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Velocity averaging – a general framework

Martin Lazar and Darko Mitrović

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Abstract. We prove that the sequence of averaged quantities $\int_{\mathbf{R}^m} u_n(\mathbf{x},\mathbf{p})\rho(\mathbf{p})d\mathbf{p}$, is strongly precompact in $L^2_{loc}(\mathbf{R}^d)$, where $\rho \in L^2_c(\mathbf{R}^m)$, and $u_n \in L^2(\mathbf{R}^m; L^s(\mathbf{R}^d))$, $s \geq 2$, are weak solutions to differential operator equations with variable coefficients. In particular, this includes differential operators of hyperbolic, parabolic or ultraparabolic type, but also fractional differential operators. If $s > 2$ then the coefficients can be discontinuous with respect to the space variable $\mathbf{x} \in \mathbf{R}^d$, otherwise, the coefficients are continuous functions. In order to obtain the result we prove a representation theorem for an extension of the H-measures.

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1. Introduction

The main subject of the paper is the following sequence of equations:

$$\mathcal{P}u_n(\mathbf{x},\mathbf{p}) = \sum_{k=1}^{d} \partial_{x_k}^{\alpha_k} \left(a_k(\mathbf{x},\mathbf{p})u_n(\mathbf{x},\mathbf{p})\right) = \partial_{\mathbf{p}}^{\boldsymbol{\kappa}} G_n(\mathbf{x},\mathbf{p}), \tag{1}$$

where u_n are weak solutions to (1) such that $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^m; L^s(\mathbf{R}^d))$, $s \geq 2$, while:

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a) $\alpha_k > 0$ are real numbers and $\partial_{x_k}^{\alpha_k}$ are (the Fourier) multiplier operators with the symbols $(2\pi i\xi_k)^{\alpha_k}$, $i^{\alpha_k} := e^{\frac{i\alpha_k \pi}{2}}$, $k = 1, \dots, d$;

b)

$$a_k \in \begin{cases} \mathrm{L}^2(\mathbf{R}^m; \mathrm{C}_b(\mathbf{R}^d)), & s = 2 \\ \mathrm{L}^2(\mathbf{R}^m; \mathrm{L}^r(\mathbf{R}^d)), & 2/s + 1/r = 1, \quad s > 2, \end{cases}$$

where $\mathrm{C}_b(\mathbf{R}^d)$ stands for a space of continuous and bounded functions;

c) $\partial_{\mathbf{p}}^{\boldsymbol{\kappa}} = \partial_{p_1}^{\kappa_1} \dots \partial_{p_m}^{\kappa_m}$ for a multi-index $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_m) \in \mathbf{N}^m$, and

$$G_n \to 0 \;\text{ in }\; \mathrm{L}^2(\mathbf{R}^m; \mathrm{W}^{-\boldsymbol{\alpha}, s'}(\mathbf{R}^d)), \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d),$$

where $\mathrm{W}^{-\boldsymbol{\alpha}, s'}(\mathbf{R}^d)$ is a dual of $\mathrm{W}^{\boldsymbol{\alpha}, s}(\mathbf{R}^d) = \{u \in \mathrm{L}^s(\mathbf{R}^d) : \partial_k^{\alpha_k} u \in \mathrm{L}^s(\mathbf{R}^d), k = 1, \dots, d\}$ (for details on anisotropic Sobolev spaces see e.g. [**40**]).

Equations (1) involve the space variable $\mathbf{x} \in \mathbf{R}^d$, with respect to which we have derivatives of solutions (u_n), and the variable $\mathbf{p} \in \mathbf{R}^m$, which is usually called the velocity variable.

Notice that if $\alpha_k \in \mathbf{N}$ then equation (1) is a standard partial differential equation. In particular, for $\alpha_1 = \dots = \alpha_d = 1$ one gets a transport equation (considered in e.g. [**16, 32**]; see more detailed discussion below). In general, we have a linear fractional differential equation.

First, we introduce a definition of a weak solution to (1). Assume for the moment that the sub-index n is removed in (1).

Definition 1. We say that a function $u \in \mathrm{L}^2(\mathbf{R}^m; \mathrm{L}^s(\mathbf{R}^d))$ is a weak solution to (1) if for every $g \in \mathrm{W}_{\mathrm{c}}^{|\boldsymbol{\kappa}|, 2}(\mathbf{R}^m; \mathrm{W}^{\boldsymbol{\alpha}, s}(\mathbf{R}^d))$ it holds

(2)
$$\int_{\mathbf{R}^{m+d}} \sum_{k=1}^{d} a_k(\mathbf{x}, \mathbf{p}) u(\mathbf{x}, \mathbf{p}) \overline{(-\partial_{x_k})^{\alpha_k}(g(\mathbf{x}, \mathbf{p}))} d\mathbf{x} d\mathbf{p} = (-1)^{|\boldsymbol{\kappa}|} \int_{\mathbf{R}^m} \Big\langle G(\cdot, \mathbf{p}), \overline{\partial_{\mathbf{p}}^{\boldsymbol{\kappa}} g(\cdot, \mathbf{p})} \Big\rangle d\mathbf{p}\,,$$

where duality on $\mathrm{W}^{\boldsymbol{\alpha}, s}(\mathbf{R}^d)$ is considered.

In this paper, we are concerned with compactness properties of sequence (u_n). It is not difficult to find examples of equations of type (1) such that the sequence (u_n) does not converge strongly in $\mathrm{L}^s_{\mathrm{loc}}(\mathbf{R}^m \times \mathbf{R}^d)$ for any $s \geq 1$. Indeed, a trivial example $u_n = \sin n\mathbf{p}$ solving (1) with coefficients being independent of $\mathbf{x} \in \mathbf{R}^d$ and $\alpha_k \in \mathbf{N}$, $k = 1, \dots, d$, does not converge strongly in L^s_{loc} for any $s \geq 1$.

Still, from the viewpoint of applications, it is almost always enough to analyse the sequence (u_n) averaged with respect to the velocity variable ($\int_{\mathbf{R}^m} \rho(\mathbf{p}) u_n(\mathbf{x}, \mathbf{p}) d\mathbf{p}$), $\rho \in \mathrm{C}_{\mathrm{c}}(\mathbf{R}^m)$ (see e.g. famous papers [**12, 24**]) which, as firstly noticed by Agoshkov [**1**] in the homogeneous hyperbolic case, can be strongly precompact in $\mathrm{L}^s_{\mathrm{loc}}(\mathbf{R}^d)$ for an appropriate $s \geq 1$ even when the sequence $(u_n(\mathbf{x}, \mathbf{p}))$ is not. Such results are usually called velocity averaging lemmas.

After Agoshkov's paper, the investigations in this directions continued rather intensively. Still, in most of the previous works on the subject, symbol $P(i\boldsymbol{\xi}, \mathbf{x}, \mathbf{p})$ of the differential operator $\mathcal{P}$ was of the first order and independent of $\mathbf{x} \in \mathbf{R}^d$. Thus the corresponding equation describes a transport process occurring in a homogeneous medium. On the other hand, most of natural phenomena take place in heterogeneous media (flow in heterogeneous porous media, sedimentation processes, blood flow, gas flow in a variable duct, etc). However, it appears that it is much more complicated to work on heterogeneous transport equations than on homogeneous ones.

This fact could be explained by the following simple observation. Assume that the coefficients in (1) do not depend on $\mathbf{x} \in \mathbf{R}^d$. If we apply the Fourier transform in $\mathbf{x} \in \mathbf{R}^d$ on equation (1), at least informally, we can separate solutions (u_n) and the known coefficients. To be more precise, let us consider the sequence of homogeneous transport equations from [**32**]:

$$\partial_t u_n + a(\mathbf{p}) \cdot \nabla_{\mathbf{x}} u_n = \sum_{j=1}^{d} \partial_{x_j} \partial_{\mathbf{p}}^{\boldsymbol{\kappa}} g_j^n, \quad (t, \mathbf{x}, \mathbf{p}) \in \mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d, \tag{3}$$

where, for some $s > 1$, $u_n \rightharpoonup 0$ weakly in $\mathrm{L}^s(\mathbf{R}^{d+1})$, while $g_j^n \to 0$ strongly in $\mathrm{L}^s_{\mathrm{loc}}(\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d)$, $j = 1, \dots, d$. The function $a : \mathbf{R}^d \to \mathbf{R}^d$ is continuous.

By finding the Fourier transform of (3) with respect to $(t, \mathbf{x}) \in \mathbf{R}^+ \times \mathbf{R}^d$ (denoted by $\hat{}$ below), we conclude from the above

$$(\tau + a(\mathbf{p}) \cdot \boldsymbol{\xi})\hat{u} = \sum_{j=1}^{d} \xi_j \partial_{\mathbf{p}}^{\boldsymbol{\kappa}} \hat{g}_j,$$

and from here, for any $\beta > 0$,

$$\hat{u} = \frac{\beta^2 |\boldsymbol{\xi}|^2 \hat{u} + \sum_{j=1}^{d} (\tau + a(\mathbf{p}) \cdot \boldsymbol{\xi}) \xi_j \partial_{\mathbf{p}}^{\boldsymbol{\kappa}} \hat{g}_j}{(\tau + a(\mathbf{p}) \cdot \boldsymbol{\xi})^2 + \beta^2 |\boldsymbol{\xi}|^2}.$$

As the term containing $\hat{u}$ on the right-hand side can be controlled by constant β, it was proved in [**32**] that the sequence of averaged quantities $(\int_{\mathbf{R}^m} \rho(\mathbf{p}) u_n(t, \mathbf{x}, \mathbf{p}) d\mathbf{p})$, $\rho \in \mathrm{L}^{s'}(\mathbf{R}^m)$, $1/s + 1/s' = 1$, converges to zero strongly in $\mathrm{L}^s(\mathbf{R}^{d+1})$.

Actually, such framework is probably the main approach used on the subject [**10, 13, 17, 36**]. Other approaches include the use of wavelet decomposition [**11**], "real-space methods" in time [**7, 39**] and "real-space methods" in space using the Radon transform [**9, 41**], X-transform [**21**], duality based dispersion estimates [**18**], etc.

In the heterogeneous case, the method applied on (3) is not at our disposal (since $\hat{u}$ can not be separated). Probably the only possible way to tackle the heterogeneous velocity averaging problem is through a variant of defect measures [**5, 16, 26, 31, 37**]. In [**16**, Theorem 2.5] the concrete application of defect measures on the averaging lemmas can be found. The result from [**16**] claims that the sequence of solutions (u_n) of equations (1) satisfying conditions a)–c) with $\alpha_1 = \alpha_2 = \cdots = \alpha_d \in \mathbf{N}$ and $s = 2$, is such that the sequence of averaged quantities $(\int u_n(\mathbf{x}, \mathbf{p}) \rho(\mathbf{p}) d\mathbf{p})$ strongly converges to zero in $\mathrm{L}^2(\mathbf{R}^d)$.

In this paper, we shall generalise Gerard's result on a wider class of equations, and we shall allow the coefficients to be discontinuous if the solutions u_n are from $\mathrm{L}^2(\mathbf{R}^m; \mathrm{L}^s(\mathbf{R}^d))$ for $s > 2$ (which is the situation in a numerous applications; e.g. [**6, 25, 29**]). We remark again that the result from [**32**] can be applied only in the case of homogeneous transport equations, but it is optimal in the sense that a sequence of solutions can belong to $\mathrm{L}^s(\mathbf{R}^{d+1} \times \mathbf{R}^d)$ for any $s > 1$ (in the current contribution, we must have $s \geq 2$).

Let us now describe defect measures that we are going to use. A defect measure is an object describing loss of compactness of a family of functions. Originally, the notion of the defect measure was systematically studied for sequences satisfying elliptic estimates by P.L.Lions [**23**]. Since elliptic estimates automatically eliminate oscillations, the defect measures used in [**23**] were not appropriate enough for studying loss of compactness caused by oscillations, which typically appear in the case of e.g. hyperbolic problems.

In order to control oscillations, a natural idea was to introduce an object which distinguishes oscillations of different frequencies. The idea was formalised by P. Gerard [**16**] and independently by L. Tartar [**37**]. P. Gerard named the appropriate defect measure as the microlocal defect measure (mdm in the sequel), while L. Tartar used the term H-measure. Let us recall Tartar's theorem introducing the H-measures.

THEOREM 2. [**37**] *If* $(\mathsf{u}_n) = ((u_n^1, \dots, u_n^r))$ *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^r)$ *such that* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^r)$, *then there exists its subsequence* $(\mathsf{u}_{n'})$ *and a positive definite matrix of complex Radon measures* $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\dots,r}$ *on* $\mathbf{R}^d \times S^{d-1}$ *such that for all* $\varphi_1, \varphi_2 \in C_0(\mathbf{R}^d)$ *and* $\psi \in C(S^{d-1})$

$$
\begin{aligned}
\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u^i_{n'})(\mathbf{x})\overline{(\varphi_2 u^j_{n'})(\mathbf{x})} d\mathbf{x} &= \langle \mu^{ij}, \varphi_1 \overline{\varphi_2} \psi \rangle \\
&= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \overline{\varphi_2(\mathbf{x})} \psi(\boldsymbol{\xi}) d\mu^{ij}(\mathbf{x}, \boldsymbol{\xi}), \quad i,j = 1, \dots, r,
\end{aligned} \tag{4}
$$

where $\mathcal{A}_\psi$ *is a multiplier operator with symbol* $\psi \in C(S^{d-1})$ *(see Definition 3).*

Gérard's approach generalises the above results to L^2-sequences taking values in an infinite-dimensional, separable Hilbert space H. In the case when $H = \mathrm{L}^2(\mathbf{R}^m)$, Gérard's mdm is an object belonging to $\mathcal{M}_+(S^\star\Omega, \mathcal{L}^1(H))$, i.e. to the space of non-negative Radon measures on the cospherical bundle $S^\star\Omega$ (the set $\Omega \times S^{d-1}$ endowed with the natural structure of manifold) with values in the space of trace class operators on H. It is important to mention an extension of H-measures in the case of sequences which (basically) have the form $(\mathrm{sgn}(\lambda - u_n(\mathbf{x})))$, $u_n \in \mathrm{L}^\infty(\mathbf{R}^d)$, given by Panov [**28**]. There, it was proved that for almost every $\lambda_1, \lambda_2 \in \mathbf{R}$ there exists a measure $\mu^{\lambda_1\lambda_2}$ defined by (4) for $u^{\lambda_i}(\mathbf{x}) = \mathrm{sgn}(\lambda_i - u_n(\mathbf{x}))$, $i = 1, 2$. This notion appeared to be very useful, and it was successfully applied in many recent papers [**2, 3, 20, 26, 31, 30, 33**]. Here, we extend Panov's results to sequences belonging to $\mathrm{L}^2(\mathbf{R}^m; \mathrm{L}^s(\mathbf{R}^d))$, $s \geq 2$.

Moreover, our result represents a generalisation of the original H-measures from two aspects. First, test functions (in applications these are given by coefficients entering equations of interest) in our case can be more general, even discontinuous with respect to the space variable. Second, our generalisation of the H-measures is constructed for use on a large class of equations (unlike original H-measures [**16, 37**] which were adapted only for hyperbolic type problems).

In a view of the last observation, remark that parabolic [**4, 5**], and ultra-parabolic [**31**] variants of the H-measures, and finally the H-measures adapted to large class of manifolds [**26**] were introduced. The last one is the main tool used in this paper. Its description, as well as the introduction to the main result (Theorem 7) is given in the next section.

In Section 3 we shall further develop the H-measure concept, which will be used in Section 4 for proving the precompactness property of a sequence of solutions to (1). The proof is based on a special (trivial) form of the variant H-measure corresponding to the sequence (u_n).

In Section 5 we shall apply our result on ultra-parabolic equations with discontinuous flux under different assumptions on coefficients than the ones from [**29**] (which is the most up-to-date result and which comprises the results from [**30**]).

2. Statement of the main result

To formulate the main result of the paper, we need to introduce the variant of H-measures that we are going to use. First, we need some auxiliary notions.

DEFINITION 3. A multiplier operator $\mathcal{A}_\psi : \mathrm{L}^2(\mathbf{R}^d) \to \mathrm{L}^2(\mathbf{R}^d)$ associated to a bounded function $\psi \in \mathrm{C}_b(\mathbf{R}^d)$ (see e.g. [**35**]), is a mapping by

$$\mathcal{A}_\psi(u) = \bar{\mathcal{F}}(\psi\hat{u}),$$

where $\hat{u}(\boldsymbol{\xi}) = \mathcal{F}(u)(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} u(x)dx$ is the Fourier transform while $\bar{\mathcal{F}}$ (or $^\vee$) is the inverse Fourier transform.

If the multiplier operator $\mathcal{A}_\psi$ satisfies

$$\|\mathcal{A}_\psi(u)\|_{\mathrm{L}^p} \leq C\|u\|_{\mathrm{L}^p}, \qquad u \in \mathrm{L}^p(\mathbf{R}^d) \cap \mathrm{L}^2(\mathbf{R}^d),$$

where C is a positive constant, then the function ψ is called the L^p-multiplier.

Let l be a minimal number such that $l\alpha_k > d$ for each k. We shall introduce the following manifolds, denoted by P and determined by the order of the derivatives from (1):

$$\mathrm{P} = \{\boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{l\alpha_k} = 1\}. \tag{5}$$

On such manifolds, which are smooth according to the choice of l, we shall define the necessary H-measures. Remark that it can seem more natural to take $\mathrm{P} = \{\boldsymbol{\xi} \in \mathbf{R}^d : \sum_{k=1}^{d} |\xi_k|^{\alpha_k} = 1\}$ but the latter manifold is not smooth enough. Namely, we shall need the following corollary of the Marzinkiewicz multiplier theorem [**35**, Theorem IV.6.6']:

LEMMA 4. *Suppose that $\psi \in \mathrm{C}^d(\mathbf{R}^d\backslash\{0\})$ is such that for some constant $C > 0$ it holds*

$$|\boldsymbol{\xi}^{\boldsymbol{\beta}}\partial^{\boldsymbol{\beta}}\psi(\boldsymbol{\xi})| \leq C, \quad \boldsymbol{\xi} \in \mathbf{R}^d\backslash\{0\} \tag{6}$$

for every multi-index $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbf{Z}^d_+$ such that $|\boldsymbol{\beta}| = \beta_1 + \beta_2 + \cdots + \beta_d \leq d$. Then, the function ψ is an L^p-multiplier for $p \in \langle 1, \infty\rangle$, and the operator norm of $\mathcal{A}_\psi$ depends only on C, p and d.

The next lemma is an easy corollary of Lemma 4. First, denote by

$$\pi_{\mathrm{P}}(\boldsymbol{\xi}) = \left(\frac{\xi_1}{\left(\xi_1^{l\alpha_1} + \cdots + \xi_d^{l\alpha_d}\right)^{1/l\alpha_1}}, \dots, \frac{\xi_d}{\left(\xi_1^{l\alpha_1} + \cdots + \xi_d^{l\alpha_d}\right)^{1/l\alpha_d}}\right), \quad \boldsymbol{\xi} \in \mathbf{R}^d\backslash\{0\},$$

a projection of $\mathbf{R}^d\backslash\{0\}$ on P. The following result holds.

Lemma 5. *For any* $\psi \in \mathrm{C}^d(\mathrm{P})$*, the composition* $\psi \circ \pi_{\mathrm{P}}$ *is an* L^p*-multiplier,* $p \in \langle 1, \infty\rangle$*, and the norm of the corresponding multiplier operator depends on* $\|\psi\|_{\mathrm{C}^d(\mathrm{P})}$*,* p *and* d*.*

Proof: Due to the Faá di Bruno formula, it is enough to prove that the conditions of Lemma 4 are satisfied for $\pi_k(\boldsymbol{\xi}) = \frac{\xi_k}{\left(\xi_1^{l\alpha_1}+\cdots+\xi_d^{l\alpha_d}\right)^{1/l\alpha_k}}$, $k = 1, \dots, d$.

The statement will be proved by the induction argument.

- $n = 1$

  In this case, we compute

$$\partial_j \pi_k(\boldsymbol{\xi}) = \begin{cases} -\frac{\alpha_j}{\alpha_k}\frac{1}{\xi_j}\pi_k(\boldsymbol{\xi})\pi_j^{l\alpha_j}(\boldsymbol{\xi}), & j \neq k \\ -\frac{1}{\xi_k}\pi_k(\boldsymbol{\xi})\left(1 - \pi_k^{l\alpha_k}(\boldsymbol{\xi})\right), & j = k. \end{cases}$$

  and it obviously holds $|\xi_j \partial_j \pi_k(\boldsymbol{\xi})| \leq C$.

- $n = m$

  Our inductive hypothesis is

$$\partial^{\boldsymbol{\beta}} \pi_k(\boldsymbol{\xi}) = \frac{1}{\boldsymbol{\xi}^{\boldsymbol{\beta}}} P_{\boldsymbol{\beta}}(\pi_1(\boldsymbol{\xi}), \dots, \pi_d(\boldsymbol{\xi})), \quad |\boldsymbol{\beta}| = m, \tag{7}$$

  for a polynomial $P_{\boldsymbol{\beta}}$.

- $n = m + 1$

  To prove that (7) holds for $|\boldsymbol{\beta}| = m + 1$ it is enough to notice that $\boldsymbol{\beta} = \mathsf{e}_j + \boldsymbol{\beta}'$, where $|\boldsymbol{\beta}'| = m$, and to notice

$$\partial^{\boldsymbol{\beta}} \pi_k(\boldsymbol{\xi}) = \partial_j \partial^{\boldsymbol{\beta}'} \pi_k(\boldsymbol{\xi}) = \partial_j \left(\frac{1}{\boldsymbol{\xi}^{\boldsymbol{\beta}'}} P_{\boldsymbol{\beta}'}(\pi_1(\boldsymbol{\xi}), \dots, \pi_d(\boldsymbol{\xi})) \right)$$

  and from here, repeating the procedure from the case $n = 1$, we conclude that (7) holds for $n = m + 1$.

From here, (6) immediately follows for π_k and consequently for $\psi \circ \pi_{\mathrm{P}}$.

❑

To proceed, we introduce a family of curves

$$\eta_k = \xi_k t^{1/l\alpha_k}, \ t \in \mathbf{R}^+, \tag{8}$$

by points $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathrm{P}$. They are disjoint and fibrate entire space $\mathbf{R}^d$. They play the same role as the rays $\boldsymbol{\xi}/|\boldsymbol{\xi}|$ in the definition of the H-measures. Moreover, we see that the curves (8) respect the scaling given by the differential operator from (1). Indeed, if we have the classical situation $\alpha_k = 1$, $k = 1, \dots, d$, then curves (8) are rays and we can use the classical H-measures [**16, 37**].

The following theorem is essentially proved in [**26**], but here we provide its more elegant proof based on the ideas of L. Tartar.

Theorem 6. *For fixed* $\alpha_k > 0$*,* $k = 1, \dots, d$*, denote by* P *the manifold given by* (5)*, and by* $\pi_{\mathrm{P}} : \mathbf{R}^d \to \mathrm{P}$ *projection on the manifold* P *along the fibres* (8)*. If* $(\mathsf{u}_n) = ((u_n^1, \dots, u_n^r))$ *is a sequence in* $\mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^r)$ *such that* $\mathsf{u}_n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there exists its subsequence* $(u_{n'})$ *and a positive definite matrix of complex Radon measures* $\boldsymbol{\mu} = \{\mu^{ij}\}_{i,j=1,\dots,d}$ *from* $\mathcal{M}_b(\mathbf{R}^d \times \mathrm{P})$ *such that for all* $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$

and $\psi \in \mathrm{C}(\mathrm{P})$

$$
\begin{aligned}
(9)\qquad \lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{A}_{\psi_{\mathrm{P}}}(\varphi_1 u^i_{n'})(\mathbf{x})\overline{(\varphi_2 u^j_{n'})(\mathbf{x})}dx &= \langle \mu^{ij},\varphi_1\overline{\varphi_2}\psi\rangle \\
&= \int_{\mathbf{R}^d\times\mathrm{P}}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu^{ij}(\mathbf{x},\boldsymbol{\xi}),\quad (\mathbf{x},\boldsymbol{\xi})\in\mathbf{R}^d\times\mathrm{P},
\end{aligned}
$$

where $\mathcal{A}_{\psi_{\mathrm{P}}}$ *is a multiplier operator with the symbol* $\psi_{\mathrm{P}} := \psi\circ\pi_{\mathrm{P}}$.

The measure $\boldsymbol{\mu}$ *we call the* H_{P}*-measure corresponding to the sequence* (u_n).

Proof: First, we shall prove that the fibration (8) satisfies conditions of the variant of the first commutation lemma [**38**, Lemma 28.2]. More precisely, we shall prove that any symbol ψ on the manifold P satisfies

$$
(10)\qquad \begin{aligned}
&\left(\forall\, r,\varepsilon\in\mathbf{R}^+\right)\quad\left(\exists\, M\in\mathbf{R}^+\right)\\
&|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|\le r,\ |\boldsymbol{\eta}_1|,|\boldsymbol{\eta}_2|>M \quad\Longrightarrow\quad |\psi(\pi_{\mathrm{P}}(\boldsymbol{\eta}_1))-\psi(\pi_{\mathrm{P}}(\boldsymbol{\eta}_2))|\le\varepsilon,
\end{aligned}
$$

where π_{P} is the projection on the manifold P along the fibres (8).

As ψ is an uniformly continuous on P, it is enough to show that for fixed r and ε, the difference $|\pi_{\mathrm{P}}(\boldsymbol{\eta}_1)-\pi_{\mathrm{P}}(\boldsymbol{\eta}_2)|$ is arbitrary small for M large enough. According to the mean value theorem

$$
|\pi_{\mathrm{P}}(\boldsymbol{\eta}_1)-\pi_{\mathrm{P}}(\boldsymbol{\eta}_2)|\le|\nabla\pi_{\mathrm{P}}(\boldsymbol{\zeta})||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|,
$$

where $\boldsymbol{\zeta}=\vartheta\boldsymbol{\eta}_1+(1-\vartheta)\boldsymbol{\eta}_2$ for some $\vartheta\in\langle 0,1\rangle$, and the statement follows as $\nabla\pi_{\mathrm{P}}(\boldsymbol{\eta})$ tends to zero when $|\boldsymbol{\eta}|$ approaches infinity.

Now, we can use [**38**, Lemma 28.2] to conclude that the mappings

$$
(\varphi_1\overline{\varphi_2},\psi)\mapsto\lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{A}_{\psi_{\mathrm{P}}}(\varphi_1 u^i_{n'})(\mathbf{x})\overline{(\varphi_2 u^j_{n'})(\mathbf{x})}d\mathbf{x},\quad i,j=1,\dots,d,
$$

form a positive definite matrix of bilinear functionals on $\mathrm{C}_0(\mathbf{R}^d)\times\mathrm{C}(\mathrm{P})$. According to the Schwartz kernel theorem, the functionals can be extended to a continuous linear functionals on $\mathcal{D}(\mathbf{R}^d\times\mathrm{P})$. Due to its non-negative definiteness, the Schwartz theorem on non-negative distributions [**34**, Theorem I.V] provides its extension on the Radon measures.

❑

Notice that, using the Plancherel theorem, (9) can be conveniently rewritten via the Fourier transform as follows:

$$
\begin{aligned}
\lim_{n'\to\infty}\int_{\mathbf{R}^d}\mathcal{F}(\varphi_1 u^i_{n'})(\boldsymbol{\xi})\overline{\mathcal{F}(\varphi_2 u^j_{n'})(\boldsymbol{\xi})}\,\psi\circ\pi_{\mathrm{P}}(\boldsymbol{\xi})d\boldsymbol{\xi} &= \langle\mu^{ij},\varphi_1\overline{\varphi_2}\psi\rangle\\
&=\int_{\mathbf{R}^d\times\mathrm{P}}\varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu^{ij}(\mathbf{x},\boldsymbol{\xi}).
\end{aligned}
$$

Now, we can formulate the main theorem of the paper.

Theorem 7. *Assume that* $u_n \rightharpoonup 0$ *weakly in* $\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^s(\mathbf{R}^d))\cap\mathrm{L}^2(\mathbf{R}^{m+d})$, $s\ge 2$, *where* u_n *represent weak solutions to* (1) *in the sense of Definition 1.*

Furthermore, for $s=2$ *we assume that for every* $(\mathbf{x},\boldsymbol{\xi})\in\mathbf{R}^d\times\mathrm{P}$

$$
(11)\qquad A(\mathbf{x},\boldsymbol{\xi},\mathbf{p}):=\sum_{k=1}^d a_k(\mathbf{x},\mathbf{p})(2\pi i\xi_k)^{\alpha_k}\neq 0\quad(\text{a.e. }\mathbf{p}\in\mathbf{R}^m)\,.
$$

If $s > 2$, the last assumption is reduced to almost every $\mathbf{x} \in \mathbf{R}^d$ and every $\boldsymbol{\xi} \in \mathrm{P}$.

Then, for any $\rho \in \mathrm{L}^2_c(\mathbf{R}^m)$,

$$\int_{\mathbf{R}^m} u_n(\mathbf{x},\mathbf{p})\rho(\mathbf{p})d\mathbf{p} \longrightarrow 0 \quad \textit{strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d).$$

Before we continue, remark that the conditions of the theorem can be relaxed by assuming that (u_n) is merely bounded in $\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^s(\mathbf{R}^d))$, while (G_n) strongly precompact in $\mathrm{L}^2(\mathbf{R}^m;\mathrm{W}^{-\boldsymbol{\alpha},s'}(\mathbf{R}^d))$. In that case there exists a subsequence $(u_{n'})$ such that for any $\rho \in \mathrm{L}^2_c(\mathbf{R}^m)$ the sequence $(\int_{\mathbf{R}^m} \rho(\mathbf{p})u_{n'}(\mathbf{x},\mathbf{p})d\mathbf{p})$ converges toward $\int_{\mathbf{R}^m} \rho(\mathbf{p})u(\mathbf{x},\mathbf{p})d\mathbf{p}$, where u denotes the weak limit of (u_n').

3. Auxiliary results

In this section, we shall extend Theorem 6 on sequences with uncountable indexing. A similar procedure we used in the case of the parabolic variant H-measures [**22**], and for the sake of completeness we reproduce some results here. These will be substantially extended by Proposition 12 and Theorem 13 containing representation results of $\mathrm{H_P}$-measures associated to sequences of functions $u_n \in \mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^s(\mathbf{R}^d))$, $s > 2$, which turn to be crucial for the proof of the main theorem.

Let us take an arbitrary sequence of functions (u_n) in variables $\mathbf{x} \in \mathbf{R}^d$ and $\mathbf{p} \in \mathbf{R}^m$, weakly converging to zero in $\mathrm{L}^2(\mathbf{R}^m \times \mathbf{R}^d)$. Introduce a regularising kernel $\omega \in \mathrm{C}^\infty_c(\mathbf{R}^m)$, where ω is a non-negative smooth function with total mass one. For $k \in \mathbf{N}$ denote $\omega_k(\mathbf{p}) = k^m\omega(k\mathbf{p})$ and convolute it with $(u_n(\mathbf{x},\mathbf{p}))$ in $\mathbf{p}$:

$$u_n^k(\mathbf{x},\mathbf{p}) := \Big(u_n(\mathbf{x},\cdot) * \omega_k\Big)(\mathbf{p}) = \int_{\mathbf{R}^m} u_n(\mathbf{x},\mathbf{y})\omega_k(\mathbf{p}-\mathbf{y})d\mathbf{y}.$$

By the Young inequality functions u_n^k are bounded in $\mathrm{L}^2(\mathbf{R}^{m+d})$ uniformly with respect to both k and n. Meanwhile, for every fixed k, sequence of functions $u_n^k(\cdot,\mathbf{p})$ is bounded in $\mathrm{L}^2(\mathbf{R}^d)$, uniformly in $\mathbf{p}$, and converges weakly to zero. Furthermore, u_n^k are Lipschitz continuous as functions from $\mathbf{R}^m$ to $\mathrm{L}^2(\mathbf{R}^d)$, with an n-independent Lipschitz constant. Having all this in mind, we can prove the following lemma.

Lemma 8. *There exists a subsequence $(u_{n'})$ of the sequence (u_n), and a family $\{\mu_k^{\mathbf{pq}} : \mathbf{p},\mathbf{q} \in \mathbf{R}^m\}$ of $\mathrm{H_P}$-measures on $\mathbf{R}^d \times \mathrm{P}$ such that for every $k \in \mathbf{N}$, $\varphi_i \in C_0(\mathbf{R}^d)$, $i = 1,2$, and $\psi \in C(\mathrm{P})$:*

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\mathcal{A}_{\psi_\mathrm{P}}\, \varphi_1 u_{n'}^k(\cdot,\mathbf{p})\Big)(\mathbf{x})\, \overline{\varphi_2(\mathbf{x})u_{n'}^k(\mathbf{x},\mathbf{q})}d\mathbf{x} = \int_{\mathbf{R}^d\times\mathrm{P}} \varphi_1(\mathbf{x})\overline{\varphi_2(\mathbf{x})}\psi(\boldsymbol{\xi})d\mu_k^{\mathbf{pq}}(\mathbf{x},\boldsymbol{\xi}). \tag{12}$$

Proof: According to Theorem 6, for fixed $\mathbf{p},\mathbf{q} \in \mathbf{R}^m$ and $k \in \mathbf{N}$, there exist a subsequence of (u_n) and corresponding complex Radon measure $\mu_k^{\mathbf{pq}}$ over $\mathbf{R}^d \times \mathrm{P}$ such that (12) holds. Using the diagonalisation procedure, we conclude that for a countable dense subset $D \times D \subset \mathbf{R}^m \times \mathbf{R}^m$ there exists a subsequence $(u_{n'}) \subset (u_n)$ such that (12) holds for every $(\mathbf{p},\mathbf{q}) \in D \times D$ and every $k \in \mathbf{N}$.

Let us take an arbitrary $k \in \mathbf{N}$ and $(\mathbf{p},\mathbf{q}) \in \mathbf{R}^m \times \mathbf{R}^m$. Let $(\mathbf{p}_m,\mathbf{q}_m)$ be a sequence in $D \times D$ converging to $(\mathbf{p},\mathbf{q})$. The sequence $(\mathbf{p}_m,\mathbf{q}_m)$ defines sequence of Radon measures $(\mu_k^{\mathbf{p}_m\mathbf{q}_m})$, which is bounded in $\mathcal{M}_b(\mathbf{R}^d \times \mathrm{P})$, due to the bounds of (u_n^k) in $\mathrm{L}^\infty(\mathbf{R}^m;\mathrm{L}^2(\mathbf{R}^d))$. Therefore, there exists a complex Radon measure $\mu_k^{\mathbf{pq}}$

such that, along a subsequence, $\mu_k^{\mathbf{p}_m\mathbf{q}_m} \rightharpoonup \mu_k^{\mathbf{pq}}$. Thus for arbitrary test functions $\varphi = \varphi_1\bar{\varphi}_2$ and ψ we have:

$$\begin{aligned}\int \varphi(\mathbf{x})\psi(\boldsymbol{\xi})\, d\mu_k^{\mathbf{pq}}(\mathbf{x},\boldsymbol{\xi}) &= \lim_m \int \varphi(\mathbf{x})\psi(\boldsymbol{\xi})\, d\mu_k^{\mathbf{p}_m\mathbf{q}_m}(\mathbf{x},\boldsymbol{\xi}) \\ &= \lim_m \lim_{n'} V_{n'}^k(\mathbf{p}_m,\mathbf{q}_m),\end{aligned} \tag{13}$$

where V_n^k denotes the function by

$$V_n^k(\mathbf{p},\mathbf{q}) := \int\limits_{\mathbf{R}^d} \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\, \varphi_1 u_n^k(\cdot,\mathbf{p})\Big)(\mathbf{x})\, \overline{\varphi_2(\mathbf{x}) u_n^k(\mathbf{x},\mathbf{q})} d\mathbf{x}\,. \tag{14}$$

On the other hand

$$\begin{aligned} V_{n'}^k(\mathbf{p}_m,\mathbf{q}_m) - V_{n'}^k(\mathbf{p},\mathbf{q}) &= V_{n'}^k(\mathbf{p}_m,\mathbf{q}_m) - V_{n'}^k(\mathbf{p},\mathbf{q}_m) + V_{n'}^k(\mathbf{p},\mathbf{q}_m) - V_{n'}^k(\mathbf{p},\mathbf{q}) \\ &\le C(k)\Big(|\mathbf{p}_m - \mathbf{p}|_{\mathbf{R}^m} + |\mathbf{q}_m - \mathbf{q}|_{\mathbf{R}^m}\Big),\end{aligned}$$

where on the last step we combined the Cauchy-Schwartz inequality, boundedness of the multiplier $\mathcal{A}_{\psi_{\mathrm{P}}}$ on $\mathrm{L}^2(\mathbf{R}^d)$, and the Lipschitz continuity of the functions u_n^k. The constant $C(k)$ appearing above is independent of n', and we can exchange limits in (13). This actually means that the functional $\mu_k^{\mathbf{pq}}$ does not depend on the defining subsequence (i.e. it is well for every $\mathbf{p},\mathbf{q} \in \mathbf{R}^m$), which completes the proof. ❑

Using the previous assertion, we prove the existence of $\mathrm{H_P}$-measures associated to functions taking values in $\mathrm{L}^2(\mathbf{R}^m)$. First, we need to recall a few basic notions of L^2 functions taking values in an arbitrary Banach space E.

We say that $f : \mathbf{R}^m \to E'$ is weakly $*$ measurable if it is measurable with respect to weak $*$ $\sigma(E', E)$ topology. The dual of $\mathrm{L}^2(\mathbf{R}^m, E)$ corresponds to the Banach space $\mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^m; E')$ of weakly $*$ measurable functions $f : \mathbf{R}^m \to E'$ such that $\int_{\mathbf{R}^m} \|f(\mathbf{x})\|^2_{E'} d\mathbf{x} < \infty$ (for details see [**14**, p. 606]).

By taking $E = \mathrm{C}_0(\mathbf{R}^d \times \mathrm{P})$, the topological dual of $\mathrm{L}^2(\mathbf{R}^{2m}; \mathrm{C}_0(\mathbf{R}^d \times \mathrm{P}))$ corresponds to the Banach space $\mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^d \times \mathrm{P}))$ of weakly $*$ measurable functions $\mu : \mathbf{R}^{2m} \to \mathcal{M}_b(\mathbf{R}^d \times \mathrm{P})$ such that $\int_{\mathbf{R}^{2m}} \|\mu(\mathbf{p},\mathbf{q})\|^2 d\mathbf{p}d\mathbf{q} < \infty$.

Theorem 9. *For the subsequence $(u_{n'}) \subseteq (u_n)$ extracted in Lemma 8, there exists a measure $\mu \in \mathrm{L}^2_{\mathrm{w}*}(\mathbf{R}^{2m}; \mathcal{M}_b(\mathbf{R}^d \times \mathrm{P}))$ such that for all $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$, $\varphi_i \in C_0(\mathbf{R}^d)$, $i = 1, 2$, and $\psi \in C(\mathrm{P})$:*

$$\begin{aligned}\lim_{n'} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} v(\mathbf{p},\mathbf{q})\Big(\mathcal{A}_{\psi_{\mathrm{P}}}\, \varphi_1 u_{n'}(\cdot,\mathbf{p})\Big)(\mathbf{x})\, \overline{\varphi_2(\mathbf{x}) u_{n'}(\mathbf{x},\mathbf{q})} d\mathbf{x} d\mathbf{p} d\mathbf{q} \\ = \int\limits_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\, \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1\bar{\varphi}_2 \otimes \psi\rangle d\mathbf{p}d\mathbf{q}.\end{aligned} \tag{15}$$

Remark 10. Notice that the new object has inherited the hermitian character of H-measures. Indeed, with the help of Plancherel's theorem, we can rewrite (15) as

$$\lim_{n'}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d} v(\mathbf{p},\mathbf{q})\psi(\boldsymbol{\xi})\mathcal{F}(\varphi_1 u_{n'}(\cdot,\mathbf{p}))(\boldsymbol{\xi})\,\overline{\mathcal{F}(\varphi_2(\cdot)u_{n'}(\cdot,\mathbf{q}))(\boldsymbol{\xi})}d\boldsymbol{\xi}d\mathbf{p}d\mathbf{q}$$
$$= \int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q},\cdot,\cdot),\varphi_1\bar\varphi_2\otimes\psi\rangle d\mathbf{p}d\mathbf{q}\,,$$

from which it easily follows that

$$\mu(\mathbf{p},\mathbf{q},\cdot,\cdot)=\overline{\mu(\mathbf{q},\mathbf{p},\cdot,\cdot)}\,.$$

Also, notice that we can take $\varphi_1\in \mathrm{C}_b(\mathbf{R}^d)$ (since $\varphi_2\in C_0(\mathbf{R}^d)$).

Proof: For fixed test functions $\varphi_{1,2}$ and ψ, similarly to (14), we denote

$$F_k(\mathbf{p},\mathbf{q}):=\lim_{n'}V^k_{n'}(\mathbf{p},\mathbf{q})=\langle\mu_k^{\mathbf{pq}},\varphi_1\bar\varphi_2\psi\rangle.$$

Due to the uniform bound of u_n^k in $\mathrm{L}^2(\mathbf{R}^{m+d})$, the functions V_n^k belong to the space $\mathrm{L}^2(\mathbf{R}^{2m})$, with norm depending on $\|\varphi_{1,2}\|_{\mathrm{L}^\infty}$ and $\|\psi\|_{\mathrm{L}^\infty}$, but not on n and k. Thus the Fatou lemma asserts the sequence (F_k) is bounded in $\mathrm{L}^2(\mathbf{R}^{2m})$, as well.

Furthermore, for a fixed k, the sequence (V_n^k) is bounded in $\mathrm{L}^\infty(\mathbf{R}^{2m})$. By taking an arbitrary $v\in\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^{2m})$, we have

$$\begin{aligned}\lim_k\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})F_k(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}&=\lim_k\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})\lim_{n'}V^k_{n'}(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}\\&=\lim_k\lim_{n'}\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})V^k_{n'}(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}\end{aligned}\tag{16}$$

where on the last step we have used the Lebesgue dominated convergence theorem.

As the functions u_n^k are uniformly bounded in $\mathrm{L}^2(\mathbf{R}^{m+d})$, the sequence of averaged quantities $\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})V_n^k(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}$ converges to $\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})V_n(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}$ uniformly with respect to n, where V_n is similarly to V_n^k, with u_n^k replaced by u_n in (14).

Thus we can exchange the limits in (16) providing

$$\lim_k\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})F_k(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}=\lim_{n'}\int_{\mathbf{R}^{2m}} v(\mathbf{p},\mathbf{q})V_{n'}(\mathbf{p},\mathbf{q})d\mathbf{p}d\mathbf{q}\,.\tag{17}$$

On the other hand, the boundedness of (F_k) in $\mathrm{L}^2(\mathbf{R}^{2m})$ enables us to define a bounded sequence of operators $\mu_k\in\mathrm{L}^2_{\mathrm{w}^*}(\mathbf{R}^{2m};\mathcal{M}_b(\mathbf{R}^d\times\mathrm{P}))$:

$$\mu_k(\mathbf{p},\mathbf{q})(\phi):=\langle\mu_k^{\mathbf{pq}},\phi\rangle,\quad \phi\in\mathrm{C}_0(\mathbf{R}^d\times\mathrm{P}).$$

Therefore, there exists a subsequence $(\mu_{k'})\subseteq(\mu_k)$ such that $\mu_{k'}\xrightarrow{\ *\ }\mu$ in $\mathrm{L}^2_{\mathrm{w}^*}(\mathbf{R}^{2m};\mathcal{M}_b(\mathbf{R}^d\times\mathrm{P}))$. By passing to the limit on the left side of (17), we get the relation (15).

❑

Remark 11. Notice that the last theorem remains valid in the case when the test functions $\varphi_{1,2}$ depend on the velocity variable ($\mathbf{p}$ or $\mathbf{q}$) as well, i.e. when $\varphi_{1,2}$ are taken from the space $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m;\mathrm{C}_0(\mathbf{R}^d))$ (with function v removed from (15)). As it is enough to prove the statement for test functions from a dense set, we take arbitrary $\varphi_{1,2} \in \mathrm{L}^2_c(\mathbf{R}^m; \mathrm{C}_0(\mathbf{R}^d))$ compactly supported in $\mathbf{x}$ and approximate them by sums $\sum\limits_{l=1}^N v_1^l(\mathbf{p})\varphi_1^l(\mathbf{x})$ and $\sum\limits_{j=1}^N v_2^j(\mathbf{q})\varphi_2^j(\mathbf{x})$ such that

$$\begin{aligned}
&\|\sum_{l=1}^N v_1^l \otimes \varphi_1^l - \varphi_1\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{C}_0(\mathbf{R}^d))} \leq 1/N,\\
&\|\sum_{j=1}^N v_2^j \otimes \varphi_2^j - \varphi_2\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{C}_0(\mathbf{R}^d))} \leq 1/N.
\end{aligned}$$

Then it holds for any $\psi \in \mathrm{C}(\mathrm{P})$

$$\begin{aligned}
&\Big| \int\limits_{\mathbf{R}^{2m}} \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1(\cdot,\mathbf{p})\bar{\varphi}_2(\cdot,\mathbf{q}) \otimes \psi\rangle d\mathbf{p}d\mathbf{q}\\
&\qquad - \lim_{n'} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,(\varphi_1 u_{n'})(\cdot,\mathbf{p})\Big)(\mathbf{x})\,\overline{(\varphi_2 u_{n'})}(\mathbf{x},\mathbf{q}) d\mathbf{x}d\mathbf{p}d\mathbf{q}\Big|\\
&\leq \Big| \int\limits_{\mathbf{R}^{2m}} \Big\langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1(\cdot,\mathbf{p})\bar{\varphi}_2(\cdot,\mathbf{q}) \otimes \psi\\
&- \Big(\sum_{l=1}^N v_1^l(\mathbf{p}) \otimes \varphi_1^l\Big)\Big(\sum_{j=1}^N \overline{v_2^j(\mathbf{q}) \otimes \varphi_2^j}\Big) \otimes \psi \Big\rangle d\mathbf{p}d\mathbf{q}\Big|\\
&+ \Big| \lim_{n'} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} \Bigg(\sum_{l,j}^N v_1^l(\mathbf{p})\bar{v}_2^j(\mathbf{q})\Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,\varphi_1^l u_{n'}(\cdot,\mathbf{p})\Big)(\mathbf{x})\,\overline{(\varphi_2^j u_{n'})}(\mathbf{x},\mathbf{q})\\
&\qquad\qquad - \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,(\varphi_1 u_{n'})(\cdot,\mathbf{p})\Big)(\mathbf{x})\,\overline{(\varphi_2 u_{n'})}(\mathbf{x},\mathbf{q})\Bigg) d\mathbf{x}d\mathbf{p}d\mathbf{q}\Big|\\
&= \lim_{n'} \Big| \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} \sum_{l,j}^N \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,\big((v_1^l\varphi_1^l - \varphi_1)u_{n'}\big)\,(\cdot,\mathbf{p})\Big)(\mathbf{x})\overline{\Big((v_2^j\varphi_2^j - \varphi_2)u_{n'}\Big)}(\mathbf{x},\mathbf{q})d\mathbf{x}d\mathbf{p}d\mathbf{q}\Big|\\
&+ \lim_{n'} \Big| \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} \sum_{l}^N \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,\big((v_1^l\varphi_1^l - \varphi_1)u_{n'}\big)\,(\cdot,\mathbf{p})\Big)(\mathbf{x})\overline{(\varphi_2 u_{n'})}(\mathbf{x},\mathbf{q})d\mathbf{x}d\mathbf{p}d\mathbf{q}\Big|\\
&+ \lim_{n'} \Big| \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} \sum_{j}^N \Big(\mathcal{A}_{\psi_{\mathrm{P}}}\,\varphi_1 u_{n'}(\cdot,\mathbf{p})\Big)(\mathbf{x})\overline{\Big((v_2^j\varphi_2^j - \varphi_2)u_{n'}\Big)}(\mathbf{x},\mathbf{q})d\mathbf{x}d\mathbf{p}d\mathbf{q}\Big| + \mathcal{O}(1/N)\\
&= \mathcal{O}(1/N),
\end{aligned}$$

which proves the remark.

Now, we shall describe the object μ in Theorem 9 more precisely by showing that it can be represented as $\mu(\mathbf{p},\mathbf{q},\cdot) = f(\mathbf{p},\mathbf{q},\cdot)\nu$, where $\nu \in \mathcal{M}_b(\mathbf{R}^d \times \mathrm{P})$ is a positive Radon measure, and $f \in \mathrm{L}^2(\mathbf{R}^{2m}; \mathrm{L}^1(\mathbf{R}^d \times \mathrm{P} : \nu))$. If we could conclude that for every $\phi \in \mathrm{C}_0(\mathbf{R}^d \times \mathrm{P})$, the function $\langle \mu(\mathbf{p},\mathbf{q},\cdot), \phi\rangle$ represents a kernel of a trace class operator, then we could rely on [**16**, Proposition A.1.] to state the latter representation. The most famous sufficient condition for a function to be a kernel of a trace class operator is given by the Mercer theorem. It demands the kernel to be continuous, symmetric and positive definite. The function $\langle\mu(\mathbf{p},\mathbf{q},\cdot),\phi\rangle)$ has the last two properties, but it is not necessarily continuous. Therefore, we need the following proposition.

PROPOSITION 12. *The operator* $\mu\in \mathrm{L}^2_{\mathrm{w}^*}(\mathbf{R}^{2m};\mathcal{M}_b(\mathbf{R}^d\times \mathrm{P}))$ *in Theorem 9 has the form*

$$\mu(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})=f(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})\nu(\mathbf{x},\boldsymbol{\xi}), \tag{18}$$

where $\nu\in\mathcal{M}_b(\mathbf{R}^d\times \mathrm{P})$ *is a non-negative scalar Radon measure, while* f *is a function from* $\mathrm{L}^2(\mathbf{R}^{2m};\mathrm{L}^1(\mathbf{R}^d\times \mathrm{P}:\nu))$ *satisfying*

$$\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d\times\mathrm{P}}\rho(\mathbf{p})\bar\rho(\mathbf{q})\phi(\mathbf{x},\boldsymbol{\xi})f(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})d\nu(\mathbf{x},\boldsymbol{\xi})d\mathbf{p}d\mathbf{q}\geq 0$$

for any $\rho\in\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m)$, $\phi\in\mathrm{C}_0(\mathbf{R}^d\times\mathrm{P})$, $\phi\geq 0$.

Proof: The proof is based on rewriting the measure $\mu(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})$ via the basis in the (Hilbert) space $\mathrm{L}^2(\mathbf{R}^{2m})$.

Accordingly, let $\{e_i\}_{i\in\mathbf{N}}$ be an orthonormal basis in $\mathrm{L}^2(\mathbf{R}^m)$. Denote by $\mu_{ij}\in\mathcal{M}_b(\mathbf{R}^d\times\mathrm{P})$ an $\mathrm{H_P}$-measure generated by the sequences $\int_{\mathbf{R}^m}e_i(\mathbf{p})u_n(\mathbf{x},\mathbf{p})d\mathbf{p}$ and $\int_{\mathbf{R}^m}e_j(\mathbf{q})u_n(\mathbf{x},\mathbf{q})d\mathbf{q}$. We claim:

$$\mu(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})=\sum_{i,j=1}^{\infty}\mu_{ij}(\mathbf{x},\boldsymbol{\xi})\bar e_i(\mathbf{p})e_j(\mathbf{q}). \tag{19}$$

Indeed, take arbitrary $\rho\in\mathrm{L}^2(\mathbf{R}^{2m})$, $\varphi_{1,2}\in\mathrm{C}_0(\mathbf{R}^d)$, $\psi\in\mathrm{C}(\mathrm{P})$, and notice that

$$\rho(\mathbf{p},\mathbf{q})=\sum_{i,j=1}^{\infty}c_{ij}e_i(\mathbf{p})\bar e_j(\mathbf{q}),$$

where $(c_{ij})_{i,j\in\mathbf{N}}$ is a square sumable sequence.

According to the definition of the functional μ, we have

$$\begin{aligned}&\int_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\langle\mu(\mathbf{p},\mathbf{q},\cdot),\varphi_1\bar\varphi_2\psi\rangle d\mathbf{p}d\mathbf{q}\\&=\lim_{n\to\infty}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d}\rho(\mathbf{p},\mathbf{q})\Big(\mathcal{A}_\psi\,\varphi_1u_n(\cdot,\mathbf{p})\Big)(\mathbf{x})\,\bar\varphi_2\bar u_n(\mathbf{x},\mathbf{q})d\mathbf{x}d\mathbf{p}d\mathbf{q}\\&=\sum_{i,j=1}^{\infty}\lim_{n\to\infty}\int_{\mathbf{R}^d}c_{ij}\left(\mathcal{A}_\psi\,\varphi_1\int_{\mathbf{R}^m}u_n(\cdot,\mathbf{p})e_i(\mathbf{p})d\mathbf{p}\right)(\mathbf{x})\,\bar\varphi_2(\mathbf{x})\int_{\mathbf{R}^m}\bar u_n(\mathbf{x},\mathbf{q})\bar e_j(\mathbf{q})d\mathbf{q}d\mathbf{x}\\&=\sum_{i,j=1}^{\infty}c_{ij}\langle\mu_{ij}(\mathbf{x},\boldsymbol{\xi}),\varphi_1\bar\varphi_2\psi\rangle=\sum_{i,j=1}^{\infty}\langle\mu_{ij}(\mathbf{x},\boldsymbol{\xi}),\varphi_1\bar\varphi_2\psi\rangle\int_{\mathbf{R}^{2m}}\rho(\mathbf{p},\mathbf{q})\bar e_i(\mathbf{p})e_j(\mathbf{q})d\mathbf{p}d\mathbf{q},\end{aligned}$$

which completes the proof of (19). Remark that in the last derivation we have used the square integrability of the sequence (c_{ij}) and the Lebesgue dominated convergence theorem.

We introduce a positive bounded measure

$$\nu(\mathbf{x},\boldsymbol{\xi})=\sum_{i=1}^{\infty}\frac{1}{2^i}\mu_{ii}(\mathbf{x},\boldsymbol{\xi}),$$

as the weighted trace of the measure matrix $(\mu_{ij})_{i,j=1,\infty}$ and claim that

$$\mu(\mathbf{p},\mathbf{q},\cdot) << \nu,$$

for almost every $\mathbf{p},\mathbf{q} \in \mathbf{R}^m$. Indeed, if $\nu(E)=0$ for some Borel set $E \subset \mathbf{R}^d \times \mathrm{P}$, then $\mu_{ii}(E)=0$ for every $i \in \mathbf{N}$. On the other hand, due to the hermitian character of matrix $\mathrm{H_P}$ measures, we have

$$|\mu_{ij}(E)| \leq \mu_{ii}(E)^{1/2}\mu_{jj}(E)^{1/2}.$$

From here, it follows that $\mu_{ij}(E)=0$ for every $i,j \in \mathbf{N}$, and thus, according to (19), $\mu(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})(E)=0$ for almost every $\mathbf{p},\mathbf{q} \in \mathbf{R}^m$.

Now, the conclusion follows from the Radon-Nikodym theorem.

❑

Next, we shall make an extension of Theorem 9.

Notice that if in Theorem 2 we assume $u_n \in \mathrm{L}^s(\mathbf{R}^d)$ for some $s>2$, then the $\mathbf{R}^d$–projection of a corresponding H-measure is absolutely continuous with respect to the Lebesgue measure (see [**37**, Corollary 1.5] and [**30**, Remark 2, a)]). Furthermore, in that case we can assume that the test function φ_1 is merely in $\mathrm{L}^r(\mathbf{R}^d)$.

The result generalises to sequences of functions taking values in a function space. More precisely, the following theorem holds.

Theorem 13. *Assume that the sequence $(u_n)=(u_n(\mathbf{x},\mathbf{p}))$, converges weakly to zero in $\mathrm{L}^2(\mathbf{R}^{m+d}) \cap \mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^s(\mathbf{R}^d))$, $s>2$. Then the $\mathbf{R}^d$ projection $\int_{\mathrm{P}} d\nu(\mathbf{x},\boldsymbol{\xi})$ of the measure ν from the last proposition can be extended to a bounded functional on $\mathrm{L}^r(\mathbf{R}^d)$, where r is the dual index of $s/2$. Furthermore, for all $\varphi_1 \in \mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^r(\mathbf{R}^d))$, $\varphi_2 \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m;\mathrm{C}_0(\mathbf{R}^d))$, and $\psi \in \mathrm{C}^d(\mathrm{P})$, it holds:*

$$\begin{aligned}\lim_{n'} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} &(\varphi_1 u_{n'})(\mathbf{x},\mathbf{p}) \Big(\overline{\mathcal{A}_{\psi_{\mathrm{P}}}\, \varphi_2 u_{n'}(\cdot,\mathbf{q})}\Big)(\mathbf{x}) d\mathbf{x} d\mathbf{p} d\mathbf{q} \\ &= \int\limits_{\mathbf{R}^{2m}} \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1(\cdot,\mathbf{p})\bar{\varphi}_2(\cdot,\mathbf{q}) \otimes \bar{\psi}\rangle d\mathbf{p} d\mathbf{q}.\end{aligned} \tag{20}$$

Proof: First, we shall show that we can extend the object μ in Theorem 9 as a continuous bilinear form on $\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^r(\mathbf{R}^d)) \times \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m;\mathrm{C}_0(\mathbf{R}^d)) \times \mathrm{C}^d(\mathrm{P})$.

Let $\varphi_1^\varepsilon \in \mathrm{C_c}(\mathbf{R}^{m+d})$, $\varepsilon>0$, be a family of continuous functions such that $\|\varphi_1 - \varphi_1^\varepsilon\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^r(\mathbf{R}^d))} \to 0$ as $\varepsilon \to 0$. By means of Remarks 10 and 11 we define

$$\begin{aligned}&\int\limits_{\mathbf{R}^{2m}} \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1\bar{\varphi}_2 \otimes \bar{\psi}\rangle d\mathbf{p} d\mathbf{q} : \\ &= \lim_{\varepsilon\to 0} \int\limits_{\mathbf{R}^{2m}} \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot), \varphi_1^\varepsilon\bar{\varphi}_2 \otimes \bar{\psi}\rangle d\mathbf{p} d\mathbf{q} \\ &= \lim_{\varepsilon\to 0} \lim_{n'\to\infty} \int\limits_{\mathbf{R}^{2m}} \int\limits_{\mathbf{R}^d} (\varphi_1^\varepsilon u_{n'})(\mathbf{x},\mathbf{p}) \Big(\overline{\mathcal{A}_{\psi_{\mathrm{P}}}\, \varphi_2 u_{n'}(\cdot,\mathbf{q})}\Big)(\mathbf{x}) d\mathbf{x} d\mathbf{p} d\mathbf{q}.\end{aligned} \tag{21}$$

The latter limit exists since for $\varepsilon_1, \varepsilon_2 > 0$ it holds

$$\Big|\int_{\mathbf{R}^{2m}} \langle \mu(\mathbf{p},\mathbf{q},\cdot,\cdot),(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})(\cdot,\mathbf{p})\bar\varphi_2(\cdot,\mathbf{q})\otimes\bar\psi\rangle d\mathbf{p}d\mathbf{q}\Big|$$

$$\leq \limsup_{n'}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d}\Big|(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})u_{n'}(\mathbf{x},\mathbf{p})\left(\overline{\mathcal{A}_{\psi_{\mathrm{P}}}\,\varphi_2 u_{n'}(\cdot,\mathbf{q})}\right)(\mathbf{x})\Big|d\mathbf{x}d\mathbf{p}d\mathbf{q}$$

$$\leq \limsup_{n'} C\,\|\psi\|_{\mathrm{C}^d(\mathrm{P})}\int_{\mathbf{R}^{2m}}\|(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})u_{n'}(\cdot,\mathbf{p})\|_{\mathrm{L}^{s'}(\mathbf{R}^d)}\|(\varphi_2 u_{n'})(\cdot,\mathbf{q})\|_{\mathrm{L}^{s}(\mathbf{R}^d)}d\mathbf{p}d\mathbf{q}$$

$$\leq \limsup_{n'} C\,\|\psi\|_{\mathrm{C}^d(\mathrm{P})}\int_{\mathbf{R}^{m}}\|(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})(\cdot,\mathbf{p})\|_{\mathrm{L}^{r}(\mathbf{R}^d)}\|(u_{n'}(\cdot,\mathbf{p})\|_{\mathrm{L}^{s}(\mathbf{R}^d)}d\mathbf{p}$$

$$\cdot\int_{\mathbf{R}^{m}}\|\varphi_2(\cdot,\mathbf{q})\|_{\mathrm{L}^{\infty}(\mathbf{R}^d)}\|u_{n'}(\cdot,\mathbf{q})\|_{\mathrm{L}^{s}(\mathbf{R}^d)}d\mathbf{q}$$

$$\leq \limsup_{n'} C\,\|\psi\|_{\mathrm{C}^d(\mathrm{P})}\ \|(\varphi_1^{\varepsilon_1}-\varphi_1^{\varepsilon_2})\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^r(\mathbf{R}^d))}$$

$$\|\varphi_2\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^\infty(\mathbf{R}^d))}\ \|u_{n'}\|^2_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^s(\mathbf{R}^d))}\,,$$

where C depends on s and d only. Since $\|\varphi_1-\varphi_1^{\varepsilon}\|_{\mathrm{L}^2(\mathbf{R}^m;\mathrm{L}^r(\mathbf{R}^d))}\to 0$, the limit in (21) exists.

The same analysis from the above implies

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d}\Big|(\varphi_1^{\varepsilon}-\varphi_1)u_{n'}(\mathbf{x},\mathbf{p})\left(\overline{\mathcal{A}_{\psi_{\mathrm{P}}}\,\varphi_2 u_{n'}(\cdot,\mathbf{q})}\right)(\mathbf{x})\Big|d\mathbf{x}d\mathbf{p}d\mathbf{q}=0$$

and the convergence is uniform with respect to n'. Thus we can exchange limits in the second line of (21), which proves (20).

In order to prove that the $\mathbf{R}^d$ projection $\int_{\mathrm{P}} d\nu(\mathbf{x},\boldsymbol{\xi})$ of the measure ν belongs to $\mathrm{L}^{r'}(\mathbf{R}^d)$ take an arbitrary $\varphi\in\mathrm{C_c}(\mathbf{R}^d)$ and consider

$$\Big|\int_{\mathbf{R}^d}\varphi(\mathbf{x})\int_{\mathrm{P}} d\nu(\mathbf{x},\boldsymbol{\xi})\Big| = \sum_{i=1}^{\infty}\Big|\langle\frac{1}{2^i}\mu^{ii},\varphi\otimes 1\rangle\Big|$$

$$\leq \sum_{i=1}^{\infty}\frac{1}{2^i}\lim_{n'\to\infty}\int_{\mathbf{R}^{2m}}\int_{\mathbf{R}^d}\Big|\varphi(\mathbf{x})u_{n'}(\mathbf{x},\mathbf{p})e_i(\mathbf{p})\bar u_{n'}(\mathbf{x},\mathbf{q})\bar e_i(\mathbf{q})\Big|d\mathbf{x}d\mathbf{p}d\mathbf{q}$$

$$\leq \sum_{i=1}^{\infty}\frac{1}{2^i}\|\varphi\|_{\mathrm{L}^r(\mathbf{R}^d)}\limsup_{n'\to\infty}\|u_{n'}\|^2_{\mathrm{L}^2(\mathbf{R}^{2m};\mathrm{L}^s(\mathbf{R}^d))}\leq C\|\varphi\|_{\mathrm{L}^r(\mathbf{R}^d)}.$$

Thus $\int_{\mathrm{P}} d\nu(\mathbf{x},\boldsymbol{\xi})$ can be extended to a bounded functional on $\mathrm{L}^r(\mathbf{R}^d)$, i.e. there exists an $h\in\mathrm{L}^{r'}(\mathbf{R}^d)$ such that $\int_{\mathrm{P}} d\nu(\mathbf{x},\boldsymbol{\xi}) = h(\mathbf{x})d\mathbf{x}$.

❑

The following statement on the measure ν now follows from results on slicing measures [**15**, Theorem 1.5.1].

LEMMA 14. *Under assumptions of the last theorem, for* a.e. $\mathbf{x}\in\mathbf{R}^d$ *there exists a Radon probability measure* $\nu_{\mathbf{x}}$ *such that* $d\nu(\mathbf{x},\boldsymbol{\xi}) = d\nu_{\mathbf{x}}(\boldsymbol{\xi})h(\mathbf{x})d\mathbf{x}$, *where* h *is a* $\mathrm{L}^{r'}$ *function introduced above. More precisely, for each* $\phi\in\mathrm{C}_0(\mathbf{R}^d\times\mathrm{P})$

$$\int_{\mathbf{R}^d\times\mathrm{P}}\phi(\mathbf{x},\boldsymbol{\xi})d\nu(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbf{R}^d}\left(\int_{\mathrm{P}}\phi(\mathbf{x},\boldsymbol{\xi})d\nu_{\mathbf{x}}(\boldsymbol{\xi})\right)h(\mathbf{x})d\mathbf{x}\,.$$

The above result is also valid if we take a test function $\phi \in \mathrm{L}^r(\mathbf{R}^d; \mathrm{C}(\mathrm{P}))$.

4. Proof of the main theorem

In this section, we shall prove Theorem 7. The proof is based on the special choice of the test function to be applied in (2), and the $\mathrm{H_P}$-measures techniques developed in the previous section.

We introduce the multiplier operator $\mathcal{I}$ with the symbol $\frac{1-\theta(\boldsymbol{\xi})}{\left(|\xi_1|^{l\alpha_1}+\cdots+|\xi_d|^{l\alpha_d}\right)^{1/l}}$, where $\theta \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ is a cut-off function, such that $\theta \equiv 1$ on a neighbourhood of the origin.

According to Lemma 5, for any $\psi \in \mathrm{C}^d(\mathrm{P})$, the multiplier operator $\mathcal{I} \circ \mathcal{A}_{\psi\circ\pi_\mathrm{P}}$: $\mathrm{L}^2(\mathbf{R}^d) \cap \mathrm{L}^s(\mathbf{R}^d) \to \mathrm{W}^{\boldsymbol{\alpha},s}(\mathbf{R}^d)$ is bounded (with L^s norm considered on the domain). Indeed, it is enough to notice that the symbol of $\partial^{\alpha}_{x_k}(\mathcal{I} \circ \mathcal{A}_{\psi\circ\pi_\mathrm{P}})$:

$$(\psi \circ \pi)(\boldsymbol{\xi}) \frac{(1-\theta(\boldsymbol{\xi}))(2\pi i \xi_k)^{\alpha_k}}{\left(|\xi_1|^{l\alpha_1} + \cdots + |\xi_d|^{l\alpha_d}\right)^{1/l}}$$

is a smooth, bounded function that satisfies conditions of Lemma 4.

Insert in (2) (with reintroduced sub-index n) the test function g_n given by (a similar procedure was firstly applied in [**33**]):

$$g_n(\mathbf{x}, \mathbf{p}) = \rho_1(\mathbf{p}) \int_{\mathbf{R}^m} (\mathcal{I} \circ \mathcal{A}_{\psi\circ\pi_\mathrm{P}})\big(\varphi u_n(\cdot, \mathbf{q})\big)(\mathbf{x}) \rho_2(\mathbf{q}) d\mathbf{q}, \tag{22}$$

where $\psi \in \mathrm{C}^d(\mathrm{P})$, $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, $\rho_1, \rho_2 \in \mathrm{C}_c^{|\boldsymbol{\kappa}|}(\mathbf{R}^m)$, and $\boldsymbol{\kappa}$ is the multi-index appearing in (1). Due to the boundedness properties of the operator $\mathcal{I} \circ \mathcal{A}_{\psi\circ\pi_\mathrm{P}}$ discussed above, the sequence (g_n) is bounded in $\mathrm{C}_c^{|\boldsymbol{\kappa}|}(\mathbf{R}^m) \times \mathrm{W}^{\boldsymbol{\alpha},s}(\mathbf{R}^d)$.

Letting $n \to \infty$ in (2), we get after taking into account Theorem 13 and the strong convergence of (G_n)

$$\int_{\mathbf{R}^{2m}} \int_{\mathbf{R}^d \times \mathrm{P}} A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p}) \overline{\rho_1(\mathbf{p})\rho_2(\mathbf{q})\varphi(x)\psi(\boldsymbol{\xi})} d\mu(\mathbf{p}, \mathbf{q}, \mathbf{x}, \boldsymbol{\xi}) d\mathbf{p} d\mathbf{q} = 0,$$

where, let it be repeated, $A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p}) = \sum_{k=1}^d (2\pi i \xi_k)^{\alpha_k} a_k$. As the test functions ρ_i, φ, and ψ are taken from dense subsets in appropriate spaces, we conclude

$$A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p}) d\mu(\mathbf{p}, \mathbf{q}, \mathbf{x}, \boldsymbol{\xi}) = 0, \quad (\text{a.e. } \mathbf{p}, \mathbf{q} \in \mathbf{R}^{2m}). \tag{23}$$

For $s = 2$ the non-degeneracy condition (11) directly implies that $\mu = 0$. In order to show the same result for $s > 2$ fix an arbitrary $\delta > 0$, and for a $\rho \in \mathrm{L}^2_\mathrm{c}(\mathbf{R}^m)$ and $\phi \in \mathrm{C}_0(\mathbf{R}^d \times \mathrm{P})$ consider the test function

$$\frac{\rho(\mathbf{p})\bar{\rho}(\mathbf{q})\phi(\mathbf{x}, \boldsymbol{\xi})\overline{A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})}}{|A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})|^2 + \delta}.$$

From (23), we obtain

$$\int_{\mathbf{R}^{2m}} \int_{\mathbf{R}^d \times \mathrm{P}} \frac{\rho(\mathbf{p})\bar{\rho}(\mathbf{q})\phi(\mathbf{x}, \boldsymbol{\xi})|A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})|^2}{|A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})|^2 + \delta} d\mu(\mathbf{p}, \mathbf{q}, \mathbf{x}, \boldsymbol{\xi}) d\mathbf{p} d\mathbf{q} = 0,$$

which by means of representation (18) and Fubini's theorem takes the form

$$\int_{\mathbf{R}^d \times \mathrm{P}} \int_{\mathbf{R}^{2m}} \frac{\rho(\mathbf{p})\bar{\rho}(\mathbf{q})\phi(\mathbf{x}, \boldsymbol{\xi})|A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})|^2}{|A(\mathbf{x}, \boldsymbol{\xi}, \mathbf{p})|^2 + \delta} f(\mathbf{p}, \mathbf{q}, \mathbf{x}, \boldsymbol{\xi}) d\mathbf{p} d\mathbf{q} d\nu(\mathbf{x}, \boldsymbol{\xi}) = 0. \tag{24}$$

Let us denote

$$I_\delta(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbf{R}^{2m}} \rho(\mathbf{p})\bar\rho(\mathbf{q})\frac{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{p})|^2}{|A(\mathbf{x},\boldsymbol{\xi},\mathbf{p})|^2+\delta} f(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})d\mathbf{p}d\mathbf{q}\,.$$

According to the non-degeneracy condition (11) and the representation of the measure ν given in Lemma 14, for $s>2$ we have

$$I_\delta(\mathbf{x},\boldsymbol{\xi}) \to \int_{\mathbf{R}^{2m}} \rho(\mathbf{p})\bar\rho(\mathbf{q}) f(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})d\mathbf{p}d\mathbf{q},$$

as $\delta\to 0$ for $\nu-$a.e. $(\mathbf{x},\boldsymbol{\xi})\in\mathbf{R}^d\times\mathrm{P}$. By using the Lebesgue dominated convergence theorem, it follows from (24) after letting $\delta\to 0$:

$$\begin{aligned}&\int_{\mathbf{R}^d\times \mathrm{P}}\int_{\mathbf{R}^{2m}} \rho(\mathbf{p})\bar\rho(\mathbf{q})\phi(\mathbf{x},\boldsymbol{\xi}) f(\mathbf{p},\mathbf{q},\mathbf{x},\boldsymbol{\xi})d\mathbf{p}d\mathbf{q}d\nu(\mathbf{x},\boldsymbol{\xi})\\ &= \int_{\mathbf{R}^{2m}} \rho(\mathbf{p})\bar\rho(\mathbf{q})\,\langle\mu(\mathbf{p},\mathbf{q},\cdot,\cdot),\phi\rangle d\mathbf{p}d\mathbf{q}=0.\end{aligned}$$

Having in mind the definition of the measure μ from Theorem 9, by putting here $\phi(\mathbf{x},\boldsymbol{\xi})=|\varphi(\mathbf{x})|^2$ for $\varphi\in\mathrm{C}_0(\mathbf{R}^d)$, we immediately obtain

$$\lim_{n'\to\infty}\int_{\mathbf{R}^d}\left|\int_{\mathbf{R}^m}\rho(\mathbf{p})u_{n'}(\mathbf{x},\mathbf{p})d\mathbf{p}\right|^2|\varphi(\mathbf{x})|^2d\mathbf{x}=0.$$

Due to arbitrariness of φ, this concludes the proof. □

Remark 15. We conclude the section by remarking that our results easily extend to equations containing mixed derivatives with respect to the space variables (see also [**16**, Theorem 2.1]):

$$\mathcal{P}u_n(\mathbf{x},\mathbf{p})=\sum_{s\in I}\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_s}\left(a_s(\mathbf{x},\mathbf{p})u_n(\mathbf{x},\mathbf{p})\right)=\partial_{\mathbf{p}}^{\boldsymbol{\kappa}}G_n(\mathbf{x},\mathbf{p}), \tag{25}$$

where I is a finite set of indices, and $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}_s}=\partial_{x_1}^{\alpha_{1s}}\dots\partial_{x_d}^{\alpha_{ds}}$, for a multi-index $\boldsymbol{\alpha}_s=(\alpha_{s1},\dots,\alpha_{sd})\in\mathbf{R}^d$.

Denote by A the principal symbol of the (pseudo-)differential operator $\mathcal{P}$, which is of the form

$$A(\boldsymbol{\xi},\mathbf{x},\mathbf{p})=\sum_{s\in I'}(2\pi i\boldsymbol{\xi})^{\boldsymbol{\alpha}_s}a_s(\mathbf{x},\mathbf{p}),$$

where the upper sum goes above all terms from (25) whose order of derivative $\boldsymbol{\alpha}_s$ is not dominated by any other multiindex from I.

For A we must additionally assume that there exist $\alpha_1,\cdots,\alpha_d\in\mathbf{R}^+$ such that for any positive $\lambda\in\mathbf{R}$, it holds

$$A(\lambda^{1/\alpha_1}\xi_1,\dots,\lambda^{1/\alpha_d}\xi_d,\mathbf{x},\mathbf{p})=\lambda A(\boldsymbol{\xi},\mathbf{x},\mathbf{p}),$$

and that it satisfies genuine non-degeneracy condition: for almost every $x\in\mathbf{R}^d$, every $\xi\in\mathrm{P}$, it holds

$$A(\mathbf{x},\boldsymbol{\xi},\mathbf{p})\neq 0\quad(\text{a.e. }\mathbf{p}\in\mathbf{R}^m)\,;$$

The proof of Theorem 7 for equation of form (25) goes along the same lines as for the equation (1).

REMARK 16. Let us finally remark that in the case when derivative orders α_k, $k = 1, \dots, d$, are non-negative integers, we can assume that the sequence u_n is only locally bounded in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^m; \mathrm{L}^s_{\mathrm{loc}}(\mathbf{R}^d))$.

In that case we simply take

$$g_n(\mathbf{x}, \mathbf{p}) = \rho_1(\mathbf{p})\varphi(\mathbf{x}) \int_{\mathbf{R}^m} (\mathcal{I} \circ \mathcal{A}_{\psi\circ\pi_{\mathrm{P}}})\big(\varphi u_n(\cdot, \mathbf{q})\big)(\mathbf{x})\rho_2(\mathbf{q})d\mathbf{q},$$

instead of g_n from (22). By repeating the rest of the procedure from this section, we conclude that the measure μ from Theorem 9 corresponding to (φu_n) equals zero. Due to arbitrariness of φ, we conclude that for any $\rho \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}^m)$, the sequence $(\int u_n(\mathbf{x}, \mathbf{p})\rho(\mathbf{p})d\mathbf{p})$ is strongly precompact in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.

5. Ultra-parabolic equation with discontinuous coefficients

In this section, we consider an ultra-parabolic equation with discontinuous coefficients in a domain Ω (an open subset of $\mathbf{R}^d$). Ultraparabolic equations (with regular coefficients) were first considered by Graetz [**19**] and Nusselt [**27**] in their investigations concerning the heat transfer. A specific situation modelled by such equations is the one when diffusion can be neglected in the directions $x_{l+1}, \dots, x_d$, $l \geq 0$. Recently, such equations were investigated in [**29**] and we aim to extend results from there.

More precisely, the equation that we are going to consider here has the form

$$\operatorname{div} \mathsf{f}(\mathbf{x}, u) - \operatorname{div}\operatorname{div} \mathbf{B}(\mathbf{x}, u) + \psi(\mathbf{x}, u) = 0, \tag{26}$$

where $\mathbf{B}(x, u) = (b_{jk})_{j,k=1,\dots,d}$ is a symmetric matrix such that for some $l < d$ it holds $(b_{jk}) \equiv 0$ for $\min(j, k) \leq l$, while $\tilde{\mathbf{B}} = (b_{jk})_{j,k=l+1,\dots,d}$ satisfies an ellipticity condition on $\mathbf{R}^{d-l}$ in the following sense: for every $\tilde{\boldsymbol{\xi}} \in \mathbf{R}^{d-l}$, $\lambda_1, \lambda_2 \in \mathbf{R}$ and $\mathbf{x} \in \Omega$,

$$(\lambda_1 > \lambda_2) \quad \Longrightarrow \quad (\tilde{\mathbf{B}}(\mathbf{x}, \lambda_1) - \tilde{\mathbf{B}}(\mathbf{x}, \lambda_2))\tilde{\boldsymbol{\xi}} \cdot \tilde{\boldsymbol{\xi}} \geq c|\tilde{\boldsymbol{\xi}}|^2, \quad c > 0.$$

Accordingly, we shall use anisotropic spaces like $\mathrm{W}^{(1,2),q}(\Omega)$, where $(1, 2) \in \mathbf{R}^d$ is a multiindex with first l components equal to 1.

Furthermore, we assume that $\psi \in \mathrm{L}^1(\Omega; \mathrm{L}^\infty(\mathbf{R}))$, while $\mathsf{f} = (f_1, \dots, f_d)$ and $\mathbf{B}$ are such that for every $j, k = 1, \dots, d$

$$\partial_\lambda f_k, \partial_\lambda b_{jk} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}; \mathrm{L}^r_{\mathrm{loc}}(\Omega)), \quad r > 1.$$

We also need to assume a kind of uniform continuity of f and $\mathbf{B}$ in the sense that there exists an increasing function w on $\mathbf{R}^+$, vanishing and continuous at 0 (i.e. w is a modulus of continuity type function), and $\sigma \in \mathrm{L}^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$, $\varepsilon > 0$ such that

$$\big|\mathsf{f}(x, \lambda_1) - \mathsf{f}(x, \lambda_2)\big|, \big|\mathbf{B}(x, \lambda_1) - \mathbf{B}(x, \lambda_2)\big| \leq w\left(\big|\lambda_1 - \lambda_2)\big|\right) \big|\sigma(x)\big|. \tag{27}$$

Concerning regularity with respect to $\mathbf{x} \in \mathbf{R}^d$ of the functions f and $\mathbf{B}$, we assume that for every $\lambda \in \mathbf{R}$

$$\operatorname{div} \mathsf{f}(\mathbf{x}, \lambda) - \operatorname{div}\operatorname{div} \mathbf{B}(\mathbf{x}, \lambda) = \gamma(\mathbf{x}, \lambda) \in \mathcal{M}(\mathbf{R}^d).$$

To proceed, denote $\gamma(\mathbf{x}, \lambda) = \omega(\mathbf{x}, \lambda)d\mathbf{x} + \gamma^s(\mathbf{x}, \lambda)$ where $\omega(\mathbf{x}, \lambda)d\mathbf{x}$ denotes the regular, and $\gamma^s(\mathbf{x}, \lambda)$ denotes the singular part of the measure γ with respect to the Lebesgue measure. The following definition is used in [**29**].

DEFINITION 17. We say that a function $u \in \mathrm{L}^\infty(\Omega)$ represents an entropy admissible weak solution to (26) if for every $\lambda \in \mathbf{R}$ it holds

$$\begin{aligned}\text{(28)}\qquad & \mathsf{div}\Big(\operatorname{sgn}(u(\mathbf{x})-\lambda)\big(\mathsf{f}(\mathbf{x},u(\mathbf{x}))-\mathsf{f}(\mathbf{x},\lambda)\big)\Big)\\ & -\mathsf{div}\,\mathsf{div}\Big(\operatorname{sgn}(u(\mathbf{x})-\lambda)\big(\mathbf{B}(\mathbf{x},u(\mathbf{x}))-\mathbf{B}(\mathbf{x},\lambda)\big)\Big)\\ & +\operatorname{sgn}(u(\mathbf{x})-\lambda)\big(\omega(\mathbf{x},\lambda)+\psi(\mathbf{x},u(\mathbf{x}))\big)-|\gamma(\mathbf{x},\lambda)|\leq 0\end{aligned}$$

in the sense of distributions on $\mathbf{R}^d$.

We shall prove a result similar to those from [**29, 30**], stating the assumptions under which a sequence of entropy solutions is strongly precompact in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$. There it is assumed that $\max\limits_{\lambda\in\langle -M,M\rangle}|\mathsf{f}(\cdot,\lambda)|, \max\limits_{\lambda\in\langle -M,M\rangle}|\mathbf{B}(\cdot,\lambda)| \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)$, for $M = \limsup_n \|u_n\|_{\mathrm{L}^\infty(\Omega)}$, while we demand $\partial_\lambda\mathsf{f}, \partial_\lambda\mathbf{B} \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};\mathrm{L}^r_{\mathrm{loc}}(\Omega))$ for an $r>1$. Remark that we have increased regularity with respect to $\lambda\in\mathbf{R}$ (there the continuity is merely assumed), but we have decreased it with respect to $\mathbf{x}\in\Omega$. However, in the case $r\geq 2$ the statement of the next theorem also follows from the more general results of [**29**].

THEOREM 18. *Assume that the coefficients of equation* (26) *satisfy the genuine nonlinearity conditions analogical to* (11)*:*

- *for every* $\boldsymbol{\xi} = (\hat{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) \in \mathrm{P} = \{(\hat{\boldsymbol{\xi}},\tilde{\boldsymbol{\xi}}) \in \mathbf{R}^l\times\mathbf{R}^{d-l} : |\hat{\boldsymbol{\xi}}|^2+|\tilde{\boldsymbol{\xi}}|^4 = 1\}$ *and almost every* $x\in\mathbf{R}^d$

$$\text{(29)}\qquad 2\pi i\sum_{k=1}^{l}\xi_k\partial_\lambda f_k(\mathbf{x},\lambda)+4\pi^2\langle\partial_\lambda\mathbf{B}(\mathbf{x},\lambda)\boldsymbol{\xi},\boldsymbol{\xi}\rangle\neq 0\quad(\text{a.e. }\lambda\in\mathbf{R}).$$

Then, a sequence of entropy solutions (u_n) *to* (26) *such that* $\|u_n\|_{\mathrm{L}^\infty(\Omega)}<M$ *for every* $n\in\mathbf{N}$ *is strongly precompact in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Proof: To prove the theorem, remark first that, according to the Schwartz theorem [**34**, Theorem I.V], for every $\lambda\in\mathbf{R}$ we can rewrite (28) as

$$\begin{aligned}\text{(30)}\qquad & \mathsf{div}\Big(\operatorname{sgn}(u_n(\mathbf{x})-\lambda)\big(\mathsf{f}(\mathbf{x},u_n(\mathbf{x}))-\mathsf{f}(\mathbf{x},\lambda)\big)\Big)\\ & -\mathsf{div}\,\mathsf{div}\Big(\operatorname{sgn}(u_n(\mathbf{x})-\lambda)\big(\mathbf{B}(\mathbf{x},u_n(\mathbf{x}))-\mathbf{B}(\mathbf{x},\lambda)\big)\Big)\\ & = G_n(\mathbf{x},\lambda)\end{aligned}$$

where $G_n(\cdot,\lambda)\in\mathcal{M}(\Omega)$ are Radon measure on Ω, locally uniformly bounded with respect to n. According to [**15**, Theorem 1.6] (see also [**29**, Proposition 7]), the sequence of measures $(G_n(\cdot,\lambda))$ is strongly precompact in $\mathrm{W}^{(-1,-2),q}_{\mathrm{loc}}(\Omega)$ for each

$q \in \langle 1, \frac{d}{d-1}\rangle$. Furthermore, for every $\varphi \in C_c^{1,2}(\Omega)$, we have according to (27)

$$
\begin{aligned}
&|\langle G_n(\cdot,\lambda_1) - G_n(\cdot,\lambda_2), \varphi\rangle| \\
&= \int_\Omega \Big| \Big(\mathrm{sgn}(u_n - \lambda_1)\big(\mathbf{B}(\cdot,u_n) - \mathbf{B}(\cdot,\lambda_1)\big) \\
&-\mathrm{sgn}(u_n - \lambda_2)\big(\mathbf{B}(\cdot,u_n) - \mathbf{B}(\cdot,\lambda_2)\big)\Big) \cdot (\nabla\otimes\nabla)\varphi \Big| d\mathbf{x} \\
&\qquad + \int_\Omega \Big| \Big(\mathrm{sgn}(u_n - \lambda_1)\big(\mathsf{f}(\cdot,u_n) - \mathsf{f}(\cdot,\lambda_1)\big) \\
&-\mathrm{sgn}(u_n - \lambda_2)\big(\mathsf{f}(\cdot,u_n) - \mathsf{f}(\cdot,\lambda_2)\big)\Big) \cdot \nabla\varphi \Big| d\mathbf{x} \\
&\leq C w\left(\left|\lambda_1 - \lambda_2\right)\right|\right) \|\varphi\|_{\mathrm{W}^{(1,2),q'}},
\end{aligned}
\tag{31}
$$

for a constant C independent of n (it depends only on f, $\mathbf{B}$, and σ) and $q \leq 1+\varepsilon$ (note that $\sigma \in \mathrm{L}^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$, $\varepsilon > 0$). Indeed, according to (27) it holds

$$
\begin{aligned}
&\big|\mathrm{sgn}(u-\lambda_1)(\mathsf{f}(\mathbf{x},u) - \mathsf{f}(\mathbf{x},\lambda_1) - \mathrm{sgn}(u-\lambda_2)(\mathsf{f}(\mathbf{x},u) - \mathsf{f}(\mathbf{x},\lambda_2)\big| \\
&\leq \begin{cases} \big|\mathsf{f}(\mathbf{x},\lambda_1) - \mathsf{f}(\mathbf{x},\lambda_2)\big|, & (u-\lambda_1)(u-\lambda_2) \geq 0 \\ \big|\mathsf{f}(\mathbf{x},u) - \mathsf{f}(\mathbf{x},\lambda_1)\big| + |\mathsf{f}(\mathbf{x},u) - \mathsf{f}(\mathbf{x},\lambda_2)|, & (u-\lambda_1)(u-\lambda_2) \leq 0 \end{cases} \\
&\leq\ 2w\left(\left|\lambda_1 - \lambda_2)\right|\right) \left|\sigma(\mathbf{x})\right|,
\end{aligned}
$$

and similarly for f replaced by $\mathbf{B}$, from where (31) immediately follows.

Take now a countable dense subset D of $\mathbf{R}$ and for every $\lambda_m \in D$ denote by $G(\cdot,\lambda_m) \in \mathcal{M}(\Omega)$ such that $G_n(\cdot,\lambda_m) \longrightarrow G(\cdot,\lambda_m)$ strongly in $\mathrm{W}^{(-1,-2),q}_{\mathrm{loc}}(\Omega)$ along a subsequence. Since D is countable, we can choose the same subsequence (which we denote the same as the original one) for every $\lambda_m \in D$. Now, we extend $G(\cdot,\lambda)$, $\lambda \in D$, by continuity on entire $\mathbf{R}$: for every $\lambda \in \mathbf{R}$, we choose a sequence (λ_m) from D converging to λ and define for every $\varphi \in C_c^{1,2}(\Omega)$:

$$
\langle G(\cdot,\lambda), \varphi\rangle := \lim_{m\to\infty} \langle G(\cdot,\lambda_m), \varphi\rangle. \tag{32}
$$

The latter is well defined since for any $\lambda_1, \lambda_2 \in D$ and any $\varepsilon > 0$ one can find an $n > 0$ such that

$$
\begin{aligned}
&\Big|\langle G(\cdot,\lambda_1) - G(\cdot,\lambda_2), \varphi\rangle\Big| \leq \\
&\Big|\langle G_n(\cdot,\lambda_1) - G(\cdot,\lambda_1), \varphi\rangle\Big| + \Big|\langle G_n(\cdot,\lambda_1) - G_n(\cdot,\lambda_2), \varphi\rangle\Big| + \\
&|\langle G_n(\cdot,\lambda_2) - G(\cdot,\lambda_2), \varphi\rangle\Big| \leq \Big(\varepsilon + Cw\left(\left|\lambda_1 - \lambda_2)\right|\right) + \varepsilon\Big) \|\varphi\|_{\mathrm{W}^{(1,2),q'}}.
\end{aligned}
$$

From here, the Cauchy criterion will provide properness of (32). Furthermore, since $G(\cdot,\lambda_m)$ are Radon measures, the functional $G(\cdot,\lambda)$ is also a Radon measure.

Using the same arguments, it is not difficult to prove that for every $\lambda \in \mathbf{R}$,

$$
G_n(\cdot,\lambda) \to G(\cdot,\lambda) \ \text{ in } \ \mathrm{W}^{(-1,-2),q}_{\mathrm{loc}}(\Omega).
$$

According to the Lebesgue dominated convergence theorem, we conclude from the latter that

$$
G_n \to G \ \text{ in } \ \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}; \mathrm{W}^{(-1,-2),q}_{\mathrm{loc}}(\Omega)). \tag{33}
$$

By finding derivative of (30) with respect to λ, we reach to (the kinetic formulation of (26); see [**8**])

$$\mathsf{div}\left(h_n(\mathbf{x},\lambda)\partial_\lambda \mathsf{f}(\mathbf{x},\lambda)\right) - \mathsf{div}\,\mathsf{div}\,(h_n(\mathbf{x},\lambda)\partial_\lambda \mathbf{B}(\mathbf{x},\lambda)) = -\partial_\lambda G_n(\mathbf{x},\lambda)$$

where $h_n(\mathbf{x},\lambda) = \operatorname{sgn}(u_n(\mathbf{x}) - \lambda)$, and this is the special case of equation (1). From here, we see that, due to Remark 16, the convergence (33), and the genuine nonlinearity conditions (29), the sequence (φh_n) satisfies conditions of Theorem 7 (see also Remark 15). Thus it follows that $(\int_{-M}^{M} h_n(\mathbf{x},\lambda)d\lambda)$ is strongly precompact in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$. Since

$$\int_{-M}^{M} h_n(\mathbf{x},\lambda)d\lambda = 2u_n(\mathbf{x}),$$

we conclude that (u_n) is strongly $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ precompact itself. ❑

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University of Dubrovnik, Department of Electrical Engineering, 20000 Dubrovnik, Croatia
E-mail address: martin.lazar@unidu.hr

University of Montenegro, Faculty of Mathematics, Cetinjski put bb, 81000 Podgorica, Montenegro
E-mail address: matematika@t-com.me