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Weighted in time energy estimates for parabolic equations with applications to non-linear and non-local problems

Nikolai Dokuchaev

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ABSTRACT. The paper suggests a modification of the contracting mapping method for non-linear and non-local parabolic equations. This modification is based on weighted in time energy estimates for the L_2-norm of the solution of a parabolic equation via a weighted version of the H^{-1}-norm of the free term such that the inverse matrix of the higher order coefficients of the parabolic equation is included into the weight. More precisely, this estimate represents the upper estimate that can be achieved via transformation of the equation by adding a constant to the zero order coefficient. The limit constant in this estimate is independent from the choice of the dimension, domain, and the coefficients of the parabolic equation.

CONTENTS

1. Introduction

The paper studies first boundary value problems for parabolic equations. A modification is suggested for the contracting mapping method for non-linear and

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non-local parabolic equations. This modification is based on special estimates for L_2-norm of the solution of a linear parabolic equation via a weighted version of H^{-1}-norm of the free term.

The classical result for parabolic equations is the so-called energy estimate for the L_2-type Sobolev norm of the solution via a H^{-1}-norm of the nonhomogeniuos term, where H^{-1} is the space being dual to the space $\overset{0}{W}{}_2^1(D)$ (see, e.g., the first energy inequality in Ladyzhenskaia [**3**]). We suggest a modification of this estimate.

We found a suboptimal upper estimate that can be achieved by varying the zero order coefficient of the original equation by adding a constant. In other words, we study the case when the original equation is transformed into a new one such that the original solution $u(x,t)$ is to be replaced by $u(x,t)e^{-Kt}$; the value of K is being varied (Theorem 3.1 and Lemma 7.1). The limit constant in these estimates is the same for all possible choices of the dimension, domain, time horizon, and the coefficients of the parabolic equations. It is why it can be called a universal estimate. These results represent an important development of the extension of the results from [**2**], where an "universal" estimate was obtained for the gradient via L_2-norm of the free term. In contrast, the present paper gives the estimate of the L_2-norm via a H^{-1}-type norm of the nonhomogeniuos term, i.e., via a weaker norm. It is shown that the estimate obtained is sharp (Theorem 6.1).

As an example of applications, this estimate was used to obtain explicit sufficient conditions of existence and regularity for a variety of non-linear and non-local parabolic equations (Theorems 5.1-5.3). The corresponding proof is based on the contracting mapping theorem.

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2. Definitions

Spaces and classes of functions. We denote by $|\cdot|$ the Euclidean norm in $\mathbf{R}^k$ and the Frobenius norm in $\mathbf{R}^{k\times m}$, and we denote by $\bar G$ denote the closure of a region $G\subset \mathbf{R}^k$.

We denote by $\|\cdot\|_X$ the norm in a linear normed space X, and $(\cdot,\cdot)_X$ denote the scalar product in a Hilbert space X. For a Banach space X, we denote by $C([a,b],X)$ the Banach space of continuous functions $x:[a,b]\to X$.

Let $G\subset\mathbf{R}^k$ be an open domain, then $W_q^m(G)$ denote the Sobolev space of functions that belong $L_q(G)$ together with the distributional derivatives up to the mth order, $q\ge 1$.

We are given an open domain $D\subseteq\mathbf{R}^n$ such that either $D=\mathbf{R}^n$ or D is bounded with C^2-smooth boundary ∂D.

Let $T>0$ be given, and let $Q\triangleq D\times(0,T)$.

Let $H^0\triangleq L_2(D)$, and let $H^1\triangleq\overset{0}{W}{}_2^1(D)$ be the closure in the $W_2^1(D)$-norm of the set of all smooth functions $u:D\to\mathbf{R}$ such that $u|_{\partial D}\equiv 0$. Let $H^2=W_2^2(D)\cap H^1$ be the space equipped with the norm of $W_2^2(D)$. The spaces H^k are Hilbert spaces, and H^k is a closed subspace of $W_2^k(D)$, $k=1,2$.

Let H^{-1} be the dual space to H^1, with the norm $\|\cdot\|_{H^{-1}}$ such that if $u\in H^0$ then $\|u\|_{H^{-1}}$ is the supremum of $(u,w)_{H^0}$ over all $w\in H^1$ such that $\|w\|_{H^1}\le 1$. H^{-1} is a Hilbert space.

We will write $(u,w)_{H^0}$ for $u\in H^{-1}$ and $w\in H^1$, meaning the obvious extension of the bilinear form from $u\in H^0$ and $w\in H^1$.

We denote by $\bar{\ell}_1$ the Lebesgue measure in $\mathbf{R}$, and we denote by $\bar{\mathcal{B}}_1$ the σ-algebra of Lebesgue sets in $\mathbf{R}^1$.

For $k=-1,0,1,2$, we introduce the spaces

$$X^k \triangleq L^2\big([0,T],\bar{\mathcal{B}}_1,\bar{\ell}_1;H^k\big),\qquad \mathcal{C}^k \triangleq C\left([0,T];H^k\right).$$

We introduce the spaces

$$Y^k \triangleq X^k\cap\mathcal{C}^{k-1},\quad k=0,1,2,$$

with the norm $\|u\|_{Y^k}\triangleq\|u\|_{X^k}+\|u\|_{\mathcal{C}^{k-1}}$.

We use the notations

$$\nabla u \triangleq \Big(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\dots,\frac{\partial u}{\partial x_n}\Big)^\top,\qquad \nabla\cdot U=\sum_{i=1}^n\frac{\partial U_i}{\partial x_i}$$

for functions $u:\mathbf{R}^n\to\mathbf{R}$ and $U=(U_1,\dots,U_n)^\top:\mathbf{R}^n\to\mathbf{R}^n$. In addition, we use the notation

$$(U,V)_{H^0}=\sum_{i=1}^n(U_i,V_i)_{H^0},\qquad \|U\|_{H^0}=(U,U)_{H^0}^{1/2}$$

for functions $U,V:D\to\mathbf{R}^n$, where $U=(U_1,\dots,U_n)$ and $V=(V_1,\dots,V_n)$.

2.0.1. *The boundary value problem.* We consider the following problem

$$\begin{array}{ll}\frac{\partial u}{\partial t}=\mathcal{A}u+\varphi, & t\in(0,T),\\ u(x,0)=0, & u(x,t)|_{x\in\partial D}=0.\end{array}\tag{2.1}$$

Here $u=u(x,t)$, $(x,t)\in Q$, and

$$\mathcal{A}y\triangleq\sum_{i=1}^n\frac{\partial}{\partial x_i}\sum_{j=1}^n\Big(b_{ij}(x,t)\frac{\partial y}{\partial x_j}(x)\Big)+\sum_{i=1}^n f_i(x,t)\frac{\partial y}{\partial x_i}(x)+\,\lambda(x,t)y(x),\tag{2.2}$$

where $b(x,t):\mathbf{R}^n\times[0,T]\to\mathbf{R}^{n\times n}$, $f(x,t):\mathbf{R}^n\times[0,T]\to\mathbf{R}^n$, and $\lambda(x,t):\mathbf{R}^n\times[0,T]\to\mathbf{R}$, are bounded measurable functions, and b_{ij},f_i,x_i are the components of b,f, and x. The matrix $b=b^\top$ is symmetric.

To proceed further, we assume that Conditions 2.1-2.2 remain in force throughout this paper.

CONDITION 2.1. *There exists a constant $\delta>0$ such that*

$$\xi^\top b(x,t)\,\xi\geq\delta|\xi|^2\quad\forall\,\xi\in\mathbf{R}^n,\ (x,t)\in Q.\tag{2.3}$$

Inequality (2.3) means that equation (2.1) is coercive.

CONDITION 2.2. *The functions $b(x,t):\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^{n\times n}$, $f(x,t):\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}^n$, $\lambda(x,t):\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$, are measurable, and*

$$\operatorname*{ess\,sup}_{(x,t)\in Q}\Big[|b(x,t)|+|b(x,t)^{-1}|+|f(x,t)|+|\lambda(x,t)|\Big]<+\infty.$$

We introduce the sets of parameters

$$\mu\triangleq(T,\,n,\ D,\ \ \delta,\ \ b,f,\lambda),$$

$$\mathcal{P}=\mathcal{P}(\mu)\triangleq\Big(T,\,n,\ D,\ \delta,\ \operatorname*{ess\,sup}_{(x,t)\in Q}\Big[|b(x,t)|+|f(x,t)|+|\lambda(x,t)|\Big]\Big).$$

We consider all possible μ such that the conditions imposed above are satisfied.

3. Special estimates for the solution

We assume that $\varphi \in X^{-1}$. This means that there exist functions

$$F = (F_1, ..., F_n) : Q \to \mathbf{R}^n$$

and $F_0 : Q \to \mathbf{R}$ such that $F_k \in X^0 = L_2(Q)$, $k = 0, 1, ..., n$, and

$$\varphi(x,t) = \nabla \cdot F(x,t) + F_0(x,t). \tag{3.1}$$

In other words, $\varphi(x,t) = \sum_{k=1}^n \frac{\partial F_k}{\partial x_k}(x,t) + F_0(x,t)$.

The classical solvability results for the parabolic equations give that there exists a unique solution $u \in Y^1$ of problem (2.1) for any $\varphi \in X^{-1}$.

Theorem 3.1. *For any μ and any $M > 0$, $\varepsilon > 0$, there exists*

$$K = K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$$

such that

$$\begin{aligned} &\sup_{s\in[0,t]} e^{-2Ks}\|u(\cdot,s)\|_{H^0}^2 + M\int_0^t e^{-2Ks}\|u(\cdot,s)\|_{H^0}^2 ds \\ &\leq \left(\frac{1}{2}+\varepsilon\right)\int_0^t e^{-2Ks}(F(\cdot,s), b(\cdot,s)^{-1}F(\cdot,s))_{H^0} ds \\ &+\varepsilon\int_0^t e^{-2Ks}\|F_0(\cdot,s)\|_{H^0}^2 ds \\ &\forall t \in [0,T],\ \ \varphi \in X^{-1}, \end{aligned} \tag{3.2}$$

where u is the solution of problem (2.1), and where $F_i \in X^0$ are such that (3.1) holds.

4. The case of non-linear and non-local equations

Let us consider a mapping $\mathcal{N}(v) : Y^1 \to X^{-1}$ such that

$$\mathcal{N}(v) \triangleq \sum_{i=1}^n \frac{\partial}{\partial x_i}\sum_{j=1}^n\Big(\widehat{b}_{ij}(v(\cdot),x,t)\frac{\partial v}{\partial x_j}(x,t)\Big) \tag{4.1}$$

$$+\sum_{i=1}^n \widehat{f}_i(v(\cdot),x,t)\frac{\partial v}{\partial x_i}(x,t) \tag{4.2}$$

$$+\widehat{\lambda}(v(\cdot),x,t)v(x,t) + \widehat{\varphi}(v(\cdot),x,t),$$

where

$$\widehat{b}(v(\cdot),x,t) : Y^1 \times Q \to \mathbf{R}^{n\times n},$$

$$\widehat{f}(v(\cdot),x,t) : Y^1 \times Q \to \mathbf{R}^n,$$

$$\widehat{\lambda}(v(\cdot),x,t) : Y^1 \times Q \to \mathbf{R},$$

are bounded functions. In (4.1), $\widehat{b}_{ij}, \widehat{f}_i, x_i$ are the components of $\widehat{b}$, $\widehat{f}$, and x. The function $\widehat{\varphi}(v(\cdot),x,t)$ defined on $Y^1 \times Q$ is such that $\widehat{\varphi}(v(\cdot),x,t) \in X^{-1}$ for any given $v(\cdot) \in Y^1$. The matrix $\widehat{b} = \widehat{b}^\top$ is symmetric.

THEOREM 4.1. *Let $u \in Y^1$ be a solution of the problem*

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \mathcal{N}(u), \qquad t \in (0,T), \\ &u(x,0) = 0, \qquad u(x,t)|_{x\in\partial D} = 0. \end{aligned} \tag{4.3}$$

such that Conditions 2.1-2.2 are satisfied for

$$b(x,t) = \widehat{b}(u(\cdot),x,t), f(x,t) = \widehat{f}(u(\cdot),x,t),$$

and $\lambda(x,t) = \widehat{\lambda}(u(\cdot),x,t)$, and such that $\varphi(x,t) \triangleq \widehat{\varphi}(u(\cdot),x,t)$ belongs to X^{-1} and is such that (3.1) holds for $F_i \in X^0$. Then, for any $M > 0$ and $\varepsilon > 0$, there exists $K = K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$ such that (3.2) holds, where $\mathcal{P}(\mu)$ is defined as above for the functions b, f, λ.

Note that the parabolic equation in (4.3) is non-linear and non-local in time and space. Moreover, the operator $\mathcal{N}(u)$ are not necessary causal with respect to time; the case of $(\mathcal{N}(v))(t)$ defined by the values $\{v(\cdot,s), s \geq t\}$ is not excluded.

Theorem 4.1 does not establish existence. Some existence results for non-local and non-linear problems are given below.

5. Applications: existence for non-linear and non-local equations

The estimates from Theorem 3.1 can be also applied to analysis of non-linear and non-local parabolic equations. These equations have many applications, and they were intensively studied (see. e.g., Ammann [**1**], Ladyzenskaya *et al* [**4**], Zheng [**5**], and references there). Theorem 3.1 gives a new way to establish conditions of solvability of these equations. This approach covers many cases when the solutions and the gradient are included into the non-local and non-linear term.

Let $B(u(\cdot)) : X^0 \to X^{-1}$ be a mapping that describes non-linear and non-local term in the equation.

Let us consider the following boundary value problem in Q:

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \mathcal{A}u + B(u) + \varphi, \qquad t \in (0,T), \\ &u(x,0) = 0, \qquad u(x,t)|_{x\in\partial D} = 0. \end{aligned} \tag{5.1}$$

Here $\mathcal{A}$ is the linear operator defined above. For $K > 0$, introduce the mappings

$$B_K(u) \triangleq e^{-Kt} B(\bar{u}_K), \quad \text{where} \quad \bar{u}_K(x,t) \triangleq e^{Kt} u(x,t). \tag{5.2}$$

THEOREM 5.1. *Assume that $B(u)$ maps X^0 into X^{-1}. Moreover, assume that there exist constants $K_* > 0$ and $C_* > 0$ such that*

$$\begin{aligned} &\|B_K(u_1) - B_K(u_2)\|_{X^{-1}} \leq C_* \|u_1 - u_2\|_{X^0} \quad \forall u_1, u_2 \in X^0, \\ &\forall K \in [K_*, +\infty). \end{aligned} \tag{5.3}$$

Then there exists a unique solution $u \in Y^1$ of problem (5.1) for any $\varphi \in X^{-1}$.

THEOREM 5.2. *Assume that*

$$\operatorname*{ess\,sup}_{(x,t)\in Q} \left(\left|\frac{\partial b}{\partial x}(x,t)\right| + \left|\frac{\partial f}{\partial x}(x,t)\right| + \left|\frac{\partial \lambda}{\partial x}(x,t)\right| + \left|\frac{\partial b}{\partial t}(x,t)\right| \right) < +\infty. \tag{5.4}$$

Further, assume that $B(u)$ maps X^1 into X^0 and that there exist constants $K_* > 0$ and $C_* > 0$ such that

$$\begin{aligned} &\|B_K(u_1) - B_K(u_2)\|_{X^0} \leq C_* \|u_1 - u_2\|_{X^1} \quad \forall u_1, u_2 \in X^1, \\ &\forall K \in [K_*, +\infty). \end{aligned} \tag{5.5}$$

Then there exists a unique solution $u \in Y^2$ of problem (5.1) for any $\varphi \in X^0$.

5.0.2. *Examples of admissible B.* Some examples covered by Theorems 5.1-5.2 are listed below.

THEOREM 5.3. *The assumptions of Theorem 5.1 hold for the following mappings $B(u)$:*

(i) A local non-linearity:

$$B(u) = \beta(u(x,t),x,t),$$

where $\beta : \mathbf{R} \times Q \to \mathbf{R}$ is a measurable function such that $\beta(0,\cdot) \in L_2(Q)$ and that there exists a constant $C_L > 0$ such that

$$|\beta(z_1,x,t) - \beta(z_2,x,t)| \le C_L|z_1 - z_2| \quad \forall z_1,z_2 \in \mathbf{R},\ x,t. \tag{5.6}$$

(ii) A distributional non-linearity:

$$B(u) \triangleq \nabla \cdot \beta(u(x,t),x,t),$$

where $\beta : \mathbf{R} \times Q \to \mathbf{R}^n$ is a measurable function such that $\beta(0,\cdot) \in L_2(Q)$ and (5.6) holds.

(iii) A non-local in space non-linearity (integral nonlinearity):

$$(B(u))(x,t) = \int_D \beta(u(y,t),x,t,y)dy,$$

where $\beta : \mathbf{R} \times Q \times D \to \mathbf{R}$ is a measurable function such that $\int_D \beta(0,x,t,y)dy \in L_2(Q)$ as a function of (x,t), and there exists a constant $C_L > 0$ such that

$$|\beta(z_1,x,t,y) - \beta(z_2,x,t,y)| \le C_L|z_1 - z_2| \quad \forall z_1,z_2 \in \mathbf{R},\ x,t,y. \tag{5.7}$$

We assume here that D is a bounded domain.

(iv) A non-local in space distributional non-linearity:

$$(B(u))(x,t) = \nabla \cdot \int_D \beta(u(y,t),x,t,y)dy,$$

where $\beta : \mathbf{R} \times Q \times D \to \mathbf{R}^n$ is a measurable function such that $\int_D \beta(0,\cdot,y)dy \in L_2(Q)$ as a function of (x,t), and (5.7) holds. We assume here that D is a bounded domain.

(v) A non-local in time and space non-linearity:

$$(B(u))(x,t) = \int_0^t ds \int_D \beta(u(y,s),x,t,y,s)dy,$$

where $\beta : \mathbf{R} \times Q^2 \to \mathbf{R}$ is a measurable function such that $\int_0^t ds \int_D \beta(0,x,t,y,s)dy \in L_2(Q)$ as a function of (x,t), and there exists a constant $C_L > 0$ such that

$$|\beta(z_1,x,t,y,s) - \beta(z_2,x,t,y,s)| \le C_L|z_1 - z_2| \quad \forall z_1,z_2 \in \mathbf{R},\ x,t,y,s. \tag{5.8}$$

We assume here that D is a bounded domain.

(vi) A non-local in time and space distributional non-linearity:

$$(B(u))(x,t) = \nabla \cdot \int_0^t ds \int_D \beta(u(y,s),x,t,y,s)dy,$$

where $\beta : \mathbf{R} \times Q^2 \to \mathbf{R}^n$ is a measurable function such that $\int_0^t ds \int_D \beta(0,\cdot,y,s)dy \in L_2(Q)$ as a function of (x,t), and (5.8) holds. We assume here that D is a bounded domain.

(vii) Nonlinear delay parabolic equations:

$$(B(u))(x,t) \triangleq \nabla \cdot \beta(u(x,\tau(t)),x,\tau(t)) + \widehat{\beta}(u(x,\tau(t)),x,\tau(t)). \tag{5.9}$$

Here $\tau(\cdot) : [0,T] \to \mathbf{R}$ is a given measurable function such that $\tau(t) \in [0,t]$, and that there exists $\theta \in [0,T)$ such that $\tau(t) = 0$ for $t < \theta$, the function $\tau(\cdot) : [\theta,T] \to \mathbf{R}$ is non-decreasing and absolutely continuous, and $\operatorname{ess\,sup}_{t\in[\theta,T]} \left|\frac{d\tau}{dt}(t)\right|^{-1} < +\infty$. The functions $\beta : \mathbf{R} \times \mathbf{R}^n \times [0,T] \to \mathbf{R}^n$ and $\widehat{\beta} : \mathbf{R} \times \mathbf{R}^n \times [0,T] \to \mathbf{R}$ are bounded and measurable. In addition, we assume that the derivative $\frac{\partial \beta}{\partial x}(x,t)$ is bounded, $\beta(0,\cdot) \in L_2(Q)$, $\widehat{\beta}(0,\cdot) \in L_2(Q)$, and there exists a constant $C_L > 0$ such that

$$\begin{aligned} &|\beta(z_1,x,t) - \beta(z_2,x,t)| + |\widehat{\beta}(z_1,x,t) - \widehat{\beta}(z_2,x,t)| \\ &\le C_L|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbf{R},\ x, t. \end{aligned} \tag{5.10}$$

(viii) Non-local term for the backward Kolmogorov equations for a jump diffusion process:

$$\begin{aligned} (Bu)(x,t) \triangleq &\int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t)\in D\}}(u(x + c(x,y,t),t) \\ &-u(x,t) - c(x,y,t)^\top \nabla u(x,t))\rho(y,t)dy. \end{aligned}$$

Here $\rho(y,t) : \mathbf{R}^n \times [0,T] \to \mathbf{R}$ is a function such that $\rho(\cdot) \in L_\infty([0,T],\ell_1,\bar{\mathcal{B}}_1,L_1(\mathbf{R}^n))$. The function $c(x,y,t) : D \times \mathbf{R}^n \times [0,T] \to \mathbf{R}^n$ is measurable, and there exists a uniquely defined function $\psi : D \times \mathbf{R}^n \times [0,T] \to \mathbf{R}^n$ such that $z = x + c(x,y,t)$ for $y = \psi(x,z,t)$. In addition, we assume that $\operatorname{ess\,sup}_{t\in[0,T]} \int_{D\times D} |r(x,z,t)|^2 dxdz < +\infty$, where the function $r(x,z,t) \triangleq \rho(\psi(x,z,t),t)\frac{\partial \psi}{\partial z}(x,z,t)$ is such that the derivative $\frac{\partial c}{\partial x}(x,y,t)$ is bounded.

THEOREM 5.4. *Assume that (5.4) holds. Then the remaining assumptions of Theorem 5.2 hold for $B(u)$ such as in Theorem 5.3(i),(iii),(v), as well as for $B(u)$ with delay such as in Theorem 5.3(vii) given that $\beta \equiv 0$.*

Clearly, linear combinations of the non-linear and non-local terms listed above are also covered, as well as terms formed as compound mappings.

The statement of Theorem 5.4 for the case of $B(u)$ with delay was presented in [**2**].

REMARK 5.1. The proof of Theorems 5.1-5.2 is based on the contraction mapping theorem. However, it is not required that the operator $B(u)$ is "small". Instead, we require that (5.3) or (5.5) is satisfied for all large enough K. For the non-local in time operators described in Theorem 5.3 (v)-(vii), this requires causality of the operator $B(u)$ with respect to time; this can be seen from the proof given below. For instance, it is not possible to replace the operator in Theorem 5.3(v) by the operator

$$(B(u))(x,t) = \int_0^T ds \int_D \beta(u(y,s),x,t,y,s)dy$$

that is not non-causal with respect to time. For this $B(u)$, the contraction mapping theorem still ensures existence but only if the function β is "small".

6. On the sharpness of the estimates

THEOREM 6.1. *There exists a set of parameters $(n, D, b(\cdot), f(\cdot), \lambda(\cdot))$ such that, for any $T > 0$, $M \geq 0$, $\varepsilon > 0$, $K > 0$,*

$$\forall T>0, c>0, K>0 \quad \exists \varphi \in X^{-1}: \\ e^{-2KT}\|u(\cdot,T)\|^2_{H^0} \geq \left(\frac{1}{2}-\varepsilon\right)\int_0^T (F(\cdot,t), b(\cdot,t)^{-1}F(\cdot,t))_{H^0}dt, \tag{6.1}$$

where u is the solution of problem (7.1) and $F_i \in X^0$ are such as presented in (3.1), $F = (F_1, ..., F_n)$.

7. Proofs

LEMMA 7.1. *For any admissible μ and any $\varepsilon > 0$, $M > 0$, there exists $\tilde{K} = \tilde{K}(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$ such that*

$$\|u(\cdot,t)\|^2_{H^0} + M\int_0^t \|u(\cdot,s)\|^2_{H^0}ds \\ \leq \left(\frac{1}{2}+\varepsilon\right)\int_0^t \Big((F(\cdot,s), b(\cdot,s)^{-1}F(\cdot,s))_{H^0}ds + \varepsilon\int_0^t \|F_0(\cdot,s)\|^2_{H^0}ds$$

for all $K \geq \tilde{K}(\varepsilon, M, \mathcal{P})$, $t \in (0,T]$, for all $\varphi \in X^{-1}$ represented as (3.1) with $F_i \in X^0$. Here $u \in Y^1$ is the solution of the boundary value problem

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \mathcal{A}u - Ku + \varphi, \quad t \in (0,T), \\ &u(x,0) = 0, \qquad u(x,t)|_{x\in\partial D} = 0. \end{aligned} \tag{7.1}$$

Uniqueness and existence of solution $u \in Y^1$ of problem (7.1) follows from the classical results (see, e.g., Ladyzhenskaia [**3**], Chapter III).

Proof of Lemma 7.1. Clearly, $\mathcal{A}u = \mathcal{A}_s u + \mathcal{A}_r u$, where

$$\mathcal{A}_s u = \nabla\cdot(b\nabla u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\sum_{j=1}^n\Big(b_{ij}\frac{\partial u}{\partial x_j}\Big), \qquad \mathcal{A}_r u = \sum_{i=1}^n f_i\frac{\partial u}{\partial x_i} + \lambda u.$$

Assume that $\varphi(\cdot,t)$ is differentiable and has a compact support inside D for all t. We have that

$$\begin{aligned} &\|u(\cdot,t)\|^2_{H^0} - \|u(\cdot,0)\|^2_{H^0} \\ &= (u(\cdot,t), u(\cdot,t))_{H^0} - (u(\cdot,0), u(\cdot,0))_{H^0} \\ &= 2\int_0^t \left(u, \frac{\partial u}{\partial s}\right)_{H^0} ds = 2\int_0^t (u, \mathcal{A}u - Ku + \varphi)_{H^0}\, ds \\ &= 2\int_0^t \big(u, \nabla\cdot(b\nabla u)\big)_{H^0} ds + 2\int_0^t (u, \mathcal{A}_r u)_{H^0}\, ds - 2K\int_0^t (u,u)_{H^0}\, ds \\ &+2\int_0^t (u,\varphi)_{H^0}\, ds. \end{aligned} \tag{7.2}$$

Let arbitrary $\varepsilon_0 > 0$ and $\widehat{\varepsilon}_0 > 0$ be given. Let $v \triangleq \sqrt{b}$, i.e., $b = v^2$, $v = v^\top$. We have that

$$\begin{aligned} 2\,(u,\varphi)_{H^0} &= 2\,(u,\nabla\cdot F)_{H^0} + 2\,(u,F_0)_{H^0} \\ &= -2\left(v\nabla u, v^{-1}F\right)_{H^0} + 2\,(u,F_0))_{H^0} \\ &\le \frac{2}{1+2\varepsilon_0}\,(v\nabla u, v\nabla u)^2_{H^0} + \left(\frac{1}{2}+\varepsilon_0\right)\left\|v^{-1}F\right\|^2_{H^0} \\ &+\frac{1}{\widehat{\varepsilon}_0}\,\|u\|^2_{H^0} + \widehat{\varepsilon}_0\,\|F_0\|^2_{H^0} \\ &= \frac{2}{1+2\varepsilon_0}\,(\nabla u, b\nabla u)^2_{H^0} + \left(\frac{1}{2}+\varepsilon_0\right)\left(F, b^{-1}F\right)_{H^0} \\ &+\frac{1}{\widehat{\varepsilon}_0}\,\|u\|^2_{H^0} + \widehat{\varepsilon}_0\,\|F_0\|^2_{H^0}\,, \end{aligned} \tag{7.3}$$

and

$$2\,(u,\nabla\cdot(b\nabla u))_{H^0} = -2\,(\nabla u, b\nabla u)_{H^0}\,. \tag{7.4}$$

In addition, we have that, under the integrals in (7.2),

$$2\,(u,\mathcal{A}_r u)_{H^0} \le \varepsilon_1^{-1}\,\|u\|^2_{H^0} + \varepsilon_1\,\|\mathcal{A}_r u\|^2_{H^0} \quad \forall\varepsilon_1 > 0.$$

By the first energy inequality, there exist constants $c'_* = c'_*(\mathcal{P}) > 0$ and $c_* = c_*(\mathcal{P}) > 0$ such that

$$\int_0^t \|u(\cdot,s)\|^2_{H^1}\,ds \le c'_* \sum_{k=0}^n \int_0^t \|F_k(\cdot,s)\|^2_{H^0}\,ds \le c_* \int_0^t \left(F, b^{-1}F\right)_{H^0} ds. \tag{7.5}$$

(See, e.g. inequality (3.14) from [**3**], Chapter III). Moreover, this constant c_* can be taken the same for all $t \in [0,T]$ and all $K > 0$. Further, there exists a constant $c_1 = c_1(\mathcal{P}) > 0$ such that

$$2\,(u,\mathcal{A}_r u)_{H^0} \le \varepsilon_1^{-1}\,\|u\|^2_{H^0} + c_1\varepsilon_1\,\|u\|^2_{H^1}\,.$$

It follows that

$$2\int_0^t (u,\mathcal{A}_r u)_{H^0}\,ds \le \varepsilon_1^{-1}\int_0^t \|u\|^2_{H^0}\,ds + \varepsilon_0\int_0^t \left(F, b^{-1}F\right)_{H^0} ds, \tag{7.6}$$

if $\varepsilon_1 > 0$ is taken such that $c_1 c_* \varepsilon_1 = \varepsilon_0$.

By (7.2)-(7.6), it follows that

$$\begin{aligned} &\|u(\cdot,t)\|^2_{H^0} + M\int_0^t \|u(\cdot,s)\|^2_{H^0} ds \\ &\le \left[\frac{2}{1+2\varepsilon_0} - 2\right]\int_0^t (\nabla u, b\nabla u)_{H^0}\,ds + [\varepsilon_1^{-1} + \widehat{\varepsilon}_0^{-1} + M - 2K]\int_0^t \|u\|^2_{H^0}\,ds \\ &+\Big(\frac{1}{2} + 2\varepsilon_0\Big)\int_0^t \left(F, b^{-1}F\right)_{H^0} ds + (\varepsilon_0 + \widehat{\varepsilon}_0)\int_0^t \|F_0(\cdot,s)\|^2_{H^0}\,ds \\ &\le \Big(\frac{1}{2} + 2\varepsilon_0\Big)\int_0^t \left(F, b^{-1}F\right)_{H^0} ds + (\varepsilon_0 + \widehat{\varepsilon}_0)\int_0^t \|F_0(\cdot,s)\|^2_{H^0}\,ds, \end{aligned}$$

if $2K > \varepsilon_1^{-1} + c'_v + M$. Then the proof of Lemma 7.1 follows. □

Proof of Theorem 3.1. Clearly, $u(x,t) = e^{Kt}u_K(x,t)$, where u is the solution of problem (2.1) and u_K is the solution of (7.1) for the nonhomogeneous term $e^{-Kt}\varphi(x,t)$. Therefore, Theorem 3.1 follows immediately from Lemma 7.1.

Proof of Theorem 4.1 follows immediately from Theorem 3.1 and Lemma 7.1 applied to the set of the coefficients (b, f, λ) that is independent from K. □

Proof of Theorem 5.1. Note that $u \in Y^1$ is the solution of the problem (5.1) if and only if $u_K(x,t) \triangleq e^{-Kt}u(x,t)$ is the solution of the problem

$$\begin{aligned} &\tfrac{\partial u_K}{\partial t} = \mathcal{A}u_K - Ku_K + B_K(u_K) + \varphi_K, \qquad t \in (0,T), \\ &u_K(x,0) = 0, \qquad u_K(x,t)|_{x\in\partial D} = 0, \end{aligned} \tag{7.7}$$

where $\varphi_K(x,t) \triangleq e^{-Kt}\varphi(x,t)$. In addition,

$$\|u\|_{Y^1} \leq e^{KT}\|u_K\|_{Y^1}, \quad \|\varphi_K\|_{X^{-1}} \leq \|\varphi\|_{X^{-1}}.$$

Therefore, the solvability and uniqueness in Y^1 of problem (5.1) follows from existence of $K > 0$ such that problem (7.7) has an unique solution in Y^1. Let us show that this K can be found.

We introduce operators $F_K : X^{-1} \to Y^1$ such that $u = F_K\varphi$ is the solution of problem (7.1).

Let $g \in X^{-1}$ be such that

$$g = \varphi + B_K(w), \quad \text{where} \quad w = F_K g. \tag{7.8}$$

In that case, $u_K \triangleq F_K g \in Y^1$ is the solution of (7.7).

Equation (7.8) can be rewritten as $g = \varphi + R_K(g)$, or

$$g - R_K(g) = \varphi, \tag{7.9}$$

where the mapping $R_K : X^{-1} \to X^{-1}$ is defined as

$$R_K(g) = B_K(F_K g).$$

Let $w = F_K h$, where $h \in X^{-1}$. By Theorem 3.1 reformulated as Lemma 7.1, for any $\varepsilon > 0$, $M > 0$, there exists $K(\varepsilon, M, \mathcal{P}(\mu)) \geq 0$ and a constant $C_0 = C_0(\mathcal{P}(\mu))$ such that

$$\sup_{t\in[0,T]} \|w(\cdot,t)\|^2_{H^0} + M\int_0^T \|w(\cdot,s)\|^2_{H^0} ds \leq C_0\|h\|^2_{X^{-1}} \qquad \forall h \in X^{-1}. \tag{7.10}$$

and, therefore,

$$M\int_0^T \|w(\cdot,s)\|^2_{H^0} ds \leq C_0\|h\|^2_{X^{-1}} \qquad \forall h \in X^{-1}.$$

Hence

$$\|F_K h\|^2_{X^0} \leq M^{-1}C_0\|h\|^2_{X^{-1}}.$$

Let us select M and K such that $\delta_* \triangleq C_* M^{-1/2}C_0^{1/2} < 1$. By (5.3), it follows that

$$\begin{aligned} \|R_K(g_1) - R_K(g_2)\|_{X^{-1}} \leq C_*\|F_K g_1 - F_K g_2\|_{X^0} &\leq C_* M^{-1/2}C_0^{1/2}\|g_1 - g_2\|_{X^{-1}} \\ &= \delta_*\|g_1 - g_2\|_{X^{-1}}. \end{aligned}$$

By the contraction mapping theorem, it follows that the equation (7.9) has an unique solution $g \in X^{-1}$. Hence problem (7.7) has an unique solution $u_K = F_K g \in Y^1$. This completes the proof of Theorem 5.1. □

Proof of Theorem 5.2. Let $w = F_K h$, where $h \in X^0$, and where F_k is the operator defined in the proof of Theorem 5.1. By (5.4), the assumptions of Lemma

7.1 from [**2**] are satisfied. By this lemma, for any $\varepsilon > 0$, $M > 0$, there exists $K = K(\varepsilon, M, b, f, \lambda) \geq 0$ and a constant $C_0 = C_0(b, f, \lambda)$ such that

$$\sup_{t\in[0,T]} \|w(\cdot,t)\|^2_{H^1} + M\int_0^T \|w(\cdot,s)\|^2_{H^1} ds \leq C_0\|h\|^2_{X^0} \qquad \forall h \in X^0$$

and

$$M\int_0^T \|w(\cdot,s)\|^2_{H^1} ds \leq C_0\|h\|^2_{X^0} \qquad \forall h \in X^0.$$

The rest of the proof of Theorem 5.2 repeats the proof of Theorem 5.1 with the replacement of Y^1 for Y^2, and X^{-1} for X^0, and with R_K being a mapping $R_K : X^0 \to X^0$. □

Proof of Theorem 5.3. The proof for (i)-(iv) represents simplified versions of the proof for (v)-(vi) given below and will be omitted.

Let us prove (v). Let $Q_t \triangleq \{(y,s) \in Q : \ s \leq t\}$. We have that

$$\begin{aligned}&|B_K(u_1)(x,t) - B_K(u_2)(x,t)|\\ &\leq e^{-Kt}\int_{Q_t} |\beta(e^{Ks}u_1(y,s),x,t,y,s) - \beta(e^{Ks}u_2(y,s),x,t,y,s)|dyds\\ &\leq e^{-Kt}C_L\int_{Q_t} e^{Ks}|u_1(y,s) - u_2(y,s)|dyds \leq C_L\ell_{n+1}(Q)^{1/2}\|u_1(\cdot) - u_2(\cdot)\|_{X^0}\end{aligned}$$

for all $u_1(\cdot), u_2(\cdot) \in X^0$. Since the domain Q is bounded, we have that

$$\|B(u_1) - B(u_2)\|_{X^{-1}} \leq \|B(u_1) - B(u_2)\|_{X^0} \leq \ell_{n+1}(Q)^{1/2}\|B(u_1) - B(u_2)\|_{L_\infty(Q)}.$$

Hence (5.3) holds.

Further, it follows from the assumptions that $B(0) \in X^0$. Hence $B(u) \in X^0$ for all $u \in X^0$. This completes the proof of statement (v).

Let us prove (vi). By the definition, $B(u) = \nabla \cdot \widehat{B}(u)$, where $\widehat{B} : X^0 \to X^0$ is a mapping similar to the one from statement (v). Then the proof is similar to the proof of statement (v).

Let us prove statement (vii). Let $C_\tau = \operatorname{ess\,sup}_{t\in[\theta,T]} \left|\frac{d\tau}{dt}(t)\right|^{-1} < +\infty$. We have that

$$\begin{aligned}&\|B_K(u_1) - B_K(u_2)\|^2_{X^{-1}}\\ &\leq \int_0^T e^{-2Kt}\|\beta(e^{K\tau(t)}u_1(\cdot,\tau(t)),\cdot,\tau(t)) - \beta(e^{K\tau(t)}u_2(\cdot,\tau(t)),\cdot,\tau(t))\|^2_{H^0} dt\\ &+ \int_0^T e^{-2Kt}\|\widehat{\beta}(e^{K\tau(t)}u_1(\cdot,\tau(t)),\cdot,\tau(t)) - \widehat{\beta}(e^{K\tau(t)}u_2(\cdot,\tau(t)),\cdot,\tau(t))\|^2_{H^0} dt\\ &\leq 2C_L^2\int_0^T \|u_1(\cdot,\tau(t))) - u_2(\cdot,\tau(t))\|^2_{H^0} dt\\ &= 2C_L^2\int_0^T \|u_1(\cdot,\tau(t)) - u_2(\cdot,\tau(t))\|^2_{H^0}\left(\frac{d\tau(t)}{dt}\right)^{-1} d\tau(t)\\ &\leq 2C_L^2C_\tau\int_{\tau(\theta)}^{\tau(T)} \|u_1(\cdot,s) - u_2(\cdot,s)\|^2_{H^0} ds \leq C_L^2C_\tau\|u_1 - u_2\|^2_{X^0}.\end{aligned}$$

By the assumptions, it follows that $B(0) \in X^{-1}$. Hence $B(u) \in X^{-1}$ for all $u \in X^0$.

Let us prove statement (viii). We have that $B_K(u) = B(u)$, i.e., it is independent from K. Further, $B(u) = \widehat{B}(u) + \tilde{B}(u)$, where

$$(\widehat{B}(u))(x,t) = \int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t)\in D\}} u(x+c(x,y,t),t)\rho(y,t)dy$$
$$= \int_D u(z,t)r(x,z,t)dz,$$

and

$$\begin{aligned}(\tilde{B}(u))(x,t) &= -u(x,t)\int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t)\in D\}}\rho(y,t)dy \\ &\quad -\left(\int_{\mathbf{R}^n} \mathbb{I}_{\{x+c(x,y,t)\in D\}} c(x,y,t)\rho(y,t)dy\right)^\top \nabla u(x,t) \\ &= -u(x,t)\int_D r(x,z,t)dz - \left(\int_D r(x,z,t)(z-x)dz\right)^\top \nabla u(x,t).\end{aligned}$$

It follows from the assumptions that $\tilde{B} : X^0 \to X^{-1}$ is a linear and continuous operator. Hence it suffices to prove that (5.3) holds for the operator $\widehat{B}$. Clearly, $\widehat{B}(0) = 0$. Further, we have that

$$\begin{aligned} &\|\widehat{B}(u_1) - \widehat{B}(u_2)\|^2_{X^{-1}} \\ \le\ &\|\widehat{B}(u_1) - \widehat{B}(u_2)\|^2_{X^0} \\ =\ &\int_Q \left(\int_D (u_1(z,t) - u_2(z,t))r(x,z,t)dz\right)^2 dxdt \\ \le\ &\int_Q \left(\int_D |u_1(z,t) - u_2(z,t)|^2 dz\right)\left(\int_D |r(x,z,t)|^2 dz\right) dxdt \\ \le\ &\int_0^T dt \left(\int_D |u_1(z,t) - u_2(z,t)|^2 dz\right)\int_D dx \int_D |r(x,z,t)|^2 dz \\ \le\ &\left(\operatorname*{ess\,sup}_{t\in[0,T]} \int_{D\times D} |r(x,z,t)|^2 dxdz\right) \|u_1 - u_2\|^2_{X^0}.\end{aligned}$$

This completes the proof of statement (viii) and the proof of Theorem 5.3. □

Proof of Theorem 5.4 repeats the proof of the corresponding statements of Theorem 5.3 with minor adjustments. □

Proof of Theorem 6.1. Repeat that $u(x,t) = e^{Kt}u_K(x,t)$, where u is the solution of problem (2.1) and u_K is the solution of (7.1) for $h_K(x,t) = e^{-Kt}h(x,t)$. Therefore, it suffices to find n, D, b, f, λ, such that

$$\forall T>0, \varepsilon>0, K>0 \quad \exists \varphi \in X^{-1}:$$
$$|u(\cdot,T)\|^2_{H^0} \ge \left(\frac{1}{2} - \varepsilon\right)\int_0^T (F(\cdot,t), b(\cdot,t)^{-1}F(\cdot,t))_{H^0}dt, \tag{7.11}$$

where u is the solution of problem (7.1) and $F_i \in X^0$ are such as presented in (3.1), $F = (F_1, ..., F_n)$.

Let us show that (7.11) holds for

$$n = 1, \qquad D = (-\pi,\pi), \qquad b(x,t) \equiv 1, \qquad f(x,t) \equiv 0, \qquad \lambda(x,t) \equiv 0.$$

In this case, (7.1) has the form

$$u'_t = u''_{xx} - Ku + h, \qquad u(x,0) \equiv 0, \qquad u(x,t)|_{x\in\partial D} = 0,$$

Let

$$\gamma = m^2 + K, \qquad \varphi_m(x,t) \triangleq m\sin(mx)e^{\gamma t}, \qquad F_m(x,t) \triangleq -\cos(mx)e^{\gamma t},$$

where $m = 1, 2, 3, \ldots$. It can be verified immediately that the solution of the boundary value problem is

$$u(x,t) = m\sin(mx)\int_0^t e^{-\gamma(t-s)+\gamma s}ds = m\sin(mx)e^{-\gamma t}\int_0^t e^{2\gamma s}ds$$
$$= m\sin(mx)e^{-\gamma t}\frac{e^{2\gamma t}-1}{2\gamma}.$$

Hence

$$\|u(\cdot,T)\|^2_{H^0} = m^2\|\sin(mx)\|^2_{H^0}e^{-2\gamma T}\left(\frac{e^{2\gamma}-1}{2\gamma}\right)^2 = m^2\pi e^{-2\gamma T}\frac{(e^{2\gamma T}-1)^2}{4\gamma^2},$$

and

$$\int_0^T \|F_m(\cdot,t)\|^2_{H^0}dt = \|\cos(mx)\|^2_{H^0}\int_0^T e^{2\gamma t}dt = \pi\frac{e^{2\gamma T}-1}{2\gamma}.$$

It follows that

$$\|u(\cdot,T)\|^2_{H^0}\left(\int_0^T \|F_m(\cdot,t)\|^2_{H^0}dt\right)^{-1} = \frac{m^2}{2\gamma}e^{-2\gamma T}(e^{2\gamma T}-1)$$
$$= \frac{m^2}{2\gamma}(1-e^{-2\gamma T}) \to \frac{1}{2}$$

as $\gamma \to +\infty$. In particular, it holds if K is fixed and $m \to +\infty$. It follows that (6.1) holds. This completes the proof of Theorem 6.1. □

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Department of Mathematics & Statistics, Curtin University, GPO Box U1987, Perth, 6845, Western Australia

E-mail address: N.Dokuchaev@curtin.edu.au

URL: http://maths.curtin.edu.au/people/nikolai.cfm