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Existence, physical sense and analyticity of solitons for a 2D Boussinesq-Benney-Luke System

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Abstract. We show the existence and the analyticity of solitons (solitary waves of finite energy) for a 2D-Boussinesq-Benney-Luke type system that emerges in the study of the evolution of long water waves with small amplitude in the presence of surface tension. We follow a variational approach by characterizing travelling waves as minimizers of some functional under a suitable constrain. Using Lion's concentration-compactness principle, we prove that any minimizing sequences converges strongly, after an appropriate translation, to a minimizer. The Boussinesq-Benney-Luke system is formally close to the Benney-Luke equation and to the Kadomtsev-Petviashivili (KP) equation. For wave speed small and surface tension large, we assure some physical sense for this water wave system by establishing that a suitable (renormailized) family of solitons of the Boussinesq-Benney-Luke system converges to a nontrivial soliton for the KP-I equation.

Contents

1. Introduction

Models for dispersive and weakly nonlinear long water waves with small amplitude in finite depth are derived from the full water wave problem through an approximation process, under the imposition of some restrictions on the parameters

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that affect the propagation of gravity water waves, as the nonlinearity (amplitude parameter) and the dispersion (long-wave parameter), and also by assuming that the free surface elevation, its derivatives, and the derivatives of the velocity potential are small quantities compared with the amplitude parameter and the long wave parameter. As it is well known, roughly speaking, the study of water waves is reduced to determine the free surface elevation and the velocity potential on the free surface. So, if the vertical variable is eliminated from the equations by using Taylor expansion approximation about some height with respect to the vertical spatial variable, then it is possible to obtain some water wave models in two spatial variables. This approach to derive dispersive water wave models through out expansions in the vertical spatial variable of the velocity potential, related with the (KP) equation, the Benney-Luke models, and Boussinesq systems, has been followed in some recent works by M. Ablowitz, A. Fokas, and H. Musslimani in [**1**], A. Montes in [**12**], J. Quintero in [**13**], J. Quintero and R. Pego in [**14**] (in the last three works including the effect of the surface tension). We also must mention the important work done by D. Benney and J. Luke in [**2**], in which the derivation of a model is performed under the assumption that the amplitude parameter and the long wave parameter were equal in the absence of surface tension. For a dispersive model without the long-wave assumption, we are aware of the work of P. Milewski and J. Keller in [**10**]. It is important to point out that some Benney-Luke models have been described using a single equation for the velocity potential about some hight, by eliminating the free surface elevation from the system describing the full water wave problem.

In this work, we will show that the analysis of the evolution of long water waves with small amplitude in the presence of surface tension is reduced to looking for a couple $(\Phi, \eta)(x, y, t)$ satisfying the Boussinesq-Benney-Luke type system

$$\text{(BBL)} \qquad \begin{cases} \left(I - \frac{\mu}{2}\Delta\right)\eta_t + \Delta\Phi - \frac{2\mu}{3}\Delta^2\Phi + \epsilon\nabla\cdot(\eta\nabla\Phi) &= 0, \\ \left(I - \frac{\mu}{2}\Delta\right)\Phi_t + \eta - \mu\sigma\Delta\eta + \frac{\epsilon}{2}|\nabla\Phi|^2 &= 0. \end{cases}$$

where ϵ is the amplitude parameter (nonlinearity coefficient), $\sqrt{\mu} = \frac{h_0}{L}$ is the long-wave parameter (dispersion coefficient), σ is the inverse of the Bond number (associated with the surface tension). The variable Φ represents the rescale nondimensional velocity potential on the bottom $z = 0$, and the variable η corresponds the rescaled free surface elevation. We want to point out that system (BBL) is rather close to one derived by M. Boussinesq (cf. system (θ) on p. 314 of [**5**], taking account of the relation at the bottom of p. 323). We consider that the Boussinesq-Benney-Luke system (BBL) could provide a better approximation, from the physical view point, to the full water wave equations than some Benney-Luke type models for long water wave with small amplitude in the sense that in the derivation of those models the surface elevation η is eliminated up to order two in ϵ and μ, in order to obtain a single equation in the variable Φ (essentially the rescaled velocity potential ϕ expanded at the bottom $z = 0$) up to order two in ϵ and μ.

The Boussinesq-Benney-Luke system has some sort of physical sense since some well known water wave models as the generalized Benney-Luke equation ([**14**]-[**16**]), a Boussinesq-KdV system ([**13**]) and the generalized Kadomtsev-Petviashivili equation emerge from this Boussinesq type system (BBL) (up to some order with respect to ϵ and μ), making the system (BBL) very interesting from the physical and numerical view points.

According with the assumption made to derive the system (BBL), it is necessary to establish for ϵ, μ and c (in some range) the existence of solitons, but also to provide some physical relevance of this model. In that direction, we will show that appropriate order-one solutions exist for arbitrarily small value of the parameters. Note that if we balance the effect of nonlinearity and dispersion ($\mu = \epsilon$), and seek a travelling-wave solution of the form $(\eta, \Phi)(x, y, t) = (v, \varphi)(x - ct, y)$ with $c > 0$, then (v, φ) should satisfy the system of the single equations,

$$(1 - c^2)v_{xx} + v_{yy} = O(\epsilon), \quad (1 - c^2)\varphi_{xx} + \varphi_{yy} = O(\epsilon).$$

This fact is indicating us that making the wave speed c close to one and having weak dependence on y could provide an appropriate regime to find physically meaningful finite-energy solutions, as done when deriving the Kadomtsev-Petviashivili equation (KP) as an approximation for the full water wave problem. One the mayor goals in this paper is to exhibit some physical relevance of the system, by establishing that in an appropriate scaling there are sequences of solitons for this Boussinesq type system that, after an appropriate translation, converge as $\epsilon \to 0$ to a soliton of the Kadomtsev-Petviashivili equation (KP-I) equation.

The paper is organized as follows. In section 2, we derive formally the Boussinesq-Benney-Luke system (BBL) from the full water problem reduced to the case of weakly nonlinear long wave propagation in shallow water. We also describe the Hamiltonian structure for the Boussinesq type system (BBL) derived. In section 3, from the Hamiltonian structure, we find the natural finite-energy space for solitons (solitary-wave solutions), and characterize solitary waves variationally as critical points of an action functional. We prove the existence solitons for the Boussinesq type system (BBL) by using the Lions' concentration-compactness principle for $\epsilon > 0$, $\mu > 0$, $\sigma > 0$ and wave speed $0 < c < c_*$. In Section 4, we show that any sequence $\{(\eta_{\epsilon,\mu,c}, \Phi_{\epsilon,\mu,c})\}_{\epsilon,\mu,c}$ of solitons for the Boussinesq-Benney-Luke system (BBL) converges strongly in the (KP) energy space, after a renormalization and an appropiate translation, to a soliton for the (KP-I) equation, provided that $\sigma > \frac{3}{8}$ and that $c \to 1^-$ as $\epsilon \to 0^+$. In section 5, we show that the solitary wave solutions for the Boussineq-Benney-Luke system (BBL) are analytic.

2. A Boussinesq-Benney-Luke system

Suppose that $\overrightarrow{u}$ represents the velocity of a particle in an irrotational, three-dimensional flow of an inviscid, incompressible fluid which at rest occupies the region $-\infty < x < \infty$, $-\infty < y < \infty$, $0 < z < h_0$, then for some distribution ϕ, the velocity potential $\overrightarrow{u}$ takes the form $\overrightarrow{u} = (\nabla\phi, \partial_z\phi)$, where $\nabla = (\partial_x, \partial_y)$. Moreover, the study of the water waves with surface tension reduces to find solutions of the linear equation

$$\Delta\phi + \phi_{zz} = 0 \quad \text{for } 0 < z < h_0 + \eta, \quad (\Delta = \partial_x^2 + \partial_y^2) \tag{1}$$

with the boundary and interface nonlinear conditions

$$\phi_z = 0 \quad \text{at } z = 0, \tag{2}$$

$$\eta_t + \nabla\eta \cdot \nabla\phi - \phi_z = 0 \quad \text{at } z = h_0 + \eta, \tag{3}$$

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}\phi_z^2 + g\eta - \frac{2TH}{\rho} = 0 \quad \text{at } z = h_0 + \eta, \tag{4}$$

where $z = 0$ represents the solid boundary, $z = h_0 + \eta$ is the disturbed free surface, T is the coefficient of surface tension, ρ is density (assumed constant), g is the gravitational acceleration, and the Mean Curvature of the free surface $z = \eta(x, y, t)$ is given by

$$H = \frac{1}{2}\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + |\nabla\eta|^2}}\right).$$

In order to study long water wave with small amplitude, we introduce the amplitude parameter ϵ and the long wave parameter $\mu = (h_0/L)^2$, where L stands for the horizontal length of motion. The long-wave regime corresponds to $\mu \ll 1$. The system is introduced through the following rescaling of the variables x, y, and z:

$$x = L\hat{x}, \quad y = L\hat{y}, \quad z = h_0\hat{z}, \quad t = L(gh_0)^{-\frac{1}{2}}\hat{t},$$

and the definition of the functions $\hat{\phi}$ and $\hat{\eta}$ as $\phi = \epsilon\frac{h_0}{\sqrt{\mu}}(gh_0)^{\frac{1}{2}}\hat{\phi}$ and $\eta = \epsilon h_0\hat{\eta}$. Note that a simple computation shows that $\partial_x\phi$ and $\partial_y\phi$ are of order $O(\epsilon)$, as long as $\partial_{\hat{x}}\hat{\phi}$ and $\partial_{\hat{y}}\hat{\phi}$ are of order $O(1)$ with respect to ϵ. Taking $T = h_0^2\rho g\sigma$ and after dropping hats, we obtain that the couple (ϕ, η) satisfies the nonlinear system

$$\mu\Delta\phi + \phi_{zz} = 0 \quad \text{for } 0 < z < 1 + \epsilon\eta, \tag{5}$$

$$\phi_z = 0 \quad \text{at } z = 0, \tag{6}$$

$$\eta_t + \epsilon\nabla\eta \cdot \nabla\phi - \frac{1}{\mu}\phi_z = 0 \quad \text{at } z = 1 + \epsilon\eta, \tag{7}$$

(8)

$$\phi_t + \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{\epsilon}{2\mu}\phi_z^2 + \eta - \mu\sigma\nabla \cdot \left(\frac{\nabla\eta}{\sqrt{1 + \epsilon^2\mu|\nabla\eta|^2}}\right) = 0 \quad \text{at} \quad z = 1 + \epsilon\eta,$$

where Δ and ∇ are the laplacian and the gradiant with respect to the variables x, y, respectively. Now, to derive the Boussinesq-Benney-Luke model we have to assume that $\nabla\phi$, η, and its derivatives with respect to variables x and y are $O(1)$ with respect to ϵ. Defining

$$\Phi(x, y, t) = \phi(x, y, z = 0, t)$$

and using Taylor expansion to the velocity potential at the bottom $z = 0$, one can see that

$$\phi = \Phi - \frac{\mu z^2}{2}\Delta\Phi + \frac{\mu^2 z^4}{4!}\Delta^2\Phi + O(\mu^3). \tag{9}$$

Plugging this in equations (7)-(8), formally we obtain the following system

$$\eta_t + \epsilon\nabla\eta \cdot \nabla\Phi + (1 + \epsilon\eta)\Delta\Phi - \frac{\mu}{6}\Delta^2\Phi = O(\epsilon^2, \mu^2), \tag{10}$$

$$\Phi_t - \frac{\mu}{2}\Delta\Phi_t + \frac{\epsilon}{2}|\nabla\Phi|^2 + \eta - \mu\sigma\Delta\eta = O(\epsilon^2, \mu^2). \tag{11}$$

This system was originally derived by J. Quintero and R. Pego (see [**14**]) in their study of the Benney-Luke equation. In this work, J. Quintero *et. al.* showed that evolution of long water waves with small amplitude can be reduced to determine the potential velocity at the bottom Φ as solution of the Benney-Luke equation,

$$\text{(BL)} \qquad \Phi_{tt} - \Delta\Phi + \mu(a\Delta^2\Phi - b\Delta\Phi_{tt}) + \epsilon(\Phi_t\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t) = 0.$$

In this work we are interested in deriving a model for long water waves of small amplitude, including the free surface elevation η. From (10) we observe that

$$\eta_t + \Delta\Phi = O(\epsilon, \mu).$$

Then by using this expression in (10) and neglecting terms of order $O(\epsilon^2, \mu^2)$ we have the system,

$$\text{(BBL)} \qquad \begin{cases} \left(I - \frac{\mu}{2}\Delta\right)\eta_t + \Delta\Phi - \frac{2\mu}{3}\Delta^2\Phi + \epsilon\nabla\cdot(\eta\nabla\Phi) &= 0, \\ \left(I - \frac{\mu}{2}\Delta\right)\Phi_t + \eta - \mu\sigma\Delta\eta + \frac{\epsilon}{2}|\nabla\Phi|^2 &= 0. \end{cases}$$

The (BBL) system is a Boussinesq type system that we will classify as a Boussinesq-Benney-Luke system for its relation with the Benney-Luke (BL) equation. From the physical view point, it turns out that the (BBL) system could be a better approximation that some Benney-Luke type model to describe the evolution of long waves of small amplitude where the surface elevation η has been eliminated up to some order with respect to the amplitude and log wave parameters.

We observe now that the system (BBL) arise as the Euler-Lagrange for the action functional

$$\hat{S} = -\int_{t_1}^{t_2}\left(\int_{\mathbb{R}^2}\Big(\Phi_t - \frac{\mu}{2}\Delta\Phi_t\Big)\eta\,dxdy + \mathcal{H}(\eta,\Phi)\right)dt$$

with Hamiltonian given by

$$(12) \qquad \mathcal{H}(\eta,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla\Phi|^2 + \eta^2 + \frac{2\mu}{3}|\Delta\Phi|^2 + \mu\sigma|\nabla\eta|^2 + \epsilon\eta|\nabla\Phi|^2\right)dxdy,$$

for which the system (BBL) takes the form

$$(\eta_t, \Phi_t) = J\nabla\mathcal{H}(\eta,\Phi) \qquad J = \left(I - \frac{\mu}{2}\Delta\right)^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A very interesting fact is that we obtain the Benney-Luke equation (BL) from the system (BBL). Also we can formally reduce this system to a Kadomtsev-Petviashvili (KP) type equation in appropriate limits. In fact, from the second equation of (BBL) we see that

$$\Phi_t + \eta = O(\epsilon, \mu),$$

then, again from the second equation,

$$\eta_t = -\Phi_{tt} - \mu\left(\sigma - \tfrac{1}{2}\right)\Delta\Phi_{tt} - \epsilon\nabla\Phi\cdot\nabla\Phi_t + O(\epsilon^2, \mu^2).$$

Plugging this expression in the first equation of (BBL) and using

$$\eta_t + \Delta\Phi = O(\epsilon, \mu),$$

we obtain the equation

$$\Phi_{tt} - \Delta\Phi + \mu\big(\tfrac{1}{6}\Delta^2\Phi + (\sigma - \tfrac{1}{2})\Delta\Phi_{tt}\big) + \epsilon\big(\Phi\Delta\Phi + 2\nabla\Phi\cdot\nabla\Phi_t\big) = O(\epsilon^2, \mu^2).$$

Thus, choosing a, b such that

$$a - b = \sigma - \frac{1}{3}, \tag{13}$$

and neglecting the terms of order $O(\epsilon^2, \mu^2)$, we obtain the so-called Benney-Luke equation (BL). This equation has been studied, see for example [**7**], [**11**], [**14**]-[**18**]. Note that we are able to show, formally, that travelling waves for the Boussinesq system (BBL) formally reduce to travelling waves for a KP type equation, when we consider waves propagating predominantly in the x-direction, weakly in the y-direction, having slowly evolving in time and balancing the effects of nonlinearity and dispersion. In fact, we start balancing the effects of nonlinearity and dispersion by setting $\mu = \epsilon$, and defining the following KP type scaling, $X = x - ct$ and $Y = \epsilon^{1/2} y$. Let (η, Φ) be a solution of the (BBL) system, take $c^2 = 1 - \epsilon$ and consider (u, v) defined as

$$(\eta, \Phi)(x, y, t) = (u, v)(X, Y).$$

Then plugging this into the (BBL) system and using $c = 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$ give us that

$$-cu_X + v_{XX} - \epsilon\left[-v_{YY} + \tfrac{2}{3}v_{XXXX} - \tfrac{1}{2}u_{XXX} - (uv_X)_X\right] = O(\epsilon^2). \tag{14}$$

$$-cv_X + u - \epsilon\left[\sigma u_{XX} - \tfrac{1}{2}v_{XXX} - \tfrac{1}{2}(v_X^2)\right] = O(\epsilon^2). \tag{15}$$

From (15) we see that $-cv_X + u = O(\epsilon)$, then using this in (14), we obtain that

$$-cu_X + v_{XX} - \epsilon\left(-v_{YY} + \tfrac{1}{6}v_{XXXX} - \left(v_X^2\right)_X\right) = O(\epsilon^2). \tag{16}$$

Multiplying by c and differentiating with respect to X the equation (15), and using (16), we get that

$$(1 - c^2)v_{XX} - \epsilon\left(-v_{YY} + \left(\sigma - \tfrac{1}{3}\right)v_{XXXX} - \tfrac{3}{2}\left(v_X^2\right)_X\right) = O(\epsilon^2).$$

Using that $\epsilon = 1 - c^2$ we obtain,

$$v_{XX} + v_{YY} - \left(\sigma - \tfrac{1}{3}\right)v_{XXXX} + \tfrac{3}{2}\left(v_X^2\right)_X = O(\epsilon). \tag{17}$$

If we differentiate (17) with respect to X and neglect the $O(\epsilon)$ term, then we find that the couple (u, v) satisfies the system

$$u = v_X \tag{18}$$

$$\left(u_X - \left(\sigma - \tfrac{1}{3}\right)u_{XXX} + 3uu_X\right)_X + u_{YY} = 0. \tag{19}$$

In other words, $u = v_X$ (up to order $O(\epsilon)$) is a solution for the travelling wave equation of a (KP-I) type equation.

3. Existence of solitary wave solutions

By a solitary wave solution we shall mean a solution (η, Φ) of (BBL) of the form

$$\eta(x, y, t) = \frac{1}{\epsilon}u\left(\frac{x - ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right), \quad \Phi(x, y, t) = \frac{\sqrt{\mu}}{\epsilon}v\left(\frac{x - ct}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right). \tag{20}$$

Then, one sees that (u, v) must satisfy

$$\begin{cases} \frac{2}{3}\Delta^2 v - \Delta v + c\left(I - \frac{1}{2}\Delta\right)u_x - \nabla\cdot(u\nabla v) = 0, \\ u - \sigma\Delta u - c\left(I - \frac{1}{2}\Delta\right)v_x + \frac{1}{2}|\nabla v|^2 = 0. \end{cases} \tag{21}$$

We establish the existence of a solution of (21) in the weak sense by using a variational approach in which weak solutions correspond to critical points of an energy under a special constrain, associated with the Hamiltonian structure given by (12) which provides the natural space to look for travelling waves. For $k \in \mathbb{R}$, let $H^k(U)$ be Sobolev space the Hilbert space defined as the closure of $C^\infty(U)$ with inner product

$$(u, v)_{H^k} = \sum_{|\alpha| \le k} \int_U D^\alpha u \cdot D^\alpha v \, dx.$$

We denote $\mathcal{V}$ the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm given by

$$\|v\|_{\mathcal{V}}^2 := \int_{\mathbb{R}^2} \left(|\nabla v|^2 + |\Delta v|^2\right) dxdy = \int_{\mathbb{R}^2} \left(v_x^2 + v_y^2 + v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2\right) dx\, dy.$$

Note that $\mathcal{V}$ is a Hilbert space with respect to the inner product

$$(v, w)_{\mathcal{V}} = (v_x, w_x)_{H^1(\mathbb{R}^2)} + (v_y, w_y)_{H^1(\mathbb{R}^2)}.$$

Also, we define the Hilbert space $\mathcal{X} = H^1(\mathbb{R}^2) \times \mathcal{V}$ with respect to the norm

$$\|(u, v)\|_{\mathcal{X}}^2 = \|u\|_{H^1(\mathbb{R}^2)}^2 + \|v\|_{\mathcal{V}}^2 = \int_{\mathbb{R}^2} \left(u^2 + |\nabla u|^2 + |\nabla v|^2 + |\Delta v|^2\right) dxdy.$$

If we multiply the travelling wave system (21) with a test couple $(w, z) \in \mathcal{X}$, then, after integrating by parts, a travelling wave solution $(u, v) \in \mathcal{X}$ satisfies the system

(22)
$$\int_{\mathbb{R}^2} \left[\begin{pmatrix} \nabla v \cdot \nabla z + \frac{2}{3}\Delta v \Delta z \\ uw + \sigma \nabla u \cdot \nabla w \end{pmatrix} + c \begin{pmatrix} -(uz_x + \frac{1}{2}u_x \Delta z) \\ -(wv_x + \frac{1}{2}w_x \Delta v) \end{pmatrix} + \begin{pmatrix} u\nabla v \cdot \nabla z \\ \frac{1}{2}w|\nabla v|^2 \end{pmatrix} \right] dxdy = 0,$$

which can be written in terms of suitable functionals A, $B_{1,c}$ and B_2 as

$$A((u, v), (w, z)) + B_{1,c}((u, v), (w, z)) + B_2((u, v), (w, z)) = 0. \tag{23}$$

Definition 3.1. *We say that (u,v) is a weak solution of (21) if for all $(w, z) \in \mathcal{X}$, the system (23) holds.*

Our strategy to prove the existence of a weak solution of (21) is to consider the following minimization problem

$$\mathcal{I}_c := \inf \left\{ I_c(u, v) \ : \ (u, v) \in \mathcal{X} \ \text{ with } \ G(u, v) = 1 \right\}, \tag{24}$$

where the energy I_c and the constrain G are functionals defined in $\mathcal{X}$ given by

$$I_c(u, v) = \int_{\mathbb{R}^2} \left(u^2 + \sigma|\nabla u|^2 + |\nabla v|^2 + \tfrac{2}{3}(\Delta v)^2 - 2cuv_x - cu_x \Delta v\right) dxdy, \tag{25}$$

$$G(u, v) = \int_{\mathbb{R}^2} u|\nabla v|^2 \, dx\, dy. \tag{26}$$

We start by showing some properties of I_c and G, assuming in this section that $\sigma > 0$, $0 < c < \min\left\{1, \frac{8\sigma}{3}\right\}$, and that C denotes a generic constant whose value may change from instance to instance.

Lemma 3.1. (1) *The functionals I_c and G are well defined in $\mathcal{X}$ and smooth.*
(2) *The functional I_c is nonnegative. Moreover, there is $C_1 = C_1(\sigma, c) > 1$ such that*

$$C_1^{-1} I_c(u, v) \le \|(u, v)\|_{\mathcal{X}}^2 \le C_1 I_c(u, v). \tag{27}$$

(3) *$\mathcal{I}_c$ is finite and positive.*

Proof. *1)* I_c is clearly well defined for $(u,v)\in\mathcal{X}$. Now, note that if $v\in\mathcal{V}$ then $v_x, v_y\in H^1(\mathbb{R}^2)$. Using the Young inequality and the fact that the embedding $H^1(\mathbb{R}^2)\hookrightarrow L^q(\mathbb{R}^2)$ is continuous for $q\geq 2$, we see that there is a constant $C>0$ such that

$$|G(u,v)|\leq C\Big(\|u\|^3_{H^1(\mathbb{R}^2)}+\|\nabla v\|^3_{H^1(\mathbb{R}^2)}\Big). \tag{28}$$

So, G is well defined.
2) Let σ be a fixed positive number and $0<c<\min\big\{1,\frac{8\sigma}{3}\big\}$. Then

$$\begin{aligned}I_c(u,v)&=\int_{\mathbb{R}^2}\left((u-cv_x)^2+\sigma u_y^2+(1-c^2)v_x^2+v_y^2\right)dxdy\\&\quad+\int_{\mathbb{R}^2}\left(\sigma\left(u_x-\frac{c}{2\sigma}\Delta v\right)^2+\left(\frac{2}{3}-\frac{c^2}{4\sigma}\right)(\Delta v)^2\right)dxdy\\&=\int_{\mathbb{R}^2}\left((1-c^2)u^2+\left(\sigma-\frac{3c^2}{8}\right)u_x^2+\sigma u_y^2+v_y^2\right)dxdy\\&\quad+\int_{\mathbb{R}^2}\left((v_x-cu)^2+\frac{2}{3}\left(\Delta v-\frac{3c}{4}u_x\right)^2\right)dxdy\geq 0.\end{aligned}\tag{29}$$

Now, using (25) and Young inequality we obtain that

$$\begin{aligned}I_c(u,v)&\leq\int_{\mathbb{R}^2}\bigg((1+c)u^2+\left(\sigma+\frac{c}{2}\right)u_x^2+\sigma u_y^2\\&\quad+(1+c)v_x^2+v_y^2+\left(\frac{2}{3}+\frac{c}{2}\right)(\Delta v)^2\bigg)dxdy\\&\leq\max\left(1+c,\frac{2}{3}+\frac{c}{2},\sigma+\frac{c}{2}\right)\|(u,v)\|^2_{\mathcal{X}}.\end{aligned}\tag{30}$$

Moreover,

$$\begin{aligned}I_c(u,v)&=\int_{\mathbb{R}^2}\bigg((1-c)u^2+\sigma(1-c)u_x^2+\sigma u_y^2+\\&\quad(1-c)v_x^2+v_y^2+\left(\frac{2}{3}-\frac{c}{4\sigma}\right)(\Delta v)^2\bigg)dxdy\\&\quad+\int_{\mathbb{R}^2}\left(c\,(u-v_x)^2+\sigma c\Big(u_x-\frac{1}{2\sigma}\Delta v\Big)^2\right)dxdy\\&\geq\min\left(1-c,\,\sigma(1-c),\frac{2}{3}-\frac{c}{4\sigma}\right)\|(u,v)\|^2_{\mathcal{X}},\end{aligned}$$

showing that the inequality (27) holds.
3) Note that there exist $(u,v)\in\mathcal{X}$ such that $G(u,v)\neq 0$. Then for some t we have that $G(tu,tv)=t^3G(u,v)=1$. On the other hand, for any $(u,v)\in\mathcal{X}$ such that $G(u,v)=1$ the inequalities (27)-(28) implies that there is $C>0$ such that

$$C\,(I_c(u,v))^{\frac{3}{2}}\geq C\left(\|u\|^3_{H^1}+\|\nabla v\|^3_{H^1}\right)\geq G(u,v)=1,$$

meaning that the infimum $\mathcal{I}_c$ is finite and positive. □

To simplify the notation, we observe that

$$I_c(u,v)=I_1(u)+I_2(v)+I_3(u,v),$$

where

$$I_1(u) = \int_{\mathbb{R}^2} \left(u^2 + \sigma|\nabla u|^2\right) dxdy, \quad I_2(v) = \int_{\mathbb{R}^2} \left(|\nabla v|^2 + \tfrac{2}{3}(\Delta v)^2\right) dxdy,$$
$$I_3(u,v) = -c\int_{\mathbb{R}^2} (2uv_x + u_x\Delta v)\, dxdy.$$

THEOREM 3.1. *If (u_0, v_0) is a minimizer for (24), then $(u,v) = -k(u_0,v_0)$ is a nontrivial weak solution of (21) for $k = \frac{2}{3}\mathcal{I}_c$.*

PROOF. By the Lagrange Theorem there is a multiplier k such that for any $(w,z) \in \mathcal{X}$,

$$I_c'(u_0,v_0)(w,z) - kG'(u_0,v_0)(w,z) = 0.$$

Now, a direct calculation shows that

$$A((u_0,v_0),(w,z)) + B_{1,c}((u_0,v_0),(w,z)) - kB_2((u_0,v_0),(w,z)) = 0. \tag{31}$$

In particular, if we put $(w,z) = (u_0,v_0)$ in the previous equation, we have that

$$I_2(v_0) + \tfrac{1}{2}I_3(u_0,v_0) - kG(u_0,v_0) = 0 \tag{32}$$

$$I_1(u_0) + \tfrac{1}{2}I_3(u_0,v_0) - \tfrac{k}{2}G(u_0,v_0) = 0. \tag{33}$$

Combining (32) and (33) we obtain,

$$I_c(u_0,v_0) - \tfrac{3}{2}kG(u_0,v_0) = 0.$$

Then $k = \frac{2}{3}\mathcal{I}_c$, and $-k(u_0,v_0)$ is a nontrivial weak solution of (21). □

3.1. Existence of minimizers. The existence of travelling waves solutions for (21) as a minimizer of the minimization problem (24) is based on the existence of a compact embedding (local) result and also on an important result by L. P. Lions, which characterizes completely the convergence of measures, known as the Concentration-Compactness principle (see [**8**], [**9**]).

THEOREM 3.2. *(L. Lions, [**8**], [**9**]) Suppose $\{\nu_m\}$ is a sequence of nonnegative measures on $\mathbb{R}^2$ such that*

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} d\nu_m = \mathcal{I}.$$

Then there is a subsequence of $\{\nu_m\}$ (which denote the same) that satisfy only one of the following properties.
Vanishing. For any $R > 0$,

$$\lim_{m\to\infty} \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_R(x,y)} d\nu_m \right) = 0, \tag{34}$$

where $B_R(x,y)$ is the ball of radius R centered at (x,y).

Dichotomy. There exist $\theta \in (0,\mathcal{I})$ such that for any $\gamma > 0$, there are $R > 0$ and a sequence $\{(x_m,y_m)\}$ in $\mathbb{R}^2$ with the following property: Given $R' > R$ there are nonnegative measures ν_m^1, ν_m^2 such that

(1) $0 \le \nu_m^1 + \nu_m^2 \le \nu_m$,

(2) *supp* $(\nu_m^1) \subset B_R(x_m,y_m)$, *supp* $(\nu_m^2) \subset \mathbb{R}^2 \setminus B_{R'}(x_m,y_m)$,

(3) $\limsup_{m\to\infty} \left(|\theta - \int_{\mathbb{R}^2} d\nu_m^1| + |(\mathcal{I}-\theta) - \int_{\mathbb{R}^2} d\nu_m^2|\right) \le \gamma.$

Compactness. There exists a sequence $\{(x_m, y_m)\}$ in $\mathbb{R}^2$ such that for any $\gamma > 0$, there is $R > 0$ with the property that

$$\int_{B_R(x_m,y_m)} d\nu_m \geq \mathcal{I} - \gamma, \quad \textit{for all } m. \tag{35}$$

In order to apply this result to our case, let us assume that $\{(u_m, v_m)\}$ in $\mathcal{X}$ is a minimizing sequence for $\mathcal{I}_c$, then we define the positive measures $\{\nu_m\}$ by $d\nu_m = \rho_m \, dxdy$, where ρ_m is defined as

$$\rho_m = u_m^2 + \sigma|\nabla u_m|^2 + |\nabla v_m|^2 + \tfrac{2}{3}(\Delta v_m)^2 - 2cu_m\partial_x v_m - c\partial_x u_m \Delta v_m, \tag{36}$$

which correspond to the integrand of $I_c(u_m, v_m)$. From the Concentration Compactness Theorem, there exists a subsequence of $\{\nu_m\}$ (which denote the same) that satisfy either *vanishing*, or *dichotomy*, or *compactness*. We will see that *vanishing* and *dichotomy* can be ruled out, and so using *compactness* we will establish that the minimizing sequence $\{(u_m, v_m)\}$ is compact in $\mathcal{X}$, up to translation, as a consequence of local compact embedding.

We will establish some technical result. The first one is related with the characterization of "vanishing sequences" in $\mathcal{X}$.

Lemma 3.2. *(Vanishing sequences) Let $R > 0$ be given and let $\{(u_m, v_m)\}_m$ be a bounded sequence in $\mathcal{X}$ such that*

$$\lim_{m\to\infty} \left(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_R(x,y)} d\nu_m \right) = 0.$$

Then we have that

$$\lim_{m\to\infty} G(u_m, v_m) = \lim_{m\to\infty} \int_{\mathbb{R}^2} u_m |\nabla v_m|^2 \, dxdy = 0.$$

In particular, if $\{(u_m, v_m)\}_m$ is a minimizing sequence for $\mathcal{I}_c$, then vanishing is ruled out.

Proof. Let $B_1 = B_1(x, y)$. Since the embedding $H^1(B_1) \hookrightarrow L^3(B_1)$ is continuous, we obtain that

$$\begin{aligned}\int_{B_1} u_m |\nabla v_m|^2 \, dxdy &\leq C\Big(\|u_m\|^3_{H^1(B_1)} + \|\nabla v_m\|^3_{H^1(B_1)}\Big) \\ &\leq C\Big(\|u_m\|^2_{H^1(B_1)} + \|\nabla v_m\|^2_{H^1(B_1)}\Big)\Big(\int_{B_1} d\nu_m\Big)^{1/2}.\end{aligned}$$

But we know that $\mathbb{R}^2$ can be covered with balls of radius 1 in such a way that each point of $\mathbb{R}^2$ is contained in at most 3 balls. Then we conclude that

$$\begin{aligned}\int_{\mathbb{R}^2} u_m |\nabla v_m|^2 \, dV &\leq 3C\Big(\|u_m\|^2_{H^1(\mathbb{R}^2)} + \|\nabla v_m\|^2_{H^1(\mathbb{R}^2)}\Big)\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_1} d\nu_m\Big)^{1/2} \\ &\leq 3CI_c(u_m, v_m)\Big(\sup_{(x,y)\in\mathbb{R}^2} \int_{B_1} d\nu_m\Big)^{1/2}.\end{aligned}$$

As a consequence of this we see that

$$\lim_{m\to\infty} G(u_m, v_m) = \lim_{m\to\infty} \int_{\mathbb{R}^2} u_m |\nabla v_m|^2 \, dxdy = 0.$$

Now assume that $\{(u_m, v_m)\}_m$ is a minimizing sequence for $\mathcal{I}_c$. Then we have that $G(u_m, v_m) = 1$, but from the previous fact we reach a contradiction. □

In order to rule out dichotomy, we will establish a splitting result for a sequence $\{(u_m, v_m)\}_m$ in $\mathcal{X}$. Fix a function $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^+)$ such that *supp* $\phi \subset B_2(0,0)$ and $\phi \equiv 1$ in $B_1(0,0)$. If $R > 0$ and $(x_0, y_0) \in \mathbb{R}^2$, we define a split for $(u, v) \in \mathcal{X}$ given by

$$u = u_R^1 + u_R^2 \quad \text{and} \quad v = v_R^1 + v_R^2,$$

where

$$u_R^1 = u\phi_R, \;\; u_R^2 = u(1-\phi_R), \quad v_R^1 = (v - a_R)\phi_R, \;\; v_R^2 = (v - a_R)(1-\phi_R) + a_R,$$

with

$$\phi_R(x,y) = \phi\Big(\frac{x - x_0}{R}, \frac{y - y_0}{R}\Big),$$

and

$$a_R = \frac{1}{\operatorname{vol}(A_R(x_0,y_0))}\int_{A_R(x_0,y_0)} v(x,y)\,dxdy, A_R(x_0,y_0) = B_{2R}(x_0,y_0)\backslash B_R(x_0,y_0).$$

We note that the decomposition of v is non standard and reflects the nature of the space $\mathcal{V}$.

In the coming result, we use the following Poincaré type inequality

$$\text{(37)} \qquad \Big(\int_{A_R(x_0,y_0)} |v - a_R|^q \,dxdy\Big)^{1/q} \le CR^{2/q}\Big(\int_{A_R(x_0,y_0)} |\nabla v|^2\,dxdy\Big)^{1/2},$$

where $2 \le q \le \infty$ and C does not depend on R (see [**4**], [**14**]).

Lemma 3.3. *(A splitting result) Let $R_m > 0$ and $(x_m, y_m) \in \mathbb{R}^2$ be sequences. Define $A(m) = A_{R_m}(x_m, y_m)$ and $\phi_m(x,y) = \phi\Big(\frac{x-x_m}{R_m}, \frac{y-y_m}{R_m}\Big)$. If*

$$\text{(38)} \qquad \limsup_{m\to\infty}\left(\int_{A(m)} d\nu_m\right) = 0,$$

then as $m \to \infty$ we have that
a.- $I_c(u_m, v_m) = I_c(u_m^1, v_m^1) + I_c(u_m^2, v_m^2) + o(1)$.
b.- $G(u_m, v_m) = G(u_m^1, v_m^1) + G(u_m^2, v_m^2) + o(1)$.

Proof. We need to recall that

$$I_c(u,v) = I_1(u) + I_2(v) + I_3(u,v).$$

We will see that

$$\text{(39)} \qquad I_j(z_m) = I_j(z_m^1) + I_j(z_m^2) + o(1), \quad \text{as } m \to \infty,$$

where $z_m = u_m$ for $j = 1$, $z_m = v_m$ for $j = 2$, and $z_m = (u_m, v_m)$ for $j = 3$.
First note that

$$\delta^{(0)}u_m := \int_{\mathbb{R}^2}\Big\{(u_m)^2 - (u_m^1)^2 - (u_m^2)^2\Big\}\,dxdy = 2\int_{A(m)} \phi_m(1-\phi_m)(u_m)^2\,dxdy.$$

Then

$$|\delta^{(0)}u_m| \le C\int_{A(m)} (u_m)^2\,dxdy \le C\int_{A(m)} d\nu_m \;\to\; 0.$$

For $i = 1, 2$, we have that

$$\begin{aligned}
\delta^{(i)} u_m &:= \int_{\mathbb{R}^2} \left\{ (\partial_i u_m)^2 - (\partial_i u_m^1)^2 - (\partial_i u_m^2)^2 \right\} dxdy \\
&= 2 \int_{A(m)} \Big\{ \phi_m (1 - \phi_m)(\partial_i u_m)^2 + (1 - 2\phi_m)(u_m \partial_i u_m)(\partial_i \phi_m) \\
&\quad - u_m^2 (\partial_i \phi_m)^2 \Big\} dxdy.
\end{aligned}$$

Consequently,

$$|\delta^{(i)} u_m| \leq C \int_{A(m)} \left\{ (u_m)^2 + (\partial_i u_m)^2 \right\} dxdy \leq C \int_{A(m)} d\nu_m \to 0.$$

Then we obtain that

$$\lim_{m \to \infty} \left[I_1(u_m) - I_1(u_m^1) - I_1(u_m^2) \right] = 0.$$

Now,

$$\begin{aligned}
\delta^{(i)} v_m &:= \int_{\mathbb{R}^2} \left\{ (\partial_i v_m)^2 - (\partial_i v_m^1)^2 - (\partial_i v_m^2)^2 \right\} dxdy \\
&= 2 \int_{A(m)} \left\{ \phi_m (1 - \phi_m)(\partial_i v_m)^2 - (v_m - a_m)^2 (\partial_i \phi_m)^2 \right. \\
&\quad \left. + (1 - 2\phi_m)(\partial_i v_m)(v_m - a_m)(\partial_i \phi_m) \right\} dxdy.
\end{aligned}$$

But $|\partial_i \phi_m| \leq C/R_m$, and using the Poincaré type inequality (37) we obtain that

$$\begin{aligned}
|\delta^{(i)} v_m| &\leq C \int_{A(m)} \left\{ (\partial_i v_m)^2 + \frac{(v_m - a_m)^2}{R_m^2} \right\} dxdy \\
&\leq C \int_{A(m)} \left\{ |\partial_i v_m|^2 + |\nabla v_m|^2 \right\} dxdy \\
&\leq C \int_{A(m)} d\nu_m \to 0.
\end{aligned}$$

If $w_m = (v_m - a_m)\partial_{ij}\phi_m + \partial_i v_m \partial_j \phi_m + \partial_j v_m \partial_i \phi_m$ a simple calculation shows that

$$\partial_{ij} v_m^1 = (\partial_{ij} v_m)\phi_m + w_m, \quad \partial_{ij} v_m^2 = (\partial_{ij} v_m)(1 - \phi_m) - w_m.$$

Therefore

$$\begin{aligned}
\delta^{(ij)} v_m &:= \int_{\mathbb{R}^2} \left\{ (\partial_{ij} v_m)^2 - (\partial_{ij} v_m^1)^2 - (\partial_{ij} v_m^2)^2 \right\} dxdy \\
&= 2 \int_{A(m)} \left\{ \phi_m (1 - \phi_m)(\partial_{ij} v_m)^2 - w_m^2 + (1 - 2\phi_m)(\partial_{ij} v_m) w_m \right\} dxdy.
\end{aligned}$$

Since $|\partial_{ij} \phi_m| \leq C/R_m^2$ we have that

$$\begin{aligned}
|\delta^{(ij)} v_m| &\leq C \int_{A(m)} \left\{ (\partial_{ij} v_m)^2 + \frac{(v_m - a_m)^2}{R_m^4} + \frac{(\partial_i v_m)^2 + (\partial_j v_m)^2}{R_m^2} \right\} dxdy \\
&\leq C \int_{A(m)} d\nu_m.
\end{aligned}$$

Then we conclude that

$$\lim_{m \to \infty} \left[I_2(v_m) - I_2(v_m^1) - I_2(v_m^2) \right] = 0.$$

Now we show the result for I_3. First note that

$$\begin{aligned}\delta_i^{(0)} &:= \int_{\mathbb{R}^2} \left\{u_m \partial_i v_m - u_m^1 \partial_i v_m^1 - u_m^2 \partial_i v_m^2\right\} dxdy \\ &= \int_{A(m)} 2\phi_m(1-\phi_m) u_m \partial_i v_m + (1-2\phi_m) u_m (v_m - a_m)\partial_i \phi_m \, dxdy.\end{aligned}$$

Then, using again that $|\partial_i \phi_m| \leq C/R_m$, Young inequality and inequality (37) we obtain that

$$|\delta_i^{(0)}| \leq C \int_{A(m)} d\nu_m \to 0.$$

On the other hand,

$$\begin{aligned}\delta_{ii}^{(1)} &:= \int_{\mathbb{R}^2} \left\{\partial_1 u_m \partial_{ii} v_m - \partial_1 u_m^1 \partial_{ii} v_m^1 - \partial_1 u_m^2 \partial_{ii} v_m^2\right\} dxdy \\ &= \int_{A(m)} \Big\{ 2\phi_m(1-\phi_m)\partial_1 u_m \partial_{ii} v_m + (1-\phi_m)(\partial_i \phi_m) u_m \partial_{ii} v_m \\ &\quad + (1-2\phi_m)(\partial_1 u_m) w_m - 2(\partial_1 \phi_m) u_m w_m \Big\} dxdy.\end{aligned}$$

In a similar fashion one checks that

$$|\delta_{ii}^{(1)}| \leq C \int_{A(m)} d\nu_m \ \to \ 0,$$

which concludes (39). Now, we will show the item (b). For $i = 1, 2$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^2} \left| u_m (\partial_i v_m)^2 - u_m^1 (\partial_i v_m^1)^2 - u_m^2 (\partial_i v_m^2)^2 \right| dxdy \\ &\quad = \int_{A(m)} \Big| 2(1-2\phi_m) u_m (\partial_i v_m)(v_m - a_m)\partial_i \phi_m \\ &\quad + 3\phi_m(1-\phi_m) u_m (\partial_i v_m)^2 - u_m (v_m - a_m)^2 (\partial_i \phi_m)^2 \Big| \, dxdy \\ &\quad \leq C \int_{A(m)} \Big\{ (u_m)^2 + (\partial_i v_m)^2 + |u_m|^3 + |\partial_i v_m|^3 + \frac{(v_m - a_m)^2}{R_m^2} \\ &\quad + \frac{|v_m - a_m|^3}{R_m^3} \Big\} dxdy \\ &\quad \leq C \left(\int_{A(m)} d\nu_m + \left(\int_{A(m)} d\nu_m \right)^{3/2} \right) \ \to \ 0.\end{aligned}$$

Then we conclude as $m \to \infty$ that

$$G(u_m, v_m) = G(u_m^1, v_m^1) + G(u_m^2, v_m^2) + o(1).$$

□

Using the previous result we have that

LEMMA 3.4. *Let $\{(u_m, v_m)\}_m$ be a minimizing sequence for $\mathcal{I}_c$. Then dichotomy is not possible.*

PROOF. Assume that dichotomy occurs, then we can choose sequences $\gamma_m \to 0$ and $R_m \to \infty$ such that

$$supp\,(\nu_m^1) \subset B_{R_m}(x_m, y_m), \quad supp(\nu_m^2) \subset \mathbb{R}^2 \setminus B_{2R_m}(x_m, y_m) \tag{40}$$

and

$$\limsup_{m\to\infty}\left(\left|\theta - \int_{\mathbb{R}^2} d\nu_m^1\right| + \left|(\mathcal{I}_c - \theta) - \int_{\mathbb{R}^2} d\nu_m^2\right|\right) = 0. \tag{41}$$

Using these facts, we have that

$$\limsup_{m\to\infty}\left(\int_{A(m)} d\nu_m\right) = 0. \tag{42}$$

In fact,

$$\begin{aligned}
\int_{A(m)} d\nu_m &= \left(\int_{\mathbb{R}^2} - \int_{B_{R_m}(x_m,y_m)} - \int_{\mathbb{R}^2\setminus B_{2R_m}(x_m,y_y)}\right) d\nu_m \\
&\le \int_{\mathbb{R}^2} d\nu_m - \int_{\mathbb{R}^2} d\nu_m^1 - \int_{\mathbb{R}^2} d\nu_m^2 \\
&\le \left(\int_{\mathbb{R}^2} d\nu_m - \mathcal{I}_c\right) + \left|\theta - \int_{\mathbb{R}^2} d\nu_m^1\right| + \left|(\mathcal{I}_c - \theta) - \int_{\mathbb{R}^2} d\nu_m^2\right|.
\end{aligned}$$

Using (42) and Lemma 3.3 we conclude that

$$\begin{aligned}
\lim_{m\to\infty}\left[I_c(u_m, v_m) - I_c(u_m^1, v_m^1) - I_c(u_m^2, v_m^2)\right] = 0, \\
\lim_{m\to\infty}\left[G(u_m, v_m) - G(u_m^1, v_m^1) - G(u_m^2, v_m^2)\right] = 0.
\end{aligned}$$

Now, let $\lambda_{m,i} = G(u_m^i, v_m^i)$, for $i = 1, 2$. We will show that $\lambda_i := \lim_{m\to\infty}\lambda_{m,i} \neq 0$. Assume that $\lim_{m\to\infty}\lambda_{m,1} = 0$, then $\lim_{m\to\infty}\lambda_{m,2} = 1$ (we proceed in a similar way in the other case). Therefore $\lambda_{m,2} > 0$, for m large enough. Then we consider

$$(w_m, z_m) = \lambda_{m,2}^{-\frac{1}{3}}(u_m^2, v_m^2).$$

So that,

$$(w_m, z_m) \in \mathcal{X}, \quad G(w_m, z_m) = 1.$$

Using that $\phi_m \equiv 1$ in $B_{R_m}(x_m, y_m)$ we have a contradiction since

$$\begin{aligned}
\mathcal{I}_c &= \lim_{m\to\infty}\left(I_c(u_m^1, v_m^1) + I_c(u_m^2, v_m^2)\right) \\
&\ge \lim_{m\to\infty}\left(\int_{B_{R_m}(x_m,y_m)} d\nu_m + \lambda_{m,2}^{\frac{2}{3}} I_c(w_m, z_m)\right) \\
&\ge \lim_{m\to\infty}\left(\int_{\mathbb{R}^2} d\nu_m^1 + \lambda_{m,2}^{\frac{2}{3}}\mathcal{I}_c\right) \\
&= \theta + \mathcal{I}_c.
\end{aligned}$$

In other words, $|\lambda_{m,i}| > 0$ for m large enough. Then we are allowed to define

$$(w_{m,i}, z_{m,i}) = \lambda_{m,i}^{-\frac{1}{3}}(u_m^i, v_m^i), \quad i = 1, 2.$$

Note that $(w_{m,i}, z_{m,i}) \in \mathcal{X}$ and $G(w_{m,i}, z_{m,i}) = 1$. So that,

$$\begin{aligned}\mathcal{I}_c &= \lim_{m\to\infty}\Big(I_c(u_m^1, v_m^1) + I_c(u_m^2, v_m^2)\Big)\\ &= \lim_{m\to\infty}\Big(|\lambda_{m,1}|^{\frac{2}{3}} I_c(w_{m,1}, z_{m,1}) \;+\; |\lambda_{m,2}|^{\frac{2}{3}} I_c(w_{m,2}, z_{m,2})\Big)\\ &\geq \Big(|\lambda_1|^{\frac{2}{3}} + |\lambda_2|^{\frac{2}{3}}\Big)\mathcal{I}_c.\end{aligned}$$

Then

$$1 \geq |\lambda_1|^{\frac{2}{3}} + |\lambda_2|^{\frac{2}{3}} \geq \big(|\lambda_1| + |\lambda_2|\big)^{\frac{2}{3}} \geq |\lambda_1 + \lambda_2|^{\frac{2}{3}} = 1.$$

Hence, $|\lambda_1| + |\lambda_2| = 1$. Using that $\lambda_1 + \lambda_2 = 1$ and $\lambda_i \neq 0$, we have that $\lambda_i > 0$ and

$$\lambda_1^{\frac{2}{3}} + \lambda_2^{\frac{2}{3}} = (\lambda_1 + \lambda_2)^{\frac{2}{3}}. \tag{43}$$

But (43) gives us a contradiction, because for $t \in \mathbb{R}^+$ the function $f(t) = t^{\frac{2}{3}}$ is strictly concave, meaning that

$$f(t_1 + t_2) > f(t_1) + f(t_2).$$

In other words, we have ruled out dichotomy.

□

Now we are in position to obtain the main result in this section: the existence of a minimizer for $\mathcal{I}_c$. Since we ruled out vanishing and dichotomy above for a minimizing sequence of $\mathcal{I}_c$, then by Lion's Concentration Compactness Theorem, there exists a subsequence of $\{\nu_m\}$ (which denote the same) satisfying *compactness*. We will see as a consequence of local compact embedding that a minimizing sequence $\{(u_m, v_m)\}$ is compact in $\mathcal{X}$, up to translation.

THEOREM 3.3. *If $\{(u_m, v_m)\}$ is a minimizing sequence for (24), then there is a subsequence (which we denote the same), a sequence of points $(x_m, y_m) \in \mathbb{R}^2$, and a minimizer $(u_0, v_0) \in \mathcal{X}$ of (24), such that the translated functions*

$$(\tilde{u}_m, \tilde{v}_m) = (u_m(\cdot + x_m, \cdot + y_m), v_m(\cdot + x_m, \cdot + y_m))$$

converge to (u_0, v_0) strongly in $\mathcal{X}$.

PROOF. Let $\{(u_m, v_m)\}$ be a minimizing sequence for (24). In other words,

$$\lim_{m\to\infty} I_c(u_m, v_m) = \mathcal{I}_c \quad \text{and} \quad G(u_m, v_m) = 1.$$

By *compactness*, there exists a sequence (x_m, y_m) in $\mathbb{R}^2$ such that for a given $\gamma > 0$, there exists $R > 0$ with the following property,

$$\int_{B_R(x_m,y_m)} d\nu_m \geq \mathcal{I}_c - \gamma, \quad \text{for all } m \in \mathbb{N}. \tag{44}$$

Using this we may localize the minimizing sequence $\{(u_m, v_m)\}_m$ around the origin by defining

$$\tilde{\rho}_m(x, y) = \rho_m(x + x_m, y + y_m), \quad (\tilde{u}_m, \tilde{v}_m)(x, y) = (u_m, v_m)(x + x_m, y + y_m).$$

Thus, we have the following localized inequality

$$\int_{B_R(0,0)} \tilde{\rho}_m dxdy = \int_{B_R(x_m,y_m)} d\nu_m \geq \mathcal{I}_c - \gamma, \quad \text{for all } m \in \mathbb{N}, \tag{45}$$

and also that

$$G(\tilde{u}_m,\tilde{v}_m)=G(u_m,v_m)=1,\quad \lim_{m\to\infty} I_c(\tilde{u}_m,\tilde{v}_m)=\lim_{m\to\infty} I_c(u_m,v_m)=\mathcal{I}_c. \tag{46}$$

Then we note that $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ is a bounded sequence in $\mathcal{X}=H^1(\mathbb{R}^2)\times\mathcal{V}$. On the other hand, since $\tilde{u}_m,\nabla\tilde{v}_m\in H^1(U)$ for any bounded open set U and the embedding $H^1(U)\hookrightarrow L^q(U)$ is compact for $q\geq 2$, then there exist a subsequence of $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ (which we denote the same) and $(u_0,v_0)\in H^1\times\mathcal{V}$ such that for $i=1,2$,

$$\tilde{u}_m\rightharpoonup u_0 \quad\text{in}\quad H^1(\mathbb{R}^2),\quad \tilde{u}_m\rightharpoonup u_0\quad\text{in}\quad L^2(\mathbb{R}^2),$$
$$\tilde{v}_m\rightharpoonup v_0\quad\text{in}\quad \mathcal{V},\quad \partial_i\tilde{v}_m\rightharpoonup v_0\quad\text{in}\quad L^2(\mathbb{R}^2)$$

and we also have that

$$\tilde{u}_m\to u_0\quad\text{in}\quad L^2_{loc}(\mathbb{R}^2),\quad \partial_i\tilde{v}_m\to\partial_i v_0\quad\text{in}\quad L^2_{loc}(\mathbb{R}^2).$$

Moreover,

$$\tilde{u}_m\to u_0\quad\text{a.e. in }\mathbb{R}^2,\quad \partial_i\tilde{v}_m\to\partial_i v_0\quad\text{a.e in }\mathbb{R}^2\quad\text{for } i=1,2.$$

Using these facts we will show that some subsequence of $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ (which we denote the same) converges strongly in $\mathcal{X}$ to a nontrivial minimizer (u_0,v_0) of (24). We first see that

$$\tilde{u}_m\to u_0,\quad \partial_i\tilde{v}_m\to\partial_i v_0\quad\text{in}\quad L^2(\mathbb{R}^2). \tag{47}$$

In fact, using (45), (46) and the definition of I_c we have that for $\gamma>0$, there exists $R>0$ such that for m large enough,

$$\int_{B_R(0,0)}|\tilde{u}_m|^2dxdy\geq\int_{\mathbb{R}^2}|\tilde{u}_m|^2\,dxdy-2\gamma.$$

Then we have that

$$\begin{aligned}\int_{\mathbb{R}^2}|u_0|^2dxdy&\leq\liminf_{m\to\infty}\int_{\mathbb{R}^2}|\tilde{u}_m|^2dxdy\\&\leq\liminf_{m\to\infty}\int_{B_R(0,0)}|\tilde{u}_m|^2dxdy+2\gamma\\&\leq\int_{B_R(0,0)}|u_0|^2dxdy+2\gamma\\&\leq\int_{\mathbb{R}^2}|u_0|^2dxdy+2\gamma.\end{aligned}$$

Therefore,

$$\liminf_{m\to\infty}\int_{\mathbb{R}^2}|\tilde{u}_m|^2dxdy=\int_{\mathbb{R}^2}|u_0|^2dxdy.$$

Thus, there exist a subsequence of $\{\tilde{u}_m\}$ such that $\tilde{u}_m\to u_0$ in $L^2(\mathbb{R}^2)$. Using a similar argument we prove the other part of (47). Moreover, also we can see that

$$\partial_i\tilde{u}_m\to\partial_i u_0,\quad \partial_{ij}\tilde{v}_m\to\partial_{ij}v_0\quad\text{in}\quad L^2(\mathbb{R}^2). \tag{48}$$

Now, using (47)-(48) and the fact that the inclusion $H^1(\mathbb{R}^2)\hookrightarrow L^4(\mathbb{R}^2)$ is continuous we have that

$$G(u_0,v_0)=\lim_{m\to\infty}G(\tilde{u}_m,\tilde{v}_m)=1. \tag{49}$$

In fact, from definition of G we see that

$$\begin{aligned} G(\tilde{u}_m,\tilde{v}_m)-G(u_0,v_0) &= \int_{\mathbb{R}^2}\left(\tilde{u}_m|\nabla\tilde{v}_m|^2-u_0|\nabla v_0|^2\right)dxdy \\ &= \int_{\mathbb{R}^2}\Big[(\tilde{u}_m-u_0)|\nabla\tilde{v}_m|^2+u_0\left(|\nabla\tilde{v}_m|^2-|\nabla v_0|^2\right)\Big]dxdy. \end{aligned}$$

But

$$\begin{aligned} \int_{\mathbb{R}^2}(\tilde{u}_m-u_0)|\nabla v_m|^2dxdy &\le \|\tilde{u}_m-u_0\|_{L^2}\|\nabla v_m\|_{L^4}^2 \\ &\le C\,\|\tilde{u}_m-u_0\|_{L^2}\|\nabla v_m\|_{H^1}^2 \\ &\le C\,I_c(\tilde{u}_m,\tilde{v}_m)\,\|\tilde{u}_m-u_0\|_{L^2} \\ &\le C\,\|\tilde{u}_m-u_0\|_{L^2}\to 0, \end{aligned}$$

and also

$$\begin{aligned} \int_{\mathbb{R}^2}u_0\left(|\nabla\tilde{v}_m|^2-|\nabla v_0|^2\right)dxdy &\le \|\nabla(\tilde{v}_m-v_0)\|_{L^2}\|\nabla(\tilde{v}_m+v_0)\|_{L^4}\|u_0\|_{L^4} \\ &\le C\|\nabla(\tilde{v}_m-v_0)\|_{L^2}\|\nabla(\tilde{v}_m+v_0)\|_{H^1}\|u_0\|_{H^1} \\ &\le C\|\nabla(\tilde{v}_m-v_0)\|_{L^2}\left(\|u_0\|_{H^1}^2+\|\nabla(\tilde{v}_m+v_0)\|_{H^1}^2\right) \\ &\le C\|\nabla(\tilde{v}_m-v_0)\|_{L^2}\left(I_c(\tilde{u}_m,\tilde{v}_m)+I_c(u_0,v_0)\right)\to 0. \end{aligned}$$

Then we conclude that (49) holds, which implies that

$$(u_0,v_0)\neq(0,0),\quad I_c(u_0,v_0)\ge\mathcal{I}_c.$$

On the other hand,

$$\lim_{m\to\infty}\big[I_c(\tilde{u}_m,\tilde{v}_m)-I_c(\tilde{u}_m-u_0,\tilde{v}_m-v_0)\big]=I_c(u_0,v_0).$$

Hence we have that

$$0\le\lim_{m\to\infty}I_c(\tilde{u}_m-u_0,\tilde{v}_m-v_0)=\lim_{m\to\infty}I_c(\tilde{u}_m,\tilde{v}_m)-I_c(u_0,v_0)=\mathcal{I}_c-I_c(u_0,v_0)\le 0.$$

Thus, we see that

$$\lim_{m\to\infty}I_c(\tilde{u}_m,\tilde{v}_m)=I_c(u_0,v_0)=\mathcal{I}_c,\quad \lim_{m\to\infty}I_c(\tilde{u}_m-u_0,\tilde{v}_m-v_0)=0.$$

Moreover, the sequence $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ converges to (u_0,v_0) in $\mathcal{X}$, since

$$\|(\tilde{u}_m-u_0,\tilde{v}_m-v_0)\|_{\mathcal{X}}\le C_1I_c(u_m-u_0,v_m-v_0).$$

Then we concluded that $\{(\tilde{u}_m,\tilde{v}_m)\}$ converges to (u_0,v_0) in $\mathcal{X}$ and (u_0,v_0) is a minimizer for $\mathcal{I}_c$. □

4. Inter-relation between (BBL) and (KP) solitons

In this Section we will establish that after an appropriate choice of (ϵ,μ,c) and a renormalization of the family of solitons

$$\left\{\left(u_{(\epsilon,\mu,c)},v_{(\epsilon,\mu,c)}\right)\right\}_{(\epsilon,\mu,c)}$$

for the system (BBL), it is possible to obtain solitons for the (KP-I) equation. We must remember that the derivation of the (BBL) system required the parameters ϵ and μ to be small, while the free surface elevation η and the derivatives of Φ to be of order one with respect to ϵ and μ. Up to now, we have not shown that the solitons for system (BBL) are physically meaningful for the water wave model. In order to give some physical sense of the solitons for the (BBL), we will prove that as the parameter ϵ, μ are small enough and c is close to 1^-, we are able to recover a soliton for a (KP-I) type equation. The reason to adjust the wave speed (c near 1^-) to obtain a soliton for the (KP-I) equation is related with the fact that in the derivation the KP model the variation in the y variable is weaker that the variation in the x variable.

We know the following characterization of solitons for the (KP-I) model given by A. de Bouard and J. C. Saut in [**3**] and [**4**]. First consider the space $\mathcal{Z}$ defined as the closure of $C_0(\mathbb{R}^2)$ with respect to the norm given by

$$\|w\|_{\mathcal{Z}}^2 = \int_{\mathbb{R}^2} \left(w_x^2 + w_y^2 + w_{xx}^2\right) dxdy,$$

and the functionals defined in $\mathcal{Z}$ by

$$J^0(w) = \int_{\mathbb{R}^2} \left(w_x^2 + w_y^2 + \left(\sigma - \tfrac{1}{3}\right) w_{xx}^2\right) dxdy,$$
$$K^0(w) = \int_{\mathbb{R}^2} w_x^3 dxdy,$$

and the minimization problem

$$\mathcal{J}^0 := \inf\{J^0(w) : w \in \mathcal{Z},\ K^0(w) = 1\}. \tag{50}$$

Then we have the next theorem.

Theorem 4.1. *([**3**], [**4**]) Let $\sigma > \frac{1}{3}$. If $\{w_m\}_{m\geq 1}$ is a minimizing sequence for $\mathcal{J}^0$, then there exist a subsequence (denoted the same) and a nonzero distribution $w_0 \in \mathcal{Z}$ such that*

$$J^0(w_0) = \mathcal{J}^0 > 0,$$

and there exists a sequence of points $\{\zeta_m\}_{m\geq 1}$ in $\mathbb{R}^2$ such that $w_m(\cdot + \zeta_m) \to w_0$ in $\mathcal{Z}$. Moreover, w_0 is a solution in the sense of distributions of the equation

$$-w_{xx} - w_{yy} + \left(\sigma - \tfrac{1}{3}\right) w_{xxxx} + (2\mathcal{J}^0) w_x w_{xx} = 0. \tag{51}$$

So that, $w = -\left(\frac{2}{3}\mathcal{J}^0\right)\partial_x w_0$ is a nontrivial solitary wave solution in the sense of distributions for the (KP-I) equation (19).

Now we introduce a (KP-I) type scaling for obtain a soliton of the (KP-I) equation from a renormalized family of solitons of the system (BBL). Set $\sigma > 0$, $\epsilon > 0$, $\mu = \epsilon$, $c^2 = 1 - \epsilon$ and for a given couple $(u, v) \in \mathcal{X}$ define the functions w and z by

$$u(x, y) = \epsilon^{\frac{1}{2}} z(X, Y), \quad v(x, y) = w(X, Y), \quad X = \epsilon^{\frac{1}{2}} x,\ \ Y = \epsilon y.$$

Then a simple calculation shows that

$$I_{c(\epsilon)}(u, v) = \epsilon^{\frac{1}{2}} I^{\epsilon}(z, w), \quad G(u, v) = G^{\epsilon}(z, w),$$

where I^ϵ and G^ϵ are given by

$$\begin{aligned} I^\epsilon(z,w) = &\int_{\mathbb{R}^2}\biggl(\epsilon^{-1}z^2+\sigma(z_x^2+\epsilon z_y^2)+\epsilon^{-1}w_x^2+w_y^2 \\ &+\frac{2}{3}\left(w_{xx}^2+2\epsilon w_{xy}^2+\epsilon^2 w_{yy}^2\right)\biggr)dxdy \\ &-c\int_{\mathbb{R}^2}\left(2\epsilon^{-1}zw_x+z_x(w_{xx}+\epsilon w_{yy})\right)dxdy, \\ G^\epsilon(z,w) = &\int_{\mathbb{R}^2} z\left(w_x^2+\epsilon w_y^2\right)dxdy. \end{aligned}$$

Note that if $\sigma > \frac{3}{8}$ then there is a family $\{(u_c,v_c)\}_c$ such that

$$I_c(u_c,v_c)=\mathcal{I}_c, \quad G(u_c,v_c)=1, \quad 0<c<1.$$

Then, if we denote

$$\mathcal{I}^\epsilon := \inf\left\{I^\epsilon(z,w) \ : \ (z,w)\in\mathcal{X} \ \text{ with } \ G^\epsilon(z,w)=1\right\},$$

there is a correspondent family $\{(z^\epsilon,w^\epsilon)\}_\epsilon$ such that

$$\mathcal{I}^\epsilon = I^\epsilon(z^\epsilon,w^\epsilon), \quad G^\epsilon(z^\epsilon,w^\epsilon)=1, \quad \mathcal{I}_c=\epsilon^{\frac{1}{2}}\mathcal{I}^\epsilon.$$

Moreover, (z^ϵ,w^ϵ) is a solution, in the sense of distributions, of the system

(52)
$$\begin{cases} \frac{2}{3}\left(w_{xxxx}+2\epsilon w_{xxyy}+\epsilon^2 w_{yyyyy}\right)-\epsilon^{-1}w_{xx}-w_{yy}+c\left(\epsilon^{-1}z_x-\frac{1}{2}(z_{xxx}+\epsilon z_{xyy})\right) \\ +(\frac{2}{3}\mathcal{I}^\epsilon)\left((zw_x)_x+\epsilon(zw_y)_y\right)=0, \\ \epsilon^{-1}z-\sigma(z_{xx}+\epsilon z_{yy})-c\left(\epsilon^{-1}w_x-\frac{1}{2}(w_{xxx}+\epsilon w_{xyy})\right)-(\frac{1}{3}\mathcal{I}^\epsilon)(w_x^2+\epsilon w_y^2)=0. \end{cases}$$

We are interested in relating the family $\{(z^\epsilon,w^\epsilon)\}_\epsilon$ as ϵ goes to zero with solitons for the (KP-I) equation. To do this, we define in $\mathcal{V}$ the functionals

$$\begin{aligned} J^\epsilon(w) &= I^\epsilon(cw_x,w) \\ &= \int_{\mathbb{R}^2}\left(w_x^2+w_y^2+\left((\sigma-1)c^2+\frac{2}{3}\right)w_{xx}^2+\epsilon\left((\sigma-1)\,c^2+\frac{4}{3}\right)w_{xy}^2\right. \\ &\quad \left.+\frac{2}{3}\epsilon^2 w_{yy}^2\right)dxdy, \\ K^\epsilon(w) &= G^\epsilon(cw_x,w) = c\int_{\mathbb{R}^2}\left(w_x^3+\epsilon w_x w_y^2\right)dxdy, \end{aligned}$$

Note that if $\sigma > \frac{1}{3}$ and ϵ small enough, the functional J^ϵ is nonnegative. Then we define the number $\mathcal{J}^\epsilon$ by

$$\mathcal{J}^\epsilon = \inf\left\{J^\epsilon(w) \ : \ w\in\mathcal{V} \ \text{ with } \ K^\epsilon(w)=1\right\}, \tag{53}$$

and establish the following important result.

LEMMA 4.1. *Let $\sigma > \frac{3}{8}$. Then we have that*

(54)
$$\lim_{\epsilon\to 0^+}\mathcal{I}^\epsilon = \lim_{\epsilon\to 0^+}\mathcal{J}^\epsilon = \mathcal{J}^0, \qquad \lim_{\epsilon\to 0^+}K^0(w^\epsilon) = \lim_{\epsilon\to 0^+}K^\epsilon(w^\epsilon) = \lim_{\epsilon\to 0^+}G^\epsilon(z^\epsilon,w^\epsilon)=1.$$

Moreover, for any sequence $\epsilon_j\to 0$, the sequence $\{\left(K^0(w^{\epsilon_j})\right)^{-1/3}w^{\epsilon_j}\}$ is a minimizing sequence for $\mathcal{J}^0$.

PROOF. Let $w \in \mathcal{V}$ be such that $\int_{\mathbb{R}^2} w_x^3\,dxdy = 1$, then for $\epsilon > 0$ small enough we have that

$$K^\epsilon(w) = c\int_{\mathbb{R}^2} \left(w_x^3 + \epsilon w_x w_y^2\right) dxdy \neq 0.$$

Then we see that

$$\mathcal{I}^\epsilon \leq I^\epsilon\left(\frac{1}{G^\epsilon(cw_x, w)^{1/3}}(cw_x, w)\right) = \frac{I^\epsilon(cw_x, w)}{G^\epsilon(cw_x, w)^{2/3}} = \frac{J^\epsilon(w)}{K^\epsilon(w)^{2/3}} \tag{55}$$

and

$$\mathcal{J}^\epsilon \leq J^\epsilon\left(\frac{1}{K^\epsilon(w)^{1/3}}w\right) = \frac{J^\epsilon(w)}{K^\epsilon(w)^{2/3}}.$$

On the other hand,

$$\lim_{\epsilon\to 0^+} J^\epsilon(w) = J^0(w), \qquad \lim_{\epsilon\to 0^+} K^\epsilon(w) = 1.$$

Hence, if $w \in \mathcal{V}$ and $\int_{\mathbb{R}^2} w_x^3\,dxdy = 1$ we conclude that

$$\limsup_{\epsilon\to 0^+} \mathcal{I}^\epsilon \leq J^0(w), \qquad \limsup_{\epsilon\to 0^+} \mathcal{J}^\epsilon \leq J^0(w).$$

So that,

$$\limsup_{\epsilon\to 0^+} \mathcal{I}^\epsilon \leq \mathcal{J}^0, \qquad \limsup_{\epsilon\to 0^+} \mathcal{J}^\epsilon \leq \mathcal{J}^0. \tag{56}$$

Moreover, for ϵ small enough we have that

$$\epsilon^{-\frac{1}{2}} I_c(u_c, v_c) = \mathcal{I}^\epsilon \leq 2\mathcal{J}^0.$$

Then, using that $c^2 = 1 - \epsilon$ and the definition of $I_{c(\epsilon)}$ (see Eq. (29)), we have that

$$\|u_c - c\partial_x v_c\|_{L^2(\mathbb{R}^2)} + \|\partial_y v_c\|_{H^1(\mathbb{R}^2)} = O(\epsilon^{\frac{1}{4}}), \quad \|\partial_x v_c\|_{L^2(\mathbb{R}^2)} = O(\epsilon^{\frac{-1}{4}}). \tag{57}$$

Now, since $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, as $\epsilon \to 0^+$ we see that

$$\begin{aligned}\epsilon\left|\int_{\mathbb{R}^2}(z^\epsilon - c\partial_x w^\epsilon)(\partial_y w^\epsilon)^2 dxdy\right| &= \left|\int_{\mathbb{R}^2}(u_c - c\partial_x v_c)(\partial_y v_c)^2 dxdy\right| \\ &\leq C\|(u_c - c\partial_x v_c)\|_{L^2(\mathbb{R}^2)}\|\partial_y v_c\|^2_{H^1(\mathbb{R}^2)} \\ &\leq C\epsilon^{\frac{3}{4}} \to 0.\end{aligned} \tag{58}$$

and also that

$$\epsilon\left|\int_{\mathbb{R}^2}\partial_x w^\epsilon(\partial_y w^\epsilon)^2 dxdy\right| \leq C\|\partial_x v_c\|_{L^2(\mathbb{R}^2)}\|\partial_y v_c\|^2_{H^1(\mathbb{R}^2)} \leq C\epsilon^{\frac{1}{4}} \to 0. \tag{59}$$

Hence, we find that

$$\begin{aligned}\lim_{\epsilon\to 0^+} K^0(w^\epsilon) &= \lim_{\epsilon\to 0^+} c\int_{\mathbb{R}^2}(\partial_x w^\epsilon)^3 dxdy \\ &= \lim_{\epsilon\to 0^+} c\int_{\mathbb{R}^2}\left((\partial_x w^\epsilon)^3 + \epsilon\partial_x w^\epsilon(\partial_y w^\epsilon)^2\right) dxdy \\ &= \lim_{\epsilon\to 0^+} G^\epsilon(c\partial_x w^\epsilon, w^\epsilon).\end{aligned}$$

We want to show that

$$\lim_{\epsilon\to 0^+} K^0(w^\epsilon) = 1.$$

Since $\lim_{\epsilon\to 0^+} G^\epsilon(z^\epsilon, w^\epsilon) = 1$, then we will establish that

$$\lim_{\epsilon\to 0^+} G^\epsilon(z^\epsilon, w^\epsilon) = \lim_{\epsilon\to 0^+} G^\epsilon(c\partial_x w^\epsilon, w^\epsilon). \tag{60}$$

For this, by using (58) it is sufficient to establish that the following limit holds.

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}(z^\epsilon - c\partial_x w^\epsilon)(\partial_x w^\epsilon)^2 dxdy = 0. \tag{61}$$

In fact, from the second equation of (52) we have that (z^ϵ, w^ϵ) satisfy, in the sense of distributions, the equation

$$\begin{aligned} c\partial_{xy}w = z_y - \epsilon\sigma(\partial_{xxy}z + \epsilon\partial_{yyy}z) + \frac{1}{2}c\epsilon(\partial_{xxxy}w + \epsilon\partial_{xyyy}w) \\ - \left(\frac{1}{3}\epsilon\mathcal{I}^\epsilon\right)\left(\left(\partial_x w^2\right)_y + \epsilon\left(\partial_y w^2\right)_y\right) = 0. \end{aligned} \tag{62}$$

Using a duality argument we will show that in $L^2(\mathbb{R}^2)$ the right hand side of (62) is of order $O(1)$. First, using that the sequence $\{I^\epsilon(z^\epsilon, w^\epsilon)\}_\epsilon$ is uniformly bounded we note that

$$\|z^\epsilon\|_{L^2(\mathbb{R}^2)} = O(1), \quad \|w^\epsilon_x\|_{L^2(\mathbb{R}^2)} = O(1), \quad \|w^\epsilon_{yy}\|_{L^2(\mathbb{R}^2)} = O(\epsilon^{-1}) \tag{63}$$

Then for any $\psi \in C_0^\infty(\mathbb{R}^2)$ we have that

$$|\left\langle z^\epsilon, \psi_y\right\rangle| \le \|\psi_y\|_{L^2(\mathbb{R}^2)}\|z^\epsilon\|_{L^2(\mathbb{R}^2)} \le C\|z^\epsilon\|_{L^2(\mathbb{R}^2)}.$$

We illustrate only some calculations: Also we have that

$$|\left\langle \partial_{yy}w^\epsilon, \psi_{xy}\right\rangle| \le \|\psi_{xy}\|_{L^2(\mathbb{R}^2)}\|w_{yy}\|_{L^2(\mathbb{R}^2)} \le C\|w_{yy}\|_{L^2(\mathbb{R}^2)}.$$

Moreover,

$$\left|\left\langle (\partial_x w^\epsilon)^2, \psi_y\right\rangle\right| \le \|\psi_y\|_\infty \|\partial_x w^\epsilon\|^2_{L^2(\mathbb{R}^2)} \le C\|\partial_x w^\epsilon\|_{L^2(\mathbb{R}^2)}.$$

Similarly one work the other terms. Then using (62)-(63) we see that

$$\|\partial_{xy}w^\epsilon\|_{L^2(\mathbb{R}^2)} = O(1).$$

In a similar way we have that

$$\|z^\epsilon - c\partial_x w^\epsilon\|_{H^1(\mathbb{R}^2)} = O(\epsilon). \tag{64}$$

We will use (64) in the proof of following theorem. Now, we also have that

$$\|z^\epsilon_x\|_{L^2(\mathbb{R}^2)} = O(1), \quad \|w^\epsilon_{xx}\|_{L^2(\mathbb{R}^2)} = O(1), \quad \|z^\epsilon_y\|_{L^2(\mathbb{R}^2)} = O(\epsilon^{\frac{-1}{2}}). \tag{65}$$

Then we see that

$$\left|\int_{\mathbb{R}^2}(z^\epsilon - c\partial_x w^\epsilon)(\partial_x w^\epsilon)^2 dxdy\right| \le C\|z^\epsilon - c\partial_x w^\epsilon\|_{L^2(\mathbb{R}^2)}\|\partial_x w^\epsilon\|^2_{H^1(\mathbb{R}^2)} \to 0,$$

as desired. Thus, we conclude that

$$\lim_{\epsilon\to 0^+} K^0(w^\epsilon) = \lim_{\epsilon\to 0^+} K^\epsilon(w^\epsilon) = \lim_{\epsilon\to 0^+} G^\epsilon(z^\epsilon, w^\epsilon) = 1.$$

Hence, we have for ϵ small enough that $K^0(w^\epsilon) \neq 0$, then it follows that

$$\mathcal{J}^0 \le J^0\left(\frac{w^\epsilon}{K^0(w^\epsilon)^{1/3}}\right) = \frac{J^0(w^\epsilon)}{K^0(w^\epsilon)^{2/3}}.$$

But, as $\epsilon \to 0$, we have that

$$J^\epsilon(w^\epsilon) - J^0(w^\epsilon) = o(1), \qquad I^\epsilon(z^\epsilon, w^\epsilon) - J^\epsilon(w^\epsilon) = o(1). \tag{66}$$

Then

$$\mathcal{J}^0 \le \liminf_{\epsilon\to 0^+}\mathcal{I}^\epsilon. \tag{67}$$

Combining (56) and (67) we obtain,

$$\lim_{\epsilon \to 0^+} \mathcal{I}^\epsilon = \mathcal{J}_0.$$

Now note that if $K^\epsilon(w) = 1$, by using (55), we have that $\mathcal{I}^\epsilon \leq J^\epsilon(w)$, so that $\mathcal{I}^\epsilon \leq \mathcal{J}^\epsilon$. Hence

$$\mathcal{J}^0 = \lim_{\epsilon \to 0^+} \mathcal{I}^\epsilon \leq \liminf_{\epsilon \to 0^+} \mathcal{J}^\epsilon.$$

From (56) we conclude that

$$\lim_{\epsilon \to 0^+} \mathcal{J}^\epsilon = \mathcal{J}^0,$$

and so the proof follows.

□

Now, we establish the main result in this section. More exactly, we will see that a translate subsequence of the renormalized sequence (z^ϵ, w^ϵ) converges weakly to a couple (z_0, w_0) that satisfies the system (18)-(19), and so $z_0 = \partial_x w_0$ is a weak solution of a (KP-I) equation.

THEOREM 4.2. *Let $\sigma > \frac{3}{8}$. For any sequence $\epsilon_j \to 0$, there is a translate subsequence (denoted the same) of $\{(z^{\epsilon_j}, w^{\epsilon_j})\}_j$ and there exist nontrivial distributions $w_0 \in \mathcal{Z}$ and $z_0 \in H^1(\mathbb{R}^2)$ such that as $j \to \infty$,*

$$w^\epsilon \to w_0 \quad \text{in } \mathcal{Z} \quad \text{and} \quad z^{\epsilon_j} - \partial_x w^{\epsilon_j} \to 0, \quad z^{\epsilon_j} \to z_0 \quad \text{in } H^1(\mathbb{R}^2). \tag{68}$$

Moreover, (z_0, w_0) is a nontrivial weak solution of the system (18)-(19), and so $z_0 = \partial_x w_0 \in H^1(\mathbb{R}^2)$, with $\partial_x w_0$ being a solution of the travelling wave equation for a (KP-I) equation in the sense of distributions.

PROOF. Let $\{\epsilon_j\}_j$ a sequence of positive number such that $\epsilon_j \to 0$. Then from Lemma 4.1 we have that $\{(K_0(w^{\epsilon_j}))^{-1/3} w^{\epsilon_j}\}_j$ is a minimizing sequence for $\mathcal{J}_0$ and

$$K_0(w^{\epsilon_j}) \to 1.$$

Considering this and Theorem 4.1 we have that there exist a translate sequence of $\{(z^{\epsilon_j}, w^{\epsilon_j})\}_j$ (denote the same) and there exist a nonzero distribution $w_0 \in \mathcal{Z}$ such that $w^\epsilon \to w_0$ in $\mathcal{Z}$ and w_0 is a solution of the equation (51). Then using (64) we have that there exist $z_0 \in H^1(\mathbb{R}^2)$ such that $z^{\epsilon_j} \to z_0$ in $H^1(\mathbb{R}^2)$. Thus, we obtain (68) and $z_0 = \partial_x w_0$.

Now, multiply by c_j and differentiating with respect to x the second equation in (52) we find that

$$
\begin{aligned}
&\epsilon_j^{-1}\left(c_j z_x^{\epsilon_j} - w_{xx}^{\epsilon_j}\right)\\
&= -\frac{2}{3} w_{xxxx}^{\epsilon_j} - \frac{4}{3}\epsilon_j w_{xxyy}^{\epsilon_j} - \frac{2\epsilon_j^2}{3} w_{yyyy}^{\epsilon_j} + w_{yy}^{\epsilon_j}\\
&+ \frac{c_j}{2} z_{xxx}^{\epsilon_j} + \frac{c_j\epsilon_j}{2} z_{xyy}\\
&- \left(\frac{2}{3}\mathcal{I}^{\epsilon_j}\right)\left(\left(z^{\epsilon_j} w_x^{\epsilon_j}\right)_x + \epsilon_j\left(z^{\epsilon_j} w_y^{\epsilon_j}\right)_y\right),
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
&c_j\epsilon_j^{-1}\left(z_x^{\epsilon_j} - c_j w_{xx}^{\epsilon_j}\right)\\
&= \sigma c_j z_{xxx}^{\epsilon_j} + \sigma c_j \epsilon_j z_{xyy}^{\epsilon_j}\\
&- \frac{c_j^2}{2} w_{xxxx}^{\epsilon_j} - \frac{c_j^2\epsilon_j}{2} w_{xxyy}^{\epsilon_j}\\
&+ \left(\frac{c_j}{3}\mathcal{I}^{\epsilon_j}\right)\left(\left(w_x^{\epsilon_j}\right)_x^2 + \epsilon_j\left(w_y^{\epsilon_j}\right)_x^2\right).
\end{aligned} \tag{70}
$$

Using $c_j^2 = 1 - \epsilon_j$ and replacing (69) in (70) we have that

$$
- w_{xx}^{\epsilon_j} + c_j\sigma z_{xxx}^{\epsilon_j} \tag{71}
$$

$$
\begin{aligned}
&+ \sigma c_j\epsilon_j z_{xyy}^{\epsilon_j} - \frac{c_j^2}{2} w_{xxxx}^{\epsilon_j} - \frac{c_j^2\epsilon_j}{2} w_{xxyy}^{\epsilon_j}\\
&+ \left(\frac{c_j}{3}\mathcal{I}^{\epsilon_j}\right)\left((w_x^{\epsilon_j})_x^2 + \epsilon_j (w_y^{\epsilon_j})_x^2\right)\\
&= -\frac{2}{3} w_{xxxx}^{\epsilon_j} - \frac{4}{3}\epsilon_j w_{xxyy}^{\epsilon_j} - \frac{2\epsilon_j^2}{3} w_{yyyy}^{\epsilon_j} + w_{yy}^{\epsilon_j} + \frac{c_j}{2} z_{xxx} + \frac{c_j\epsilon_j}{2} z_{xyy}\\
&- \left(\frac{2}{3}\mathcal{I}^{\epsilon_j}\right)\left((z^{\epsilon_j} w_x^{\epsilon_j})_x + \epsilon_j (z^{\epsilon_j} w_y^{\epsilon_j})_y\right).
\end{aligned} \tag{72}
$$

For any test function $\psi \in C_0^\infty(\mathbb{R}^2)$ we have that

$$
\begin{aligned}
\epsilon_j\left|\left\langle c_j\left(\sigma - \frac{1}{2}\right) z_{xyy}^{\epsilon_j} + \left(\frac{4}{3} - \frac{c_j^2}{2}\right) w_{xxyy}^{\epsilon_j} + \frac{2\epsilon_j}{3} w_{yyyy}^{\epsilon_j}, \psi\right\rangle\right|&\\
\le C\epsilon_j\left(\|z_y^{\epsilon_j}\|_2 + \|w_{xy}^{\epsilon_j}\| + \epsilon_j\|w_{yy}^{\epsilon_j}\|_2\right)\|\psi_y\|_{H^1(\mathbb{R}^2)} &\to 0.
\end{aligned}
$$

Since $\lim_{j\to 0}\mathcal{I}^{\epsilon_j} = \mathcal{J}^0$, also we have that

$$
\begin{aligned}
&\frac{\epsilon_j\mathcal{I}^{\epsilon_j}}{3}\Big(c_j\left|\left\langle (w_y^{\epsilon_j})^2, \psi_x\right\rangle\right|\\
&+ 2\left|\left\langle z^{\epsilon_j} w_y^{\epsilon_j}, \psi_y\right\rangle\right|\Big)\\
&\le C\epsilon_j \|w_y^{\epsilon_j}\|_2\left(\|w_y^{\epsilon_j}\|_2 + \|z^{\epsilon_j}\|_2\right)\|\nabla\psi\|_\infty \to 0.
\end{aligned}
$$

On the other hand, using (68) we find that

$$
\lim_{j\to\infty}\left\langle \frac{c_j}{3}\mathcal{I}^{\epsilon_j}(w_x^{\epsilon_j})^2 + \frac{2}{3}\mathcal{I}^{\epsilon_j}(z^{\epsilon_j} w_x^{\epsilon_j}), \psi_x\right\rangle = \mathcal{J}^0\left\langle (\partial_x w_0)^2, \psi_x\right\rangle.
$$

So that,

$$
\lim_{j\to\infty}\left\langle \frac{c_j}{3}\mathcal{I}^{\epsilon_j}(w_x^{\epsilon_j})_x^2 + \frac{2}{3}\mathcal{I}^{\epsilon_j}(z^{\epsilon_j} w_x^{\epsilon_j})_x, \psi\right\rangle = 2\mathcal{J}^0\left\langle \partial_x w_0 \partial_{xx} w_0, \psi\right\rangle.
$$

Finally, we see that

$$\lim_{j\to\infty}\Big\langle -w_{xx}^{\epsilon_j} - w_{yy}^{\epsilon_j} + \Big(\frac{2}{3} - \frac{c_j^2}{2}\Big) w_{xxxx}^{\epsilon_j} + c_j\Big(\sigma - \frac{1}{2}\Big) z_{xxx}^{\epsilon_j}, \psi\Big\rangle$$
$$= \Big\langle -\partial_{xx} w_0 - \partial_{yy} w_0 + \Big(\sigma - \frac{1}{3}\Big)\partial_{xxxx} w_0, \psi\Big\rangle.$$

Then since we assume that $\sigma > \frac{3}{8} > \frac{1}{3}$, using (71) we concluded that the nonzero distribution w_0 is a nontrivial solution of the equation

$$-w_{xx} - w_{yy} + \Big(\sigma - \frac{1}{3}\Big) w_{xxxx} + \left(2\mathcal{J}^0\right)(w_x w_{xx}) = 0.$$

In particular, the couple $(z^0, w^0) = -\Big(\frac{2}{3}\mathcal{J}^0\Big)(z_0, w_0)$ is a nontrivial solution of the system (18)-(19), and so $z^0 = \partial_x w^0$ is a solution for the equation (19) in the sense of distributions.

□

5. Analyticity of Solitons

In this section, we will establish that travelling wave solutions of (21) are analytic functions. We have the following result.

THEOREM 5.1. *Let $\sigma > \frac{3}{8}$ and $0 < c < 1$. If $(u, v) \in \mathcal{X}$ is travelling wave solution of system (21) then u and v are analytic.*

PROOF. First we will establish that $u, v_x, v_y \in H^k(\mathbb{R}^2)$ for any $k \geq 1$, if $(u, v) \in \mathcal{X}$ is a weak solution of (21). Since $u, v_x, v_y \in H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, then we have for $i = 1, 2$ that the functions

$$h_i = -u\partial_i v, \ \ g_i = -\frac{1}{2}\left(\partial_i v\right)^2$$

belong to $L^2(\mathbb{R}^2)$. Taking Fourier transform on the system (21) we obtain that $(\widehat{u}, \widehat{v})(\xi, \eta)$ satisfies the system

(73)
$$\begin{cases} \left[(\xi^2+\eta^2)\left(\frac{2}{3}(\xi^2+\eta^2)+1\right)\right]\widehat{v} + ic\xi(1+\frac{1}{2}(\xi^2+\eta^2))\widehat{u} + i(\xi\widehat{h_1} + \eta\widehat{h_2}) &= 0,\\ \left(1+\sigma(\xi^2+\eta^2)\right)\widehat{u} - ic\xi\left(1+\frac{1}{2}(\xi^2+\eta^2)\right)\widehat{v} - (\widehat{g_1}+\widehat{g_2}) &= 0. \end{cases}$$

Then solving this system, we have that

$$\widehat{u}(\xi,\eta) = \frac{c\xi\left(1+\frac{1}{2}(\xi^2+\eta^2)\right)(\xi\widehat{h_1}+\eta\widehat{h_2}) + \left[(\xi^2+\eta^2)\left(\frac{2}{3}(\xi^2+\eta^2)+1\right)\right](\widehat{g_1}+\widehat{g_2})}{P(\xi,\eta)}$$

where P is a polynomial of sixth degree given by

$$P(\xi,\eta) = (\xi^2+\eta^2)\left[\tfrac{2}{3}(\xi^2+\eta^2)+1\right]\left(1+\sigma(\xi^2+\eta^2)\right) - c^2\xi^2\left(1+\tfrac{1}{2}(\xi^2+\eta^2)\right)^2.$$

Since $\sigma > \frac{3}{8}$ and $0 < c < 1$, we notice that there exist $M = M(\sigma, c)$ such that

$$P(\xi,\eta) \geq M(\xi^2+\eta^2)\left(1+(\xi^2+\eta^2)\right)^2,$$

then there exist $M_1 > 0$ such that

$$|\widehat{u}| \leq M_1\left(\frac{|\widehat{h_1}|+|\widehat{h_2}|+|\widehat{g_1}|+|\widehat{g_2}|}{1+\xi^2+\eta^2}\right). \tag{74}$$

From inequality (74) it follows that $u \in H^2(\mathbb{R}^2)$. On the other hand, from the second equation of (73) we note that

$$\widehat{v_x} = \frac{\left(1+\sigma(\xi^2+\eta^2)\right)\widehat{u} - (\widehat{g_1}+\widehat{g_2})}{c\left(1+\frac{1}{2}(\xi^2+\eta^2)\right)}.$$

Then there is $M_2 = M_2(\sigma, c) > 0$ such that

$$|\widehat{v_x}| \le M_2\left(|\widehat{u}| + \frac{|\widehat{g_1}|+|\widehat{g_2}|}{1+\xi^2+\eta^2}\right).$$

This implies that $v_x \in H^2(\mathbb{R}^2)$. By a similar argument we have that $v_y \in H^2(\mathbb{R}^2)$, and a simple bootstrapping argument then yields that $u, v_x, v_y \in H^k(\mathbb{R}^2)$ for all $k \ge 1$.

Now, we will prove the analyticity of (u, v). First we establish the result under the assumption of the existence of $R > 0$ such that for all $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, with $|\alpha| = \alpha_1 + \alpha_2 \ge 1$,

$$\|\partial^\alpha u\|_{H^1(\mathbb{R}^2)} + \|\partial^\alpha v\|_{H^2(\mathbb{R}^2)} \le C\frac{|\alpha|!}{|\alpha|+2}R^{|\alpha|}, \tag{75}$$

where $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$. If $(x_0, y_0) \in \mathbb{R}^2$, we will show that there exists $r > 0$ such that we have the following Taylor expansion for u and v in $B_r(x_0, y_0)$,

$$u(x,y) = \sum_\alpha \frac{\partial^\alpha u(x_0,y_0)}{\alpha!}(x-x_0, y-y_0)^\alpha$$
$$v(x,y) = \sum_\alpha \frac{\partial^\alpha v(x_0,y_0)}{\alpha!}(x-x_0, y-y_0)^\alpha.$$

We establish the result for v. If we set $\zeta_0 = (x_0, y_0)$ and $\zeta = (x, y) - \zeta_0$, then by the Taylor Theorem (with remainder) we have that

$$v(x,y) = \sum_{k=0}^{N-1}\sum_{|\alpha|=k,\alpha\in\mathbb{N}^2}\frac{\partial^\alpha v(\zeta_0)}{\alpha!}\zeta^\alpha + \mathcal{E}_N(x,y),$$

where

$$\mathcal{E}_N(x,y) = \sum_{|\alpha|=N,\alpha\in\mathbb{N}^2}\frac{\partial^\alpha v(\zeta_0+t\zeta)}{\alpha!}\zeta^\alpha.$$

On the other hand, for $|\alpha| \ge 1$ we have that

$$|\partial^\alpha v(x,y)| \le ||\partial^\alpha v||_{H^2(\mathbb{R}^2)} \le CR^{|\alpha|}\frac{|\alpha|!}{|\alpha|+2}. \tag{76}$$

If we take $r > 0$ in such a way that $2^2 r^2 R < 1$. Then using that $\frac{|\alpha|!}{\alpha!} \leq 2^{|\alpha|}$, we conclude for $||\zeta|| < r$ that

$$
\begin{aligned}
|\mathcal{E}_N(x,y)| &\leq C \sum_{|\alpha|=N, \alpha\in\mathbb{N}^2} \frac{|\alpha|!\, R^{|\alpha|}}{(|\alpha|+2)\alpha!} \|\zeta\|^{2|\alpha|} \\
&\leq C \sum_{|\alpha|=N, \alpha\in\mathbb{N}^2} \frac{2^N R^N}{N+2} r^{2N} \\
&\leq C \frac{2^N R^N (N+1) r^{2N}}{(N+2)} \\
&\leq C(2Rr^2)^N \\
&\leq C 2^{-N}.
\end{aligned}
$$

In other words, the Taylor series for v converges in $B_r(x_0, y_0)$. In order to show that the Taylor series for u converges, we only need to follow the same steps used for the function v, after noting that we have an estimate like (76) for the function u given by

$$
|\partial^\alpha u(x,y)| \leq ||\partial^\alpha u||_{H^2(\mathbb{R}^2)} \leq C R^{|\alpha|+1} \frac{(|\alpha|+1)!}{|\alpha|+3}.
$$

Now, to complete the proof, we only need to prove that there exists $R > 0$ such that for all $\alpha \in \mathbb{N}^2$, with $|\alpha| \geq 1$,

$$
\|\partial^\alpha u\|_{H^1(\mathbb{R}^2)} + \|\partial^\alpha v\|_{H^2(\mathbb{R}^2)} \leq C \frac{|\alpha|!}{|\alpha|+2} R^{|\alpha|}. \tag{77}
$$

We will argue by induction on $|\alpha|$. First assume that $|\alpha| = 1$. Since $u, v_x, v_y \in H^k(\mathbb{R}^2)$ for all $k \geq 1$ we have the result. Now, suppose that (75) holds for $|\alpha|$ and R (which will be choosen later). Applying the operator ∂^α to system (21) and computing the L^2- inner product with $(\partial^\alpha u, \partial^\alpha v)$, we obtain that

$$
\begin{aligned}
I_c(\partial^\alpha u, \partial^\alpha v) = &- \langle \partial^\alpha (u v_x), \partial^\alpha v_x \rangle - \langle \partial^\alpha (u v_y), \partial^\alpha v_y \rangle \\
&- \frac{1}{2} \left\langle \partial^\alpha \left(v_x^2\right), \partial^\alpha u \right\rangle - \frac{1}{2} \left\langle \partial^\alpha \left(v_y^2\right), \partial^\alpha u \right\rangle.
\end{aligned}
$$

Then applying inequality (27) and Hölder inequality, we obtain that

$$
\begin{aligned}
\|\partial^\alpha u\|_{H^1(\mathbb{R}^2)} + \|\partial^\alpha \nabla v\|_{H^1(\mathbb{R}^2)} \leq C_1 \Big(&\|\partial^\alpha (u v_x)\|_2 \\
&+ \|\partial^\alpha (u v_y)\|_2 + \|\partial^\alpha \left(v_x^2\right)\|_2 + \|\partial^\alpha \left(v_y^2\right)\|_2 \Big).
\end{aligned}
$$

Note that if $u, w \in H^l$ for any $l \geq 1$, we have for $\alpha = (\alpha_1, \alpha_2)$ that

$$
\partial^\alpha (uw) = (\partial^\alpha u) w + u \partial^\alpha w + \sum_{k=1}^{|\alpha|-1} \sum_{\substack{\beta < \alpha \\ |\beta| = k}} \binom{\alpha_1}{\beta_1} \binom{\alpha_2}{\beta_2} (\partial^{\alpha-\beta} u)(\partial^\beta w).
$$

Then we conclude that

(78)
$$\partial^\alpha(u\partial_i v) = \partial^\alpha u \partial_i v + \sum_{k=1}^{|\alpha|-1} \sum_{\substack{\beta<\alpha\\ |\beta|=k}} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} (\partial^{\alpha-\beta}u)(\partial^\beta(\partial_i v)) + u\partial^\alpha(\partial_i v).$$

Since, $H^r(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $r > 1$, then

$$\|(\partial^\alpha u)w\|_2 \le \|\partial^\alpha u\|_2 \|w\|_\infty \le \|\partial^\alpha u\|_2 \|w\|_{H^2(\mathbb{R}^2)}$$

and

$$\|(\partial^\alpha w)u\|_2 \le \|\partial^\alpha w\|_2 \|u\|_\infty \le \|\partial^\alpha w\|_2 \|u\|_{H^2(\mathbb{R}^2)}.$$

Moreover, we also have that

$$\|(\partial^{\alpha-\beta}u)(\partial^\beta w)\|_2 \le \|\partial^{\alpha-\beta}u\|_2 \|\partial^\beta w\|_\infty \le \|\partial^{\alpha-\beta}u\|_2 \|\partial^\beta w\|_{H^2(\mathbb{R}^2)}$$

Using the previous inequalities with $w = \partial_i v$ and using the induction hypothesis on the right hand side, we conclude that

$$\begin{aligned}\|(\partial^\alpha u)\partial_i v\|_2 &\le \|\partial^\alpha u\|_2 \|\partial_i v\|_{H^2(\mathbb{R}^2)}\\ &\le C^2\left(\frac{(|\alpha|-1)!}{3(|\alpha|+1)}\right)R^{|\alpha|}\\ &\le \left(CR^{|\alpha|+1}\frac{(|\alpha|+1)!}{|\alpha|+3}\right)\left(CR^{-1}\left(\frac{|\alpha|+3}{|\alpha|+1}\right)\right). \qquad (79)\end{aligned}$$

Similarly we obtain that

$$\begin{aligned}\|(\partial^\alpha \partial_i v)u\|_2 &\le \|\partial^\alpha \partial_i v\|_2 \|u\|_{H^2(\mathbb{R}^2)}\\ &\le C^2\left(\frac{(|\alpha|-1)!}{3(|\alpha|+1)}\right)R^{|\alpha|}\\ &\le \left(CR^{|\alpha|+1}\frac{(|\alpha|+1)!}{|\alpha|+3}\right)\left(CR^{-1}\left(\frac{|\alpha|+3}{|\alpha|+1}\right)\right). \qquad (80)\end{aligned}$$

We also have that

$$\begin{aligned}\|(\partial^{\alpha-\beta}u)(\partial^\beta\partial_i v)\|_2 &\le \|\partial^{\alpha-\beta}u\|_2 \|\partial^\beta \partial_i v\|_{H^2(\mathbb{R}^2)}\\ &\le C^2\left(\frac{(|\alpha|-|\beta|-1)!(|\beta|+1)!}{(|\alpha|-|\beta|+1)(|\beta|+3)}\right)R^{|\alpha|}.\end{aligned}$$

Using the induction hypothesis and the previous inequality, we have that

$$\begin{aligned}&\sum_{\substack{\beta<\alpha\\ |\beta|=k}} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \left\|(\partial^{\alpha-\beta}u)(\partial^\beta(\partial_i v))\right\|_2\\ &\qquad \le C^2 \sum_{\substack{\beta<\alpha\\ |\beta|=k}} \frac{\alpha_1!\alpha_2!(|\alpha|-k-1)!(k+1)!R^{|\alpha|}}{(\alpha_1-\beta_1)!(\alpha_2-\beta_2)!\beta_1!\beta_2!(|\alpha|-k+1)(k+3)}.\end{aligned}$$

But we know that for any $(n_1, n_2) \in \mathbb{N}^2$ (see Proposition 16 (b) in Soriano [**15**]),

$$|\alpha|! = \sum_{A(\alpha,n_1,n_2)} \frac{\alpha!|\rho_1|!|\rho_2|!}{\rho_1!\rho_2!},$$

where $A(\alpha, n_1, n_2) = \{(\rho_1, \rho_2) : \rho_1 + \rho_2 = \alpha, \quad |\rho_i| = n_i, \text{ for } 1 \leq i \leq 2\}$. Note that for $\alpha = (\alpha_1, \alpha_2)$, $\rho_1 = (\alpha_1 - \beta_1, \alpha_2 - \beta_2)$ and $\rho_2 = (\beta_1, \beta_2)$, we have that $\rho_1 + \rho_2 = \alpha$, that $k = \beta_1 + \beta_2 = |\rho_2|$, and also that $|\alpha| - k = |(\alpha_1 - \beta_1, \alpha_2 - \beta_2)| = |\rho_1|$. Using the above property, we conclude that

$$\frac{\alpha!|\rho_1|!|\rho_2|!}{\rho_1!\rho_2!} = \frac{\alpha_1!\alpha_2!(|\alpha| - k)!k!}{(\alpha_1 - \beta_1)!(\alpha_2 - \beta_2)!\beta_1!\beta_2!} \leq |\alpha|!.$$

and also that

$$\frac{\alpha_1!\alpha_2!(|\alpha| - k - 1)!(k + 1)!R^{|\alpha|}}{(\alpha_1 - \beta_1)!(\alpha_2 - \beta_2)!\beta_1!\beta_2!(|\alpha| - k + 1)(k + 3)} \leq \frac{|\alpha|!(k + 1)R^{|\alpha|}}{(|\alpha| - k)(|\alpha| - k + 1)(k + 3)}.$$

From these estimates, we get that

$$\begin{aligned}
&\sum_{\substack{\beta < \alpha \\ |\beta| = k}} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \left\|(\partial^{\alpha-\beta}u)(\partial^{\beta}(\partial_i v))\right\|_2 \\
&\leq C^2 \sum_{\substack{\beta < \alpha \\ |\beta| = k}} \frac{|\alpha|!(k + 1)R^{|\alpha|}}{(|\alpha| - k)(|\alpha| - k + 1)(k + 3)} \\
&\leq C^2|\alpha|!R^{|\alpha|}\sum_{k=0}^{\infty}\frac{1}{k^2},
\end{aligned}$$

which implies that

$$\sum_{k=1}^{|\alpha|-1} \sum_{\substack{\beta < \alpha \\ |\beta| = k}} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \left\|(\partial^{\alpha-\beta}u)(\partial^{\beta}(\partial_i v))\right\|_2 \leq C^2(|\alpha|!)|\alpha|R^{|\alpha|}\sum_{k=0}^{\infty}\frac{1}{k^2} \tag{81}$$

$$\leq C^2(|\alpha| + 1)!R^{|\alpha|}\sum_{k=0}^{\infty}\frac{1}{k^2} \tag{82}$$

So, taking R large enough such that

$$CR^{-1}\left(\frac{|\alpha| + 3}{|\alpha| + 2}\right)\left(\sum_{k\geq 0}^{\infty}\frac{1}{k^2}\right) < 1.$$

and using estimates (79), (80) and (81), we conclude that,

$$\sum_{k=1}^{|\alpha|-1} \sum_{\substack{\beta < \alpha \\ |\beta| = k}} \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \left\|(\partial^{\alpha-\beta}u)(\partial^{\beta}(\partial_i v))\right\|_2 \leq C\frac{(|\alpha| + 1)!}{|\alpha| + 3}R^{|\alpha|+1}$$

and also that

$$\|\partial^{\alpha}(u\partial_i v)\|_2 \leq C\frac{(|\alpha| + 1)!}{|\alpha| + 3}R^{|\alpha|+1}.$$

Now note that $\partial^\alpha\left((\partial_i v)^2)\right) = \partial^\alpha\left(w\partial_i v\right)$ for $w = \partial_i v$. Then the couple (w, v) satisfies the induction hypothesis (75), since

$$\|\partial^\alpha w\|_{H^1(\mathbb{R}^2)} = \|\partial^\alpha \partial_i v\|_{H^1(\mathbb{R}^2)} \le \|\partial^\alpha v\|_{H^2(\mathbb{R}^2)} \le C\left(\frac{|\alpha|!}{|\alpha|+2}\right) R^{|\alpha|}.$$

Then we conclude that

$$\|\partial^\alpha\left((\partial_i v)^2)\right)\|_2 \le C\frac{(|\alpha|+1)!}{|\alpha|+3}R^{|\alpha|+1}.$$

In other words, we have shown that

$$\begin{aligned}&\|\partial^\alpha u\|_{H^1(\mathbb{R}^2)} + \|\partial^\alpha \nabla v\|_{H^1(\mathbb{R}^2)}\\ &\le C_1\left(\|\partial^\alpha(uv_x)\|_2 + \|\partial^\alpha(uv_y)\|_2 + \|\partial^\alpha\left(v_x^2\right)\|_2 + \|\partial^\alpha\left(v_y^2\right)\|_2\right) \le C\frac{(|\alpha|+1)!}{|\alpha|+3}R^{|\alpha|+1}.\end{aligned}$$

Now we have to estimate terms of the form:

$$||\partial^\alpha\partial_i u||_{H^1(\mathbb{R}^2)},\ \ ||\partial^\alpha\partial_i\nabla v||_{H^1(\mathbb{R}^2)},\quad \text{for}\ \ i = 1, 2.$$

To do this, we apply operator $\partial^\alpha\partial_i$ to equation (21) and compute the L^2- inner product with $(\partial^\alpha\partial_i u, \partial^\alpha\partial_i v)$. Thus, we have that

$$\begin{aligned}I_c(\partial^\alpha\partial_i u, \partial^\alpha\partial_i v) = &-\langle\partial^\alpha\partial_i(uv_x), \partial^\alpha\partial_i v_x\rangle - \langle\partial^\alpha\partial_i(uv_y), \partial^\alpha\partial_i v_y\rangle\\ &-\frac{1}{2}\left\langle\partial^\alpha\partial_i(v_x^2), \partial^\alpha\partial_i u\right\rangle - \frac{1}{2}\left\langle\partial^\alpha\partial_i(v_y^2), \partial^\alpha\partial_i u\right\rangle.\end{aligned} \tag{83}$$

To illustrate the type of computation, we only consider the typical term

$$\langle\partial^\alpha\partial_i(uv_x), \partial^\alpha\partial_i v_x\rangle$$

to exhibit the calculation

$$\begin{aligned}|\langle\partial^\alpha\partial_i(uv_x), \partial^\alpha\partial_i v_x\rangle| &= \left|\left\langle\partial^\alpha(uv_x), \partial^\alpha\partial_i^2 v_x\right\rangle\right|\\ &\le \|\partial^\alpha(uv_x)\|_2\|\partial^\alpha\partial_i^2 v_x\|_2\\ &\le \|\partial^\alpha(uv_x)\|_2||\partial^\alpha\partial_x\nabla v||_{H^1(\mathbb{R}^2)}.\end{aligned}$$

We observe that the left hand side of (83) can be written as

$$I_c(\partial^\alpha\partial_i u, \partial^\alpha\partial_i v) \simeq \|\partial^\alpha\nabla u\|^2_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_x\nabla v||^2_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_y\nabla v||^2_{H^1(\mathbb{R}^2)},$$

and so the therm $||\partial^\alpha\partial_x\nabla v||_{H^1(\mathbb{R}^2)}$ is absorbed in the right hand side of previous equivalence. Using this fact, we end up with the estimate

$$\begin{aligned}&\|\partial^\alpha\nabla u\|_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_x\nabla v||_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_y\nabla v||_{H^1(\mathbb{R}^2)}\\ &\qquad\le C_1\left(\|\partial^\alpha(uv_x)\|_2 + \|\partial^\alpha(uv_y)\|_2 + \|\partial^\alpha(v_x^2)\|_2 + \|\partial^\alpha(v_y^2)\|_2\right).\end{aligned}$$

Putting the previous estimates together, we conclude for R large enough that

$$\begin{aligned}&\|\partial^\alpha\nabla u\|_{H^1(\mathbb{R}^2)} + \|\partial^\alpha\nabla v\|_{H^2(\mathbb{R}^2)}\\ &\le \|\partial^\alpha\nabla u\|^2_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_x\nabla v||^2_{H^1(\mathbb{R}^2)} + ||\partial^\alpha\partial_y\nabla v||^2_{H^1(\mathbb{R}^2)} \le C\frac{(|\alpha|+1)!}{|\alpha|+3}R^{|\alpha|+1},\end{aligned}$$

as desired. □

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