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Dynamics of non-autonomous equations of non-Newtonian fluid on 2D unbounded domains

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Abstract. This paper studies the asymptotic behavior of solutions for a non-autonomous incompressible non-Newtonian fluid on two-dimensional unbounded domains. We first prove the existences of the L^2-regularity uniform attractors $\mathcal{A}^H_{\mathcal{H}(g_0)}$ and H^2-regularity uniform attractor $\mathcal{A}^V_{\mathcal{H}(g_0)}$, respectively. Then we establish the regularity of the uniform attractors by showing

$$\mathcal{A}^H_{\mathcal{H}(g_0)} = \mathcal{A}^V_{\mathcal{H}(g_0)},$$

which implies the uniform (with respect to the external forces) asymptotic smoothing effect of the non-autonomous fluid in the sense that the solutions become eventually more regular than the initial data.

Contents

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1. Introduction

In this paper, we study the existence and regularity of the uniform attractors for the following non-autonomous incompressible non-Newtonian fluid on two-dimensional (2D) unbounded channel-like domains

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nabla \cdot \tau(e(u)) + \nabla p = g(x,t), \tag{1.1}$$

$$\operatorname{div} u = \nabla \cdot u = 0,\ x = (x_1, x_2) \in \Omega, \tag{1.2}$$

where $\Omega = \mathbb{R} \times (-L, L) \subset \mathbb{R}^2$ and $L > 0$ is a positive constant. Equations (1.1)-(1.2) describe the motion of an isothermal incompressible viscous fluid, where u denotes the velocity field of the fluid, g is the time-dependent external force function, the scalar function p is the pressure, and $\tau(e(u)) = (\tau_{ij}(e(u)))_{2\times 2}$, which is usually called the extra stress tensor of the fluid, is a matrix of order 2×2 defined as

$$\tau_{ij}(e(u)) = 2\mu_0(\varepsilon + |e(u)|^2)^{-\alpha/2} e_{ij}(u) - 2\mu_1 \Delta e_{ij}(u),\ \ i,j = 1,2, \tag{1.3}$$

where

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad |e(u)|^2 = \sum_{i,j=1}^{2} |e_{ij}(u)|^2, \tag{1.4}$$

and $\mu_0, \mu_1, \alpha, \varepsilon$ are parameters associated to the fluid. In this paper we assume that $\mu_0, \mu_1, \varepsilon$ are positive constants and $\alpha \in (0,1)$.

In equation (1.3) if $\tau_{ij}(e(u))$ depends linearly on $e_{ij}(u)$ then we say the corresponding fluid is a Newtonian one. Generally speaking, gases, water, motor oil, alcohols, and simple hydrocarbon compounds tend to be Newtonian fluids and their motions can be described by the Navier-Stokes equations. If the relation between $\tau_{ij}(e(u))$ and $e_{ij}(u)$ is nonlinear, then the fluid is called to be non-Newtonian. For instance, molten plastics, polymer solutions and paints tend to be non-Newtonian fluids. One can refer to [**4, 5, 6, 21, 24, 28**] and the references therein for detailed physical significance. Factually, equations (1.1)-(1.3) were firstly formulated by Ladyzhenskaya as a modification to the Navier-Stokes equations when the gradient $|\nabla u|$ of the velocity is relatively large ([**21**]). Clearly, equations (1.1)-(1.3) reduce into Navier-Stokes equations when $\alpha = \mu_1 = 0$ and into Euler equations as $\mu_1 = \mu_0 = 0$.

The first objective of this paper is to prove the existence of uniform attractors $\mathcal{A}^H_{\mathcal{H}(g_0)}$ in space H and $\mathcal{A}^V_{\mathcal{H}(g_0)}$ in space V (see notations in section 2) for the family of processes corresponding to equations (1.1)-(1.3), respectively. In studying time asymptotic behavior of solutions of PDEs defined on unbounded spatial domains, one will find a considerable obstacle. If the spatial domain is unbounded, we loose the compactness of the Sobolev embedding related to the phase spaces. This absence of compactness is also the main difficulty when we prove the existence of the uniform attractor in the present paper. For example, we have $V \hookrightarrow H$, but the embedding is not compact because that the spatial domain Ω is unbounded.

To obtain the existence of the uniform attractor $\mathcal{A}^H_{\mathcal{H}(g_0)}$ in space H, we use the truncation function and decomposition of spatial domain, as well as the compact Sobolev embedding on bounded spatial domain, to prove the asymptotic compactness of the associated family of processes. The technique of truncation function has been successfully used by some researchers, see e.g. [**2, 31, 32, 41**].

To obtain the existence of the uniform attractor in space V, we use the approach of enstrophy equation of the fluid to prove the asymptotic compactness of the associated family of processes. The idea of energy equation was essentially due to Ball [**3**] and it has been extended and generalized in some directions [**11, 19, 20, 22, 33, 34**]. This technique was later presented by Moise, Rosa and Wang in a systematic way and in a general abstract framework in [**26, 27**]. Also, this technique has been successfully extended to study the pullback asymptotic behavior of non-autonomous systems (see e.g. [**8, 9, 10**]).

We want to point out here that the method of energy equation seems difficult to be used in the present paper to obtain the asymptotic compactness of the associated family of processes in space H. The obstacle comes from the nonlinear term $\mu_0(\varepsilon+|e(u)|^2)^{-\alpha/2}e_{ij}(u)$. We are not easy to prove the corresponding convergence of this term in the energy equation. This is the reason that we use the truncation function in space H.

The second purpose of this paper is to establish the regularity of the uniform attractors. We prove that $\mathcal{A}^H_{\mathcal{H}(g_0)}=\mathcal{A}^V_{\mathcal{H}(g_0)}\subset V$. There are two conclusions can be concluded from this result. The one is that the uniform attractor associated to equations (1.1)-(1.3) does not depend on the energy space chosen for the mathematical studying; the other is the uniform (with respect to (w.r.t. for short) the external forces) asymptotic smoothing effect of the fluid in the sense that the solutions become eventually more regular (possessing H^2-regularity) than the initial data (possessing L^2-regularity).

There are some results on the regularity of global attractors for autonomous dynamical systems, see e.g. [**15, 16, 17, 23, 25, 35**]. However, to our knowledge, there are only a little of reference on the regularity of uniform attractor for non-autonomous dynamical systems in the unbounded spatial domain case.

Other than the global attractor of a semigroup in the autonomous case, the uniform attractor of a family of processes does not possess the invariance property. We will first use the minimality of the uniform attractor to show that $\mathcal{A}^V_{\mathcal{H}(g_0)}=\mathcal{A}^H_{\mathcal{H}(g_0)}\subset H$. Then we utilize the Uniform Gronwall Lemma to establish that the solutions of (1.1)-(1.3) with initial values in any bounded set of H will enter a bounded set of V after large enough time. And then by the structure of the uniform attractor, we show that $\mathcal{A}^H_{\mathcal{H}(g_0)}$ is indeed a bounded set of space V. So we get $\mathcal{A}^V_{\mathcal{H}(g_0)}=\mathcal{A}^H_{\mathcal{H}(g_0)}\subset V$.

The paper is organized as follows. Section 2 is preliminaries. We first introduce some notations, and then we show the unique existence, as well as some *a priori* estimates of solutions. In Section 3, we prove some properties and the existence of uniform attractor for the family of processes corresponding to the non-Newtonian fluid in space H. In Section 4, we verify some properties and the existence of uniform attractor in space V. In Section 5, we establish the regularity of the uniform attractors.

2. Preliminaries

In this paper, we will use the following notations.

$\mathbb{L}^p(\Omega)=L^p(\Omega)\times L^p(\Omega)$-the 2D vector Lebesgue space with norm $\|\cdot\|_{\mathbb{L}^p(\Omega)}$; particularly, $\|\cdot\|_{\mathbb{L}^2(\Omega)}=\|\cdot\|$;

$\mathbb{H}^m(\Omega) = H^m(\Omega) \times H^m(\Omega)$–the 2D vector Sobolev space $\{\phi = (\phi_1, \phi_2) \in \mathbb{L}^2(\Omega), \nabla^k \phi \in \mathbb{L}^2(\Omega), k \leq m\}$ with norm $\|\cdot\|_{\mathbb{H}^m(\Omega)}$;(see [**1**])
$\mathbb{H}_0^1(\Omega) = \overline{\{\phi : \phi = (\phi_1, \phi_2) \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega)\}}^{\mathbb{H}^1(\Omega)}$;
$\mathcal{V} = \{\phi \in \mathcal{C}_0^\infty(\Omega) \times \mathcal{C}_0^\infty(\Omega) : \phi = (\phi_1, \phi_2), \ \nabla \cdot \phi = 0\}$;
$H = \overline{\mathcal{V}}^{\mathbb{L}^2(\Omega)}$ with norm $\|\cdot\|$ and dual space H';
$V = \overline{\mathcal{V}}^{\mathbb{H}^2(\Omega)}$ with norm $\|\cdot\|_V$ and dual space V';
$\overline{X}^Y$-the closure of space X under the norm of Y;
$(\cdot, \cdot)$-the inner product in H, $\langle \cdot, \cdot \rangle$–the dual pairing between V and V';
$\mathcal{B}(H)$-the union of all bounded sets of H;
$\mathcal{B}(V)$-the union of all bounded sets of V;
$L_b^2(\mathbb{R}; H)$-the set of functions $g \in L_{loc}^2(\mathbb{R}; H)$ satisfying

$$\|g\|_{L_b^2}^2 = \|g\|_{L_b^2(\mathbb{R};H)}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|g(s)\|^2 \mathrm{d}s < +\infty; \tag{2.1}$$

$L_c^2(\mathbb{R}; H)$-the set of functions $g(s) \in L_{loc}^2(\mathbb{R}; H)$ satisfying
$\overline{\{g(s+h) : h \in \mathbb{R}\}}^{L_{loc}^2(\mathbb{R};H)}$ is compact in $L_{loc}^2(\mathbb{R}; H)$; (see [**7**], P_{101})
$\mathcal{H}(g_0) = \overline{\{g_0(t + \cdot) : g_0 \in L_c^2(\mathbb{R}; H)\}}^{L_{loc}^2(\mathbb{R};H)}$;
$\mathbb{R}_\tau = [\tau, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$;
$\mathrm{dist}_M(X, Y)$-the Hausdorff semidistance between $X \subset M$ and $Y \subset M$ defined as $\mathrm{dist}_M(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_M$;
"$\longrightarrow$" denotes convergence in strong topology;
"$\rightharpoonup$" denotes convergence in weak topology;
"$\hookrightarrow$" denotes embedding between spaces;
c is the generic constant that can take different values in different places.

To put equations (1.1)-(1.3) into an abstract form, we now introduce some operators. Firstly, we set

$$a(u, v) = \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k} \frac{\partial e_{ij}(v)}{\partial x_k} \mathrm{d}x, \ \ u, v \in V. \tag{2.2}$$

Lemma 2.1 (Bloom and Hao [**5**]). *There exist two positive constants c_1 and c_2 which depends only on Ω such that*

$$c_1 \|u\|_V^2 \leq a(u, u) \leq c_2 \|u\|_V^2, \ \ \forall\, u \in V. \tag{2.3}$$

From the definition of $a(\cdot, \cdot)$ and Lemma 2.1 we see that $a(\cdot, \cdot)$ defines a positive definite symmetric bilinear form on V. By the Lax-Milgram Lemma, we obtain an isometric operator $A \in \mathcal{L}(V, V')$ via

$$\langle Au, v \rangle = a(u, v), \ \ \forall\, u, \, v \in V. \tag{2.4}$$

Moreover, let $D(A) = \{u \in V : Au \in H\}$, then $D(A)$ is a Hilbert space. Indeed, $A = \mathbb{P}\Delta^2$, where $\mathbb{P}$ is the Leray projector from $\mathbb{L}^2(\Omega)$ to H. Also by Lemma 2.1, we have

$$c_1 \|u\|_V^2 \leq a(u, u) = \langle Au, u \rangle = (Au, u) \leq \|Au\| \, \|u\| \leq \|Au\| \, \|u\|_V, \forall\, u \in D(A), \tag{2.5}$$

thus

$$c_1 \|u\|_V \leq \|Au\|, \quad \forall\, u \in D(A). \tag{2.6}$$

Secondly, we define a continuous trilinear form on $\mathbb{H}_0^1(\Omega)\times\mathbb{H}_0^1(\Omega)\times\mathbb{H}_0^1(\Omega)$ as

$$b(u,v,w)=\sum_{i,j=1}^{2}\int_{\Omega} u_i\frac{\partial v_j}{\partial x_i}w_j\,\mathrm{d}x,\ \ u,v,w\in\mathbb{H}_0^1(\Omega).$$

Since $V\subset\mathbb{H}_0^1(\Omega)$, $b(\cdot,\cdot,\cdot)$ is continuous on $V\times V\times V$ and one can check

$$b(u,v,w)=-b(u,w,v),\quad b(u,v,v)=0,\ \forall\,u,v,w\in V. \tag{2.7}$$

Now we can define a continuous mapping $B(u)$ from $V\times V$ to V' by

$$\langle B(u),v\rangle=b(u,u,v),\ \forall\,v\in V. \tag{2.8}$$

Finally, we set

$$\mu(u)=2\mu_0(\varepsilon+|e(u)|^2)^{-\alpha/2},\quad \forall u\in V,$$

and define the mapping $N(\cdot)$ as

$$\langle N(u),v\rangle=\sum_{i,j=1}^{2}\int_{\Omega}\mu(u)e_{ij}(u)e_{ij}(v)\,\mathrm{d}x,\ \forall\,v\in V. \tag{2.9}$$

Then $N(\cdot)$ is continuous from V to V'. When $u\in D(A)$, $N(u)$ can be extended to H via

$$\langle N(u),v\rangle=-\int_{\Omega}\{\nabla\cdot[\mu(u)e(u)]\}\cdot v\,\mathrm{d}x,\ \ \forall\,v\in H. \tag{2.10}$$

From the viewpoint of physics, the initial boundary value problem of (1.1)-(1.3) can be formulated as follows:

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u-\nabla\cdot\Big(2\mu_0(\varepsilon+|e|^2)^{-\alpha/2}e-2\mu_1\Delta e\Big)+\nabla p=g(x,t), \tag{2.11}$$

$$\nabla\cdot u=0, \tag{2.12}$$

$$u=0,\ \ \tau_{ijl}n_jn_l=0,\ \ x\in\partial\Omega, \tag{2.13}$$

$$u|_{t=\tau}=u_\tau,\ \ x\in\Omega,\ \ \tau\in\mathbb{R}, \tag{2.14}$$

where $\tau_{ijl}=2\mu_1\dfrac{\partial e_{ij}}{\partial x_l}$ $(i,j,l=1,2)$ and $\mathbf{n}=(n_1,n_2)$ denotes the exterior unit normal to the boundary $\partial\Omega$. The first condition in (2.13) represents the usual no-slip condition associated with a viscous fluid, while the second one expresses the fact that the first moments of the traction vanish on $\partial\Omega$. It is a direct consequence of the principle of virtual work. We refer to [**4, 5, 6, 21, 24, 28**] and the references therein for detailed physical background. There are many works concerning the unique existence, regularity and long-term behavior of solutions to equations (2.11)-(2.14) or its associated versions (see e.g. [**4, 5, 6, 12, 13, 14, 18, 21, 24, 28, 34, 36, 37, 38, 39, 40, 41, 42**]).

Excluding the pressure p, we can express the weak form of equations (2.11)-(2.14) in the solenoidal vector fields as follows (see [**5, 36**]):

$$\frac{\partial u}{\partial t}+2\mu_1Au+B(u)+N(u)=g(x,t), \tag{2.15}$$

$$u|_{t=\tau}=u_\tau\in H,\ \tau\in\mathbb{R}. \tag{2.16}$$

We now take in equation (2.15) an external force function $g_0(x,t) \in L^2_c(\mathbb{R}; H)$ and take $(\Sigma, \sigma) = (\mathcal{H}(g_0), g)$ as the symbol space. Note that $L^2_c(\mathbb{R}; H) \subset L^2_b(\mathbb{R}; H)$ and for any $g \in L^2_c(\mathbb{R}; H)$ we have (see [**7**])

$$\|g\|_{L^2_b} \le \|g_0\|_{L^2_b}. \tag{2.17}$$

Lemma 2.2 (**I**) *If $g_0 \in L^2_b(\mathbb{R}; H)$, then for $\forall\, g \in \mathcal{H}(g_0)$ and $\forall\, u_\tau \in H$, problem (2.15)-(2.16) admits a unique solution u satisfying*

$$u \in \mathcal{C}(\mathbb{R}_\tau; H) \cap L^\infty(\mathbb{R}_\tau; H) \cap L^2_{loc}(\mathbb{R}_\tau; V), \;\; \partial_t u \in L^2_{loc}(\mathbb{R}_\tau; V'), \tag{2.18}$$

and

$$\|u(t)\|^2 \le \|u_\tau\|^2 e^{-c_1\mu_1(t-\tau)} + \frac{1}{c_1\mu_1}(1 + \frac{1}{c_1\mu_1})\|g_0\|^2_{L^2_b}, \forall\, t \ge \tau, \tag{2.19}$$

$$\|u(t)\|^2 + c_1\mu_1 \int_\tau^t \|u(s)\|^2_V \mathrm{d}s \le \|u_\tau\|^2 + \frac{1}{c_1\mu_1} \int_\tau^t \|g(s)\|^2 \mathrm{d}s, \tag{2.20}$$

hereafter the positive constant c_1 comes from Lemma 2.1.

(**II**) *Suppose $g_0 \in L^2_b(\mathbb{R}; H)$, then for $\forall\, g \in \mathcal{H}(g_0)$ and $\forall u_\tau \in V$, problem (2.15)-(2.16) possesses a unique solution u satisfying*

$$u \in \mathcal{C}(\mathbb{R}_\tau; V) \cap L^\infty(\mathbb{R}_\tau; V) \cap L^2_{loc}(\mathbb{R}_\tau; D(A)), \;\; \partial_t u \in L^2_{loc}(\mathbb{R}_\tau; H). \tag{2.21}$$

Moreover,

$$(t-\tau)\|u(t)\|^2_V \le Q\left(t-\tau, \|u_\tau\|^2, \int_\tau^t \|g(s)\|^2 \mathrm{d}s\right), \;\; \forall\, t \ge \tau,\, \tau \in \mathbb{R}, \tag{2.22}$$

where $Q(z_1, z_2, z_3)$ is an increasing continuous function of $z_1 = t - \tau, z_2$ and z_3.

Proof. The unique existence of solutions, (2.18) and (2.21) can be proved similarly to that of [**5**] by using Galerkin approximations and some *a priori* estimates. The inequalities (2.19),(2.20) and (2.22) can be established analogously to that of [**37**]. We omit the detailed proof here. □

Remark 2.1 From Lemma 2.2 (II), we see that for all $g \in \mathcal{H}(g_0)$ and $\forall\, t \ge \tau$, there holds

$$(t-\tau)\|u(t)\|^2_V \le Q\left(t-\tau, \|u_\tau\|^2, (t-\tau)\|g_0\|^2_{L^2_b}\right). \tag{2.23}$$

The bound in the right hand side of (2.23) is independent of $g \in \mathcal{H}(g_0)$. In fact, for any given u_τ and $\forall\, g \in \mathcal{H}(g_0)$, denote by $u_g(t) = U_g(t,\tau)u_\tau$ the solution corresponding to initial value u_τ and symbol g. Then we have for each $T > \tau$ that

$$\sup_{t \in [\tau, T]} \|u_g(t)\|^2_V < +\infty, \quad \forall\, g \in \mathcal{H}(g_0), \tag{2.24}$$

and

$$\int_\tau^T \|Au_g(t)\|^2 \mathrm{d}t < +\infty, \quad \forall\, g \in \mathcal{H}(g_0). \tag{2.25}$$

3. Existence of the uniform attractor in space H

The aim of this section is to prove the existence of uniform attractor for the family of processes corresponding to problem (2.15)-(2.16) in space H. We will establish the existence of uniform (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set and the $(H \times \mathcal{H}(g_0), H)$-continuity of the family of processes. Then we use the truncation of functions, as well as decomposition of spatial domain, to verify the asymptotic compactness of the family processes, which plays an important role when we prove the existence of uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor in space H.

We first define a natural translation semigroup $\{S(t)\}_{t\geq 0}$ on $\mathcal{H}(g_0)$ as

$$S(h)g(\cdot) = g(\cdot + h),\ \forall h \geq 0,\ g \in \mathcal{H}(g_0). \tag{3.1}$$

By Lemma 2.2, we see that for each $g \in \mathcal{H}(g_0)$, the process $\{U_g(t,\tau)\}_{t\geq\tau}$: $U_g(t,\tau)u_\tau = u(t)$, is well defined from H to H, where $u_\tau \in H$ is arbitrarily given and $u(t)$ is the solution of problem (2.15)-(2.16) with initial value u_τ and with symbol g. Analogously, the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ is well defined from H to H. Moreover, we have the following translation identities in space H

$$U_{S(h)g}(t,\tau) = U_g(t+h,\tau+h),\ \forall\, h \geq 0, t \geq \tau,\ \tau \in \mathbb{R}, g \in \mathcal{H}(g_0), \tag{3.2}$$

$$U_g(\tau,\tau) = Id(\text{identity operator}),\ \tau \in \mathbb{R}, g \in \mathcal{H}(g_0), \tag{3.3}$$

$$U_g(t,s)U_g(s,\tau) = U_g(t,\tau),\ \forall\, t \geq s \geq \tau,\ g \in \mathcal{H}(g_0). \tag{3.4}$$

We now introduce some definitions.

Definition 3.1 *A set $\mathcal{B}_0 \subset H$ is said to be a uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set for the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$, if for $\forall \mathcal{B} \in \mathcal{B}(H)$ and $\forall \tau \in \mathbb{R}$, there exists a $t_0 = t_0(\mathcal{B},\tau) \geq \tau$ such that $\bigcup_{g\in\mathcal{H}(g_0)} U_g(t,\tau)\mathcal{B} \subseteq \mathcal{B}_0$ for all $t \geq t_0$.*

Definition 3.2 *The family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ is said to be $(H \times \mathcal{H}(g_0), H)$-continuous if for any fixed $t, \tau \in \mathbb{R}$ $(t \geq \tau)$, the mapping $(u, g) \mapsto U_g(t,\tau)u$ is continuous from $H \times \mathcal{H}(g_0)$ to H.*

Definition 3.3 *The family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ is said to be asymptotically compact in H if $\{U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\}_{n=1}^\infty$ is pre-compact in H, whenever $\{u_\tau^{(n)}\}_{n=1}^\infty$ is bounded in H, $\{g^{(n)}\}_{n=1}^\infty \subset \mathcal{H}(g_0)$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_\tau$ with $t_n \to +\infty$ as $n \to \infty$.*

Definition 3.4 *A set $\Lambda \subset H$ is said to be the uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) attracting set of $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ in H if for $\forall \mathcal{B} \in \mathcal{B}(H)$ and any fixed $\tau \in \mathbb{R}$,*

$$\lim_{t\to+\infty} \sup_{g\in\mathcal{H}(g_0)} \mathrm{dist}_H\left(U_g(t,\tau)\mathcal{B}, \Lambda\right) = 0.$$

The family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ possessing a compact uniformly attracting set in H is said to be uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in H.

Definition 3.5 *A closed set $\Lambda \subset H$ is said to be the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor of $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ if Λ satisfies*

(i) *(Uniformly attracting property) For $\forall \mathcal{B} \in \mathcal{B}(H)$ and any fixed $\tau \in \mathbb{R}$, there holds*

$$\lim_{t\to+\infty} \sup_{g\in\mathcal{H}(g_0)} \mathrm{dist}_H\left(U_g(t,\tau)\mathcal{B}, \Lambda\right) = 0.$$

(ii) *(Minimal property) Λ is the minimal set (for inclusion relation) among the closed sets satisfying* **(i)**.

Lemma 3.1 *Let $g_0 \in L^2_b(\mathbb{R}; H)$, then the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ corresponding to problem (2.15)-(2.16) possesses a bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set $\mathcal{B}^H_0 \subset H$, where*

$$\mathcal{B}^H_0 = \left\{ u \in H : \|u\|^2 \leq \frac{2}{c_1\mu_1}(1 + \frac{1}{c_1\mu_1})\|g_0\|^2_{L^2_b} \doteq R^2_0 \right\}. \tag{3.5}$$

Proof. We see from (2.19) that for arbitrarily given $\mathcal{B} \in \mathcal{B}(H)$, $\forall u_\tau \in \mathcal{B}$ and $\forall g \in \mathcal{H}(g_0)$, the corresponding solution of (2.15)-(2.16) satisfies

$$\|U_g(t,\tau)u_\tau\|^2 \leq \|u_\tau\|^2 e^{-c_1\mu_1(t-\tau)} + \frac{1}{c_1\mu_1}(1 + \frac{1}{c_1\mu_1})\|g_0\|^2_{L^2_b}, \forall\, t \geq \tau,$$

which implies that there exists a time $t_0 \doteq t_0(\tau, \mathcal{B}) \geq \tau$ such that

$$\|U_g(t,\tau)u_\tau\|^2 \leq \frac{2}{c_1\mu_1}(1 + \frac{1}{c_1\mu_1})\|g_0\|^2_{L^2_b} \doteq R^2_0, \forall\, t \geq t_0.$$

The proof is complete. $\square$

Lemma 3.2 *Let $g_0 \in L^2_c(\mathbb{R}; H)$. Then the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ corresponding to problem (2.15)-(2.16) is $(H \times \mathcal{H}(g_0), H)$-continuous.*

The proof is similar to Lemma 3.2 of [**39**] with the bounded spatial domain replaced by the unbounded spatial domain. We omit the detailed proof here.

Lemma 3.3 *Let $g_0 \in L^2_c(\mathbb{R}; H)$ and $u^{(n)}_\tau \rightharpoonup u_\tau$ weakly in H. Set $\Omega_r = \{x \in \Omega : |x| < r\}$ for $r > 0$, $\{g^{(n)}\}^\infty_{n=1} \subset \mathcal{H}(g_0)$ and $g^{(n)} \longrightarrow g$ strongly in $\mathcal{H}(g_0)$. Then*

$$U_{g^{(n)}}(t,\tau)u^{(n)}_\tau \rightharpoonup U_g(t,\tau)u_\tau \text{ weakly in } H, \tag{3.6}$$

$$U_{g^{(n)}}(\cdot,\tau)u^{(n)}_\tau \rightharpoonup U_g(\cdot,\tau)u_\tau \text{ weakly in } L^2(\tau,T;V),\ \forall\, T > \tau. \tag{3.7}$$

Proof. We get (3.6) directly from Lemma 3.2. Also, we can show that the sequence $\{U_{g^{(n)}}(t,\tau)u^{(n)}_\tau\}^\infty_{n=1}$ is bounded in $L^\infty(\mathbb{R}_\tau; H) \cap L^2_{loc}(\mathbb{R}_\tau; V)$ and the sequence $\{\frac{\partial}{\partial t} U_{g^{(n)}}(t,\tau)u^{(n)}_\tau\}^\infty_{n=1}$ is bounded in $L^2_{loc}(\mathbb{R}_\tau; V')$. The rest proofs of (3.7) are essentially the same as that of Lemma 2.1 of [**29**]. The proof is complete. $\square$

Lemma 3.4 *Let $g_0 \in L^2_c(\mathbb{R}; H)$ and $\mathcal{B} \in \mathcal{B}(H)$. Then for any $\epsilon > 0$, there exists an $r_0 > 0$ and a time $T_* \doteq T_*(\tau, \mathcal{B}) > \tau$, such that*

$$\|U_g(t,\tau)u_\tau\|^2_{\mathbb{L}^2(\Omega\setminus\Omega_r)} \leq \epsilon, \forall\, g \in \mathcal{H}(g_0),\ \forall\, u_\tau \in \mathcal{B}, \forall r > r_0, t > T_*. \tag{3.8}$$

Proof. Let $\chi(\cdot) \in \mathcal{C}^2(\mathbb{R}^2)$ such that

$$\chi(x) = \left\{ \begin{array}{ll} 0, & |x| < 1, \\ 1, & |x| \geq 2. \end{array} \right.$$

Set $\chi_r(x) = \chi(\frac{x}{r})$, $r \geq 1$. Then $\|\nabla\chi_r\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leq cr^{-1}$, $\|D^2\chi_r\|_{\mathbb{L}^\infty(\mathbb{R}^2)} \leq cr^{-2}$, where $c > 0$. Assume that p is the corresponding pressure. We remark that the following deduction will be rigorous for the solutions of problem (2.15)-(2.16) with initial data $u_\tau \in V$. By passing limit and the fact that $U_g(t,\tau)$ is continuous in H for any $g \in \mathcal{H}(g_0)$, it is also true for $u_\tau \in H$. For any given $g \in \mathcal{H}(g_0)$ and $u_\tau \in \mathcal{B}$, set $u = u(t) = U_g(t,\tau)u_\tau$. We see from (2.11) that

$$\Delta p = -\sum^2_{i,j=1} \frac{\partial^2}{\partial x_i x_j}(u_i u_j) + \sum^2_{i,j=1} \frac{\partial^2}{\partial x_i x_j}\left(2\mu_0(\varepsilon + |e|^2)^{-\alpha/2} e_{ij}(u)\right).$$

The right hand side of the above equation is at least in $L^2_{loc}(\mathbb{R}_\tau;\mathbb{H}^{-2}(\Omega))$, thus $p\in L^2_{loc}(\mathbb{R}_\tau;L^2(\Omega))$ and for each fixed $T>0$,

$$\int_t^{t+T}\|p\|^2_{L^2(\Omega)}\,\mathrm{d}t\le c(T),\quad \forall\, t\ge\tau. \tag{3.9}$$

Taking the inner product of (2.11) with χ_r^2u, noting $\nabla\cdot u=0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r u\|^2+\sum_{i,j,k=1}^{2}\Big(2\mu_1\int_\Omega\frac{\partial e_{ij}(\chi_r u)}{\partial x_k}\frac{\partial e_{ij}(\chi_r u)}{\partial x_k}\,\mathrm{d}x+\int_\Omega u_i\frac{\partial u_j}{\partial x_i}\chi_r^2u_j\,\mathrm{d}x\Big)\\
=&\int_\Omega g\chi_r^2u\,\mathrm{d}x+2\int_\Omega p\chi_r\nabla\chi_r\cdot u\,\mathrm{d}x-2\sum_{i,j,k=1}^{2}\int_\Omega\mu(u)e_{ij}(u)e_{ij}(\chi_r^2u)\,\mathrm{d}x\\
&+2\mu_1\int_\Omega\Phi(\chi_r,u)\,\mathrm{d}x,
\end{aligned}\tag{3.10}$$

where

$$\Phi(\chi_r,u)=\sum_{i,j,k=1}^{2}\left(\frac{\partial e_{ij}(\chi_r u)}{\partial x_k}\frac{\partial e_{ij}(\chi_r u)}{\partial x_k}-\frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(\chi_r^2u)}{\partial x_k}\right).$$

Using integrating by parts, we have

$$\int_\Omega u_i\frac{\partial u_j}{\partial x_i}\chi_r^2u_j\,\mathrm{d}x=-\int_\Omega u_iu_j\chi_r^2\frac{\partial u_j}{\partial x_i}\,\mathrm{d}x-2\int_\Omega u_iu_ju_j\chi_r\frac{\partial\chi_r}{\partial x_i}\,\mathrm{d}x.$$

Thus

$$\begin{aligned}
\left|\int_\Omega u_i\frac{\partial u_j}{\partial x_i}\chi_r^2u_j\,\mathrm{d}x\right| &= \left|\int_\Omega u_iu_ju_j\chi_r\frac{\partial\chi_r}{\partial x_i}\,\mathrm{d}x\right|\le\|\chi_r\nabla\chi\|_{\mathbb{L}^\infty(\Omega)}\|u\|^3_{\mathbb{L}^3(\Omega)}\\
&\le cr^{-1}\|u\|^{5/2}\|u\|^{1/2}_{\mathbb{H}^2(\Omega)}\le cr^{-1}\left(\|u\|^{10/3}+\|u\|^2_{\mathbb{H}^2(\Omega)}\right)\\
&\le cr^{-1}+cr^{-1}\|u\|^2_{\mathbb{H}^2(\Omega)}.\qquad (t>t_0(\tau,\mathcal{B}))
\end{aligned}\tag{3.11}$$

Since

$$\begin{aligned}
e_{ij}(\chi_r^2u)&=\chi_re_{ij}\left(\chi_ru\right)+\frac{1}{2}\chi_r\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right),\\
\chi_re_{ij}(u)&=e_{ij}(\chi_ru)-\frac{1}{2}\chi_r\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right),
\end{aligned}$$

we get

$$\begin{aligned}
&\left|-2\int\limits_{\Omega}\mu(u)e_{ij}(u)e_{ij}(\chi_r^2 u)\,\mathrm{d}x\right|\\
=\ &\left|-2\sum_{i,j=1}^{2}\int\limits_{\Omega}\mu(u)\left[(e_{ij}(\chi_r u))^2-\frac{1}{4}\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)^2\right]\mathrm{d}x\right|\\
\leq\ &\frac{1}{2}\sum_{i,j=1}^{2}\int\limits_{\Omega}\mu(u)\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)^2\mathrm{d}x\\
\leq\ &\frac{1}{2}\mu_0\epsilon^{-\alpha/2}\|u\|^2\|\nabla\chi_r\|^2_{\mathbb{L}^\infty(\Omega)}\leq cr^{-2}. \qquad (t>t_0(\tau,\mathcal{B}))
\end{aligned}\tag{3.12}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial e_{ij}(\chi_r^2 u)}{\partial x_k} &= \chi_r\frac{\partial}{\partial x_k}\left\{e_{ij}(\chi_r u)+\frac{1}{2}\left(u_j\frac{\partial\chi_r}{\partial x_j}+u_i\frac{\partial\chi_r}{\partial x_j}\right)\right\}\\
&\quad+\frac{\partial\chi_r}{\partial x_k}\left\{e_{ij}(\chi_r u)+\frac{1}{2}\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)\right\},\\
\chi_r\frac{\partial e_{ij}(u)}{\partial x_k} &= \frac{\partial}{\partial x_k}\left\{e_{ij}(\chi_r u)-\frac{1}{2}\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)\right\}-\frac{\partial\chi_r}{\partial x_k}e_{ij}(u),\\
\Phi(\chi_r,u) &= \frac{1}{4}\sum_{i,j,k=1}^{2}\left\{\frac{\partial}{\partial x_k}\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)\right\}^2\\
&\quad-\frac{\partial\chi_r}{\partial x_k}e_{ij}(u)\frac{\partial}{\partial x_k}\left(u_j\frac{\partial\chi_r}{\partial x_i}+u_i\frac{\partial\chi_r}{\partial x_j}\right)\\
&\quad+\frac{\partial\chi_r}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}\left(u_i\frac{\partial\chi_r}{\partial x_j}+u_j\frac{\partial\chi_r}{\partial x_i}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left|\int\limits_{\Omega}\Phi(\chi_r,u)\,\mathrm{d}x\right|\\
\leq\ &2\|u\|^2_{\mathbb{H}^2(\Omega)}(\|\nabla\chi_r\|^2_{\mathbb{L}^\infty(\Omega)}+\|D^2\chi_r\|^2_{\mathbb{L}^\infty(\Omega)})\\
&+\|\nabla\chi_r\|^2_{\mathbb{L}^2(\Omega)}\|u\|^2_{\mathbb{H}^1(\Omega)}(\|\nabla\chi_r\|_{\mathbb{L}^\infty(\Omega)}+\|\nabla\chi_r\|^2_{\mathbb{L}^\infty(\Omega)})\\
\leq\ &cr^{-2}\|u\|^2_{\mathbb{H}^2(\Omega)}. \qquad (t>t_0(\tau,\mathcal{B}))
\end{aligned}\tag{3.13}$$

Note that on the spatial domain Ω we have the Poincaré inequality

$$\lambda_1\int\limits_{\Omega}|u|^2\mathrm{d}x\leq\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x,\quad \forall\, u\in\mathbb{H}^1_0(\Omega),$$

so we have

$$\lambda_1^2\int\limits_{\Omega}|u|^2\mathrm{d}x\leq\int\limits_{\Omega}|\Delta u|^2\mathrm{d}x\leq\|u\|^2_{\mathbb{H}^2(\Omega)},\quad \forall\, u\in\mathbb{H}^1_0(\Omega)\cap\mathbb{H}^2_0(\Omega),$$

where λ_1 is a positive constant depending only on Ω. The rest terms in (3.10) are estimated as

$$\left|\int_\Omega g\chi_r^2 u \,\mathrm{d}x\right| \leq \|\chi_r g\|\,\|\chi_r u\| \leq \frac{\mu_1\lambda_1^2}{2}\|\chi_r u\|^2 + c\|\chi_r g\|^2, \tag{3.14}$$

$$\begin{aligned}\left|\int_\Omega 2p\chi_r\nabla\chi_r\cdot u\,\mathrm{d}x\right| &\leq 2\|p\|_{L^2(\Omega)}\,\|\chi_r u\|\,\|\nabla\chi_r\|_{\mathbb{L}^\infty(\Omega)}\\ &\leq \frac{\mu_1\lambda_1^2}{2}\|\chi_r u\|^2 + c\|\nabla\chi_r\|^2_{\mathbb{L}^\infty(\Omega)}\,\|p\|^2_{L^2(\Omega)}\\ &\leq \frac{\mu_1}{2}\|\chi_r u\|^2_{\mathbb{H}^2(\Omega)} + cr^{-2}\|p\|^2_{L^2(\Omega)},\end{aligned} \tag{3.15}$$

where λ_1 is the constant in the Poincaré inequality. Therefore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r u\|^2 + 2\mu_1\lambda_1^2\|\chi_r u\|^2\\ \leq\ &\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r u\|^2 + 2\mu_1\|\chi_r u\|^2_{\mathbb{H}^2(\Omega)}\\ \leq\ &c\|\chi_r g\| + cr^{-2}\left(\|u\|^2_{\mathbb{H}^2(\Omega)} + \|p\|^2_{L^2(\Omega)}\right) + cr^{-1}.\end{aligned} \tag{3.16}$$

Setting $\eta = 2\mu_1\lambda_1^2$, we get by using Gronwall inequality

$$\begin{aligned}\|\chi_r u\|^2 \leq\ &\frac{c}{r^2}\int_\tau^t e^{-\eta(t-s)}\left(\|u(s)\|^2_{\mathbb{H}^2(\Omega)} + \|p(s)\|^2_{L^2(\Omega)} + \|\chi_r g(s)\|^2\right)\mathrm{d}s\\ &+\|\chi_r u_\tau\|e^{-\eta(t-\tau)} + \frac{c}{\eta r}.\end{aligned} \tag{3.17}$$

Now (2.20) shows that for any $T>0$

$$\int_\tau^{\tau+T}\|u(s)\|^2_{\mathbb{H}^2(\Omega)} \leq c\|u_\tau\|^2 + c\int_\tau^{\tau+T}\|g(s)\|^2\mathrm{d}s \leq c(\|u_\tau\|^2 + \|g_0\|^2_{L_b^2}T).$$

Combining (3.9) (fixing $T=1$), we get

$$\begin{aligned}
&\int_\tau^t e^{-\eta(t-s)}\left(\|u(s)\|^2_{\mathbb{H}^2(\Omega)}+\|p(s)\|^2_{L^2(\Omega)}+\|\chi_r g(s)\|^2\right)\mathrm{d}s\\
\le\;& \int_{t-1}^t e^{-\eta(t-s)}\left(\|u(s)\|^2_{\mathbb{H}^2(\Omega)}+\|p(s)\|^2_{L^2(\Omega)}+\|\chi_r g(s)\|^2\right)\mathrm{d}s\\
&+\int_{t-2}^{t-1} e^{-\eta(t-s)}\left(\|u(s)\|^2_{\mathbb{H}^2(\Omega)}+\|p(s)\|^2_{L^2(\Omega)}+\|\chi_r g(s)\|^2\right)\mathrm{d}s\\
&+\int_{t-3}^{t-2} e^{-\eta(t-s)}\left(\|u(s)\|^2_{\mathbb{H}^2(\Omega)}+\|p(s)\|^2_{L^2(\Omega)}+\|\chi_r g(s)\|^2\right)\mathrm{d}s\\
&+\cdots\\
\le\;& (1+e^{-\eta}+e^{-2\eta}+e^{-3\eta}+\cdots)(c+\|\chi_r g\|^2_{L^2_b})\\
\le\;& \frac{1}{1-e^{-\eta}}(c+\|\chi_r g\|^2_{L^2_b})\le\frac{c+\|\chi_r g_0\|^2_{L^2_b}}{1+\eta}. \qquad (3.18)
\end{aligned}$$

(3.17) and (3.18) imply

$$\|\chi_r u\|^2\le\|\chi_r u_\tau\|^2 e^{-\eta(t-\tau)}+\frac{c}{\eta r}+\frac{c}{r^2}\frac{c+\|\chi_r g_0\|^2_{L^2_b}}{1+\eta},\quad \forall\, t>\tau. \qquad (3.19)$$

Since $g_0\in L^2_c(\mathbb{R};H)$, we have $\lim_{r\to+\infty}\|\chi_r g_0\|_{L^2_b}=0$. Hence, by (3.19) we see that for any $\epsilon>0$, there exists an $r_0>0$ and a $T_*>\max\{t_0(\tau,\mathcal{B}),\tau\}$ such that $\|\chi_{\frac{r_0}{2}}u\|^2<\epsilon$ for $t>T_*$. When $r>r_0$

$$\|u\|^2_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\le\|\chi_{\frac{r_0}{2}}u\|^2<\epsilon,\quad t>T_*.$$

The proof is complete. $\square$

Lemma 3.5 *Let $g_0\in L^2_c(\mathbb{R};H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\ge\tau,g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) is asymptotically compact in space H.

Proof. Let $\{u_\tau^{(n)}\}_{n=1}^\infty$ be a bounded sequence in H, $\{g^{(n)}\}_{n=1}^\infty\subset\mathcal{H}(g_0)$ and $\{t_n\}_{n=1}^\infty\subset\mathbb{R}_\tau$ with $t_n\to+\infty$ as $n\to\infty$. Without loss of generality we may assume that $t_n>t_0$ (see Lemma 3.1). Let $\{u_{g^{(n)}}(t_n)=U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\}_{n=1}^\infty$ be the corresponding solution sequence. Then $\{u_{g^{(n)}}(t_n)\}_{n=1}^\infty$ is bounded in H. We can also prove that $\{u_{g^{(n)}}(t)\}_{n=1}^\infty$ is bounded in V (see Lemma 5.2 later). Thus $\{u_{g^{(n)}}(t_n)\}_{n=1}^\infty$ converges weakly to some $v\in V$ in space V. Obviously, for any $\epsilon>0$, there exists an $r_1>0$ such that

$$\|v\|_{\mathbb{L}^2(\Omega\setminus\Omega_{r_1})}<\frac{\epsilon}{3}.$$

By Lemma 3.4, for above $\epsilon>0$ there exists a $T_*>t_0$ and an $r_2>0$ such that

$$\|u_{g^{(n)}}(t_n)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}<\frac{\epsilon}{3},\quad\forall\, t_n>T_*,\quad\forall\, r>r_2. \qquad (3.20)$$

Clearly, $\{u_{g^{(n)}}(t_n)|_{\Omega_r}\}_{n=1}^{\infty}$ is bounded in $\mathbb{H}^2(\Omega_r)$ for any given $r > 0$. Set $r = r_1 + r_2 + 1$, then the embedding $\mathbb{H}^2(\Omega_r) \hookrightarrow \mathbb{L}^2(\Omega_r)$ is compact and the sequence $\{u_{g^{(n)}}(t_n)|_{\Omega_r}\}_{n=1}^{\infty}$ is pre-compact in $\mathbb{L}^2(\Omega_r)$. It is easy to see that $\{u_{g^{(n)}}(t_n)|_{\Omega_r}\}_{n=1}^{\infty}$ converges strongly to $v|_{\Omega_r}$ in $\mathbb{L}^2(\Omega_r)$ and there exists an $n_0 > 0$ such that

$$\|u_{g^{(n)}}(t_n) - v\|_{\mathbb{L}^2(\Omega_r)} < \frac{\epsilon}{3}, \quad \forall\, n > n_0. \tag{3.21}$$

Therefore, we have when n is large enough that

$$\begin{aligned}
& \|u_{g^{(n)}}(t_n) - v\|_{\mathbb{L}^2(\Omega)} \\
\leq\ & \|u_{g^{(n)}}(t_n) - v\|_{\mathbb{L}^2(\Omega_r)} + \|u_{g^{(n)}}(t_n) - v\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)} \\
\leq\ & \|u_{g^{(n)}}(t_n) - v\|_{\mathbb{L}^2(\Omega_r)} + \|u_{g^{(n)}}(t_n)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)} + \|v\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)} \\
\leq\ & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{3.22}$$

Thus the sequence $\{u_{g^{(n)}}(t_n)\}_{n=1}^{\infty}$ converges strongly to v in H, which implies that the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ is asymptotically compact in space H. The proof is complete. $\square$

Lemma 3.6 *Let $g_0 \in L_c^2(\mathbb{R}; H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) is uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in space H.

Proof. It suffices to prove that the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ possesses a compact uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) attracting set in H. We claim that the set

$$\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H) \doteq \bigcap_{t\geq\tau} \overline{\bigcup_{g\in\mathcal{H}(g_0)} \bigcup_{s\geq t} U_g(s,\tau)\mathcal{B}_0^H}^{\mathbb{L}^2(\Omega)} \quad \text{for each } \tau \in \mathbb{R}, \tag{3.23}$$

is a compact uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) attracting set for $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ in H. In fact, the set $\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ defined by (3.23) can be characterized, similarly to the semigroup case, as follows:

$$\left\{\begin{array}{l}
w \in \omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H) \Longleftrightarrow \\
\text{there exist } \{w^{(n)}\}_{n=1}^{\infty} \subset \mathcal{B}_0^H, \{g^{(n)}\}_{n=1}^{\infty} \subset \mathcal{H}(g_0), \\
\text{and } \{t_n\} \subset \mathbb{R}_\tau \text{ with } \ t_n \to +\infty \text{ as } n \to \infty \\
\text{such that } U_{g^{(n)}}(t_n,\tau)w^{(n)} \longrightarrow w \text{ strongly in } H \text{ as } n \to \infty.
\end{array}\right. \tag{3.24}$$

Indeed, (3.24) implies that $\omega_{0,\mathcal{H}(g_0)}(\mathcal{B}_0^H) = \omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ for each $\tau \in \mathbb{R}$, in other words, $\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ is independent of τ. The rest proofs of this lemma are similar to those of Proposition 4.1, 4.2 and 4.3(iii) in [**19**]. Here we only sketch the main steps and omit the detailed proofs.

Step 1. $\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ is a nonempty compact set in H. This assertion can be established by the uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing property (Lemma 3.1), asymptotic compactness property (Lemma 3.5) of $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$ in H and the characterization of $\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ described by (3.24).

Step 2. For $\forall\mathcal{B} \in \mathcal{B}(H)$ and any fixed $\tau \in \mathbb{R}$,

$$\lim_{t\to+\infty} \sup_{g\in\mathcal{H}(g_0)} \operatorname{dist}_H\left(U_g(t,\tau)\mathcal{B}, \omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)\right) = 0. \tag{3.25}$$

(3.25) could be proved by contradiction and using of Lemma 3.5 and (3.24).

Step 3. For $\forall \mathcal{B} \in \mathcal{B}(H)$,

$$\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}) \subseteq \omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H). \tag{3.26}$$

(3.26) can be proved by using (3.4), (3.24) and Lemma 3.1. (3.25) and (3.26) imply the uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) attracting property of $\omega_{\tau,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ in H. The proof of Lemma 3.6 is complete. $\square$

Combining Lemmas 3.1, 3.2, 3.6 and Theorem 5.1 of [**7**], we now can state the main result of this section.

Theorem 3.1 *Let $g_0 \in L_c^2(\mathbb{R}; H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$$

possesses a compact uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor $\mathcal{A}^H_{\mathcal{H}(g_0)}$ in space H, which has the following structure

$$\mathcal{A}^H_{\mathcal{H}(g_0)} = \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{K}_g^H(s) = \omega_{s,\mathcal{H}(g_0)}(\mathcal{B}_0^H), \quad s \in \mathbb{R}, \tag{3.27}$$

where $\mathcal{K}_g^H(s)$ is the kernel section at time moment $t = s$, $\mathcal{K}_g^H$ is the kernel of the process $\{U_g(t,\tau)\}_{t\geq\tau}$ and $\mathcal{K}_g^H$ is nonempty for all $g \in \mathcal{H}(g_0)$, $\mathcal{B}_0^H$ is the bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set defined by (3.5) and $\omega_{s,\mathcal{H}(g_0)}(\mathcal{B}_0^H)$ is its uniform (w.r.t. $g \in \mathcal{H}(g_0)$) ω-limit set.

4. Existence of the uniform attractor in space V

The aim of this section is to prove the existence of the uniform attractor for the family of processes corresponding to problem (2.15)-(2.16) in space V. We will establish the existence of the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set and the $(V \times \mathcal{H}(g_0), V)$-continuity of the family of processes. Then we use the approach of enstrophy equation to verify the asymptotic compactness of the processes, which plays an important role when we establish the existence of the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor in space V.

By Lemma 2.2 (II), we see that for each $g \in \mathcal{H}(g_0)$, the process $\{U_g(t,\tau)\}_{t\geq\tau}$: $U_g(t,\tau)u_\tau = u(t)$, is well defined on V, where $u_\tau \in V$ is arbitrarily given and $u(t)$ is the solution of problem (2.15)-(2.16) with initial value u_τ and with symbol g. Analogously, the family of processes $\{U_g(t,\tau)\}_{t\geq\tau,g\in\mathcal{H}(g_0)}$ is well defined on V. Moreover, the identities (3.2)-(3.4) also hold true in space V.

The definitions of the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set, $(V \times \mathcal{H}(g_0), V)$-continuity, asymptotic compactness, uniform (w.r.t. $g \in \mathcal{H}(g_0)$ asymptotic compactness and uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor for the family of processes $\{U_g(t,\tau)\}_{t\geq\tau,g\in\mathcal{H}(g_0)}$ in space V are similar with Definitions 3.1-3.5.

Lemma 4.1 *Let $g_0 \in L_b^2(\mathbb{R}; H)$, then the family of processes*

$$\{U_g(t,\tau)\}_{t\geq\tau,g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) possesses a bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set $\mathcal{B}_0^V \subset V$, where

$$\mathcal{B}_0^V = \left\{ u \in V : \|u\|_V^2 \leq Q\left(1, R_0^2, \|g_0\|^2_{L_b^2}\right) \doteq R_1^2 \right\}, \tag{4.1}$$

and $Q(\cdot,\cdot,\cdot)$ is the function from Lemma 2.2 (II).

Proof. Set

$$\mathcal{B}_0^V = \bigcup_{g\in\mathcal{H}(g_0)} \bigcup_{\tau\in\mathbb{R}} U_g(\tau+1,\tau)\mathcal{B}_0^H. \tag{4.2}$$

Then we can derive from (3.5) that $\mathcal{B}_0^V$ is bounded in V. Precisely, we have by (2.23) that

$$\|u\|_V^2 \le Q\left(1, R_0^2, \|g_0\|_{L_b^2}^2\right) \doteq R_1^2, \ \ \forall\, u \in \mathcal{B}_0^V. \tag{4.3}$$

Clearly, $\mathcal{B}_0^V \subset V$ is the bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set of the family of processes $\{U_g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$, which uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbs any bounded sets of H (also of V) in norm of V. The proof is complete. □

Lemma 4.2 *Let $g_0 \in L_c^2(\mathbb{R}; H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) is $(V\times\mathcal{H}(g_0), V)$ continuous.

Proof. Recall that if for any fixed t and τ, the mapping $(u,g)\mapsto U_g(t,\tau)u$ is continuous from $V\times\mathcal{H}(g_0)$ to V, then the family of processes $\{U_g(t,\tau)\}_{t\ge\tau,g\in\mathcal{H}(g_0)}$ is said to be $(V\times\mathcal{H}(g_0), V)$-continuous. Let $\{(u_\tau^{(n)}, g^{(n)})\}_{n=1}^\infty \subset V\times\mathcal{H}(g_0)$ be a sequence that converges strongly to some $(u_\tau, g)\in V\times\mathcal{H}(g_0)$, $\{u^{(n)}(t)\}_{n=1}^\infty$ and $u(t)$ be the corresponding solutions of equations (2.15)-(2.16) with symbols $\{g^{(n)}\}_{n=1}^\infty$ and g, and with initial data $\{u_\tau^{(n)}\}_{n=1}^\infty$ and u_τ, respectively. Set

$$w^{(n)}(t) = u(t) - u^{(n)}(t) = U_g(t,\tau)u_\tau - U_{g^{(n)}}(t,\tau)u_\tau^{(n)}, \ \ n = 1,2,\cdots.$$

For each n we see that $w^{(n)}(t)$ is a solution of the following problem:

$$\frac{\partial w^{(n)}}{\partial t} + 2\mu_1 A w^{(n)} + B(u) - B(u^{(n)}) + N(u) - N(u^{(n)}) = g - g^{(n)}, \tag{4.4}$$

$$w^{(n)}|_{t=\tau} = w_\tau^{(n)} = u_\tau - u_\tau^{(n)}, \ \ \tau\in\mathbb{R}. \tag{4.5}$$

Multiplying (4.4) with $Aw^{(n)}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw^{(n)}, w^{(n)}) + 2\mu_1\|Aw^{(n)}\|^2 + \langle B(u) - B(u^{(n)}), Aw^{(n)}\rangle\\ &= (g - g^{(n)}, Aw^{(n)}) - \langle N(u) - N(u^{(n)}), Aw^{(n)}\rangle.\end{aligned} \tag{4.6}$$

Now by the property of the operator $B(\cdot) = b(\cdot,\cdot)$, (2.6), Cauchy inequality, Hölder inequality and the embedding $\mathbb{H}^2(\Omega)\hookrightarrow\mathbb{L}^\infty(\Omega)$, we get

$$\begin{aligned}
& |\langle B(u) - B(u^{(n)}), Aw^{(n)}\rangle|\\
=\ & |b(u,u,Aw^{(n)}) - b(u^{(n)}, u^{(n)}, Aw^{(n)})|\\
=\ & |b(u-u^{(n)}, u, Aw^{(n)}) + b(u^{(n)}, u, Aw^{(n)}) - b(u^{(n)}, u^{(n)}, Aw^{(n)})|\\
=\ & |b(w^{(n)}, u, Aw^{(n)}) + b(u^{(n)}, w^{(n)}, Aw^{(n)})|\\
\le\ & c\|u\|_{\mathbb{L}^\infty(\Omega)}\|w^{(n)}\|\|Aw^{(n)}\| + c\|u^{(n)}\|_{\mathbb{L}^\infty(\Omega))}\|\nabla w^{(n)}\|\|Aw^{(n)}\|\\
\le\ & c(\|u\|_V\|w^{(n)}\|_V + \|u^{(n)}\|_V\|w^{(n)}\|_V)\|Aw^{(n)}\|\\
\le\ & c(\|u\|_V^2 + \|u^{(n)}\|_V^2)\|w^{(n)}\|_V^2 + \frac{\|Aw^{(n)}\|^2}{2\mu_1}.
\end{aligned} \tag{4.7}$$

Also, using the similar derivations of (3.11) in [**37**], we have

$$\begin{aligned} |\langle N(u)-N(u^{(n)}), Aw^{(n)}\rangle| &\leq c\|w^{(n)}\|_V\|Aw^{(n)}\| \\ &\leq c\|w^{(n)}\|_V^2+\frac{\|Aw^{(n)}\|^2}{2\mu_1}. \end{aligned} \tag{4.8}$$

Combining (4.6)-(4.8), Cauchy inequality and Lemma 2.1, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Aw^{(n)}, w^{(n)}) \\ \leq\ & (c+\|u\|_V^2+\|u^{(n)}\|_V^2)\|w^{(n)}\|_V^2+\frac{\|g-g^{(n)}\|^2}{4\mu_1} \\ \leq\ & \frac{1}{c_1}(c+\|u\|_V^2+\|u^{(n)}\|_V^2)(Aw^{(n)}, w^{(n)})+\frac{\|g-g^{(n)}\|^2}{4\mu_1}. \end{aligned} \tag{4.9}$$

By Gronwall inequality and (2.20), we get for $t \geq \tau$ that

$$\begin{aligned} &(Aw^{(n)}(t), w^{(n)}(t)) \\ \leq\ & \left((Aw_\tau^{(n)}, w_\tau^{(n)})+\frac{1}{2\mu_1}\int_\tau^t \|g(s)-g^{(n)}(s)\|^2\mathrm{d}s\right) \\ & \times\exp\left\{\int_\tau^t \frac{1}{c_1}(c+\|u\|_V^2+\|u^{(n)}\|_V^2)\mathrm{d}s\right\} \\ \leq\ & \left((Aw_\tau^{(n)}, w_\tau^{(n)})+\frac{1}{2\mu_1}\int_\tau^t \|g(s)-g^{(n)}(s)\|^2\mathrm{d}s\right) \\ & \times\exp\left\{\frac{1}{c_1}(c+\|u_\tau\|^2+\|u_\tau^{(n)}\|^2+\|g_0\|_{L_b^2}^2)(t-\tau)\right\}. \end{aligned} \tag{4.10}$$

Since $\{u_\tau^{(n)}\}$ converges strongly to u_τ, $\|u_\tau^{(n)}\|$ is bounded. Thus by Lemma 2.1 we get

$$\begin{aligned} \|u^{(n)}(t)-u(t)\|_V^2 &\leq \frac{1}{c_1}(Aw^{(n)}(t), w^{(n)}(t)) \\ &\leq \frac{1}{c_1}\left(c_2\|u_\tau^{(n)}-u_\tau\|_V^2+2\mu_1\int_\tau^t \|g(s)-g^{(n)}(s)\|^2\mathrm{d}s\right) \\ &\quad\times\exp\left\{\left[\frac{1}{c_1}(c+\|u_\tau\|^2+\|u_\tau^{(n)}\|^2+\|g_0\|_{L_b^2}^2)\right](t-\tau)\right\}, \end{aligned} \tag{4.11}$$

from which we can obtain the $(V\times\mathcal{H}(g_0), V)$-continuity of the family of processes $\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$. The proof is complete. □

Lemma 4.3 *Let $u_\tau^{(n)} \longrightarrow u_\tau$ strongly in H and $g^{(n)} \longrightarrow g$ strongly in $L^2_{loc}(\mathbb{R}; H)$, $u(t)=U_g(t,\tau)u_\tau$, $u^{(n)}(t)=U_{g^{(n)}}(t,\tau)u_\tau^{(n)}$ be the corresponding solutions. Then for $\forall T>\tau$, we have*

$$u^{(n)}(t) \longrightarrow u(t) \quad \textit{strongly in } L^2(\tau, T; V). \tag{4.12}$$

Proof. From (2.5) we obtain

$$\|u^{(n)}-u\|_V^2 \leq c\|u^{(n)}-u\|\,\|A(u^{(n)}-u)\|. \tag{4.13}$$

Integrating (4.13) from τ to T, we have by Young inequality that

$$(4.14)\qquad \int_\tau^T \|u^{(n)}-u\|_V^2 \mathrm{d}t \le c\left\{\int_\tau^T \|u^{(n)}-u\|^2\mathrm{d}t\right\}^{1/2}\left\{\int_\tau^T \|A(u^{(n)}-u)\|^2\mathrm{d}t\right\}^{1/2}.$$

From the $(H\times\mathcal{H}(g_0),H)$-continuity of $\{U_g(t,\tau)\}_{t\ge\tau}$ (see Lemma 3.2), we see that $u^{(n)}\longrightarrow u$ strongly in H. We also have from (2.25) that

$$\left\{\int_\tau^T \|A(u^{(n)}-u)\|^2\mathrm{d}t\right\}^{1/2} < +\infty.$$

Therefore, we obtain from (4.14) that $\lim\limits_{n\to\infty}\int_\tau^T \|u_n-u\|_V^2\mathrm{d}t=0$, so $u^{(n)}(t)\longrightarrow u(t)$ strongly in $L^2(\tau,T;V)$. The proof is complete. $\square$

Lemma 4.4 *Let $u_\tau^{(n)}\rightharpoonup u_\tau$ weakly in V and $g^{(n)}\longrightarrow g$ strongly in $L^2_{loc}(\mathbb{R};H)$. Then*

$$(4.15)\qquad U_{g^{(n)}}(t,\tau)u_\tau^{(n)}\rightharpoonup U_g(t,\tau)u_\tau \text{ weakly in } V \text{ for } \forall\, t>\tau,$$

and

$$(4.16)\qquad U_{g^{(n)}}(\cdot,\tau)u_\tau^{(n)}\rightharpoonup U_g(\cdot,\tau)u_\tau \text{ weakly in } L^2(\tau,T;D(A)) \text{ for } \forall\, T>\tau.$$

The proof of this lemma is very similar with that of Lemma 2.2 in [**20**] and it is omitted here. We next use the idea of enstrophy equation to prove the asymptotic compactness of the family of processes in space V.

Lemma 4.5 *Let $g_0\in L^2_c(\mathbb{R};H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\ge\tau,g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) is asymptotically compact in space V.

Proof. Let $\{u_\tau^{(n)}\}_{n=1}^\infty$ be a bounded sequence in V, $\{g^{(n)}\}_{n=1}^\infty\subset\mathcal{H}(g_0)$ and $\{t_n\}_{n=1}^\infty\subset\mathbb{R}_\tau$ with $t_n\to+\infty$ as $n\to\infty$. For any given $\tau\in\mathbb{R}$, we see from Lemma 4.1 that there exists a time $t_1(\tau,R)\ge\tau$ (where R is a constant satisfying $\|u_\tau^{(n)}\|_V\le R$) such that for all $t_n\ge t_1$, $\{U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\}\subset\mathcal{B}_0^V$, where $\mathcal{B}_0^V$ is the bounded uniformly (w.r.t. $g\in\mathcal{H}(g_0)$) absorbing set in V. Thus $\{U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\}_{n=1}^\infty$ is weakly pre-compact in V and there is a subsequence (still denote by $\{U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\}_{n=1}^\infty$) such that

$$(4.17)\qquad U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)}\rightharpoonup u \text{ weakly in } V \text{ as } n\to\infty$$

for some $u\in V$. Similarly for each $T>0$ and $t_n\ge t_1+T$, we have

$$(4.18)\qquad u_T^{(n)}\doteq U_{g^{(n)}}(t_n-T,\tau)u_\tau^{(n)}\in\mathcal{B}_0^V.$$

Thus $\{u_T^{(n)}\}_{n=1}^\infty$ is weakly pre-compact in V and there exists a subsequence (still denote by $\{u_T^{(n)}\}_{n=1}^\infty$) such that

$$(4.19)\qquad u_T^{(n)}\rightharpoonup u_T \text{ weakly in } V \text{ as } n\to\infty$$

for some $u_{_T} \in V$. Using (3.2)-(3.4), we get

$$\begin{aligned} U_{g^{(n)}}(t_n, \tau) &= U_{g^{(n)}}(t_n, t_n - T)U_{g^{(n)}}(t_n - T, \tau) \\ &= U_{S(t_n - T)g^{(n)}}(T, 0)U_{g^{(n)}}(t_n - T, \tau), \quad t_n - T \geq \tau. \end{aligned} \tag{4.20}$$

Setting $g_{_T}^{(n)} = S(t_n - T)g^{(n)}$, we have by (4.18) and (4.20) for $\forall\, T > 0$ and $t_n - T \geq \tau$ that,

$$U_{g^{(n)}}(t_n, \tau)u_\tau^{(n)} = U_{g_T^{(n)}}(T, 0)U_{g^{(n)}}(t_n - T, \tau)u_\tau^{(n)} = U_{g_T^{(n)}}(T, 0)u_T^{(n)}. \tag{4.21}$$

Since $\{g_{_T}^{(n)}\}_{n=1}^\infty \subset \mathcal{H}(g_0)$ and $\mathcal{H}(g_0)$ is compact in $L^2_{loc}(\mathbb{R}; H)$, there exist a subsequence of $\{g_{_T}^{(n)}\}_{n=1}^\infty$ (still denote by $\{g_{_T}^{(n)}\}_{n=1}^\infty$) and some $g_{_T} \in \mathcal{H}(g_0)$ such that

$$g_{_T}^{(n)} \longrightarrow g_{_T} \text{ strongly in } L^2_{loc}(\mathbb{R}; H) \text{ as } n \to \infty, \quad \text{for every } T > 0. \tag{4.22}$$

Taking (4.17)-(4.19), (4.21)-(4.22), the $(V \times \mathcal{H}(g_0), V)$-continuity and Lemma 4.4 into account, we obtain

$$u = U_{g_{_T}}(T, 0)u_{_T} \text{ for every } T > 0, \tag{4.23}$$

where we also used the uniqueness of the limit. Now it follows from (4.17)-(4.18), (4.21) and the equivalence between the norm $\|\cdot\|_V$ and $(A\cdot, \cdot)$ (see Lemma 2.1) that

$$\begin{aligned} &\liminf_n (AU_{g^{(n)}}(t_n, \tau)u_\tau^{(n)}, U_{g^{(n)}}(t_n, \tau)u_\tau^{(n)}) \\ = &\liminf_n (AU_{g_T^{(n)}}(T, 0)u_T^{(n)}, U_{g_T^{(n)}}(T, 0)u_T^{(n)}) \\ \geq &(Au, u). \end{aligned} \tag{4.24}$$

Next we prove

$$\begin{aligned} &\liminf_n (AU_{g^{(n)}}(t_n, \tau)u_\tau^{(n)}, U_{g^{(n)}}(t_n, \tau)u_\tau^{(n)}) \\ = &\liminf_n (AU_{g_T^{(n)}}(T, 0)u_T^{(n)}, U_{g_T^{(n)}}(T, 0)u_T^{(n)}) \\ \leq &(Au, u). \end{aligned} \tag{4.25}$$

To this end, we use the argument of enstrophy equation of the non-Newtonian fluid in space V. First we define a bilinear operator $[\![\cdot, \cdot]\!] : D(A) \times D(A) \mapsto \mathbb{R}$ as

$$[\![u, v]\!] = 2\mu_1(Au, Av) - \gamma(Au, v), \quad \forall\, u, v \in D(A). \tag{4.26}$$

where $\gamma = \frac{c_1^2\mu_1}{c_2}$. Setting $[\![u, u]\!] = [\![u]\!]^2$, we use Lemma 2.1 to obtain

$$\begin{aligned} \mu_1\|Au\|^2 &= 2\mu_1\|Au\|^2 - \mu_1\|Au\|^2 \leq 2\mu_1\|Au\|^2 - c_1^2\mu_1\|u\|_V^2 \\ &\leq 2\mu_1\|Au\|^2 - \frac{c_1^2\mu_1}{c_2}(Au, u) \\ &= [\![u]\!]^2 \leq 2\mu_1\|Au\|^2. \end{aligned} \tag{4.27}$$

Then $\sqrt{[\![\cdot]\!]^2}$ defines a norm in space $D(A)$ which is equivalent to $\|A\cdot\|$. Now for any solution $u(t)$ of problem (2.15)-(2.16) corresponding to initial data u_τ and symbol g, we use Au to multiply equation (2.15) and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Au, u\rangle + 2\mu_1\|Au\|^2 + \langle B(u), Au\rangle + \langle N(u), Au\rangle = (g, Au). \tag{4.28}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Au, u\rangle + 2\gamma\langle Au, u\rangle = 2(g, Au) - 2\langle B(u), Au\rangle - 2\langle N(u), Au\rangle - 2[\![u]\!]^2.$$

By the formula of constant variation, we get the enstrophy equation of the non-Newtonian fluid as

$$(Au(t),u(t)) = (Au_\tau,u_\tau)e^{-2\gamma(t-\tau)} + \int_\tau^t e^{-2\gamma(t-s)}G(g(s),u(s))\mathrm{d}s,\ t\ge\tau, \tag{4.29}$$

where $G(g,u) = 2(g,Au) - 2\langle B(u),Au\rangle - 2\langle N(u),Au\rangle - 2[\![u]\!]^2$. We now apply the enstrophy equation to $U_{g_T^{(n)}}(T,0)u_T^{(n)}$ to obtain

$$\begin{aligned}
&(AU_{g_T^{(n)}}(T,0)u_T^{(n)},U_{g_T^{(n)}}(T,0)u_T^{(n)})\\
=\ &(Au_T^{(n)},u_T^{(n)})e^{-2\gamma T}\\
&+2\int_0^T e^{-2\gamma(T-s)}\left((g_T^{(n)}(s),AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right)\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\left\langle B(U_{g_T^{(n)}}(s,0)u_T^{(n)}),AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right\rangle\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\left\langle N(U_{g_T^{(n)}}(s,0)u_T^{(n)}),AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right\rangle\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}[\![U_{g_T^{(n)}}(s,0)u_T^{(n)}]\!]^2\mathrm{d}s.
\end{aligned}\tag{4.30}$$

By (4.18), there exists a time T^* such that if $t_n-T>T^*$ then

$$(Au_T^{(n)},u_T^{(n)})e^{-2\gamma T}\le c_2\|u_T^{(n)}\|_V^2\le c_2R_1^2e^{-2\gamma T}. \tag{4.31}$$

By Lemma 4.4, we have

$$U_{g_T^{(n)}}(\cdot,0)u_T^{(n)}\rightharpoonup U_{g_T}(\cdot,0)u_T \text{ weakly in } L^2(0,T;D(A)), \tag{4.32}$$

and

$$AU_{g_T^{(n)}}(\cdot,0)u_T^{(n)}\rightharpoonup AU_{g_T}(\cdot,0)u_T \text{ weakly in } L^2(0,T;H), \tag{4.33}$$

which, together with (4.22), gives

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T e^{-2\gamma(T-s)}\left(g_T^{(n)}(s),AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right)\mathrm{d}s\\
=\ &\int_0^T e^{-2\gamma(T-s)}\left(g_T(s),AU_{g_T}(s,0)u_T\right)\mathrm{d}s.
\end{aligned}\tag{4.34}$$

Since $\sqrt{[\![\cdot]\!]^2}$ defines a norm in space $D(A)$ which is equivalent to $\|A\cdot\|$ and $0<e^{-2\gamma T}\le e^{-2\gamma(T-s)}\le 1$ for any $s\in[0,T]$, we see that $\left\{\int_0^T e^{-2\gamma(T-s)}[\![\cdot]\!]^2\mathrm{d}s\right\}^{1/2}$ is

a norm in $L^2(0,T;D(A))$ equivalent to the usual norm. Thus by (4.32) we get

$$\liminf_{n\to\infty}\int_0^T e^{-2\gamma(T-s)}[\![U_{g_T^{(n)}}(s,0)u_T^{(n)}]\!]^2\mathrm{d}s \geq \int_0^T e^{-2\gamma(T-s)}[\![U_{g_T}(s,0)u_T]\!]^2\mathrm{d}s. \tag{4.35}$$

We next prove

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T e^{-2\gamma(T-s)}\left\langle B(U_{g_T^{(n)}}(s,0)u_T^{(n)}), AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right\rangle \mathrm{d}s\\ &= \int_0^T e^{-2\gamma(T-s)}\left\langle B(U_{g_T}(s,0)u_T), AU_{g_T}(s,0)u_T\right\rangle \mathrm{d}s\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T e^{-2\gamma(T-s)}\left\langle N(U_{g_T^{(n)}}(s,0)u_T^{(n)}), AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right\rangle \mathrm{d}s\\ &= \int_0^T e^{-2\gamma(T-s)}\left\langle N(U_{g_T}(s,0)u_T), AU_{g_T}(s,0)u_T\right\rangle \mathrm{d}s.\end{aligned} \tag{4.37}$$

To prove (4.36), we set

$$\begin{aligned}
I_1^{(n)} &= \Bigg|\int_0^T e^{-2\gamma(T-s)}b(U_{g_T^{(n)}}(s,0)u_T^{(n)}, U_{g_T^{(n)}}(s,0)u_T^{(n)}, AU_{g_T^{(n)}}(s,0)u_T^{(n)})\mathrm{d}s\\
&\quad -\int_0^T e^{-2\gamma(T-s)}b(U_{g_T}(s,0)u_T, U_{g_T^{(n)}}(s,0)u_T^{(n)}, AU_{g_T^{(n)}}(s,0)u_T^{(n)})\mathrm{d}s\Bigg|,\\
I_2^{(n)} &= \Bigg|\int_0^T e^{-2\gamma(T-s)}b(U_{g_T}(s,0)u_T, U_{g_T^{(n)}}(s,0)u_T^{(n)}, AU_{g_T}(s,0)u_T)\mathrm{d}s\\
&\quad -\int_0^T e^{-2\gamma(T-s)}b(U_{g_T}(s,0)u_T, U_{g_T}(s,0)u_T, AU_{g_T}(s,0)u_T)\mathrm{d}s\Bigg|,\\
I_3^{(n)} &= \Bigg|\int_0^T e^{-2\gamma(T-s)}b(U_{g_T}(s,0)u_T, U_{g_T}(s,0)u_T, AU_{g_T^{(n)}}(s,0)u_T^{(n)})\mathrm{d}s\\
&\quad -\int_0^T e^{-2\gamma(T-s)}b(U_{g_T}(s,0)u_T, U_{g_T}(s,0)u_T, AU_{g_T}(s,0)u_T)\mathrm{d}s\Bigg|.
\end{aligned}$$

Then we have

$$
\begin{aligned}
&\left|\int_0^T e^{-2\gamma(T-s)}\left\langle B(U_{g_T^{(n)}}(s,0)u_T^{(n)}), AU_{g_T^{(n)}}(s,0)u_T^{(n)}\right\rangle \mathrm{d}s\right.\\
&\left.\quad-\int_0^T e^{-2\gamma(T-s)}\left\langle B(U_{g_T}(s,0)u_T), AU_{g_T}(s,0)u_T\right\rangle \mathrm{d}s\right|\\
&\le I_1^{(n)}+I_2^{(n)}+I_3^{(n)}. \qquad (4.38)
\end{aligned}
$$

Note $\mathbb{H}^2(\Omega)\hookrightarrow\mathbb{L}^\infty(\Omega)$ and $L^2(0,T;V)\hookrightarrow L^2(0,T;H)$, we obtain by using (2.24), (2.25), (4.12) and Hölder inequality that

$$
\begin{aligned}
I_1^{(n)} &\le c\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\|\\
&\qquad\times\|U_{g_T^{(n)}}(s,0)u_T^{(n)}\|_{\mathbb{L}^\infty(\Omega)}\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\|\mathrm{d}s\\
&\le c\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\|\\
&\qquad\times\|U_{g_T^{(n)}}(s,0)u_T^{(n)}\|_V\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\|\mathrm{d}s\\
&\le c\left\{\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\|^2\|U_{g_T^{(n)}}(s,0)u_T^{(n)}\|^2_{\mathbb{H}^2(\Omega)}\mathrm{d}s\right\}^{1/2}\\
&\qquad\times\left\{\int_0^T\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\|^2\mathrm{d}s\right\}^{1/2}\\
&\le c\left\{\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\|^2\mathrm{d}s\right\}^{1/2}\longrightarrow 0,\ n\to\infty. \qquad (4.39)
\end{aligned}
$$

Similarly, we have

$$
I_2^{(n)}\le c\left\{\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\|^2\mathrm{d}s\right\}^{1/2}\longrightarrow 0,\ n\to\infty. \qquad (4.40)
$$

Now from (2.24) and (2.25) we see that $B(u)\in L^2(0,T;H)$ for $\forall\, u\in D(A)$, this fact together with (4.33) gives

$$
\begin{aligned}
\lim_{n\to\infty} I_3^{(n)} &= \left|\lim_{n\to\infty}\int_0^T\langle B(U_{g_T}(s,0)u_T), A(U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T)\rangle\mathrm{d}s\right|\\
&= 0. \qquad (4.41)
\end{aligned}
$$

Then (4.36) can be deduced from (4.38)-(4.41). To prove (4.37), we denote

$$
\begin{aligned}
I_4^{(n)} &= \int_0^T\int_\Omega e^{-2\gamma(T-s)}\Big\{\nabla\cdot\Big[\mu(U_{g_T^{(n)}}(s,0)u_T^{(n)})e(U_{g_T^{(n)}}(s,0)u_T^{(n)})\\
&\qquad -\mu(U_{g_T}(s,0)u_T)e(U_{g_T}(s,0)u_T)\Big]\Big\}\cdot AU_{g_T}(s,0)u_T\,\mathrm{d}x\mathrm{d}s. \qquad (4.42)\\
I_5^{(n)} &= \int_0^T\int_\Omega e^{-2\gamma(T-s)}\Big\{\nabla\cdot\Big[\mu(U_{g_T^{(n)}}(s,0)u_T^{(n)})e(U_{g_T^{(n)}}(s,0)u_T^{(n)})\Big]\Big\}\\
&\qquad \cdot A\Big(U_{g_T^{(n)}}(s,0)u_T^{(n)}-U_{g_T}(s,0)u_T\Big)\,\mathrm{d}x\mathrm{d}s. \qquad (4.43)
\end{aligned}
$$

Then by (2.10), we have

$$
\begin{aligned}
&\int_0^T e^{-2\gamma(T-s)}\Big\langle N(U_{g_T^{(n)}}(s,0)u_T^{(n)}), AU_{g_T^{(n)}}(s,0)u_T^{(n)}\Big\rangle\,\mathrm{d}s\\
&\qquad -\int_0^T e^{-2\gamma(T-s)}\Big\langle N(U_{g_T}(s,0)u_T), AU_{g_T}(s,0)u_T\Big\rangle\,\mathrm{d}s\\
&=\int_0^T\int_\Omega e^{-2\gamma(T-s)}\Big\{\nabla\cdot[\mu(U_{g_T^{(n)}}(s,0)u_T^{(n)})e(U_{g_T^{(n)}}(s,0)u_T^{(n)})]\Big\}\\
&\qquad\cdot AU_{g_T^{(n)}}(s,0)u_T^{(n)}\mathrm{d}x\mathrm{d}s\\
&-\int_0^T\int_\Omega e^{-2\gamma(T-s)}\Big\{\nabla\cdot[\mu(U_{g_T}(s,0)u_T)e(U_{g_T}(s,0)u_T)]\Big\}\cdot AU_{g_T}(s,0)u_T\mathrm{d}x\mathrm{d}s\\
&= I_4^{(n)}+I_5^{(n)}. \qquad (4.44)
\end{aligned}
$$

To show the convergence of $I_4^{(n)}$ and $I_5^{(n)}$, we set $F(s)=2\mu_0(\varepsilon+|s|^2)^{-\alpha/2}s$, where

$$
s=\begin{pmatrix} s_1 & s_2\\ s_3 & s_4\end{pmatrix}\in\mathcal{M}_{2\times2},\ |s|^2=\sum_{i=1}^4 s_i^2,\quad s_i\in\mathbb{R},\ \ i=1,2,3,4,
$$

and $\mathcal{M}_{2\times 2}$ is the matrix of order 2×2. By some computations we see that the first order and second order Fréchet derivatives of $F(s)$ satisfy

$$
\|DF(s)\|+\|D^2F(s)\|\le c,\ \ \forall\, s_i\in\mathbb{R}, i=1,2,3,4, \qquad (4.45)
$$

where c is a positive constant depending only on μ_0,ε and α. For any $a,b\in\mathcal{M}_{2\times2}$, we have

$$
F(b)-F(a)=\int_0^1 DF(a+\tau(b-a))(b-a)\,\mathrm{d}\tau.
$$

Taking $a = (e_{ij}(U_{g_T^{(n)}}(s,0)u_T^{(n)}))_{2\times 2}, b = (e_{ij}(U_{g_T}(s,0)u_T))_{2\times 2}$, applying the integration by parts first and then the inequality (4.45), we get

$$\begin{aligned}
|I_4^{(n)}| &= \left| \int_0^T \int_\Omega e^{-2\gamma(T-s)} \left\{ \nabla \cdot \left[\mu(U_{g_T^{(n)}}(s,0)u_T^{(n)}) e(U_{g_T^{(n)}}(s,0)u_T^{(n)}) \right.\right.\right. \\
&\qquad \left.\left.\left. -\mu(U_{g_T}(s,0)u_T) e\left(U_{g_T}(s,0)u_T\right)\right]\right\} \cdot AU_{g_T}(s,0)u_T \mathrm{d}x\mathrm{d}s \right| \\
&\le c\int_0^T (\|U_{g_T}(s,0)u_T\|_V^2 + \|U_{g_T^{(n)}}(s,0)u_T^{(n)}\|_V^2) \\
&\qquad \times \|e\left(U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\right)\|_{\mathbb{L}^\infty(\Omega)} \\
&\qquad \times \|AU_{g_T}(s,0)u_T\|\mathrm{d}s, \qquad (4.46)
\end{aligned}$$

where (2.24) is also used. By the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}
&\|e\left(U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\right)\|_{\mathbb{L}^\infty(\Omega)} \|AU_{g_T}(s,0)u_T\| \\
\le\ & c\|U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T)\|^{1/2} \\
&\times (\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\| + \|AU_{g_T}(s,0)u_T\|)^{1/2} \|AU_{g_T}(s,0)u_T\| \\
\le\ & c\|U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\|^{1/2} \\
&\times (\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\| + \|AU_{g_T}(s,0)u_T\|)^{3/2}. \qquad (4.47)
\end{aligned}$$

Using (2.24), (2.25), (4.46), (4.47) and the Hölder inequality, we have

$$\begin{aligned}
|I_4^{(n)}| &\le c\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\|^{1/2} \\
&\qquad \times (\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\| + \|AU_{g_T}(s,0)u_T\|)^{3/2}\mathrm{d}s \\
&\le c\left(\int_0^T \|U_{g_T^{(n)}}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\|^2 \mathrm{d}s\right)^{1/2} \\
&\qquad \times \left(\int_0^T (\|AU_{g_T^{(n)}}(s,0)u_T^{(n)}\| + \|AU_{g_T}(s,0)u_T\|)^2\mathrm{d}s\right)^{3/4} \\
&\longrightarrow 0, n\to\infty, \forall T>0. \qquad (4.48)
\end{aligned}$$

Note that

$$e^{-2\gamma(T-s)}\left\{\nabla\cdot\left[\mu(U_{g_T}(s,0)u_T)e(U_{g_T}(s,0)u_T)\right]\right\} \in L^2(0,T;H). \qquad (4.49)$$

Then (4.33) and (4.49) imply that for any $T > 0$

$$\begin{aligned}
I_5^{(n)} &= \int_0^T \int_\Omega e^{-2\gamma(T-s)} \left\{\nabla\cdot\left[\mu(U_{g_T^{(n)}}(s,0)u_T^{(n)})e(U_{g_T^{(n)}}(s,0)u_T^{(n)})\right]\right\} \\
&\qquad \cdot A\left(U_{g_T}(s,0)u_T^{(n)} - U_{g_T}(s,0)u_T\right)\mathrm{d}x\mathrm{d}s \longrightarrow 0, n\to\infty. \qquad (4.50)
\end{aligned}$$

Thus (4.37) can be deduced from (4.44), (4.48) and (4.50). Now, we use (4.30), (4.31) and (4.34)-(4.37) to obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\Big(AU_{g_T^{(n)}}(T,0)u_T^{(n)},U_{g_T^{(n)}}(T,0)u_T^{(n)}\Big)\\
\leq\ & c_2R_1^2e^{-2\gamma T}+2\int_0^T e^{-2\gamma(T-s)}\Big(g_T(s),AU_{g_T}(s,0)u_T\Big)\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\Big(B(U_{g_T}(s,0)u_T),AU_{g_T}(s,0)u_T\Big)\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\Big\langle N(U_{g_T}(s,0)u_T),AU_{g_T}(s,0)u_T\Big\rangle\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}[\![U_{g_T}(s,0)u_T]\!]^2\mathrm{d}s. \qquad (4.51)
\end{aligned}$$

At the same time, we apply the enstrophy equation to $u=U_{g_T}(T,0)u_T$ and obtain

$$\begin{aligned}
&\Big(AU_{g_T}(T,0)u_T,U_{g_T}(T,0)u_T\Big)\\
=\ & (Au_T,u_T)e^{-2\gamma T}\\
&+2\int_0^T e^{-2\gamma(T-s)}\Big(g_T(s),AU_{g_T}(s,0)u_T\Big)\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\Big\langle B(U_{g_T}(s,0)u_T),AU_{g_T}(s,0)u_T\Big\rangle\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}\Big\langle N(U_{g_T}(s,0)u_T),AU_{g_T}(s,0)u_T\Big\rangle\,\mathrm{d}s\\
&-2\int_0^T e^{-2\gamma(T-s)}[\![U_{g_T}(s,0)u_T]\!]^2\mathrm{d}s. \qquad (4.52)
\end{aligned}$$

(4.51) and (4.52) gives us for any $T>0$ that

$$\begin{aligned}
&\liminf_{n\to\infty}\Big(AU_{g_T^{(n)}}(T,0)u_T^{(n)},U_{g_T^{(n)}}(T,0)u_T^{(n)}\Big)\\
\leq\ & (c_2R_1^2-(Au_T,u_T))e^{-2\gamma T}+\Big(AU_{g_T}(T,0)u_T,U_{g_T}(T,0)u_T\Big). \qquad (4.53)
\end{aligned}$$

Recall that $u_T\in\mathcal{B}_0^V$ is bounded. Lemma 2.1 implies (Au_T,u_T) is bounded. Letting $T\to+\infty$ in (4.53), we have

$$\begin{aligned}
&\liminf_{n\to\infty}\Big(AU_{g_T^{(n)}}(T,0)u_T^{(n)},U_{g_T^{(n)}}(T,0)u_T^{(n)}\Big)\\
\leq\ & \Big(AU_{g_T}(T,0)u_T,U_{g_T}(T,0)u_T\Big)\\
=\ & (Au,u), \qquad (4.54)
\end{aligned}$$

i.e. (4.25) holds. (4.24) and (4.25) immediately imply

$$\lim_{n\to\infty}\left(AU_{g_T^{(n)}}(T,0)u_T^{(n)}, U_{g_T^{(n)}}(T,0)u_T^{(n)}\right) = (Au, u)\,. \tag{4.55}$$

Because V is a Hilbert space, we deduce from (4.17) and (4.55) that

$$\lim_{n\to\infty}\|U_{g_T^{(n)}}(T,0)u_T^{(n)} - u\|_V = \lim_{n\to\infty}\|U_{g^{(n)}}(t_n,\tau)u_\tau^{(n)} - u\|_V = 0.$$

Lemma 4.5 is eventually proved. $\square$

Lemma 4.6 *Let $g_0 \in L_c^2(\mathbb{R}; H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$$

corresponding to problem (2.15)-(2.16) is uniform (w.r.t. $g \in \mathcal{H}(g_0)$) asymptotically compact in space V.

The proof of Lemma 4.6 is almost the same as that as Lemma 3.6 with the small modification that the space V replaces the space H.

Combining Lemmas 4.1, 4.2, 4.6 and Theorem 5.1 of [**7**], we now can state the main result of this section as follows.

Theorem 4.1 *Let $g_0 \in L_c^2(\mathbb{R}; H)$. Then the family of processes*

$$\{U_g(t,\tau)\}_{t\geq\tau, g\in\mathcal{H}(g_0)}$$

possesses a compact uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor $\mathcal{A}^V_{\mathcal{H}(g_0)}$ in space V, which has the following structure

$$\mathcal{A}^V_{\mathcal{H}(g_0)} = \bigcup_{g\in\mathcal{H}(g_0)} \mathcal{K}_g^V(s) = \omega_{s,\mathcal{H}(g_0)}(\mathcal{B}_0^V) = \bigcap_{t\geq\tau} \overline{\bigcup_{g\in\mathcal{H}(g_0)} \bigcup_{s\geq t} U_g(s,\tau)\mathcal{B}_0^V}^{\mathbb{H}^2(\Omega)}, \tag{4.56}$$

for $\forall\, s \in \mathbb{R}$, where $\mathcal{K}_g^V(s)$ is the kernel section at time moment $t = s$, $\mathcal{K}_g^V$ is the kernel of the process $\{U_g(t,\tau)\}_{t\geq\tau}$ in space V and $\mathcal{K}_g^V$ is nonempty for all $g \in \mathcal{H}(g_0)$; $\mathcal{B}_0^V$ is the bounded uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbing set defined by (4.2) and $\omega_{s,\mathcal{H}(g_0)}(\mathcal{B}_0^V)$ is its uniform (w.r.t. $g \in \mathcal{H}(g_0)$) ω-limit set.

5. Regularity of the uniform attractors

The purpose of this section is to prove $\mathcal{A}^H_{\mathcal{H}(g_0)} = \mathcal{A}^V_{\mathcal{H}(g_0)} \subset V$ for inclusion relation. To this end, we utilize the Uniform Gronwall Lemma to establish that the solutions of (2.15)-(2.16) with initial values in any bounded set of H will enter a bounded set of V after large enough time.

Lemma 5.1 (Uniform Gronwall Lemma [**30**]). *Let $\Upsilon(t), \Phi(t), \Psi(t)$ be three positive locally integrable functions on $\mathbb{R}_\tau$ such that Φ' is locally integrable on $\mathbb{R}_\tau$ and*

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \leq \Upsilon(t)\Phi(t) + \Psi(t) \quad \text{for} \quad t \geq \tau,$$

$$\int_t^{t+r} \Upsilon(s)\mathrm{d}s \leq a_1, \quad \int_t^{t+r} \Psi(s)\mathrm{d}s \leq a_2, \quad \int_t^{t+r} \Phi(s)\mathrm{d}s \leq a_3, \quad \text{for } t \geq \tau,$$

where r, a_1, a_2 and a_3 are positive constants. Then

$$\Phi(t+r) \leq \left(\frac{a_3}{r} + a_2\right)e^{a_1}, \quad \forall\, t \geq \tau.$$

We next use Lemma 5.1 to prove the following lemma.

Lemma 5.2. *Let $g_0 \in L_c^2(\mathbb{R}; H)$ and $\mathcal{B} \in \mathcal{B}(H)$ be arbitrary. Let $u(t) = U_g(t,\tau)u_\tau$ be the corresponding solution of problem (2.15)-(2.16) with any given*

$u_\tau \in \mathcal{B}$ and any given $g \in \mathcal{H}(g_0)$. Then there is a time $T_0(\tau, \mathcal{B})$ and a positive constant K such that

$$\|u(t)\|_V = \|U_g(t,\tau)u_\tau\|_V \leq K, \quad \forall\, t \geq T_0(\tau, \mathcal{B}). \tag{5.1}$$

Proof. Multiplying (2.15) by u_t and then integrating the resulting equality over Ω, we obtain

$$\|u_t\|^2 + 2\mu_1 a(u, u_t) + \langle B(u), u_t\rangle + \langle N(u), u_t\rangle = (g, u_t). \tag{5.2}$$

Set

$$\Gamma(|e(u)|^2) = \int_0^{|e(u)|^2} \mu_0(\varepsilon + s)^{-\alpha/2}\mathrm{d}s,$$

then

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 2\sum_{i,j=1}^{2} \mu(u)e_{ij}(u)\frac{\partial e_{ij}(u)}{\partial t} = 2\sum_{i,j=1}^{2} \mu(u)e_{ij}(u)e_{ij}(u_t).$$

Thus,

$$\langle N(u), u_t\rangle = \sum_{i,j=1}^{2}\int_\Omega \mu(u)e_{ij}(u)e_{ij}(u_t)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \Gamma(|e(u)|^2)\mathrm{d}x\right). \tag{5.3}$$

Substituting (5.3) into (5.2), we obtain

$$\begin{aligned}
&\|u_t\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_1 a(u,u) + \frac{1}{2}\int_\Omega \Gamma(|e(u)|^2)\mathrm{d}x\right) \\
= \quad & -\langle B(u), u_t\rangle + (g(t), u_t) \\
\leq \quad & \left|\sum_{i,j=1}^{2}\int_\Omega u_i \frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial t}\mathrm{d}x\right| + \|g(t)\|^2 + \frac{1}{4}\|u_t\|^2 \\
\leq \quad & \|u\|_{\mathbb{L}^4(\Omega)}\|\nabla u\|_{\mathbb{L}^4(\Omega)}\|u_t\| + \|g(t)\|^2 + \frac{1}{4}\|u_t\|^2. \qquad (5.4)
\end{aligned}$$

By the Gagliardo-Nirenberg inequality, we get

$$\|u\|_{\mathbb{L}^4(\Omega)}\|\nabla u\|_{\mathbb{L}^4(\Omega)}\|u_t\| \leq c\|\Delta u\|^2\|u_t\| \leq c\|u\|_V^4 + \frac{1}{4}\|u_t\|^2. \tag{5.5}$$

Inserting (5.5) into (5.4), we have by using Lemma 2.1 that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu_1 a(u,u) + \frac{1}{2}\int_\Omega \Gamma(|e(u)|^2)\mathrm{d}x\right) \quad &\leq \quad c\|u\|_V^4 + \|g(t)\|^2 \\
&\leq \quad c\|u\|_V^2\frac{\mu_1 a(u,u)}{c_1\mu_1} + \|g(t)\|^2. \qquad (5.6)
\end{aligned}$$

So we have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} \leq \Upsilon(t)\Phi(t) + \Psi(t), \tag{5.7}$$

where

$$\begin{aligned}\Phi(t) &= \mu_1 a(u(t),u(t)) + \frac{1}{2}\int_\Omega \Gamma(|e(u)|^2)\mathrm{d}x,\\ \Upsilon(t) &= \frac{c}{c_1\mu_1}\|u(t)\|_V^2, \quad \Psi(t)=\|g(t)\|^2.\end{aligned}$$

Now taking the inner product $(\cdot,\cdot)$ of (2.15) with u and integrating the resulting equality over $[t,t+1]$, we obtain by using the facts $\langle B(u),u\rangle = 0$ and $\langle N(u),u\rangle \geq 0$ that

$$\begin{aligned}\int_t^{t+1} 2\mu_1 a(u(s),u(s))\mathrm{d}s &\leq \int_t^{t+1}\|u(s)\|\|g(s)\|\mathrm{d}s + \frac{1}{2}\|u(t)\|^2\\ &\leq \int_t^{t+1}(\|u(s)\|^2+\|g(s)\|^2)\mathrm{d}s + \frac{1}{2}\|u(t)\|^2.\end{aligned}\tag{5.8}$$

Let $t \geq t_0(\tau,\mathcal{B})$ (see Lemma 3.1), then from (5.8) it follows

$$2c_1\mu_1\int_t^{t+1}\|u(s)\|_V^2\mathrm{d}t \leq \frac{3}{2}R_0^2 + \int_t^{t+1}\|g(s)\|^2\mathrm{d}s.\tag{5.9}$$

Hence, by Lemma 2.1,

$$\begin{aligned}\int_t^{t+1}\Upsilon(s)\mathrm{d}s &= \int_t^{t+1}\frac{c}{c_1\mu_1}\|u(s)\|_V^2\mathrm{d}s \leq \frac{c}{2c_1^2\mu_1^2}\left(\int_t^{t+1}\|g(s)\|^2\mathrm{d}s + \frac{3}{2}R_0^2\right)\\ &\leq \frac{c}{2c_1^2\mu_1^2}\left(\|g_0\|_{L_b^2}^2 + \frac{3}{2}R_0^2\right) \doteq a_1, \quad \forall\, t \geq t_0(\tau,\mathcal{B}).\end{aligned}$$

Clearly, we have

$$\int_t^{t+1}\Psi(s)\mathrm{d}s = \int_t^{t+1}\|g(s)\|^2\mathrm{d}s \leq \|g_0\|_{L_b^2}^2 \doteq a_2. \quad \forall\, t\geq t_0(\tau,\mathcal{B}).\tag{5.10}$$

where a_1, a_2 are positive constants. We next show that there exists a positive constant a_3 such that

$$\int_t^{t+1}\Phi(s)\mathrm{d}s \leq a_3, \quad \forall\, t \geq t_0(\tau,\mathcal{B}).$$

From (5.9) and Lemma 2.1 we obtain

$$\begin{aligned}\int_t^{t+1}\mu_1 a(u(s),u(s))\mathrm{d}s &\leq \frac{c_2}{2c_1}\left(\int_t^{t+1}\|g(s)\|^2\mathrm{d}s + \frac{3}{2}R_0^2\right)\\ &\leq \frac{c_2}{2c_1}(\|g_0\|_{L_b^2}^2 + \frac{3}{2}R_0^2), \forall\, t\geq t_0(\tau,\mathcal{B}).\end{aligned}\tag{5.11}$$

At the same time, we have $0 < (\varepsilon + s)^{-\alpha/2} \le \varepsilon^{-\alpha/2}$ with $s \ge 0$ and $0 < \alpha < 1$. Thus

$$\Gamma(|e(u)|^2) = \int_0^{|e(u)|^2} \mu_0(\varepsilon + s)^{-\alpha/2} \mathrm{d}s \le \mu_0 \varepsilon^{-\alpha/2} |e(u)|^2,$$

from which and (5.11) we get

$$\begin{aligned}
&\int_t^{t+1}\int_\Omega \Gamma(|e(u)|^2)\mathrm{d}x\mathrm{d}s \\
\le\ & \mu_0\varepsilon^{-\alpha/2}\int_t^{t+1}\int_\Omega |e(u)|^2 \mathrm{d}x\mathrm{d}s \le 4\mu_0\varepsilon^{-\alpha/2}\int_t^{t+1}\|u(s)\|_V^2\mathrm{d}s \\
\le\ & 4\mu_0\varepsilon^{-\alpha/2}\int_t^{t+1}\frac{\mu_1 a(u(s),u(s))}{c_1\mu_1}\mathrm{d}s \\
\le\ & \frac{2c_2\mu_0\varepsilon^{-\alpha/2}}{c_1^2\mu_1}(\|g_0\|_{L_b^2}^2 + \frac{3}{2}R_0^2), \forall\, t \ge t_0(\tau, \mathcal{B}),
\end{aligned} \tag{5.12}$$

where we have used the fact that $\int_\Omega |e(u)|^2\mathrm{d}x \le 4\|u\|_V^2$. It follows from (5.11) and (5.12) that

$$\int_t^{t+1}\Phi(s)\mathrm{d}s \le \left(\frac{c_2}{2c_1} + \frac{2c_2\mu_0\varepsilon^{-\alpha/2}}{c_1^2\mu_1}\right)\left(\|g_0\|_{L_b^2}^2 + \frac{3}{2}R_0^2\right) \doteq a_3, \forall\, t \ge t_0(\tau, \mathcal{B}). \tag{5.13}$$

Taking Lemma 5.1, (5.7), (5.10), (5.11) and (5.13) into account, we obtain

$$\Phi(t) \le (a_3 + a_2)e^{a_1}, \qquad \forall\, t \ge t_0(\tau, \mathcal{B}) + 1.$$

Therefore,

$$\|u(t)\|_V^2 \le \frac{1}{c_1}a(u(t),u(t)) \le \frac{\Phi(t)}{c_1\mu_1} \le \frac{(a_3+a_2)e^{a_1}}{c_1\mu_1} \doteq K, \quad t \ge t_0(\tau, \mathcal{B}) + 1.$$

The proof of Lemma 5.2 is complete. □

Theorem 5.1. *Let $g_0 \in L_c^2(\mathbb{R}; H)$, then*

$$\mathcal{A}_{\mathcal{H}(g_0)}^H = \mathcal{A}_{\mathcal{H}(g_0)}^V \subset V. \tag{5.14}$$

Proof. On the one hand, $\mathcal{A}_{\mathcal{H}(g_0)}^V \subset V \hookrightarrow H$ is clear, and by Lemma 4.1 we see that $\mathcal{B}_{\mathcal{H}(g_0)}^V$ uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) absorbs any bounded sets of H in the norm of V (also in the norm of H). Thus $\mathcal{A}_{\mathcal{H}(g_0)}^V$ can be regarded as the uniform (w.r.t. $g \in \mathcal{H}(g_0)$) attractor for the family of processes $\{U_g(t,\tau)\}_{t\ge\tau, g\in\mathcal{H}(g_0)}$ in space H. By the minimality and thus the uniqueness of the uniform attractor, we get $\mathcal{A}_{\mathcal{H}(g_0)}^V = \mathcal{A}_{\mathcal{H}(g_0)}^H \subset H$.

On the other hand, the kernel $\mathcal{K}_g^H$ of the process $\{U_g(t,\tau)\}_{t\ge\tau}$ in space H consists of all bounded complete trajectories of equation (2.15) with time symbol $g \in \mathcal{H}(g_0)$. In fact,

$$\mathcal{K}_g^H = \{u(\cdot) : u(t) = U_g(t,\tau)u(\tau), \mathrm{dist}_H(u(t), u(0)) \le C_u, \forall t \ge \tau, \forall\, \tau \in \mathbb{R}\}.$$

From Lemma 5.2, we see that $\mathcal{A}^H_{\mathcal{H}(g_0)}$ is indeed a bounded set of V. Therefore, we get $\mathcal{A}^V_{\mathcal{H}(g_0)} = \mathcal{A}^H_{\mathcal{H}(g_0)} \subset V$. The proof is complete. $\square$

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