the corresponding eigenvalues by  $\beta_1(t) < \beta_2(t) < \dots$ . It is well-known that

$$(1.2) \quad \beta_n(t) \in [\alpha_n^-, \alpha_n^+] \quad \forall \quad (n, t) \in \mathbb{N} \times \mathbb{T}.$$

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Now we can recall the famous trace formula, i.e., the identity given by

$$(1.3) \quad q(t) = \alpha_0^+ + \sum_{n=1}^{\infty} (\alpha_n^+ + \alpha_n^- - 2\beta_n(t)) \quad \forall t \in \mathbb{T}.$$

Dubrovin [6] and Its – Matveev [7] determined the trace formula (1.3) for the so-called  $N$  band potential, when  $\beta_n(t) = \alpha_n^- = \alpha_n^+$  for all  $n \geq N, t \in \mathbb{T}$  and some  $N \geq 1$ . In this case the sum in (1.3) is finite. McKean – van Moerbeke [11] and Trubowitz [23] determined the trace formula (1.3) for sufficiently smooth potential. Korotyaev [8] determined the trace formula (1.3) for the case  $q \in L^2(\mathbb{T})$ .

We discuss the trace formula for the fourth order operator. Introduce the Sobolev space  $W_m^s(A)$ , where  $A$  is the finite interval or the circle, by

$$(1.4) \quad W_m^s(A) = \{f, f^{(m)} \in L^s(A)\}, \quad s \geq 1, \quad m = 0, 1, 2, \dots$$

Our main goal is to extend the trace formula (1.3) for the fourth order equation

$$(1.5) \quad y'''' + 2(py')' + qy = \lambda y, \quad \lambda \in \mathbb{C},$$

where the 1-periodic real functions  $p, q$  satisfy the conditions

$$(1.6) \quad p \in W_4^1(\mathbb{T}), \quad q \in W_2^1(\mathbb{T}).$$

Consider the equation (1.5) with the 2-periodic boundary conditions. Let

$$\lambda_0^+, \lambda_n^\pm, n \in \mathbb{N},$$

be the corresponding spectrum labeled by  $\lambda_0^+ \leq \lambda_1^- \leq \lambda_1^+ \leq \lambda_2^- \leq \dots$ , counted with multiplicities. Note that the eigenvalues have multiplicity  $\leq 4$ . The following asymptotics hold true:

$$(1.7) \quad \lambda_n^\pm = (\pi n)^4 - 2p_0(\pi n)^2 + \frac{p_0^2 - \|p\|^2}{2} + q_0 + o(n^{-\frac{3}{2}}),$$

as  $n \rightarrow +\infty$ , see [1], where

$$f_0 = \int_0^1 f(t) dt, \quad \|f\|^2 = \int_0^1 |f(t)|^2 dt.$$

Consider the equation (1.5) with the shifted coefficients by any fix parameter  $t \in \mathbb{R}$ :

$$(1.8) \quad y'''' + 2(p(x+t)y')' + q(x+t)y = \lambda y, \quad \lambda \in \mathbb{C},$$

with the Dirichlet type boundary conditions

$$(1.9) \quad y(0) = y''(0) = y(1) = y''(1) = 0.$$

Let  $\mu_n(t), n \geq 1$ , be the corresponding spectrum labeled by  $\mu_1(t) \leq \mu_2(t) \leq \mu_3(t) \leq \dots$ , counted with multiplicities. Note that the eigenvalues have multiplicity  $\leq 2$ . The following asymptotics hold true:

$$(1.10) \quad \mu_n(t) = (\pi n)^4 - 2p_0(\pi n)^2 + \frac{p_0^2 - \|p\|^2}{2} + q_0 + \frac{O(1)}{n^2}$$

as  $n \rightarrow +\infty$  uniformly on  $t \in \mathbb{T}$ , see [2].

Now we present trace formulas, which are similar to the second order case (1.3).

**Theorem 1.1.** *Let  $(p, q) \in W_4^1(\mathbb{T}) \times W_2^1(\mathbb{T})$ . Then the following identity*

$$(1.11) \quad q(x) - \frac{p''(x)}{2} = \lambda_0^+ + \sum_{n=1}^{\infty} (\lambda_n^+ + \lambda_n^- - 2\mu_n(x)) \quad \forall x \in \mathbb{T},$$

*holds true, where the series converges absolutely and uniformly on  $x \in \mathbb{T}$ .*

*In particular, if  $p = \text{const}$ , then*

$$(1.12) \quad q(x) = \lambda_0^+ + \sum_{n=1}^{\infty} (\lambda_n^+ + \lambda_n^- - 2\mu_n(x)) \quad \forall x \in \mathbb{T},$$

*where the series converges absolutely and uniformly on  $x \in \mathbb{T}$ .*

There are numerous results about the trace formulas for the higher order operators, see McKean [10], Ostensson [16], Nazarov, Stolyarov and Zatitskiy [15], Sadovnichii [20], [21], Sadovnichii and Podol'skii [22] and see references therein. The inverse spectral results for the fourth order operators on a finite interval were obtained by Caudill, Perry and Schueller [5], McLaughlin [12], Papanicolaou and Kravvaritis [19]. The spectral problem for the fourth and higher order operator with periodic coefficients considered by Badanin and Korotyaev [3], [4], Mikhalets and Molyboga [14], Papanicolaou [17], [18], Tkachenko [24].

## 2. The resolvents

Let  $\mathcal{B}$ ,  $\mathcal{B}_1$ ,  $\mathcal{B}_2$  be the set of all bounded, trace class and Hilbert-Schmidt class operators on  $L^2(\mathbb{R})$ , respectively. The norms of  $\mathcal{B}$ ,  $\mathcal{B}_1$ ,  $\mathcal{B}_2$  are denoted by  $\|\cdot\|$ ,  $\|\cdot\|_1$ ,  $\|\cdot\|_2$ , respectively.

**2.1. The second order operators.** We recall the following well known results about the second order operators  $h_1, h_2$ , see, for instance, in [13], [9].

Firstly, we consider the operator  $h_2 y = -y'' - py$  acting in  $L^2(0, 2)$  with the 2-periodic boundary conditions. The spectrum of  $h_2$  is discrete and its eigenvalues  $\alpha_0^+, \alpha_n^\pm, n \in \mathbb{N}$ , are simple or have multiplicity two. They can be enumerated, counting with multiplicity, so that  $\alpha_0^+ < \alpha_1^- \leq \alpha_1^+ < \alpha_2^- \leq \alpha_2^+ < \dots$ . The following asymptotics hold true:

$$(2.1) \quad \alpha_n^\pm = (\pi n)^2 + O(1) \quad \text{as } n \rightarrow +\infty.$$

Secondly, we consider the operator  $h_1$  acting in  $L^2(0, 1)$  with the Dirichlet boundary conditions given by

$$(2.2) \quad h_1 y = -y'' - py, \quad y(0) = y(1) = 0.$$

All eigenvalues  $\beta_n, n \in \mathbb{N}$ , of the operator  $h_1$  are simple. We enumerate them in the increasing order  $\beta_1 < \beta_2 < \dots$ . Recall that  $\beta_n \in [\alpha_n^-, \alpha_n^+]$  for all  $n \geq 1$ . Moreover, the following trace formula holds true:

$$(2.3) \quad p^2(0) + \frac{p''(0)}{2} = (\alpha_0^+)^2 + \sum_{n \geq 1} ((\alpha_n^+)^2 + (\alpha_n^-)^2 - 2\beta_n^2)$$

where the series converges absolutely, see [11, p.254].

**2.2. The fourth order operators.** Introduce the self-adjoint operator  $H_2$  in  $L^2(0, 2)$  by

$$(2.4) \quad H_2 = \partial^4 + 2\partial p \partial + q \quad \text{on } [0, 2],$$

with 2-periodic boundary conditions.

It is sufficiently to prove identity (1.11) at  $x = 0$ . Thus we need to define the operator  $H_1$  in  $L^2(0, 1)$  by

$$H_1 = \partial^4 + 2\partial p \partial + q \quad \text{on } [0, 1],$$

with the Dirichlet type boundary conditions

$$f(0) = f''(0) = f(1) = f''(1) = 0.$$

Using the identity

$$(2.5) \quad \partial^4 + 2\partial p \partial + q = (-\partial^2 - p)^2 + V, \quad V = q - p'' - p^2,$$

and  $V \in L^\infty(0, 1)$ , we obtain

$$(2.6) \quad H_j = h_j^2 + V, \quad j = 1, 2,$$

where the operators  $h_1$  and  $h_2$  are introduced above. Let the operator  $h_j^0, j = 1, 2$ , be equal to the operator  $h_j$  at  $p = 0$ . Introduce the resolvents

$$(2.7) \quad R_j(\lambda) = (H_j - \lambda)^{-1}, \quad \mathcal{R}_j(\lambda) = (h_j^2 - \lambda)^{-1}, \quad \mathcal{R}_j^0(\lambda) = ((h_j^0)^2 - \lambda)^{-1},$$

where  $j = 1, 2$ . We have

$$(2.8) \quad \mathcal{R}_j = \mathcal{R}_j^0 - \mathcal{R}_j(-h_j^0 p - p h_j^0 + p^2) \mathcal{R}_j^0.$$

Due to asymptotics (1.7), (1.10), (2.1) all resolvents satisfy

$$(2.9) \quad \mathcal{R}_j(\lambda), R_j(\lambda), h_j^0 \mathcal{R}_j^0(\lambda) \in \mathcal{B}_1,$$

on the corresponding resolvent sets. Define the contours  $K_n \subset \mathbb{C}$  by

$$(2.10) \quad K_n = \left\{ \lambda \in \mathbb{C} : |\lambda|^{\frac{1}{4}} = \pi \left( n + \frac{1}{2} \right) \right\}, \quad n \geq 1.$$

We present results about the asymptotics, proved in Section 3.

**Lemma 2.1.** *Let  $j = 1, 2$  and let  $n \rightarrow \infty$ . Then the following asymptotics hold true:*

$$(2.11) \quad \|\mathcal{R}_j^0(\lambda)\|_2 + \|\mathcal{R}_j(\lambda)\|_2 + \|R_j(\lambda)\|_2 = O(n^{-3}),$$

$$(2.12) \quad \|h_j^0 \mathcal{R}_j^0(\lambda)\|_2 = O(n^{-1})$$

uniformly on the contours  $K_n$ , and

$$(2.13) \quad \oint_{K_n} \left( \lambda \operatorname{Tr} V \mathcal{R}_j^2(\lambda) + \operatorname{Tr} V \mathcal{R}_j^0(\lambda) \right) d\lambda = o(1).$$

**2.3. Proof of the main results.** Introduce the function

$$(2.14) \quad \Phi = \text{Tr}(R_2 - \mathcal{R}_2) - 2 \text{Tr}(R_1 - \mathcal{R}_1).$$

**Lemma 2.2.** *The following identity holds true:*

$$(2.15) \quad \lim_{n \rightarrow +\infty} \frac{1}{2\pi i} \oint_{K_n} \lambda \Phi(\lambda) d\lambda = -V(0).$$

**Proof.** Substituting the identities

$$R_j = \mathcal{R}_j - R_j V \mathcal{R}_j = \mathcal{R}_j - \mathcal{R}_j V \mathcal{R}_j + R_j V \mathcal{R}_j V \mathcal{R}_j, \quad j = 1, 2,$$

into (2.14) we obtain

$$(2.16) \quad \Phi = \Phi_0 + \Phi_1,$$

where

$$(2.17) \quad \Phi_0 = -\text{Tr} V \mathcal{R}_2^2 + \text{Tr} 2V \mathcal{R}_1^2, \quad \Phi_1 = \text{Tr} V \mathcal{R}_2 V \mathcal{R}_2 R_2 - \text{Tr} 2V \mathcal{R}_1 V \mathcal{R}_1 R_1.$$

Let  $n \rightarrow \infty$ . Identities (2.17), (2.13) yield

$$\oint_{K_n} (\lambda \Phi_0(\lambda) - F(\lambda)) d\lambda = o(1),$$

where

$$(2.18) \quad F = \text{Tr} V \mathcal{R}_2^0 - 2 \text{Tr} V \mathcal{R}_1^0.$$

Estimates (2.11) give

$$|\text{Tr} V \mathcal{R}_j(\lambda) V \mathcal{R}_j(\lambda) R_j(\lambda)| \leq \|R_j(\lambda)\|_2 \|\mathcal{R}_j(\lambda)\|_2^2 \|V\|^2 = O(n^{-9}), \quad j = 1, 2,$$

uniformly on contours  $K_n$ , which yields  $\Phi_1(\lambda) = O(n^{-9})$  and then

$$\oint_{K_n} \lambda \Phi_1(\lambda) d\lambda = o(1).$$

Then identity (2.16) gives

$$(2.19) \quad \oint_{K_n} (\lambda \Phi(\lambda) - F(\lambda)) d\lambda = o(1).$$

We have the Fourier series

$$(2.20) \quad V(x) = V_0 + 2 \sum_{n=1}^{\infty} (V_{cn} \cos 2\pi n x + V_{sn} \sin 2\pi n x),$$

where

$$V_{sn} = \int_0^1 V(x) \sin 2\pi n x dx, \quad V_{cn} = \int_0^1 V(x) \cos 2\pi n x dx, \quad V_0 = \int_0^1 V(x) dx.$$

Let  $e_n = \frac{1}{\sqrt{2}} e^{i\pi n x}$ ,  $n \in \mathbb{Z}$ , and  $s_n = \sin \pi n x$ ,  $n \in \mathbb{N}$ . Define the scalar products  $\langle f, g \rangle = \int_0^2 f \bar{g} dx$  and  $(f, g) = \int_0^1 f \bar{g} dx$  in  $L^2(0, 2)$  and  $L^2(0, 1)$  respectively. Identity (2.18) gives

$$\begin{aligned} F(\lambda) &= \frac{\langle V e_0, e_0 \rangle}{-\lambda} + \sum_{n=1}^{\infty} \frac{\langle V e_n, e_n \rangle + \langle V e_{-n}, e_{-n} \rangle - 4(V s_n, s_n)}{(\pi n)^4 - \lambda} \\ &= -\frac{V_0}{\lambda} + \sum_{n=1}^{\infty} \frac{2V_{cn}}{(\pi n)^4 - \lambda}, \end{aligned}$$

since we have the identities

$$\begin{aligned} \langle V e_n, e_n \rangle &= V_0 \quad \forall n \in \mathbb{Z}, \\ (V s_n, s_n) &= \int_0^1 V \sin^2 \pi n x dx = \frac{V_0 - V_{cn}}{2}, \quad n \geq 1. \end{aligned}$$

Then

$$\frac{1}{2\pi i} \oint_{K_N} F(\lambda) d\lambda = -V_0 - 2 \sum_{n=1}^N V_{cn}$$

for all  $N \geq 1$ . Identity (2.19) give

$$\frac{1}{2\pi i} \oint_{K_N} \lambda \Phi(\lambda) d\lambda = -\left(V_0 + 2 \sum_{n=1}^N V_{cn}\right) + o(1) \quad \text{as } N \rightarrow \infty,$$

then (2.20) yields (2.15). ■

**Proof of Theorem 1.1.** Asymptotics (1.7), (1.10) show that the series

$$S = \lambda_0^+ + \sum_{n \geq 1} (\lambda_n^+ + \lambda_n^- - 2\mu_n)$$

converges absolutely. Let  $S_0$  be given by

$$S_0 = (\alpha_0^+)^2 + \sum_{n \geq 1} ((\alpha_n^+)^2 + (\alpha_n^-)^2 - 2\beta_n^2).$$

Then due to (2.14) we have

$$S - S_0 = -\frac{1}{2\pi i} \lim_{n \rightarrow +\infty} \oint_{K_n} \lambda \Phi(\lambda) d\lambda.$$

Identity (2.3) yields

$$(2.21) \quad S - p^2(0) - \frac{p''(0)}{2} = -\frac{1}{2\pi i} \lim_{n \rightarrow +\infty} \oint_{K_n} \lambda \Phi(\lambda) d\lambda.$$

Substituting identity (2.15) into (2.21) we obtain

$$S - p^2(0) - \frac{p''(0)}{2} = V(0).$$

Substituting (2.5) into the last identity we have  $S = q(0) - \frac{p''(0)}{2}$ , which yields (1.11). ■

### 3. Proof of Lemma 2.1

**Proof of Lemma 2.1.** We will prove (2.11) for  $R_1(\lambda)$ . The proof for other  $R_2, \mathcal{R}_j, \mathcal{R}_j^0$  is similar. Let

$$a_k = |\mu_k|^{\frac{1}{4}}, \quad a = |\lambda|^{\frac{1}{4}}, \quad \lambda \in K_n, \quad k, n \geq 1.$$

Then we have

$$(3.1) \quad |\mu_k - \lambda| \geq ||\mu_k| - |\lambda|| = |a_k - a|(a_k + a)(a_k^2 + a^2) \geq |a_k - a|a^3.$$

Asymptotics (1.10) yields

$$(3.2) \quad a_k = \pi k + \pi \varepsilon_k, \quad \text{where } |\varepsilon_k| < \frac{1}{4} \quad \forall k > N,$$

$$(3.3) \quad |a_k - a| \geq 1 \quad \forall k \leq N$$

for  $n \in \mathbb{N}$  large enough and for some  $N \in \mathbb{N}$  large enough. We have

$$(3.4) \quad |a_k - a| = \pi \left| k - n - \frac{1}{2} + \varepsilon_k \right| \geq \pi \left( |k - n| - \frac{1}{4} \right) \quad \forall \quad k > N.$$

Estimates (3.1) – (3.4) give

$$\begin{aligned} \|R_1(\lambda)\|_2^2 &= \sum_{k=1}^{\infty} \frac{1}{|\mu_k - \lambda|^2} \leq \frac{1}{a^6} \sum_{k=1}^{\infty} \frac{1}{|a_k - a|^2} \\ &\leq \frac{1}{a^6} \left( N + \sum_{k=N+1}^{\infty} \frac{1}{\pi^2 (|k - n| - \frac{1}{4})^2} \right) \leq \frac{C}{a^6}, \end{aligned}$$

where  $C = N + \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}} (k - \frac{1}{4})^{-2} < \infty$  and this yields (2.11) for  $R_1(\lambda)$ .

We will prove (2.12) for  $j = 1$ . The proof for  $j = 2$  is similar. We have

$$(3.5) \quad \begin{aligned} |(\pi k)^4 - \lambda| &\geq |(\pi k)^4 - |\lambda|| = |\pi k - a|(\pi k + a)((\pi k)^2 + a^2) \\ &\geq a|\pi k - a|((\pi k)^2 + a^2). \end{aligned}$$

This estimate implies

$$\|h_j^0 \mathcal{R}_j^0(\lambda)\|_2^2 = \sum_{k=1}^{\infty} \frac{(\pi k)^4}{|(\pi k)^4 - \lambda|^2} \leq \frac{1}{a^2} \sum_{k=1}^{\infty} \frac{1}{|\pi k - a|^2} \leq \frac{1}{\pi^2 a^2} \sum_{k=1}^{\infty} \frac{1}{|k - n - \frac{1}{2}|^2} \leq \frac{C_1}{a^2},$$

where  $C_1 = \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}} (k - \frac{1}{2})^{-2} < \infty$ , which yields (2.12) for  $j = 1$ .

We will prove asymptotics (2.13) for  $j = 2$ . The proof for  $j = 1$  is similar. The integration by parts gives

$$\oint_{K_n} \lambda \operatorname{Tr} V \mathcal{R}_2^2(\lambda) d\lambda = \oint_{K_n} \lambda \operatorname{Tr} V (\mathcal{R}_2(\lambda))' d\lambda = - \oint_{K_n} \operatorname{Tr} V \mathcal{R}_2(\lambda) d\lambda$$

for all  $n \in \mathbb{N}$ . Using identity (2.8) we obtain

$$(3.6) \quad \begin{aligned} \oint_{K_n} \left( \lambda \operatorname{Tr} V \mathcal{R}_2^2(\lambda) + \operatorname{Tr} V \mathcal{R}_2^0(\lambda) \right) d\lambda &= \oint_{K_n} \operatorname{Tr} V (\mathcal{R}_2^0(\lambda) - \mathcal{R}_2(\lambda)) d\lambda \\ &= \oint_{K_n} \operatorname{Tr} (V \mathcal{R}_2(\lambda) Q \mathcal{R}_2^0(\lambda)) d\lambda, \quad \text{where } Q = -h_2^0 p - p h_2^0 + p^2. \end{aligned}$$

Asymptotics (2.11), (2.12) give

$$(3.7) \quad \begin{aligned} &| \operatorname{Tr} (\mathcal{R}_2^0(\lambda) Q(\lambda) \mathcal{R}_2^0(\lambda) Q(\lambda) \mathcal{R}_2(\lambda)) | \\ &\leq \| \mathcal{R}_2^0(\lambda) Q(\lambda) \|_2^2 \| \mathcal{R}_2(\lambda) \|_2 = O(n^{-5}) \quad \text{as } n \rightarrow \infty \end{aligned}$$

uniformly on all contours  $K_n$ . Substituting (2.8) into (3.6) and using (3.7) we obtain

$$(3.8) \quad \begin{aligned} &\oint_{K_n} \left( \lambda \operatorname{Tr} V \mathcal{R}_2^2(\lambda) + \operatorname{Tr} V \mathcal{R}_2^0(\lambda) \right) d\lambda \\ &= \oint_{K_n} \operatorname{Tr} V \mathcal{R}_2^0(\lambda) Q \mathcal{R}_2^0(\lambda) d\lambda + o(1) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus we need to consider  $\text{Tr } V\mathcal{R}_2^0(\lambda)Q\mathcal{R}_2^0(\lambda)$ , where  $Q = p^2 - h_2^0 p - ph_2^0$ . Firstly, we consider the case  $p^2$ . Estimate (2.11) as  $n \rightarrow \infty$  yields

$$(3.9) \quad \begin{aligned} & \oint_{K_n} \text{Tr } V\mathcal{R}_2^0(\lambda)p^2\mathcal{R}_2^0(\lambda)d\lambda = O(1) \oint_{K_n} \|\mathcal{R}_2^0(\lambda)\|_2^2 |d\lambda| \\ & = \oint_{K_n} O(n^{-6}) |d\lambda| = O(n^{-2}). \end{aligned}$$

Secondly, we consider the case  $-h_2^0 p - ph_2^0$ . Using  $\widehat{f}_k = \int_0^2 f(x)e^{i\pi kx} dx$  and the identity

$$\langle (-h_2^0 p - ph_2^0)e_m, e_k \rangle = \int_0^2 e^{-i\pi kx} (\partial_x^2 p(x) + p(x)\partial_x^2) e^{i\pi mx} dx = -\pi^2(k^2 + m^2)\widehat{p}_{m-k},$$

where  $e_m = \frac{1}{\sqrt{2}}e^{i\pi mx}$  and  $\langle f, g \rangle = \int_0^2 f\bar{g}dx$ , we have

$$(3.10) \quad \begin{aligned} \text{Tr}(V\mathcal{R}_2^0(h_2^0 p + ph_2^0)\mathcal{R}_2^0) &= - \sum_{m,k=-\infty}^{\infty} F(k, m, \lambda) \\ &= - \sum_{k=-\infty}^{\infty} \frac{8\pi^2 V_0 p_0 k^2}{((\pi k)^4 - \lambda)^2} - \sum_{\substack{m,k=-\infty \\ m \neq k}}^{\infty} F(k, m, \lambda), \\ F(k, m, \lambda) &= \frac{\pi^2(k^2 + m^2)\widehat{V}_{m-k}\widehat{p}_{m-k}}{((\pi m)^4 - \lambda)((\pi k)^4 - \lambda)}, \end{aligned}$$

where the series converge uniformly on each contour  $K_n, n \in \mathbb{N}$ . Moreover, the identity  $\oint_{K_n} \frac{d\lambda}{((\pi k)^4 - \lambda)^2} = 0$  and the decomposition  $\{|k| \neq |m|\} = D_1 \cup D_2 \cup D_3$ , where  $D_1, D_2, D_3$  are given by

$$\begin{aligned} D_1 &= \{|k| \neq |m|, |k| \leq n, |m| \leq n\} \cup \{|k| \neq |m|, |k| > n, |m| > n\}, \\ D_2 &= \{|k| \leq n, |m| > n\}, \quad D_3 = \{|k| > n, |m| \leq n\}, \end{aligned}$$

give

$$(3.11) \quad \begin{aligned} & \oint_{K_n} \text{Tr}(V\mathcal{R}_2^0(h_2^0 p + ph_2^0)\mathcal{R}_2^0)d\lambda \\ &= \oint_{K_n} \sum_{\substack{m,k=-\infty \\ m \neq k}}^{\infty} F(k, m, \lambda)d\lambda = I_1(\lambda) + I_2(\lambda) + I_3(\lambda) \end{aligned}$$

for all  $n \in \mathbb{N}$ , where

$$I_j(\lambda) = \oint_{K_n} \sum_{D_j} F(k, m, \lambda)d\lambda.$$

We have  $I_1(\lambda) = 0$  and thus we need to consider  $I_2, I_3$ .

Consider  $I_3$ , the proof for  $I_2$  is similar. Identity (3.11) gives

$$(3.12) \quad I_3(\lambda) = \oint_{K_n} \sum_{D_3} F(k, m, \lambda)d\lambda = \frac{2}{i\pi} \sum_{D_3} \frac{\widehat{V}_{m-k}\widehat{p}_{m-k}}{m^2 - k^2}.$$



Consider the case  $k > n, |m| \leq n$ , the proof for the other case is similar. Using  $V \in W_2^1(0, 2)$  and  $p \in W_4^1(0, 2)$  we obtain

$$(3.13) \quad \frac{|\widehat{V}_{m-k}\widehat{p}_{m-k}|}{|m^2 - k^2|} \leq \frac{C}{|k - m|^6|m^2 - k^2|} \leq \frac{C}{|k - m|^7(|m| + |k|)}$$

for some constant  $C$ . Define  $k' = k - n \geq 1$  and  $m' = n - m \in [0, 2n]$ . Then we obtain

$$\frac{1}{|k - m|^7(|m| + |k|)} \leq \frac{1}{|k' + m'|^7 n}.$$

This yields

$$\sum_{k > n, |m| \leq n} \frac{|\widehat{V}_{m-k}\widehat{p}_{m-k}|}{|m^2 - k^2|} \leq \sum_{k' \geq 1, m' \geq 0} \frac{C}{|k' + m'|^7 n} = \frac{C_1}{n}, \quad C_1 = \sum_{k' \geq 1, m' \geq 0} \frac{C}{|k' + m'|^7}.$$

Similar arguments show that

$$\sum_{k < -n, |m| \leq n} \frac{|\widehat{V}_{m-k}\widehat{p}_{m-k}|}{|m^2 - k^2|} \leq \frac{C_2}{n}$$

for some  $C_2 > 0$  and then (3.12) yields  $I_3(\lambda) = O(n^{-1})$  as  $n \rightarrow \infty$ . Similar estimates yield  $I_2(\lambda) = O(n^{-1})$ . Then (3.11) gives

$$\oint_{K_n} \text{Tr}(V\mathcal{R}_2^0(h_2^0 p + ph_2^0)\mathcal{R}_2^0)d\lambda = O(n^{-1}).$$

Substituting this asymptotics and (3.9) into (3.8) we obtain (2.13). ■

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NORTHERN (ARCTIC) FEDERAL UNIVERSITY, NORTHERN DVINA EMB, 17, ARKHANGELSK, 163002, RUSSIA

MATHEMATICAL PHYSICS DEPARTMENT, FACULTY OF PHYSICS, ULIANOVSKAYA 2, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, 198904, RUSSIA