Dynamics of PDE, Vol.10, No.3, 255-265, 2013

Global wellposedness of an inviscid $2D$ Boussinesq system with nonlinear thermal diffusivity

Dong Li and Xiaojing Xu

Communicated by Y. Charles Li, received February 25, 2013.

Abstract. We consider a two-dimensional inviscid Boussinesq system with temperature-dependent thermal diffusivity. We prove global wellposedness of strong solutions for arbitrarily large initial data in Sobolev spaces.

Contents

1. Introduction

In this paper we consider the following two dimensional inviscid Boussinesq system

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \theta e_2, \quad e_2 = (0,1), \\ \partial_t \theta + (u\cdot\nabla)\theta = \nabla\cdot(\kappa(\theta)\,\nabla\theta), \quad (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \nabla\cdot u = 0, \\ u(0,x) = u_0, \quad \theta(0,x) = \theta_0(x), \quad x\in\mathbb{R}^2, \end{cases} \tag{1.1}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x)) : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2$, denotes the velocity field of a two-dimensional incompressible fluid. The term $p = p(t,x) : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2$ denotes the usual pressure which can be recovered from the first and the third equation in (1.1) by taking the divergence and then inverting the Laplacian operator. The scalar function $\theta = \theta(t,x)$ quantifies the temperature variation in a gravity field. It enters the first equation in (1.1) as $\theta\, e_2$ which represents the

1991 *Mathematics Subject Classification.* 35, 76.
Key words and phrases. Boussinesq system, global wellposedness, Sobolev spaces.

buoyancy force. In the second equation of (1.1), the parameter $\kappa = \kappa(\theta)$ represents the thermal diffusivity, we shall assume κ is a smooth function satisfying

$$\frac{1}{C_0} \leq \kappa(z) \leq C_0, \qquad \text{for any} \quad z \in \mathbb{R},$$

where $C_0 > 0$ is some fixed positive constant.

The main purpose of this note is to investigate global wellposedness of smooth solutions to (1.1) for arbitrarily large initial data.

The system (1.1) is a special case of the more general Boussinesq system which takes the form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = \nabla \cdot (\nu(\theta) \nabla u) + \theta e_2, \quad e_2 = (0,\, 1), \\ \partial_t \theta + (u \cdot \nabla)\theta = \nabla \cdot (\kappa(\theta)\, \nabla \theta), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ \nabla \cdot u = 0, \\ u(0,\, x) = u_0, \quad \theta(0,\, x) = \theta_0(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{1.2}$$

here $\nu \geq 0$ represents the viscosity which is allowed to depend on the temperature in general.

There is by now an extensive literature on various cases of the general system (1.1). If ν and κ are positive constants which do not depend the temperature, then global wellposedness in 2-D can be established by classical methods (see Cannon and DiBenedetto [**3**]). When both ν and κ depend on the temperature and satisfy the constraint

$$\frac{1}{C_0} \leq \nu(z) \leq C_0, \quad \frac{1}{C_0} \leq \kappa(z) \leq C_0, \qquad \textit{for any} \quad z \in \mathbb{R}, \tag{1.3}$$

for some $C_0 > 0$, Lorca and Boldrini [**13**] obtained the global wellposedness for small initial data. For the partially viscous Boussinesq system (1.2) (with constant viscosity), that is,

$$\nu > 0 \quad \textit{is a positive constant}, \qquad \kappa = 0;$$

or

$$\kappa > 0 \quad \textit{is a positive constant}, \qquad \nu = 0,$$

Chae [**4**] and Hou-Li [**8**] independently settled global regularity for large initial data. In both works, a key observation is the use of the following Brezis-Wainger inequality [**2**] (to control $\|\nabla\theta\|_{L^p} + \|\nabla^2 u\|_{L^p}$, for any $p > 2$)

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + \|\nabla f\|_{L^2(\mathbb{R}^2)})[1 + \log(10 + \|\nabla f\|_{L^p})]^{\frac{1}{2}} + C\|f\|_{L^2(\mathbb{R}^2)}, \tag{1.4}$$

for $f \in L^2(\mathbb{R}^2) \cap W^{1,p}(\mathbb{R}^2)$, $p > 2$. [1] In [**10**], Lai, Pan and Zhao settled the solvability of the initial-boundary value problem for the two-dimensional viscous Boussinesq equations in a bounded domain. In [**9**], Karch and Prioux constructed a nontrival family of self-similar solutions to the two-dimensional viscous Boussinesq system.

In a series of two papers [**6**] & [**7**], Hmidi, Keraani and Rousset used a novel diagonalization approach and proved global wellposedness of fractional diffusion

[1] The inequality (1.4) can be easily proved using Littlewood-Paley decomposition and the L^2 -summability of $\|\nabla P_N f\|_{L^2}$.

Boussinesq models with critical partial dissipation in either velocity equation or temperature equation, i.e.

$$(1.5)\qquad \begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \theta e_2, \quad e_2 = (0,\,1), \\ \partial_t \theta + (u\cdot\nabla)\theta + |\nabla|\theta = 0, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ \nabla\cdot u = 0, \\ u(0,\,x) = u_0, \quad \theta(0,\,x) = \theta_0(x), \quad x\in\mathbb{R}^2, \end{cases}$$

or

$$(1.6)\qquad \begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p + |\nabla| u = \theta e_2, \quad e_2 = (0,\,1), \\ \partial_t \theta + (u\cdot\nabla)\theta = 0, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ \nabla\cdot u = 0, \\ u(0,\,x) = u_0, \quad \theta(0,\,x) = \theta_0(x), \quad x\in\mathbb{R}^2. \end{cases}$$

Here for any $s>0$, $|\nabla|^s = (-\Delta)^{\frac{s}{2}}$ is the fractional Laplacian operator defined by

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi), \qquad \xi\in\mathbb{R}^2.$$

Note that both systems (1.5)–(1.6) are much more degenerate than the partially viscous Boussinesq system with full Laplacian dissipation considered by Chae [**4**] and Hou-Li[**8**]. The deep observation of Hmidi-Keraani-Rousset is to utilize a maximum-principle type structure hidden in (1.5) (resp. (1.6)). Namely, Let the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ and one gets from (1.5) the system

$$\begin{cases} \partial_t w = \partial_1\theta, \\ \partial_t\theta = -|\nabla|\theta. \end{cases}$$

Diagonalizing the above system gives the equation

$$\partial_t(\omega + |\nabla|^{-1}\partial_1\theta) = 0$$

from which one can derive new a priori estimates. At present the super-critical case where $|\nabla|$ is replaced by $|\nabla|^s$, $s<1$ in (1.5) and (1.6) is still open. For the Boussinesq system with partial vertical dissipation, we refer the interested readers to [**1**] and references therein.

In a recent paper [**14**], Wang and Zhang considered the general system (1.2) with non-degenerate viscosity and thermal diffusivity (i.e. (1.3) holds). Their main result reads as follows:

Theorem 1.1. *Let $s>2$ and $(u_0,\,\theta_0)\in H^s(\mathbb{R}^2)$. Then the Boussinesq system (1.2)-(1.3) has a unique global solution $(u,\,\theta)\in C_t^0 H^s \cap L^2_{t,\,loc} H^{s+1}([0,\,\infty)\times\mathbb{R}^2)$.*

To prove Theorem 1.1, it suffices to establish the a priori estimate

$$(1.7)\qquad \int_0^T \Big(\|\nabla u(t)\|_\infty^2 + \|\nabla\theta(t)\|_\infty^2\Big)\,dt < \infty, \quad for\ any\ T>0.$$

By performing an L^2-estimate on (1.2) and using the fact that the viscosity is non-degenerate, one obtains $u\in C_t^0 L_x^2 \cap L_t^2 H^1([0,\,\infty)\times\mathbb{R}^2)$. Interpolation then gives $u\in L^4_{t,\,x}([0,\,T]\times\mathbb{R}^2)$. The key argument in Wang-Zhang [**14**] is the following Hölder estimate on a (linear) transport equation:

PROPOSITION 1.2. *Let* $u \in L^4_{t,x}([0, T] \times \mathbb{R}^2)$ *be a given divergence-free vector field. Assume* $\theta \in L^\infty_t L^2_x \cap L^2_t H^1([0, T] \times \mathbb{R}^2)$ *is a weak solution of the linear equation*

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta = \nabla \cdot (\kappa \nabla \theta), \\ \theta(0, x) = \theta_0(x). \end{cases}$$

Then there exists $\alpha > 0$ *such that* $\theta \in C^\alpha((0, T] \times \mathbb{R}^2)$ *with*

$$\|\theta\|_{L^\infty_t(\delta, T; C^\alpha(\mathbb{R}^2))} \leq C(\delta, \|u\|_{L^4_{t,x}((0,T)\times\mathbb{R}^2)}, \|\theta_0\|_{L^2(\mathbb{R}^2)}),$$

for any $\delta \in (0, T)$.

The main advantage of Proposition 1.2 is that one can obtain the a priori control of Hölder norm of θ which is in some sense a super-critical quantity. Using this C^α estimate[2], one can then[3] perform a simple H^1 estimate of (u, θ) and derive

$$\|(u, \theta)\|_{L^\infty_t H^1([0,T]\times\mathbb{R}^2)} + \|(\Delta u, \Delta\theta)\|_{L^2_{tx}([0,T]\times\mathbb{R}^2)} < \infty,$$

for any $T > 0$. We should point it out that in this H^1 estimate, we utilized the following identity which only holds for smooth two-dimensional incompressible flows:

$$\int_{\mathbb{R}^2} [(u \cdot \nabla)u] \cdot \Delta u dx = 0. \tag{1.8}$$

The identity (1.8) can be easily proved using the vorticity formulation, see (3.6). Bootstrapping the H^1-estimates then easily yields global wellposedness. The details of the above simplified argument is given in Section 3.

Besides giving a simplified proof of Theorem 1.1, the main objective of this paper is to prove global wellposedness of strong solutions with large initial data for the degenerate system (1.1), i.e. the velocity equation has no dissipation on the right hand side. In some sense this complements the analysis of Wang-Zhang. Our main result is the following

THEOREM 1.3 (Global regularity). *Let the initial data* $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ *for some* $s > 2$. *Then the Boussinesq system* (1.1) *has a unique global solution satisfying* $(u, \theta) \in C^0_t H^s([0, \infty) \times \mathbb{R}^2)$, $\theta \in L^2_{t,loc} H^{s+1}([0, T] \times \mathbb{R}^2)$ *for any* $0 < T < \infty$.

The proof of Theorem 1.3 relies heavily on certain parabolic estimates on the temperature θ. To derive the bound (1.7), we carry out the estimate on a carefully chosen quantity $\|\nabla\omega\|_{2+\epsilon} + \|\nabla\theta\|_2$. The advantage of this approach is that one can avoid completely dyadic-type estimates which is unnecessary to the matter. Alternatively one could give a proof using Hmidi-Keraani-Rousset's diagonalization approach. We leave the details to interested readers.

---

[2] For example in the nonlinear estimates, instead of the usual interpolation inequality $\|\nabla\theta\|_{L^4(\mathbb{R}^2)} \lesssim \|\theta\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \|\Delta\theta\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}$, one can use $\|\nabla\theta\|_{L^4(\mathbb{R}^2)} \lesssim \|\theta\|_2^{2\epsilon} \cdot \|\theta\|^{\frac{1}{2}-\epsilon}_{C^\alpha(\mathbb{R}^2)} \cdot \|\Delta\theta\|^{\frac{1}{2}-\epsilon}_{L^2(\mathbb{R}^2)}$ for some $\epsilon > 0$. See (3.4) for the precise form and (3.3) where it is used. This is one of the key steps to close the H^1 estimate.

[3] The analysis in [**14**] proceeds in a different manner and is much more involved. Namely the authors first prove the estimate $\|\nabla\theta\|_{L^2_t L^\infty_x} \lesssim 1$ by working directly with the temperature equation using the fact that $u \in L^2_t H^1$. This is done by a Littlewood-Paley analysis using a Chemin-type space. See Proposition 5.1 therein for more details.

The rest of this article is organized as follows. In the following section we give the proof of Theorem 1.3. In the last section we give the details of the simplified proof of Theorem 1.1.

Notations and Preliminaries.

- For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some harmless constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim_{Z_1, Z_2, \cdots, Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $(Z_1, \cdots, Z_k)$. Similarly we define $\gtrsim_{Z_1,\cdots,Z_k}$ and $\sim_{Z_1,\cdots,Z_k}$.
- Let Ω be an open set in $\mathbb{R}^d$. For any $1 \le p \le \infty$ we use $\|f\|_p$ (when there is no confusion), $\|f\|_{L^p(\Omega)}$, or $\|f\|_{L^p_x(\Omega)}$ to denote the Lebesgue norm on Ω. We write $f \in L^p_{loc}(\Omega)$ if $f \in L^p(K)$ for any compact $K \subset \Omega$. The Sobolev space $H^1(\mathbb{R}^d)$ is defined in the usual way as the completion of C_c^∞ functions under the norm $\|f\|_{H^1} = \|f\|_2 + \|\nabla f\|_2$. For any $s \in \mathbb{R}$, we define the homogeneous Sobolev norm
$$\|f\|_{\dot{H}^s} = \Big(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$
For any $0 < \alpha \le 1$, the Hölder norm $\|\cdot\|_{C^\alpha}$ is defined as
$$\|f\|_{C^\alpha(\mathbb{R}^d)} := \|f\|_\infty + \sup_{x \ne y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$
- We will occasionally need to use the Littlewood–Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \le 2$ and equal to one on the ball $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood–Paley operators
$$\begin{aligned} \widehat{P_{\le N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\ \widehat{P_{> N} f}(\xi) &:= [1 - \varphi(\xi/N)]\hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi). \end{aligned}$$
Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers.
- We recall the following Bernstein estimates: for any $1 \le p \le q \le \infty$ and dyadic $N > 0$,
$$\|P_N f\|_{L^q_x(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p_x(\mathbb{R}^d)}.$$
Similar inequalities also hold when P_N is replaced by $P_{<N}$ or $P_{\le N}$.
- We recall the following logarithmic Sobolev interpolation inequality:

  For any $f \in H^1(\mathbb{R}^2)$ with $\nabla f \in L^p(\mathbb{R}^2)$ for some $2 < p < \infty$, we have
$$\sum_{i,j=1}^2 \|\Delta^{-1}\partial_i\partial_j f\|_\infty \lesssim \|f\|_2 + \|f\|_\infty \log(10 + \|\nabla f\|_p). \tag{1.9}$$
  We sketch the proof of (1.9) here for the sake of completeness.

  Proof of (1.9). Let $N_0 \ge 4$ be a dyadic number whose value will be chosen later. Splitting the function f into low and high frequencies and using the Bernstein inequality, we have

$$
\begin{aligned}
\text{LHS of (1.9)} &\lesssim \|P_{<1}f\|_2 + \sum_{1<N<N_0} \|P_N f\|_\infty + \sum_{N>N_0} \|P_N f\|_\infty \\
&\lesssim \|f\|_2 + (\log N_0)\|f\|_\infty + \sum_{N>N_0} N^{-1+\frac{2}{p}} \|P_N \nabla f\|_p \\
&\lesssim \|f\|_2 + (\log N_0)\|f\|_\infty + N_0^{-1+\frac{2}{p}} \|\nabla f\|_p.
\end{aligned}
$$

Choosing $N_0 \sim (10 + \|\nabla f\|_p)^{\frac{p}{p-2}}$ then yields the result. □

Acknowledgements. D. Li was supported in part by NSF under agreement No. DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. D. Li was also supported in part by an NSERC discovery grant. X. Xu was partially supported by NSFC(No.11171026) and BNSF(No.2112023).

2. Proof of Theorem 1.3

We begin with the local wellposedness and blowup/continuation criteria. Albeit standard, we state it here for the sake of completeness.

Proposition 2.1. *Let the initial data $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$, $s > 2$. Then there exists unique local solution such that $(u, \theta) \in C_t^0 H^s$, $\theta \in L^2_{t,loc} H^{s+1}$. Let the maximal lifespan of (u, θ) be T^*. If $T^* < \infty$, then*

$$
\lim_{t \nearrow T^*} \int_0^t \Big(\|\nabla u(s)\|_{L^\infty}^2 + \|\nabla \theta(s)\|_{L^\infty}^2 \Big) ds = +\infty.
$$

We shall omit the proof of Proposition 2.1 since it is a simple exercise of the standard energy method. One can refer to Theorem 3.1 in [**14**] or [**5**], [**11**] for the construction of local solutions by various approximation schemes.

With Proposition 2.1 in hand, we are now ready to complete the

Proof of Theorem 1.3. By Proposition 2.1, we only need to control the quantity

$$
\int_0^T \Big(\|\nabla u(t)\|_{L^\infty}^2 + \|\nabla \theta(t)\|_{L^\infty}^2 \Big) dt \tag{2.1}
$$

for any $T > 0$.

To simplify the notations, we shall use the letter C to denote a generic constant which may vary from line to line. The dependence of C on other parameters is usually clear from the context and we shall explicitly specify it whenever necessary. For any quantity $X = X(t)$, we shall write

$$
X(t) \lesssim 1 \quad \text{or } X \lesssim 1 \tag{2.2}
$$

if

$$
X(t) \le C(t, \theta_0, u_0) < \infty, \quad \forall\, t \ge 0.
$$

Here $C(t, \theta_0, u_0)$ is some constant depending only on the initial data (θ_0, u_0), the elliptic constant (in $\kappa = \kappa(\theta)$) and the time t. It is of course possible to keep track of the constants and obtain explicit growth rate of various Sobolev norms of (θ, u). But we shall not dwell on this issue here for simplicity.

Let the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. Taking the curl on the first equation of (1.1), we obtain

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = \theta_{x_1}, \\ \partial_t \theta + (u \cdot \nabla)\theta = \nabla \cdot (\kappa \nabla \theta). \end{cases} \tag{2.3}$$

An L^2-estimate on (ω, θ) using $\mathrm{div}(u) = 0$ then gives us

$$\begin{aligned} \frac{d}{dt}\Big(\|\omega(t)\|_2^2 + \|\theta(t)\|_2^2\Big) &= \int_{\mathbb{R}^2} \theta_{x_1} \omega dx + \int_{\mathbb{R}^2} \nabla \cdot (\kappa \nabla \theta)\theta dx \\ &\le \frac{1}{100C_0}\|\theta_{x_1}(t)\|_2^2 + 100C_0\|\omega(t)\|_2^2 - \frac{1}{C_0}\|\nabla\theta(t)\|_2^2 \\ &\le 100C_0\|\omega(t)\|_2^2. \end{aligned}$$

Therefore we have

$$\|\omega(t)\|_2^2 + \|\theta(t)\|_2^2 \lesssim 1, \quad \forall\, t \ge 0. \tag{2.4}$$

Consequently

$$\|u(t)\|_{H^1} \lesssim 1, \quad \forall\, t \ge 0. \tag{2.5}$$

Now using (2.5) and a standard parabolic estimate on the second equation in (2.3) (cf. Theorem 4.8 of [**12**][4]), we have for some $0 < \alpha < 1$,

$$\|\nabla\theta\|_{L_t^\infty C_x^\alpha([\delta,T]\times\mathbb{R}^2)} \le C(\delta, u_0, \theta_0) < \infty,$$

where $0 < \delta < T$ is arbitrary. Consequently

$$\|\nabla\theta(t)\|_{C_x^\alpha} \lesssim 1, \quad \forall\, t \ge 0. \tag{2.6}$$

Hence we have already settled the $\|\nabla\theta(t)\|_\infty$-part in (2.1). It remains to control $\|\nabla u(t)\|_\infty$. Plugging the estimate (2.6) into the first equation of (2.3), we obtain

$$\|\omega(t)\|_{L_x^\infty} \lesssim 1, \quad \forall\, t \ge 0. \tag{2.7}$$

Therefore for any $2 < p < \infty$, we have

$$\|\nabla u(t)\|_{L_x^p} \lesssim 1, \quad \forall\, t \ge 0. \tag{2.8}$$

Now let $g = D\theta$. Differentiating the second equation of (2.3), we obtain

$$\partial_t g + (u \cdot \nabla) g = \nabla \cdot (\kappa \nabla g) - (Du \cdot \nabla)\theta + \nabla \cdot (\kappa'(\theta)(D\theta)\nabla\theta). \tag{2.9}$$

By using (2.6), (2.8) and a parabolic estimate on (2.9), we then obtain

$$\|g\|_{L_t^\infty C_x^\alpha([\delta,T]\times\mathbb{R}^2)} \le C(\delta, u_0, \theta_0) < \infty, \quad \forall\, 0 < \delta < T,$$

and consequently

$$\sum_{i,j=1}^{2} \|\partial_i \partial_j \theta(t)\|_{C_x^\alpha} \lesssim 1, \quad \forall\, t \ge 0. \tag{2.10}$$

[4]Strictly speaking, to check the hypothesis of Theorem 4.8 therein, one needs to perform a standard C^α-estimate first since the thermal diffusivity $\kappa = \kappa(\theta)$. For this one can invoke the standard parabolic theory since the velocity u satisfies the strong estimate (2.5).

Now take some $p = 2 + \epsilon$, $\epsilon > 0$ is sufficiently small such that $H^{s-1} \hookrightarrow W^{1,p}$ (here $s > 2$ is the same exponent as in Theorem 1.3). Using (2.10) and the first equation in (2.3), we estimate $\|\nabla\omega(t)\|_p$ as:

$$\begin{aligned}\frac{d}{dt}\Big(\|\nabla\omega(t)\|_p^p\Big) &\lesssim \|\nabla u(t)\|_\infty \cdot \|\nabla\omega(t)\|_p^p + \int_{\mathbb{R}^2} |\nabla\theta_{x_1}||\nabla\omega|^{p-1} dx \\ &\lesssim (\|\nabla u(t)\|_\infty + 1)\|\nabla\omega(t)\|_p^p + \|\Delta\theta(t)\|_p^p. \qquad (2.11)\end{aligned}$$

To estimate $\|\Delta\theta(t)\|_p$ we proceed as follows. Let $R > 0$ be a dyadic number which will be taken to be sufficiently large. Obviously

$$\begin{aligned}\|\Delta P_{\le R}\theta(t)\|_p &\lesssim R^{2+2(\frac{1}{2}-\frac{1}{p})}\|\theta(t)\|_2 \\ &\lesssim_R 1.\end{aligned}$$

On the other hand by (2.10),

$$\begin{aligned}\|\Delta P_{>R}\theta(t)\|_p &\le \|\Delta P_{>R}\theta(t)\|_2^{\frac{2}{p}} \cdot \|\Delta P_{>R}\theta(t)\|_\infty^{\frac{p-2}{p}} \\ &\lesssim \|\Delta\theta(t)\|_2^{\frac{2}{p}} \cdot R^{-\alpha(\frac{p-2}{p})}\|\nabla^2\theta(t)\|_{C^\alpha}^{\frac{p-2}{p}} \\ &\le \Big(\frac{1}{100C_0}\Big)^{\frac{1}{p}} \cdot \|\Delta\theta(t)\|_2^{\frac{2}{p}},\end{aligned}$$

where in the last inequality we need to take R sufficiently large.

Plugging the last two estimates into (2.11), we obtain

$$\frac{d}{dt}\Big(\|\nabla\omega(t)\|_p^p\Big) \le C \cdot (\|\nabla u(t)\|_\infty + 1)\|\nabla\omega(t)\|_p^p + \frac{1}{100C_0}\|\Delta\theta(t)\|_2^2 + C. \qquad (2.12)$$

Now we perform an H^1-estimate on the temperature θ in the second equation of (2.3), and we get

$$\begin{aligned}\frac{d}{dt}\Big(\|\nabla\theta(t)\|_2^2\Big) &\le (\|\nabla u(t)\|_\infty + 1)\|\nabla\theta(t)\|_2^2 - \frac{1}{C_0}\|\Delta\theta(t)\|_2^2 \\ &\quad + \int_{\mathbb{R}^2} |\kappa'(\theta)| \cdot |\nabla\theta|^2 \cdot |\Delta\theta| dx \\ &\le (\|\nabla u(t)\|_\infty + 1)\|\nabla\theta(t)\|_2^2 - \frac{1}{2C_0}\|\Delta\theta(t)\|_2^2 + C\|\nabla\theta(t)\|_4^4. \qquad (2.13)\end{aligned}$$

By interpolation and (2.6), we have

$$\begin{aligned}\|\nabla\theta(t)\|_4^4 &\lesssim \|\nabla\theta(t)\|_2^2 \cdot \|\nabla\theta(t)\|_\infty^2 \\ &\lesssim \|\nabla\theta(t)\|_2^2, \quad \forall\, t \ge 0. \qquad (2.14)\end{aligned}$$

By the usual logarithmic Sobolev interpolation inequality (1.9) (note here that $p > 2$!), (2.4) and (2.7), we have

$$\begin{aligned}\|\nabla u(t)\|_\infty &\lesssim \|\omega(t)\|_\infty \cdot \log(\|\nabla\omega(t)\|_p + 10) + \|\omega(t)\|_2 \\ &\lesssim \log(\|\nabla\omega(t)\|_p^p + 10). \qquad (2.15)\end{aligned}$$

Now adding together (2.12), (2.13) and using (2.14)–(2.15), we obtain

$$\frac{d}{dt}X(t) \lesssim X(t)\log X(t), \qquad \forall\, t \ge 0,$$

where

$$X(t) = \|\nabla\omega(t)\|_p^p + \|\nabla\theta(t)\|_2^2 + 10.$$

A log-Gronwall argument then easily yields

$$\|\nabla\omega(t)\|_p^p + \|\nabla\theta(t)\|_2^2 \lesssim 1, \quad \forall\, t \ge 0.$$

Since $p > 2$, Sobolev embedding then gives (here $\nabla^\perp = (-\partial_2, \partial_1)$)

$$\begin{aligned} \|\nabla u(t)\|_\infty &= \|\Delta^{-1}\nabla^\perp\nabla\omega(t)\|_\infty \\ &\lesssim \|\nabla\omega(t)\|_p + \|\omega(t)\|_2 \\ &\lesssim 1, \quad \forall\, t \ge 0. \end{aligned}$$

This concludes the estimate of $\|\nabla u(t)\|_\infty$. □

3. A Simplified Proof of Theorem 1.1

We shall adopt the same notation "$\lesssim$" as in the proof of Theorem 1.3 (See (2.2)).

By Proposition 1.2, we have

$$\|\theta(t)\|_{C^\alpha(\mathbb{R}^2)} \lesssim 1, \quad \text{for any} \quad t \ge 0. \tag{3.1}$$

Multiplying the second equation in (1.2) by $-\Delta\theta$ and integrating by parts, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\|\nabla\theta(t)\|_2^2\right) &\le C\int_{\mathbb{R}^2} |u|\,|\nabla\theta|\,|\Delta\theta|\,dx - \frac{1}{C_0}\|\Delta\theta\|_2^2 + C\int_{\mathbb{R}^2} |\nabla\theta|^2|\,|\Delta\theta|\,dx \\ &\le C\|u\|_4^4 + C\|\nabla\theta\|_4^4 - \frac{1}{100C_0}\|\Delta\theta\|_2^2. \end{aligned} \tag{3.2}$$

Let $N_0 \ge 1$ be a parameter whose value will be chosen momentarily. By splitting into low and high frequencies, the Bernstein inequality, (3.1) and the Hölder inequality, we have

$$\begin{aligned} \|\nabla\theta\|_4 &\lesssim \|P_{<N_0}\theta\|_4 + \sum_{N\ge N_0} N\|P_N\theta\|_2^{\frac12}\|P_N\theta\|_\infty^{\frac12} \\ &\lesssim N_0^{\frac12}\|\theta\|_2 + (\|\Delta\theta\|_2\|\theta\|_{C^\alpha})^{\frac12}\sum_{N\ge N_0} N^{-\frac{\alpha}{2}} \\ &\lesssim N_0^{\frac12}\|\theta\|_2 + (\|\Delta\theta\|_2\|\theta\|_{C^\alpha})^{\frac12} N_0^{-\frac{\alpha}{2}}. \end{aligned}$$

By choosing N_0 sufficiently large, we then obtain

$$\|\nabla\theta(t)\|_4^4 \le C + \frac{1}{200C_0}\|\Delta\theta\|_2^2. \tag{3.3}$$

In fact, it is not difficult to check that the calculation preceding (3.3) gives a proof of the inequality

$$\|\nabla\theta\|_4 \lesssim \|\theta\|_{C^\alpha}^{\frac{1}{2(1+\alpha)}}\|\Delta\theta\|_2^{\frac{1}{2(1+\alpha)}}\|\theta\|_2^{\frac{\alpha}{1+\alpha}}. \tag{3.4}$$

It is also possible to replace the RHS norms by weaker Besov norms but we shall not need this generality here.

Plugging (3.3) into (3.2) and using the fact that $\|u\|_{L^4_{t,x}([0,T]\times\mathbb{R}^2)} \lesssim 1$ for any $T > 0$, we obtain

$$\|\theta(t)\|_{L_t^\infty H^1([0,T]\times\mathbb{R}^2)} + \|\nabla\theta\|_{L^4_{t,x}([0,T]\times\mathbb{R}^2)} + \|\Delta\theta\|_{L^2_{t,x}([0,T]\times\mathbb{R}^2)} \lesssim 1, \quad \text{for all} \quad T > 0. \tag{3.5}$$

Now multiplying the first equation in (1.2) by $-\Delta u$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u(t)\|_2^2) \le \int_{\mathbb{R}^2} [(u\cdot\nabla)u]\cdot\Delta u\,dx - \int_{\mathbb{R}^2} \nu(\theta)|\Delta u|^2\,dx + \int_{\mathbb{R}^2} |\nu'(\theta)|\,|\nabla\theta|\,|\nabla u|\,|\Delta u|\,dx.$$

Note that in $2D$, by using the fact that $\Delta u = \nabla^\perp\omega$ (recall $\nabla^\perp = (-\partial_2, \partial_1)$) and

$$\nabla^\perp \cdot ((u\cdot\nabla)u) = (u\cdot\nabla)\omega,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^2} [(u\cdot\nabla)u]\cdot\Delta u\,dx &= \int_{\mathbb{R}^2} [(u\cdot\nabla)u]\cdot\nabla^\perp\omega\,dx\\ &= -\int_{\mathbb{R}^2} (u\cdot\nabla)\omega\cdot\omega\,dx\\ &= -\int_{\mathbb{R}^2} (u\cdot\nabla)(\frac{1}{2}\omega^2)\,dx = 0. \end{aligned}\tag{3.6}$$

By the above fact and the interpolation inequality $\|\nabla u\|_4 \lesssim \|\nabla u\|_2^{\frac{1}{2}}\|\Delta u\|_2^{\frac{1}{2}}$, we have

$$\begin{aligned}\frac{d}{dt}(\|\nabla u(t)\|_2^2) \le& -\frac{1}{2C_0}\|\Delta u(t)\|_2^2 + C\int_{\mathbb{R}^2} |\nabla\theta(t,x)|^2|\nabla u(t,x)|^2\,dx\\ \le& -\frac{1}{2C_0}\|\Delta u(t)\|_2^2 + C\|\nabla\theta(t)\|_4^2\|\nabla u(t)\|_2\|\Delta u(t)\|_2\\ \le& -\frac{1}{4C_0}\|\Delta u(t)\|_2^2 + C\|\nabla\theta(t)\|_4^4\|\nabla u(t)\|_2^2.\end{aligned}\tag{3.7}$$

By (3.5) and integrating (3.7) in time, we obtain

$$\|\nabla u\|_{L_t^\infty L_x^2([0,T]\times\mathbb{R}^2)} + \|\Delta u\|_{L_{t,x}^2([0,T]\times\mathbb{R}^2)} \lesssim 1, \quad \text{for any} \quad T>0.\tag{3.8}$$

The strong estimate (3.8) is enough to yield the $\|\nabla\theta\|_{C^\alpha(\mathbb{R}^2)}$ estimate. The rest of the proof now proceeds in a very similar spirit as in the proof of Theorem 1.3. We omit further details.

References

[1] D. Adhikari, C. Cao, J. Wu, *Global regularity results for the 2D Boussinesq equations with vertical dissipation.* J. Differential Equations 251 (2011), no. 6, 1637–1655.

[2] H. Brezis, S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Comm. Partial Differential Equations 5 (7) (1980) 773–789.

[3] J. R. Cannon, E. DiBenedetto, *The initial problem for the Boussinesq equations with data in L^p*, Lecture Note in Mathematics, 771 (1980), 129–144.

[4] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Advances in Mathematics, 203 (2006), 497–515.

[5] H. Dong and D. Li, On a one-dimensional α-patch model with nonlocal drift and fractional dissipation. arXiv:1207.0957.

[6] T. Hmidi, S. Keraani, S., F. Rousset. *Global well-posedness for Euler-Boussinesq system with critical dissipation.* Comm. Partial Differential Equations 36 (2011), no. 3, 420–445.

[7] T. Hmidi, S. Keraani, S., F. Rousset. *Global well-posedness for a Boussinesq-Navier-Stokes system with critical dissipation.* J. Differential Equations 249 (2010), no. 9, 2147–2174.

[8] H.-Y. Hou, C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete Continuous Dynamical System, 12(1) (2005), 1–12.

[9] G. Karch and N. Prioux, *Self-similarity in viscous Boussinesq equations.* Proc. Amer. Math. Soc. 136 (2008), no. 3, 879–888

[10] M. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations.* Arch. Ration. Mech. Anal. 199 (2011), no. 3, 739–C-760.

[11] D. Li and T. Li, *Shock formation in a traffic flow model with Arrhenius look-ahead dynamics.* Netw. Heterog. Media 6 (2011), no. 4, 681–694.

[12] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific Publishing Co., Inc., River Edge, NJ (1996).

[13] S.A. Lorca, J.L. Boldrini, *The initial value problem for a generalized Boussinesq model: regularity and global existence of strong solutions*, Mat. Contemp. 11 (1996) 71–94.

[14] C. Wang, Z. Zhang, *Global well-posedness for the 2-D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity,* Adv. Math., 228(2011), 43–62.

Department of Mathematics, University of British Columbia, Vancouver BC Canada V6T 1Z2

E-mail address: `mpdongli@gmail.com`

School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China

E-mail address: `xjxu@bnu.edu.cn`