Dynamics of PDE, Vol.9, No.3, 261-271, 2012

On the Hausdorff dimension of singular sets for the Leray-α Navier-Stokes equations with fractional regularization

Hani Ali and Zied Ammari

Communicated by Sergey Zelik, received October 27, 2011.

Abstract. We consider a family of Leray-α models with periodic boundary conditions in three space dimensions. Such models are a regularization, with respect to a parameter θ, of the Navier-Stokes equations. In particular, they share with the original equation (NS) the property of existence of global weak solutions. We establish an upper bound on the Hausdorff dimension of the time singular set of those weak solutions when θ is subcritical. The result is an interpolation between the bound proved by Scheffer for the Navier-Stokes equations and the regularity result proved in [**1**].

Contents

1991 *Mathematics Subject Classification.* 35Q30, 35Q35, 76F60.
Key words and phrases. Turbulence models, weak solution, singular set, Hausdorff measure.

1. Introduction

We consider, for $\alpha > 0$ and $0 < \theta < 1/4$, the Leray-α equations in the 3-dimensional flat Torus $\mathbb{T}_3$

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\overline{\boldsymbol{u}} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{in } \mathbb{R}^+ \times \mathbb{T}_3, \\ \overline{\boldsymbol{u}} = (1-\alpha^2\Delta)^{-\theta}\boldsymbol{u}, \quad \text{in } \mathbb{T}_3, \\ \nabla \cdot \boldsymbol{u} = 0, \ \displaystyle\int_{\mathbb{T}_3} \boldsymbol{u} = 0, \\ \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0. \end{cases} \tag{1.1}$$

Here the unknowns are the velocity vector field $\boldsymbol{u}$ and the scalar pressure p. The viscosity ν, the initial velocity vector field $\boldsymbol{u}_0$ and the external force $\boldsymbol{f}$, with $\nabla\cdot\boldsymbol{f} = 0$, are given.
The nonlocal operator $M_\theta = (1-\alpha^2\Delta)^{-\theta}$, acting on $L^2(\mathbb{T}_3, \mathbb{R}^3)$, is defined through the Fourier transform on the torus

$$\widehat{M_\theta \boldsymbol{u}}(\boldsymbol{k}) = (1+\alpha^2|\boldsymbol{k}|^2)^{-\theta}\, \widehat{\boldsymbol{u}}(\boldsymbol{k}) \ , \quad \boldsymbol{k} \in \mathbb{Z}^3. \tag{1.2}$$

The Leray-α equations are among the simplest models of turbulence, introduced nearly a decade ago for numerical simulation purposes. When $\theta = 0$, (1.1) reduces to the Navier Stokes equations for an incompressible fluid. It is well known that weak solutions for the Navier Stokes equations, which additionally satisfies a form of the energy inequality, exist globally in time, either in $\mathbb{R}^3$, a result due to Leray [**12**], or on a bounded domain [**21**].

Although the question of uniqueness and regularity of those weak solutions remains open some important intermediate results are known. Among this is the weak-strong uniqueness property which means that a weak solution coincide with a possible strong solution with the same initial data as long as the latter exists. A classical result due to Prodi [**15**] and Serrin [**19**] gives weak-strong uniqueness under the assumption $\boldsymbol{u} \in L^r([0,T], L^s(\mathbb{T}_3)^3)$, where $\frac{3}{s} + \frac{2}{r} = 1$ with $3 < s < \infty$. The scaling in Serrin's condition was improved in [**13, 20**]. There are further improvements of this criterion in the modern literature (see e.g. [**4, 5, 9, 11**]). Another aspect is the Hausdorff measure of the time singular set of weak solutions. We know, thanks to Scheffer's work [**16, 17**], that if $\boldsymbol{u}$ is a weak Leray solution of the Navier-Stokes equations then the $\frac{1}{2}$-dimensional Hausdorff measure of the time singular set of $\boldsymbol{u}$ is zero. A more thorough study of the singularity set in space and time of suitable weak solutions was initiated by Scheffer [**18, 17**] and improved later on in the seminal paper of Caffarelli et al. [**3**].

The Leray-α models are approximation of the Navier-Stokes equations with a regularized velocity vector field $\overline{\boldsymbol{u}} = M_\theta \boldsymbol{u}$ when $\theta > 0$. Actually, a crude regularization (or filtering) already appeared in the early work of Leray [**12**] where a mollifier was used (i.e., $\overline{\boldsymbol{u}} = \phi_\varepsilon * \boldsymbol{u}$) instead of the operator M_θ.

As expected the regularized equation (1.1) have several properties in common with (NS) and in a certain sense it has a more regular behavior depending on the values of θ. In particular, the Leray-α models (1.1) have existence of weak solutions for arbitrary time and large initial-data (Theorem 2.1) and have existence and uniqueness of strong solutions for small time (Theorem 2.2). Furthermore, a weak-strong uniqueness result also holds true (Proposition 2.1).

Our goal, in this short note, is to establish an upper bound for the Hausdorff dimension of the time singular set $\mathcal{S}_\theta(\boldsymbol{u})$ of weak solutions $\boldsymbol{u}$ of (1.1). When $\theta = \frac{1}{4}$, the author in [**1**] proved the existence of a unique regular weak solution to the Leray-α model (1.1). Therefore, it is intersecting to understand how the potential time singular set $\mathcal{S}_\theta(\boldsymbol{u})$ may depend on the regularization parameter θ. In fact, we will prove that $\frac{1-4\theta}{2}$-dimensional Hausdorff measure of the time singular set $\mathcal{S}_\theta(\boldsymbol{u})$ of any weak solution $\boldsymbol{u}$ of (1.1) is zero. This result is stated in Theorem 3.1. The Hausdorff dimension of the time singular set of weak solutions for another modification of the Navier-Stokes equations was studied in [**2**].

We only focus on the Leray-α equations (1.1) but the same results hold for other models of turbulence as the magnetohydrodynamics MHD-α equations. Moreover, it was observed in [**10**] that the qualitative properties of the equation (1.1) relies on the regularization effect of the operator M_θ rather than its explicit form. This fact can also be checked for the Hausdorff dimension of the time singular set $\mathcal{S}_\theta(\boldsymbol{u})$.

2. Preliminaries

Before giving some preliminary results we fix some notations and give a precise definition of weak and strong solutions of the Cauchy problem (1.1). For $p \in [1, \infty)$, the Lebesgue spaces $L^p(\mathbb{T}_3)$, the Sobolev spaces $W^{1,p}(\mathbb{T}_3)$ and the Bochner spaces $L^p(0, T; X)$, $C(0, T;\, X)$, X being a Banach space, are defined in a standard way. In addition for $s \geq -1$, we introduce the following spaces

$$\boldsymbol{V}^s = \left\{ \boldsymbol{u} \in W^{s,2}(\mathbb{T}_3)^3,\ \int_{\mathbb{T}_3} \boldsymbol{u} = 0,\ \operatorname{div} \boldsymbol{u} = 0 \right\},$$

endowed with the norms

$$\|\boldsymbol{u}\|^2_{\boldsymbol{V}^s} = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} |\boldsymbol{k}|^{2s} \, |\widehat{\boldsymbol{u}}(\boldsymbol{k})|^2 \,.$$

For the sake of simplicity we set

$$\boldsymbol{H} := \boldsymbol{V}^0 \quad \text{and} \quad \boldsymbol{V} := \boldsymbol{V}^1 \,.$$

By $C_{weak}([0, T]; \boldsymbol{H})$ we denote the vector space of all functions $\boldsymbol{v} : [0, T] \to \boldsymbol{H}$ such that for any $\boldsymbol{h} \in \boldsymbol{H}$, the function

$$t \mapsto \int_{\mathbb{T}_3} \langle \boldsymbol{v}(t), \boldsymbol{h} \rangle d\boldsymbol{x}$$

is continuous on $[0, T]$.

Definition 2.1. *For $T > 0$, $\boldsymbol{f} \in L^2([0, T], \boldsymbol{V}^{-1})$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$, a Leray-Hopf weak solution of the Cauchy problem* (1.1) *in* $[0, T] \times \mathbb{T}_3$ *is a velocity vector field* $\boldsymbol{u} : [0, T] \times \mathbb{T}_3 \to \mathbb{R}^3$ *satisfying:*

(i) $\boldsymbol{u} \in C_{weak}([0, T]; \boldsymbol{H}) \cap L^2([0, T]; \boldsymbol{V})$,

(ii) *for all* $t, s \in [0, T]$ *and* $\boldsymbol{v} \in V$

$$\langle \boldsymbol{u}(t), \boldsymbol{v} \rangle + \int_s^t \nu \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle + \langle (\overline{\boldsymbol{u}} \cdot \nabla) \boldsymbol{u}, \boldsymbol{v} \rangle \, d\tau = \langle \boldsymbol{u}(s), \boldsymbol{v} \rangle + \int_s^t \langle \boldsymbol{f}, \boldsymbol{v} \rangle \, d\tau \,, \tag{2.1}$$

(iii) *the velocity field $\boldsymbol{u}$ verifies the energy inequality in the distribution sense in* $]0,T[$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_{\boldsymbol{H}}^2+\nu\|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2\leq\langle\boldsymbol{f},\boldsymbol{u}(t)\rangle\ , \tag{2.2}$$

(iv) *the initial data is attained in the following sense*

$$\lim_{t\to 0+}\|\boldsymbol{u}(t)-\boldsymbol{u}_0\|_{\boldsymbol{H}}^2=0.$$

We say that a Leray-Hopf weak solution $\boldsymbol{u}$ is a strong solution if, moreover

$$\boldsymbol{u}\in C([0,T];V).$$

REMARK 2.1. *1) Once the velocity vector field $\boldsymbol{u}$ is known a scalar pressure p can be found such that $(\boldsymbol{u},p)$ satisfies a weak formulation of the Cauchy problem* (1.1) *(see e.g.* [**8**]*).*
2) The above definition make sense in any bounded interval I.
3) If $T=\infty$ the definition still make sense by assuming $\boldsymbol{u}\in C_{weak}(\mathbb{R}_+;\boldsymbol{H})\cap L^2_{loc}(\mathbb{R}_+;\boldsymbol{V})$ instead of (i).
4) The energy inequality (2.2) *can be replaced by the assumption that for almost $t'\in[0,T]$ the velocity vector field satisfies*

$$\frac{1}{2}||\boldsymbol{u}(t)||_{\boldsymbol{H}}^2+\nu\int_{t'}^{t}||\boldsymbol{u}(s)||_{\boldsymbol{V}}^2\,ds\leq\frac{1}{2}||\boldsymbol{u}(t')||_{\boldsymbol{H}}+\int_{t'}^{t}\langle\boldsymbol{f},\boldsymbol{u}(s)\rangle ds\,,$$

for all $t\in]t',T]$, where the allowed times $t^{'}$ are characterized as the points of strong continuity from the right, in $\boldsymbol{H}$, for $\boldsymbol{u}$, (see [**7**, Remark 1]*).*

2.1. A priori estimates. The essential feature of the operator M_θ is the following regularization effect.

LEMMA 2.1. *Let $\theta\in\mathbb{R}_+, s\geq -1$ and assume that $\boldsymbol{u}\in\boldsymbol{V}^s$. Then $M_\theta\boldsymbol{u}\in\boldsymbol{V}^{s+2\theta}$ and*

$$\|M_\theta\boldsymbol{u}\|_{\boldsymbol{V}^{s+2\theta}}\leq\frac{1}{\alpha^{2\theta}}\|\boldsymbol{u}\|_{\boldsymbol{V}^s}.$$

Next, we prove some a priori estimates for the Cauchy problem (1.1) in the same manner as for the Navier Stokes equations (see [**21, 7**]). Those estimates combined with a Galerkin method yields the existence of Leray-Hopf weak solutions (see Theorem 2.1). We suppose that $\boldsymbol{u}$ is a sufficient regular Leray-Hopf weak solution of (1.1).

2.1.1. *A priori estimates in $\boldsymbol{H}$.*

LEMMA 2.2. *Let $T>0$, $\boldsymbol{f}\in L^2([0,T],\boldsymbol{V}^{-1})$ and $\boldsymbol{u}_0\in\boldsymbol{H}$, then any sufficently regular Leray-Hopf weak solution $\boldsymbol{u}$ of (1.1) satisfies*

$$\|\boldsymbol{u}\|_{L^2([0,T],\boldsymbol{V})}^2\leq\frac{1}{\nu}\left(\|\boldsymbol{u}_0\|_{\boldsymbol{H}}^2+\frac{1}{\nu}\|\boldsymbol{f}\|_{L^2([0,T],\boldsymbol{V}^{-1})}^2\right) \tag{2.3}$$

and

$$\|\boldsymbol{u}\|_{L^\infty([0,T],\boldsymbol{H})}^2\leq\|\boldsymbol{u}_0\|_{\boldsymbol{H}}^2+\frac{1}{\nu}\|\boldsymbol{f}\|_{L^2([0,T],\boldsymbol{V}^{-1})}^2\ . \tag{2.4}$$

Proof Taking the L^2-inner product of the first equation of (1.1) with $\boldsymbol{u}$ and integrating by parts. Using the incompressibility of the velocity field and the duality relation we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2_{\boldsymbol{H}} + \nu\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{H}} \quad = \quad \int_{\mathbb{T}_3} \boldsymbol{f}\ \boldsymbol{u} dx \le \|\boldsymbol{f}\|_{\boldsymbol{V}^{-1}}\|\boldsymbol{u}\|_{\boldsymbol{V}}.$$

Using Young inequality we get

$$\frac{d}{dt}\|\boldsymbol{u}\|^2_{\boldsymbol{H}} + \nu\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{H}} \quad \le \quad \frac{1}{\nu}\|\boldsymbol{f}\|^2_{\boldsymbol{V}^{-1}}.$$

Integration with respect to time gives the desired estimates. ■

2.1.2. *A priori estimates in* $\boldsymbol{V}$. Now we use the regularization effect of Lemma 2.1 to prove the following a priori estimate on the existence time of strong solutions.

LEMMA 2.3. *Let* $T > 0$, $\boldsymbol{f} \in L^2([0,T],\boldsymbol{H})$ *and* $\boldsymbol{u}_0 \in \boldsymbol{V}$. *Assume that* $0 < \theta < 1/4$, *then there exists a constant* $C(\alpha,\theta,\nu,f) > 0$ *such that a strong solution* $\boldsymbol{u}$ *of* (1.1) *satisfies*

$$\sup_{t\in[0,T_*[} \|\boldsymbol{u}\|^2_{\boldsymbol{V}} \le 2(1+\|\boldsymbol{u}_0\|^2_{\boldsymbol{V}})\,,$$

with

$$T_* := \frac{C(\alpha,\theta,\nu,f)}{(1+\|\boldsymbol{u}_0\|^2_{\boldsymbol{V}})^{\frac{2}{1+4\theta}}}. \tag{2.5}$$

Proof Taking the L^2-inner product of the first equation of (1.1) with $-\Delta\boldsymbol{u}$ and integrating by parts. Using the incompressibility of the velocity field and the duality relation combined with Hölder inequality and Sobolev injection, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{H}} + \nu\|\Delta\boldsymbol{u}\|^2_{\boldsymbol{H}} \quad &\le \quad \int_{\mathbb{T}_3} |(\overline{\boldsymbol{u}}\cdot\nabla)\boldsymbol{u}\Delta\boldsymbol{u}|dx + \int_{\mathbb{T}_3} |\boldsymbol{f}\Delta\boldsymbol{u}|dx \\ &\le \quad \alpha^{-2\theta}\|\nabla\boldsymbol{u}\|_{\boldsymbol{H}}\|\nabla\boldsymbol{u}\|_{\boldsymbol{V}^{\frac{1}{2}-2\theta}}\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}} + \|\boldsymbol{f}\|_{\boldsymbol{H}}\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}}.\end{aligned}$$

Interpolating between $\boldsymbol{V}^1$ and $\boldsymbol{V}^2$ we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{H}} + \nu\|\Delta\boldsymbol{u}\|^2_{\boldsymbol{H}} \quad \le \quad \alpha^{-2\theta}\|\nabla\boldsymbol{u}\|^{\frac{3}{2}+2\theta}_{\boldsymbol{H}}\|\Delta\boldsymbol{u}\|^{\frac{3}{2}-2\theta}_{\boldsymbol{H}} + \|\boldsymbol{f}\|_{\boldsymbol{H}}\|\Delta\boldsymbol{u}\|_{\boldsymbol{H}}.$$

Using Young inequality we get

$$\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{H}} + \nu\|\Delta\boldsymbol{u}\|^2_{\boldsymbol{H}} \quad \le \quad \frac{1}{\nu}\|\boldsymbol{f}\|^2_{\boldsymbol{H}} + C(\alpha,\theta)\|\nabla\boldsymbol{u}\|^{\frac{2(3+4\theta)}{1+4\theta}}_{\boldsymbol{H}}. \tag{2.6}$$

We get a differential inequality

$$Y^{'} \le C(\alpha,\theta,\nu,f)\ Y^{\gamma}, \tag{2.7}$$

where

$$Y(t) = 1 + \|\boldsymbol{u}\|^2_{\boldsymbol{V}} \quad \text{and} \quad \gamma = \frac{3+4\theta}{1+4\theta}\,.$$

We conclude that

$$Y(t) \le \frac{Y(0)}{(1-2Y(0)^{\gamma-1}C(\alpha,\theta,\nu,f)\,t)^{\frac{1}{\gamma-1}}}$$

as long as $t < \dfrac{1}{2Y(0)^{\gamma-1}C(\alpha,\theta,\nu,f)}$. ■

2.2. Existence and uniqueness results.

2.2.1. *Existence result.* The next two theorems collect the most typical results for the Leray-α models of turbulence (see [**1**], [**14**]). The proofs of these two theorems follow by combination of the above a priori estimates with a Galerkin method. This is a classical argument which we avoid its repetition. For further information, we refer the reader to [**21**], [**1**] and the references therein.

THEOREM 2.1. *Let $T > 0$ and assume that $0 \leq \theta < 1/4$. For any $\boldsymbol{f} \in L^2([0,T], \boldsymbol{V}^{-1})$ and $\boldsymbol{u}_0 \in \boldsymbol{H}$ there exists at least one Leray-Hopf weak solution (Definition 2.1) of the Cauchy problem* (1.1). *Moreover,* $(\boldsymbol{u}, p)$ *satisfies*

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^{\frac{5}{3-2\theta}}([0,T]; W^{-1,\frac{5}{3-2\theta}}(\mathbb{T}_3)^3), \quad p \in L^{\frac{5}{3-2\theta}}([0,T], L^{\frac{5}{3-2\theta}}(\mathbb{T}_3)) .$$

REMARK 2.2. *If $\theta = \frac{1}{4}$ a weak solution to Leray-α model is called regular weak solution* [**1**]*, in addition, the solution is unique and it satisfies*

$$\boldsymbol{u} \in C([0,T]; \boldsymbol{H}) \cap L^2([0,T]; \boldsymbol{V}), \quad \frac{\partial \boldsymbol{u}}{\partial t} \in L^2([0,T]; \boldsymbol{V}^{-1}), \quad \textit{and} \quad p \in L^2([0,T], L^2(\mathbb{T}_3)).$$

In this case one also has energy equality in (2.2) instead of inequality.

THEOREM 2.2. *Let $\boldsymbol{u}_0 \in \boldsymbol{V}$, $\boldsymbol{f} \in L^2([0,T], \boldsymbol{H})$ and assume that $0 \leq \theta < 1/4$. Then there exists $T_* := T_*(\boldsymbol{u}_0)$, determined by (2.5), and there exists a unique strong solution $\boldsymbol{u}$ to (1.1) on $[0, T_*[$ satisfying:*

$$\boldsymbol{u} \in C([0,T_*[; \boldsymbol{V}) \cap L^2([0,T_*[; \boldsymbol{V}^2),$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2([0,T_*[; L^2(\mathbb{T}_3)^3) \quad \textit{and} \quad p \in L^2([0,T_*[, W^{1,2}(\mathbb{T}_3)) .$$

REMARK 2.3. *If $\theta = \frac{1}{4}$ the strong solution to the Leray-α model exists for any arbitrary time $T > 0$. Indeed, when $\theta = \frac{1}{4}$, $\gamma = \frac{3+4\theta}{1+4\theta} = 2$, the differential inequality* (2.7) *becomes*

$$Y^{'} \leq C(\alpha, \theta, \nu, f)\, Y^2 . \tag{2.8}$$

Thus, using Gronwall's inequality ($Y \in L^1[0,T]$) we obtain the desired result.

2.2.2. *Weak-strong uniqueness.* In this subsection, we show a Serrin type criterion for the weak-strong uniqueness result for the Leray-α models (1.1). More precisely if $\boldsymbol{u}$ and $\boldsymbol{v}$ are respectively a Leray-Hopf weak solution and a strong solution of (1.1) with the same initial data then $\boldsymbol{u}$ and $\boldsymbol{v}$ coincide as long as the latter exists.

PROPOSITION 2.1. *Let $0 < \theta < \frac{1}{4}$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two Leray-Hopf weak solutions to (1.1) corresponding to the same initial data. Assume additionally that*

$$v \in L^r((0,T), L^s(\mathbb{T}_3)^3), \tag{2.9}$$

with

$$2/r + 3/s = 1 + 2\theta, \quad s \in \left(\frac{3}{1+2\theta}, \frac{3}{2\theta} \right] .$$

Then $\boldsymbol{u} = \boldsymbol{v}$ a.e. in $(0,T) \times \mathbb{T}_3$.

Proof. Let us denote by $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and $\overline{\boldsymbol{w}} = \overline{\boldsymbol{u}} - \overline{\boldsymbol{v}}$. Formally, we subtract the equation for $\boldsymbol{u}$ from the one for $\boldsymbol{v}$ and test it with $\boldsymbol{w}$ (This can be done using the same approximation procedure as in [**8**]), we get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_{\boldsymbol{H}}^2 + \nu\|\nabla\boldsymbol{w}\|_{\boldsymbol{H}}^2 = \int_{\mathbb{T}_3} (\overline{\boldsymbol{v}}\cdot\nabla\boldsymbol{v} - \overline{\boldsymbol{u}}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{w}\, d\boldsymbol{x}, \tag{2.10}$$

using integration by parts and divergence free constrain on $\boldsymbol{u}$ and $\boldsymbol{v}$ we deduce from the above equation that

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_{\boldsymbol{H}}^2 + \nu\|\nabla\boldsymbol{w}\|_{\boldsymbol{H}}^2 = \int_{\mathbb{T}_3} \overline{\boldsymbol{w}}\otimes\boldsymbol{v} : \nabla\boldsymbol{w}\, d\boldsymbol{x}. \tag{2.11}$$

Using hölder inequality we bound the nonlinear term in the right hand side of (2.11) as

$$|\int_{\mathbb{T}_3} \overline{\boldsymbol{w}}\otimes\boldsymbol{v} : \nabla\boldsymbol{w} d\boldsymbol{x}\,| \leq \|\overline{\boldsymbol{w}}\|_{L^{\frac{2s}{s-2}}}\|\boldsymbol{v}\|_{L^s}\|\nabla\boldsymbol{w}\|_{L^2}, \tag{2.12}$$

Then by using the Sobolev embedding of $\boldsymbol{H}^{\frac{3}{s}}$ in $L^{\frac{2s}{s-2}}$ and the following interpolation inequality $\|\boldsymbol{u}\|_{\boldsymbol{H}^{\frac{3}{s}}} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^{2\theta}}^{1-\frac{3}{s}+2\theta}\|\boldsymbol{u}\|_{\boldsymbol{H}^{1+2\theta}}^{\frac{3}{s}-2\theta}$, we get for all $\frac{3}{1+2\theta} \leq s \leq \frac{3}{2\theta}$

$$\begin{aligned} |\int_{\mathbb{T}_3} \overline{\boldsymbol{w}}\otimes\boldsymbol{v} : \nabla\boldsymbol{w} d\boldsymbol{x}\,| &\leq C\|\overline{\boldsymbol{w}}\|_{\boldsymbol{H}^{2\theta}}^{1-\frac{3}{s}+2\theta}\|\overline{\boldsymbol{w}}\|_{\boldsymbol{H}^{1+2\theta}}^{\frac{3}{s}-2\theta}\|\boldsymbol{v}\|_{L^s}\|\nabla\boldsymbol{w}\|_{L^2} \\ &\leq C(\alpha)\|\boldsymbol{w}\|_{\boldsymbol{H}}^{1-\frac{3}{s}+2\theta}\|\boldsymbol{w}\|_{\boldsymbol{V}}^{1+\frac{3}{s}-2\theta}\|\boldsymbol{v}\|_{L^s}. \end{aligned} \tag{2.13}$$

Hence using Young inequality, we get for $\frac{3}{1+2\theta} < s \leq \frac{3}{2\theta}$

$$|\int_{\mathbb{T}_3} \overline{\boldsymbol{w}}\otimes\boldsymbol{v} : \nabla\boldsymbol{w} d\boldsymbol{x}\,| \leq \frac{C(\alpha)}{\nu}\|\boldsymbol{w}\|_{\boldsymbol{H}}^2\|\boldsymbol{v}\|_{L^s}^{\frac{2s}{s-3+2\theta s}} + \frac{\nu}{2}\|\boldsymbol{w}\|_{\boldsymbol{V}}^2. \tag{2.14}$$

By combining (2.10) and (2.14) we get

(2.15)
$$\|\boldsymbol{w}(t)\|_{\boldsymbol{H}}^2 + \nu\int_0^t \|\nabla\boldsymbol{w}\|_{\boldsymbol{H}}^2 \leq \|\boldsymbol{w}_0\|_{\boldsymbol{H}}^2 + \frac{C(\alpha)}{\nu}\int_0^t \|\boldsymbol{w}\|_{\boldsymbol{H}}^2\|\boldsymbol{v}\|_{L^s}^r d\tau, \text{ for all } t\in[0,T],$$

Thus using the assumption (2.9) for $\boldsymbol{v}$ and Gronwall lemma we obtain the continuous dependence of the solutions on the initial data in the $L^\infty([0,T],\boldsymbol{H})$ norm. In particular, if $\boldsymbol{w}_0 = 0$ then $\boldsymbol{w} = 0$ and the solutions are unique for all $t\in[0,T]$. ∎

Remark 2.4. *1) When $\theta = 0$ we get the well-known serrin condition to the Navier Stokes equations $2/r + 3/s = 1, s \in (3,\infty]$.*
2) when $\theta = 1/4$ we get that $2/r + 3/s = 3/2, s\in(2,6]$, which is true for any weak solution of (1.1).

3. The Main Result And Its Proof

The basic facts about Hausdorff measures can be found for instance in [**6**]. We recall here the definition of those measures.

Definition 3.1. *Let X be a metric space and let $a > O$. The a-dimensional Hausdorff measure of a subset Y of X is*

$$\mu_a(Y) = \lim_{\epsilon\searrow o}\mu_{a,\epsilon}(Y) = \sup_{\epsilon>0}\mu_{a,\epsilon}(Y)$$

where

$$\mu_{a,\epsilon}(Y) = \inf \sum_j (\text{diameter } B_j)^a,$$

the infimum being taken over all the coverings of Y *by balls* B_j *such that* diameter $B_j \leq \epsilon$.

Definition 3.2. *For* $T > 0$ *we define the time singular set* $\mathcal{S}_\theta(\boldsymbol{u})$ *of a Leray-Hopf weak solution* $\boldsymbol{u}(t)$ *of (1.1), given by Theorem 2.1, as the set of* $t \in [0,T]$ *such that* $\boldsymbol{u}(t) \notin \boldsymbol{V}$.

The main result of the paper is the following theorem.

Theorem 3.1. *Let* $\boldsymbol{u}$ *be any Leray-Hopf weak solution of (1.1) on* $[0,T]$ *given by Theorem 2.1 with an external force* $\boldsymbol{f} \in L^2([0,T],\boldsymbol{H})$. *Then the* $\frac{1-4\theta}{2}$*-dimensional Hausdorff measure of the time singular set* $\mathcal{S}_\theta(\boldsymbol{u})$ *of* $\boldsymbol{u}$ *is zero.*

The rest of the paper is devoted to the proof of the main Theorem. The following Lemma characterizes the structure of the time singularity set of a Leray-Hopf weak solution of (1.1).

Lemma 3.1. *We assume that* $\boldsymbol{u}_0 \in \boldsymbol{H}$, $\boldsymbol{f} \in L^2([0,T],\boldsymbol{H})$ *and* $\boldsymbol{u}$ *is any Leray-Hopf weak solution of (1.1) given by Theorem 2.1. Then there exist an open set* $\mathcal{O}$ *of* $(0,T)$ *such that:*
(i) For all $t \in \mathcal{O}$ *there exist* t_1, t_2 *such that* $(t_1,t_2) \subseteq (0,T)$ *and* $\boldsymbol{u} \in C((t_1,t_2),\boldsymbol{V})$.
(ii) The Lebesgue measure of $[0,T] \setminus \mathcal{O}$ *is zero.*

Proof. Since $\boldsymbol{u} \in C_{weak}([0,T];\boldsymbol{H})$, $\boldsymbol{u}(t)$ is well defined for every t and we can define

$$\Sigma = \{t \in [0,T], \boldsymbol{u}(t) \in \boldsymbol{V}\},$$
$$\Sigma^c = \{t \in [0,T], \boldsymbol{u}(t) \notin \boldsymbol{V}\},$$
$$\mathcal{O} = \{t \in (0,T), \exists \epsilon > 0, \boldsymbol{u} \in C((t-\epsilon, t+\epsilon),\boldsymbol{V})\}.$$

It is clear that $\mathcal{O}$ is open. Since $\boldsymbol{u} \in L^2([0,T];\boldsymbol{V})$, Σ^c has Lebesgue measure zero. Let $t_0 \in \Sigma \setminus \mathcal{O}$ then according to Theorem 2.2 there exists a local strong solution $\boldsymbol{v} \in C([t_0, t_0+\epsilon),\boldsymbol{V})$, with $\epsilon > 0$, such that $u(t_0) = v(t_0)$. Thanks to the weak-strong uniqueness result (Proposition 2.1) such a strong solution is unique among all Leray-Hopf weak solutions. Hence $\boldsymbol{u} \in C((t_0, t_0+\epsilon),\boldsymbol{V})$ and therefore t_0 is the left end of one of the connected components of $\mathcal{O}$. Thus $\Sigma \setminus \mathcal{O}$ is countable and consequently $[0,T] \setminus \mathcal{O}$ has Lebesgue measure zero. This finishes the proof. ■

Remark 3.1. *We deduce from Theorem 2.2 that, if* (α_i, β_i) , $i \in I$, *is one of the connected components of* $\mathcal{O}$, *then*

$$\lim_{t \to \beta_i} \|\boldsymbol{u}(t)\|_{\boldsymbol{V}} = +\infty.$$

Indeed, otherwise Theorem 2.2 would show that there exist an $\epsilon > 0$ *such that* $\boldsymbol{u} \in C((\beta_i, \beta_i+\epsilon),\boldsymbol{V})$ *and* β_i *would not be the end point of a connected component of* $\mathcal{O}$.

Lemma 3.2. *Under the same notations of Lemma 3.1. Let* (α_i, β_i), $i \in I$, *be the connected components of* $\mathcal{O}$. *Then*

$$\sum_{i \in I} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} < \infty \tag{3.1}$$

Proof. Let (α_i, β_i) be one of these connected components and let $t \in (\alpha_i, \beta_i) \subseteq \mathcal{O}$. Since $\boldsymbol{u} \in C_{weak}([0,T]; \boldsymbol{H}) \cap L^2([0,T]; \boldsymbol{V})$, $\boldsymbol{u}(t)$ is well defined for every $t \in (\alpha_i, \beta_i)$ and t can be chosen such that $\boldsymbol{u}(t) \in \boldsymbol{V}$. According to Theorem 2.2, inequality (2.5) and the fact that $\|\boldsymbol{u}(\beta_i)\|_{\boldsymbol{V}} = +\infty$, we have for $t \in (\alpha_i, \beta_i)$

$$\beta_i - t \geq \frac{1}{C(\alpha, \theta, \nu, f)} \frac{1}{(1 + \|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2)^{\gamma - 1}},$$

where we have used that $\gamma = \frac{3+4\theta}{1+4\theta} > 1$. Thus

$$\frac{C(\alpha, \theta, \nu, f)}{(\beta_i - t)^{\frac{1}{\gamma-1}}} \leq 1 + \|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2.$$

Then we integrate on (α_i, β_i) to obtain

$$C(\alpha, \theta, \nu, f)(\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq (\beta_i - \alpha_i) + \int_{\alpha_i}^{\beta_i} \|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2 dt,$$

Adding all these relations for $i \in I$ we obtain

$$C(\alpha, \theta, \nu, f) \sum_{i \in I} (\beta_i - \alpha_i)^{\frac{-1}{\gamma-1}+1} \leq T + \int_0^T \|\boldsymbol{u}(t)\|_{\boldsymbol{V}}^2 dt.$$

■

Proof of Theorem 3.1. We set $\mathcal{S} = \mathcal{S}_\theta(\boldsymbol{u}) = [0,T] \setminus \mathcal{O}$. We have to prove that the $\frac{1-4\theta}{2}$-dimensional Hausdorff measure of $\mathcal{S}$ is zero. Since the Lebesgue measure of $\mathcal{O}$ is finite ,i.e.

$$\sum_{i \in I} (\beta_i - \alpha_i) < \infty, \tag{3.2}$$

it follows from Lemma 3.2 that for every $\epsilon > 0$ there exist a finite part $I_\epsilon \subset I$ such that

$$\sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i) \leq \epsilon \tag{3.3}$$

and

$$\sum_{i \in I \setminus I_\epsilon} (\beta_i - \alpha_i)^{\frac{1-4\theta}{2}} \leq \epsilon \tag{3.4}$$

Note that $\mathcal{S} \subset [0,T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$ and the set $[0,T] \setminus \bigcup_{i \in I_\epsilon} (\alpha_i, \beta_i)$ is the union of finite number of mutually disjoint closed intervals, say B_j, for $j = 1, ..., N$. Our aim now is to show that the diameter $B_j \leq \epsilon$. Since the intervals (α_i, β_i) are mutually disjoint, each interval (α_i, β_i), $i \in I \setminus I_\epsilon$, is included in one, and only one, interval B_j. We denote by I_j the set of indice i such that $(\alpha_i, \beta_i) \subset B_j$. It is clear that $I_\epsilon, I_1, ..., I_N$ is a partition of I and we have $B_j = (\bigcup_{i \in I_j} (\alpha_i, \beta_i)) \cup (B_j \cap \mathcal{S})$ for all $j = 1, ..., N$. It follows from (3.2) that

$$\text{diameter}\, B_j = \sum_{i \in I_j} (\beta_i - \alpha_i) \leq \epsilon. \tag{3.5}$$

Finally in virtue of the definition 3.1 and estimates (3.5), (3.4) and since $l^\delta \hookrightarrow l^1$ for all $0 < \delta < 1$ we have

$$
\begin{aligned}
\mu_{\frac{1-4\theta}{2},\epsilon}(\mathcal{S}) &\leq \sum_{j=1}^{N}(\text{diameter } B_j)^{\frac{1-4\theta}{2}} \\
&\leq \sum_{j=1}^{N}\left(\sum_{i\in I_j}(\beta_i-\alpha_i)\right)^{\frac{1-4\theta}{2}} \\
&\leq \sum_{j=1}^{N}\sum_{i\in I_j}(\beta_i-\alpha_i)^{\frac{1-4\theta}{2}} \\
&= \sum_{i\in I\setminus I_\epsilon}(\beta_i-\alpha_i)^{\frac{1-4\theta}{2}} \leq \epsilon.
\end{aligned}
\tag{3.6}
$$

Letting $\epsilon \to 0$, we find $\mu_{\frac{1-4\theta}{2}}(\mathcal{S}) = 0$ and this completes the proof. ■

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E-mail address: hani.ali@univ-rennes1.fr

IRMAR, UMR 6625, Université Rennes 1, Campus Beaulieu, 35042 Rennes cedex, FRANCE

E-mail address: zied.ammari@univ-rennes1.fr

IRMAR, UMR 6625, Université Rennes 1, Campus Beaulieu, 35042 Rennes cedex, FRANCE