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On the new concept of solutions and existence results for impulsive fractional evolution equations

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Abstract. In this paper we discuss the existence of PC-mild solutions for Cauchy problems and nonlocal problems for impulsive fractional evolution equations involving Caputo fractional derivative. By utilizing the theory of operators semigroup, probability density functions via impulsive conditions, a new concept on a PC-mild solution for our problem is introduced. Our main techniques based on fractional calculus and fixed point theorems. Some concrete applications to partial differential equations are considered.

Contents

1. Introduction

The theory of impulsive differential equations has recently years been an object of increasing interest because of its wide applicability in biology, in medicine and in more and more fields. The reason for this applicability arises from the fact that impulsive differential problems are an appropriate model for describing process

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which at certain moments change their state rapidly and which cannot be described using the classical differential problems. For a wide bibliography and exposition on this object see for instance the monographs of [**1, 2, 3, 4**] and the papers [**5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19**].

On the other hand, the first definition of the fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus was mentioned already in 1695 by Leibniz and L'Hospital. However, only in the late 1960s engineers started to be interested in this idea when the fact that descriptions of some systems are more accurate in "fractional language" appeared. Since that time fractional calculus is increasingly used to model behaviors of real systems in various fields of science and engineering. Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details, one can see the monographs of [**20, 21, 22, 23, 24, 25**] and the survey [**26, 27**].

A pioneering work on some probability densities and fundamental solutions of Caputo fractional evolution equations has been reported by El-Borai [**28, 29**]. Particular, Wang et al. [**30, 31, 32, 33, 34, 35**] and Zhou et al. [**36, 37**] also introduced two characteristic solutions operators and give a suitable concept on a mild solution by applying Laplace transform and probability density functions and discussed the optimal control problems for some classes of fractional controlled systems.

In this paper we continue to combine these two areas and our works [**16, 17, 33, 36**] to extend the study to the Cauchy problems for impulsive fractional evolution equations:

$$
\begin{cases}
{}^{C}D_t^{\alpha}x(t) = Ax(t) + f\left(t, x(t)\right), \ \alpha \in (0,1], \ t \in J = [0,b], \ t \neq t_k, \\
x(0) = x_0, \\
x(t_k^+) = x(t_k^-) + y_k, \ k = 1, 2, \cdots, \delta,
\end{cases}
\tag{1.1}
$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order α, A: $D(A) \subseteq X \to X$ is the generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a Banach space X, f: $J \times X \to X$ is continuous, x_0, y_k are the element of X, $0 = t_0 < t_1 < t_2 < \cdots < t_\delta < t_{\delta+1} = b$, $x(t_k^+) = \lim_{h \to 0^+} = x(t_k + h)$ and $x(t_k^-) = x(t_k)$ represent respectively the right and left limits of $x(t)$ at $t = t_k$.

To study fractional evolution equations with nonlinear impulsive conditions, Mophou firstly introduced a concept on a mild solution (Definition 3.2, [**14**]) which was inspired by Jaradat et al. [**38**]. However, it does not incorporate the memory effects involved in fractional calculus and impulsive conditions. Thus, this kind of definition is not suitable enough for these settings although it has been utilized by several authors.

One of the main novelty of this paper is the concept on a PC-mild solution (Section 3, Definition 3.1) for system (1.1). Then, utilizing some fixed point theorems such as Banach contraction mapping principle, Schauder's fixed point theorem, Schaefer's fixed point theorem and Krasnoselskii's fixed point theorem, we derive many existence and uniqueness results concerning the PC-mild solution for system (1.1) under the different assumptions on the nonlinear terms.

Naturally, we extend to study the nonlocal Cauchy problems for impulsive fractional evolution equations:

$$\begin{cases} {}^{C}D_t^{\alpha}x(t) = Ax(t) + f(t, x(t)), \ t \in J, \ t \neq t_k, \\ x(0) = x_0 + g(x), \\ x(t_k^+) = x(t_k^-) + y_k, \ k = 1, 2, \cdots, \delta, \end{cases} \tag{1.2}$$

where A, f, y_k are defined as above, g is a given function and constitutes a nonlocal Cauchy problem. We adopt the ideas given in [**11, 17, 39**] and obtained some new existence and uniqueness results for system (1.2) under the different assumptions on the nonlocal terms.

The rest of this paper is organized as follows. In Section 2, some notations and preparation results are given. In section 3, a suitable concept on a PC-mild solution for our problems is introduced. In Section 4, the existence results of PC-mild solutions for impulsive Cauchy problems are obtained. In Section 5, the existence results of PC-mild solutions for impulsive nonlocal Cauchy problems are also showed. At last, some interesting examples are presented to illustrate the theory.

2. Preliminaries

Let $L_b(X)$ be the Banach space of all linear and bounded operators on X. For a C_0-semigroup $\{T(t), t \geq 0\}$ on X, we set $M = \sup_{t\in J} \|T(t)\|_{L_b(X)}$. Let $C(J, X)$ be the Banach space of all X-valued continuous functions from $J = [0, b]$ into X endowed with the norm $\|x\|_C = \sup_{t\in J} \|x(t)\|$. We also introduce the set of functions $PC(J, X) = \{x : J \to X \mid x \text{ is continuous at } t \in J\backslash\{t_1, t_2, \cdots, t_\delta\}$, and x is continuous from left and has right hand limits at $t \in \{t_1, t_2, \cdots, t_\delta\}\}$. Endowed with the norm

$$\|x\|_{PC} = \max\left\{\sup_{t\in J} \|x(t+0)\|, \ \sup_{t\in J} \|x(t-0)\|\right\},$$

it is easy to see $(PC(J, X), \|\cdot\|_{PC})$ is a Banach space.

Let us recall the following known definitions. For more details, see [**21**].

Definition 2.1. *The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as*

$${}^{L}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. *The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as*

$${}^{C}D^{\gamma} f(t) = {}^{L}D^{\gamma}\left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right), \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. *(i) If $f(t) \in C^n[0,\infty)$, then*

$$^C D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t),\ t > 0,\ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Let's recall the following definition of mild solutions for fractional evolution equations.

Definition 2.5. *(Lemma 3.1 and Definition 3.1, [**37**]) By the mild solution of the following system*

$$\begin{cases} ^C D_t^\alpha x(t) = Ax(t) + h(t),\ t \in J, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

we mean that the function $x \in C(J,X)$ which satisfies the following integral equation

$$x(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds,\ t \in J,$$

where

$$\mathscr{T}(t) = \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \quad \mathscr{S}(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta) \geq 0,\ \theta \in (0,\infty) \text{ and } \int_0^\infty \xi_\alpha(\theta)d\theta = 1.$$

Remark 2.6. *Since $\mathscr{T}(\cdot)$ and $\mathscr{S}(\cdot)$ are associated with the α, there are no analogue of the semigroup property, i.e., $\mathscr{T}(t+s) \neq \mathscr{T}(t)\mathscr{T}(s)$, $\mathscr{S}(t+s) \neq \mathscr{S}(t)\mathscr{S}(s)$ for $t,s > 0$.*

The following results will be used throughout this paper.

Lemma 2.7. *A measurable function $f\colon J \to X$ is Bochner integrable if $\|f\|$ is Lebesuge integrable.*

Lemma 2.8. *For $\sigma \in (0,1]$ and $0 < a \leq b$, we have $|a^\sigma - b^\sigma| \leq (b-a)^\sigma$.*

Lemma 2.9. *(Lemma 3.2–3.4, [**37**]) The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(i) For any fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear and bounded operators, i.e., for any $x \in X$,

$$\|\mathscr{T}(t)x\| \leq M\|x\| \text{ and } \|\mathscr{S}(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|.$$

(ii) $\{\mathscr{T}(t), t \geq 0\}$ and $\{\mathscr{S}(t), t \geq 0\}$ are strongly continuous.

(iii) For every $t > 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are also compact operators if $T(t)$ is compact.

3. New concept of solutions

In this section, we will introduce a new concept of solutions for our problems. We first consider an nonhomogeneous impulsive linear fractional equation of the form

$$\tag{3.1} \begin{cases} {}^C D_t^\alpha x(t) = Ax(t) + h(t),\ \alpha \in (0,1),\ t \in J = [0,b],\ t \neq t_k, \\ x(0) = x_0, \\ x(t_k^+) = x(t_k^-) + y_k,\ k = 1,2,\cdots,\delta, \end{cases}$$

where $h \in PC(J,X)$. We observe that $x(\cdot)$ can be decomposed to $v(\cdot)+w(\cdot)$ where v is the continuous mild solution for

$$\tag{3.2} \begin{cases} {}^C D_t^\alpha v(t) = Av(t) + h(t),\ t \in J, \\ v(0) = x_0, \end{cases}$$

on J, and w is the PC-mild solution for

$$\tag{3.3} \begin{cases} {}^C D_t^\alpha w(t) = Aw(t),\ t \in J,\ t \neq t_k, \\ w(0) = 0, \\ w(t_k^+) = w(t_k^-) + y_k,\ k = 1,2,\cdots,\delta. \end{cases}$$

Indeed, by adding together (3.2) with (3.3), it follows (3.1). Note v is continuous, so $v(t_k^+) = v(t_k^-)$, $k = 1,2,\cdots,\delta$. On the other hand, any solution of (3.1) can be decomposed to (3.2) and (3.3).

By Definition 2.5, a mild solution of (3.2) is given by

$$v(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds,\ t \in J.$$

Now we rewrite system (3.3) in the equivalent integral equation

$$\tag{3.4} w(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aw(s)ds, \text{ for } t \in [0,t_1], \\ y_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aw(s)ds, \quad \text{for } t \in (t_1,t_2], \\ y_1 + y_2 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aw(s)ds, \text{ for } t \in (t_2,t_3], \\ \vdots \\ \sum_{i=1}^{\delta} y_i + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aw(s)ds, \text{ for } t \in (t_\delta,b]. \end{cases}$$

The the above equation (3.4) can be expressed as

$$\tag{3.5} w(t) = \sum_{i=1}^{\delta} \chi_i(t)y_i + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Aw(s)ds,\ \text{for } t \in J,$$

where

$$\chi_i(t) = \begin{cases} 0, \text{ for } t \in [0,t_i), \\ 1, \text{ for } t \in [t_i,b]. \end{cases}$$

We adopt the idea used in our previous work [**36**] and apply the Laplace transformation for (3.5) to get

$$u(\lambda) = \sum_{i=1}^{\delta} \frac{e^{-t_i\lambda}}{\lambda}y_i + \frac{1}{\lambda^\alpha}Au(\lambda),$$

which implies

$$u(\lambda) = \sum_{i=1}^{\delta} e^{-t_i\lambda}\lambda^{\alpha-1}\left(\lambda^\alpha I - A\right)^{-1}y_i.$$

Note the Laplace transform of $\mathscr{T}(t)y_i$ is $\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} y_i$. Thus we can derive the mild solution of (3.3) as

$$w(t) = \sum_{i=1}^{\delta} \chi_i(t)\mathscr{T}(t-t_i)y_i.$$

Summarizing, the mild solution of (3.1) is given by

$$x(t) = \mathscr{T}(t)x_0 + \sum_{i=1}^{\delta} \chi_i(t)\mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds,$$

i.e.,

$$x(t) = \begin{cases} \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds, \text{ for } t \in [0,t_1], \\ \mathscr{T}(t)x_0 + \mathscr{T}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds, \text{ for } t \in (t_1,t_2], \\ \vdots \\ \mathscr{T}(t)x_0 + \sum_{i=1}^{\delta} \mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)ds, \text{ for } t \in (t_\delta, b]. \end{cases}$$

By using the above results, we can introduce the following definition of the mild solution for system (1.1).

Definition 3.1. *By a PC-mild solution of the system (1.1) we mean that a function $x \in PC(J,X)$ which satisfies the following integral equation*

$$x(t) = \begin{cases} \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f(s,x(s))\,ds, \text{ for } t \in [0,t_1], \\ \\ \mathscr{T}(t)x_0 + \mathscr{T}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f(s,x(s))\,ds, \\ \text{for } t \in (t_1,t_2], \\ \\ \vdots \\ \mathscr{T}(t)x_0 + \sum_{i=1}^{\delta} \mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f(s,x(s))\,ds, \\ \text{for } t \in (t_\delta, b]. \end{cases}$$

4. Existence results for impulsive Cauchy problems

In this section, we will derive some existence and uniqueness results concerning the PC-mild solution for system (1.1) under the different assumptions on f.

Case 1. f is Lipschitz

Let us list the following hypotheses:

[HA]: A is the infinitesimal generator of a compact semigroup $\{T(t), t \geq 0\}$ in X.

[HF1]: f: $J \times X \to X$ is continuous and there exists a constant $q_1 \in (0,\alpha)$ and a real-valued function $L_f(t) \in L^{\frac{1}{q_1}}(J,R^+)$ such that

$$\|f(t,x) - f(t,y)\| \leq L_f(t)\|x-y\|, \; t \in J, \; x,y \in X.$$

For brevity, let us take

$$T^* = \left[\left(\frac{1-q_1}{\alpha-q_1}\right) b^{\frac{\alpha-q_1}{1-q_1}}\right]^{1-q_1} \|L_f\|_{L^{\frac{1}{q_1}}(J,R^+)}.$$

THEOREM 4.1. *Let [HA] and [HF1] be satisfied. Then for every $x_0 \in X$, the system (1.1) has a unique PC-mild solution on J provided that*

$$0 < \frac{\alpha M T^*}{\Gamma(1+\alpha)} < 1. \tag{4.1}$$

Proof. Let $x_0 \in X$ be fixed. Define an operator Q on $PC(J,X)$ by

$$(Qx)(t) = \begin{cases} \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s) \\ f(s,x(s))\,ds, \text{for } t\in[0,t_1], \\ \mathscr{T}(t)x_0 + \mathscr{T}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s)f(s,x(s))\,ds, \text{for } t\in(t_1,t_2], \\ \vdots \\ \mathscr{T}(t)x_0 + \sum_{i=1}^{\delta}\mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s)f(s,x(s))\,ds, \\ \text{for } t\in(t_\delta,b]. \end{cases} \tag{4.2}$$

By our assumptions and Lemma 2.7, Q is well defined on $PC(J,X)$.

Step 1. We prove that $Qx \in PC(J,X)$ for $x \in PC(J,X)$.

For $0 \le \tau < t \le t_1$, taking into account the imposed assumptions and applying Lemma 2.8 and Lemma 2.9(i), we obtain

$$\begin{aligned} & \|(Qx)(t)-(Qx)(\tau)\| \\ \le\ & \|\mathscr{T}(t)x_0-\mathscr{T}(\tau)x_0\| + \int_\tau^t (t-s)^{\alpha-1}\|\mathscr{S}(t-s)f(s,x(s))\|\,ds \\ & + \int_0^\tau (t-s)^{\alpha-1}\|\mathscr{S}(t-s)f(s,x(s)) - \mathscr{S}(\tau-s)f(s,x(s))\|\,ds \\ & + \int_0^\tau |(t-s)^{\alpha-1}-(\tau-s)^{\alpha-1}|\,\|\mathscr{S}(\tau-s)f(s,x(s))\|\,ds \\ \le\ & \|\mathscr{T}(t)-\mathscr{T}(\tau)\|\|x_0\| + \frac{\alpha M}{\Gamma(1+\alpha)}\int_\tau^t (t-s)^{\alpha-1}\|f(s,x(s))\|ds \\ & + \sup_{s\in[0,\tau]}\|\mathscr{S}(t-s)-\mathscr{S}(\tau-s)\|\int_0^\tau (t-s)^{\alpha-1}\|f(s,x(s))\|ds \\ & + \frac{\alpha M\|f\|_{C([0,t_1],X)}}{\Gamma(1+\alpha)}\left|\int_0^\tau (\tau-s)^{\alpha-1}ds - \int_0^\tau (t-s)^{\alpha-1}ds\right| \\ \le\ & \|\mathscr{T}(t)-\mathscr{T}(\tau)\|\|x_0\| \\ & + \frac{t_1^\alpha\|f\|_{PC}}{\alpha}\sup_{s\in[0,\tau]}\|\mathscr{S}(t-s)-\mathscr{S}(\tau-s)\| \\ & + \frac{3M\|f\|_{PC}(t-\tau)^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

where we use the inequality $t^\alpha - \tau^\alpha \le (t-\tau)^\alpha$. Keeping in mind of Lemma 2.9(iii), the first and second terms tend to zero as $t \to \tau$. Moreover, it is obvious that the last terms tends to zero too as $t \to \tau$. Thus, we can deduce that $Qx \in C([0,t_1],X)$.

For $t_1 \leq \tau < t < t_2$, keeping in mind our assumptions and applying Lemma 2.8 and Lemma 2.9(i) again, we have

$$\begin{aligned}
& \|(Qx)(t)-(Qx)(\tau)\| \\
\leq\ & \|\mathscr{T}(t)-\mathscr{T}(\tau)\|\|x_0\| + \|\mathscr{T}(t-t_1)-\mathscr{T}(\tau-t_1)\|\|y_1\| \\
& + \frac{t_2^\alpha \|f\|_{PC}}{\alpha} \sup_{s\in[0,\tau]} \|\mathscr{S}(t-s)-\mathscr{S}(\tau-s)\| \\
& + \frac{3M\|f\|_{PC}(t-\tau)^\alpha}{\Gamma(1+\alpha)}.
\end{aligned}$$

As $t \to \tau$, the right hand side of the above inequality tends to zero. Thus, we can deduce that $Qx \in C((t_1,t_2],X)$.

Similarly, we can also obtain that $Qx \in C((t_2,t_3],X), \cdots, Qx \in C((t_\delta,b],X)$. That is, $Qx \in PC(J,X)$.

Step 2. We show that Q is contraction on $PC(J,X)$.

For each $t \in [0,t_1]$, it comes from our assumptions and Lemma 2.9 that

$$\begin{aligned}
& \|(Qx)(t)-(Qy)(t)\| \\
\leq\ & \frac{\alpha M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} L_f(s)\|x(s)-y(s)\| ds \\
\leq\ & \frac{\alpha M\|x-y\|_{PC}}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} L_f(s) ds \\
\leq\ & \frac{\alpha M\|x-y\|_{PC}}{\Gamma(1+\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q_1}} ds\right)^{1-q_1} \|L_f\|_{L^{\frac{1}{q_1}}([0,t_1],R^+)} \\
\leq\ & \frac{\alpha M\|x-y\|_{PC}}{\Gamma(1+\alpha)} \left[\left(\frac{1-q_1}{\alpha-q_1}\right) t_1^{\frac{\alpha-q_1}{1-q_1}}\right]^{1-q_1} \|L_f\|_{L^{\frac{1}{q_1}}([0,t_1],R^+)}.
\end{aligned}$$

In general, for each $t \in (t_k,t_{k+1}]$, using our assumptions and Lemma 2.9 again,

$$\begin{aligned}
& \|(Qx)(t)-(Qy)(t)\| \\
\leq\ & \frac{\alpha M\|x-y\|_{PC}}{\Gamma(1+\alpha)} \left[\left(\frac{1-q_1}{\alpha-q_1}\right) t_{k+1}^{\frac{\alpha-q_1}{1-q_1}}\right]^{1-q_1} \|L_f\|_{L^{\frac{1}{q_1}}([t_k,t_{k+1}],R^+)}.
\end{aligned}$$

Thus,

$$\|Qx-Qy\|_{PC} \leq \frac{\alpha MT^*}{\Gamma(1+\alpha)}\|x-y\|_{PC}.$$

Hence, the condition (4.1) allows us to conclude in view of the Banach contraction mapping principle, that Q has a unique fixed point $x \in PC(J,X)$ which is just the unique PC-mild solution of system (1.1). □

Case 2. f is not Lipschitz

We make the following assumptions.

[C1]: f: $J \times X \to X$ is continuous and maps a bounded set into a bounded set.

[C2]: For each $x_0 \in X$, there exists a constant $r > 0$ such that

$$M\left[\|x_0\| + \sum_{k=1}^{\delta}\|y_k\| + \frac{b^\alpha}{\Gamma(1+\alpha)} \sup_{s\in J,\phi\in Y_\Gamma} \|f(s,\phi(s))\|\right] \leq r,$$

where

$$Y_\Gamma = \{\phi \in PC(J,X) \mid \|\phi\| \leq r \text{ for } t \in J\}.$$

Theorem 4.2. *Suppose that [HA], [C1] and [C2] are satisfied. Then for every $x_0 \in X$, the system (1.1) has at least a PC-mild solution on J.*

Proof. Let $x_0 \in X$ be fixed. We introduce that map

$$Q : PC(J,X) \to PC(J,X)$$

by

$$(Qv)(t) = (Q_1 v)(t) + (Q_2 v)(t)$$

where

$$(4.3)\quad (Q_1 v)(t) = \mathscr{T}(t)x_0 + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s) f(s, v(s))\, ds,\ t \in J \setminus \{t_1, t_2, \cdots, t_\delta\},$$

and

$$(4.4)\quad (Q_2 v)(t) = \begin{cases} 0,\ t \in [0, t_1], \\ \sum_{i=1}^{k} \mathscr{T}(t-t_i) y_i,\ t \in (t_k, t_{k+1}],\ k = 1, \cdots, \delta. \end{cases}$$

For each $t \in [0, t_1]$, $v \in Y_\Gamma$,

$$\begin{aligned} \|(Qv)(t)\| &\le \|(Q_1 v)(t)\| + \|(Q_2 v)(t)\| \\ &\le M\|x_0\| + \frac{b^\alpha M}{\Gamma(1+\alpha)} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|. \end{aligned}$$

For each $t \in (t_k, t_{k+1}]$, $v \in Y_\Gamma$,

$$\begin{aligned} \|(Qv)(t)\| &\le \|(Q_1 v)(t)\| + \|(Q_2 v)(t)\| \\ &\le M\|x_0\| + M\sum_{k=1}^{\delta}\|y_k\| + \frac{b^\alpha M}{\Gamma(1+\alpha)} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|. \end{aligned}$$

Noting that the condition [C2], we see that $Q : Y_\Gamma \to Y_\Gamma$.

Step 1. We prove that Q is a continuous mapping from Y_Γ to Y_Γ.

In order to derive the continuity of Q, we only check that Q_1 and Q_2 are all continuous.

For this purpose, we assume that $v_n \to v$ in Y_Γ. It comes from the continuity of f that $(\cdot - s)^{\alpha-1} f(s, v_n(s)) \to (\cdot - s)^{\alpha-1} f(s, v(s))$, as $n \to \infty$. Noting that

$$\begin{aligned} &(t-s)^{\alpha-1}\|f(s, v_n(s)) - f(s, v(s))\| \\ &\le (t-s)^{\alpha-1} \sup_{s \in J, \phi \in Y_\Gamma} \|f(s, \phi(s))\|,\ \text{for } s \in [0,t],\ t \in J, \end{aligned}$$

by means of Lebesgue dominated convergence theorem, we obtain that

$$\int_0^t (t-s)^{\alpha-1}\|f(s, v_n(s)) - f(s, v(s))\| ds \quad \to \quad 0,\ \text{as } n \to \infty.$$

It is easy to see that for each $t \in J$,

$$\begin{aligned} \|(Q_1 v_n)(t) - (Q_1 v)(t)\| &\le \frac{\alpha M}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1}\|f(s, v_n(s)) - f(s, v(s))\| ds \\ &\to 0,\ \text{as } n \to \infty. \end{aligned}$$

Thus, Q_1 is continuous. On the other hand, it is obvious that Q_2 is continuous.

Since Q_1 and Q_2 are continuous, Q is continuous.

Step 2. We show that Q is a compact operator, or Q_1 and Q_2 are compact operators.

The compactness of Q_2 is clear since it is a constant map (see (4.4)).

Now we prove the compactness of Q_1. For each $t \in J$, the set $\{\mathscr{T}(t)x_0\}$ is precompact in X since $\mathscr{T}(t)$, $t > 0$ is compact.

Also, for each $t \in J$, arbitrary $b > h > 0$, $\varepsilon > 0$, the set

$$\begin{aligned} &\left\{T(h^\alpha\varepsilon)\int_0^{t-h}(t-s)^{q-1}\left(\alpha\int_\varepsilon^\infty \theta\xi_\alpha(\theta)T((t-s)^\alpha\theta - h^\alpha\varepsilon)d\theta\right)\right. \\ &\left. f\left(s, v(s)\right)ds \mid v \in Y_\Gamma\right\} \\ = &\left\{\alpha\int_0^{t-h}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, v(s)\right)d\theta ds \mid v \in Y_\Gamma\right\}\end{aligned}$$

is precompact in X since $T(h^\alpha\varepsilon)$ is compact.

Proceeding as in the proof of Theorem 3.1 in our previous work [**37**], one can obtain

$$\begin{aligned} &\alpha\int_0^{t-h}\int_\varepsilon^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, v(s)\right)d\theta ds \\ \to\ &\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, v(s)\right)d\theta ds,\end{aligned}$$

as $h \to 0,\ \varepsilon \to 0$.

Thus, we can conclude that

$$\begin{aligned} &\left\{\int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f\left(s, v(s)\right)ds \mid v \in Y_\Gamma\right\} \\ = &\left\{\alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T((t-s)^\alpha\theta)f\left(s, v(s)\right)d\theta ds \mid v \in Y_\Gamma\right\}\end{aligned}$$

is precompact in X.

Therefore, the set

$$\left\{\mathscr{T}(t)x_0 + \sum_{i=1}^k \mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f\left(s, v(s)\right)ds \mid v \in Y_\Gamma\right\}$$

is precompact in X.

Thus, for each $t \in J$, $\{(Q_1v)(t) \mid v \in Y_\Gamma\}$ is precompact in X.

Next, we show the equicontinuity of $\mathcal{M} = \{(Q_1v)(\cdot) \mid v \in Y_\Gamma\}$.

The equicontinuity of $\{\mathscr{T}(t)x_0 \mid\ t \in J \setminus \{t_1, t_2, \cdots, t_\delta\}\}$, can be shown using the fact of $\mathscr{T}(\cdot)$ is continuous.

Now, we only need to check the equicontinuity of the second term in $\mathcal{M}$.

For $t \in J$, let $0 \le t' < t'' \le t_1$, set

$$\begin{aligned} I_1 &= \left\|\int_{t'}^{t''}(t''-s)^{\alpha-1}\mathscr{S}(t''-s)f(s, v(s))ds\right\|, \\ I_2 &= \left\|\int_0^{t'}[(t''-s)^{\alpha-1} - (t'-s)^{\alpha-1}]\mathscr{S}(t''-s)f(s, v(s))ds\right\|, \\ I_3 &= \left\|\int_0^{t'}(t'-s)^{\alpha-1}[\mathscr{S}(t''-s) - \mathscr{S}(t'-s)]f(s, v(s))ds\right\|.\end{aligned}$$

After some computation, we have

$$\begin{aligned} & \left\| \int_0^{t''} (t''-s)^{\alpha-1}\mathscr{S}(t''-s)f(s,v(s))ds \right. \\ & \left. - \int_0^{t'} (t'-s)^{\alpha-1}\mathscr{S}(t'-s)f(s,v(s))ds \right\| \\ \leq\ & I_1+I_2+I_3. \end{aligned}$$

Now repeating the previous discussion in Theorem 3.1 of [**37**], we derive that I_1, I_2, I_2 tend to zero as $t'' \to t'$.

Accordingly, we see that the functions in $\mathcal{M}$ are equicontinous. Therefore, Q_1 is a compact operator by the Arzela-Ascoli theorem, and hence Q is also a compact operator. Now, Schauder's fixed point theorem implies that Q has a fixed point, which gives rise to a PC-mild solution. □

To end this section, we make the following assumptions.

[D1]: f: $J\times X \to X$ is continuous and there exists a function $m(\cdot) \in L^\infty(J, R^+)$ such that

$$\|f(t,x)\| \leq m(t), \text{ for all } x \in X \text{ and } t \in J.$$

Theorem 4.3. *Suppose that [HA] and [D1] are satisfied. Then system (1.1) has at least a PC-mild solution on J.*

Proof. We defined that $Q : PC(J,X) \to PC(J,X)$ as in Theorem 4.2 by $(Qv)(t) = (Q_1v)(t) + (Q_2v)(t)$. Then we proceed in several steps.

Step 1. We prove that Q is a continuous mapping from $PC(J,X)$ to $PC(J,X)$.

Let $\{v_n\}$ be a sequence in $PC(J,X)$ such that $v_n \to v$ in $PC(J,X)$. It comes from [D1] that $(\cdot - s)^{\alpha-1}f(s,v_n(s)) \to (\cdot - s)^{\alpha-1}f(s,v(s))$, as $n \to \infty$, and note that

$$\begin{aligned} & (t-s)^{\alpha-1}\|f(s,v_n(s)) - f(s,v(s))\| \\ & \leq 2m(s)(t-s)^{\alpha-1} \in L^1(J,R^+), \text{ for } s\in[0,t],\ t\in J. \end{aligned}$$

Similar to the discussion in Theorem 4.2, one can prove that Q is a continuous mapping from $PC(J,X)$ to $PC(J,X)$.

Step 2. Q maps bounded sets into bounded sets in $PC(J,X)$.

So, let us prove that for any $r > 0$ there exits a $M^* > 0$ such that for each $v \in B_r = \{v \in PC(J,X) \mid \|v\|_{PC} \leq r\}$, we have $\|Qv\|_{PC} \leq M^*$.

Indeed, for any $v \in B_r$,

$$\begin{aligned} \|(Qv)(t)\| &\leq \|(Q_1v)(t)\| + \|(Q_2v)(t)\| \\ &\leq M\|x_0\| + M\sum_{i=1}^{\delta}\|y_i\| + \frac{b^\alpha M}{\Gamma(1+\alpha)}\|m\|_{L^\infty(J,R^+)}, \end{aligned}$$

which implies

$$\|Qv\|_{PC} \leq M\|x_0\| + M\sum_{i=1}^{\delta}\|y_i\| + \frac{b^\alpha M}{\Gamma(1+\alpha)}\|m\|_{L^\infty(J,R^+)} \equiv M^*.$$

Step 3. Q is a compact operator.

In order to verify that Q is a compact operator, one can repeat the same process in Step 2 of Theorem 4.2 only need replace $\sup_{s\in J,\phi\in Y_\Gamma}\|f(s,\phi(s))\|$ by $\|m\|_{L^\infty(J,R^+)}$.

Step 4. The set $\Theta = \{x \in PC(J,X) \mid x = \lambda Qx, \lambda \in [0,1]\}$ is bounded.

Let $v \in \Theta$. Then $v(t) = \lambda(Qv)(t)$ for some $\lambda \in [0,1]$. Thus, for $t \in J$, directly calculation implies that $\|Qv\|_{PC} \leq M^*$. Hence, we deduce that Θ is a bounded set.

Since we have already proven that Q is continuous and compact, thanks to the Schaefer's fixed point theorem, Q has a fixed point which is a PC-mild solution of system (1.1) on J. □

Remark 4.4. *In the assumption [D1], the condition* $m(\cdot) \in L^{\infty}(J,R^+)$ *can be replaced by* $m(\cdot) \in L^{\frac{1}{q_2}}(J,R^+)$ *where* $\frac{1}{q_2} \in [0,\alpha)$. *The norm of* m *is defined by*

$$\|m\|_{L^{\frac{1}{q_2}}(J,R^+)} = \begin{cases} \left(\int_J \|m(t)\|^{\frac{1}{q_2}} dt\right)^{q_2}, & \text{if } 1 < \dfrac{1}{q_2} < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \left\{ \sup\limits_{t \in J-\bar{J}} \|m(t)\| \right\}, & \text{if } \frac{1}{q_2} = \infty, \end{cases}$$

where $\mu(\bar{J})$ *is the Lebesgue measure on* $\bar{J}$.

5. Existence results for impulsive nonlocal Cauchy problems

In this section, we extend the results obtained in Section 3 to nonlocal problems for impulsive fractional evolution equations. More precisely, we will prove the existence and uniqueness of the PC-mild solutions for system (1.2). As we all known, the nonlocal conditions has a better effect on the solution and is more precise for physical measurements than the classical initial condition alone. For the nonlocal impulsive Cauchy problems, we refer the readers to [**9, 10, 11, 17**] and the references therein.

Definition 5.1. *By a PC-mild solution of the system (1.2) we mean that a function* $x \in PC(J,X)$ *which satisfies the following integral equation*

$$x(t) = \begin{cases} \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) \\ f(s,x(s))\, ds, \text{for } t \in [0,t_1], \\ \mathscr{T}(t)[x_0 + g(x)] + \mathscr{T}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s) f(s,x(s))\, ds, \text{for } t \in (t_1,t_2], \\ \vdots \\ \mathscr{T}(t)[x_0 + g(x)] + \sum_{i=1}^{\delta} \mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s) f(s,x(s))\, ds, \text{for } t \in (t_\delta, b]. \end{cases} \tag{5.1}$$

Case 1. g **is Lipschitz**

[Hg1]: g: $PC(J,X) \to X$ and there exists a constant $L_g > 0$ such that

$$\|g(x) - g(y)\| \leq L_g \|x - y\|_{PC}, \ x,y \in PC(J,X).$$

Theorem 5.2. *Let [HA], [HF1] and [Hg1] be satisfied. Then for every* $x_0 \in X$, *the system (1.2) has a unique PC-mild solution on J provided that*

$$0 < \mu' := ML_g + \frac{\alpha M T^*}{\Gamma(1+\alpha)} < 1. \tag{5.2}$$

Proof. Define an operator $\mathcal{F} : PC(J,X) \to PC(J,X)$ by

$$(\mathcal{F}x)(t) = \begin{cases} \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) \\ f(s, x(s))\, ds, \text{ for } t \in [0, t_1), \\ \mathscr{T}(t)[x_0 + g(x)] + \mathscr{T}(t-t_1)y_1 + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s) f(s, x(s))\, ds, \text{ for } t \in (t_1, t_2), \\ \vdots \\ \mathscr{T}(t)[x_0 + g(x)] + \sum_{i=1}^{\delta} \mathscr{T}(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1} \\ \mathscr{S}(t-s) f(s, x(s))\, ds, \text{ for } t \in (t_\delta, b]. \end{cases}$$

It is obvious that $\mathcal{F}$ is well defined on $PC(J,X)$.

Step 1. We prove that $\mathcal{F}x \in PC(J,X)$ for $x \in PC(J,X)$.

For $0 \le \tau < t \le t_1$, by our assumptions and Lemma 2.9,

$$\|\mathscr{T}(t)g(x) - \mathscr{T}(\tau)g(x)\| \le \|\mathscr{T}(t) - \mathscr{T}(\tau)\|(L_g\|x\|_{PC} + \|g(0)\|).$$

As $t \to \tau$, the right hand side of the above inequality tend to zero due to Lemma 2.9(iii) again. Recall the Step 1 in Theorem 4.1, we know that $\mathcal{F}x \in PC(J,X)$.

Step 2. $\mathcal{F}$ is contraction.

We only take $t \in (t_k, t_{k+1}]$, then we have

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \quad \le \quad \left[ML_g + \frac{\alpha M T^*}{\Gamma(1+\alpha)} \right] \|x - y\|_{PC}.$$

So we get

$$\|\mathcal{F}x - \mathcal{F}y\|_{PC} \le \mu' \|x - y\|_{PC}.$$

where

$$\mu' = ML_g + \frac{\alpha M T^*}{\Gamma(1+\alpha)}.$$

Hence, the condition (5.2) allows us to conclude, in view of the Banach contraction mapping principle again, that $\mathcal{F}$ has a unique fixed point $x \in PC(J,X)$ which is the PC-mild solution of system (1.2). □

THEOREM 5.3. *Suppose that [HA], [D1] and [Hg1] are satisfied. If $ML_g < \frac{1}{2}$ then system (1.2) has at least a PC-mild solution on J.*

Proof. Choose

$$\sigma \quad \ge \quad 2M\left[(\|x_0\| + \|g(0)\|) + \sum_{k=1}^{\delta} \|y_k\| + \frac{b^\alpha}{\Gamma(1+\alpha)} \|m\|_{L^\infty(J,R^+)} \right].$$

Consider $B_\sigma = \{x \in PC(J,X) \mid \|x\|_{PC} \le \sigma\}$. Define the operators $\mathcal{N}$ on B_σ by

$$(\mathcal{N}x)(t) = (\mathcal{N}_1 x)(t) + (\mathcal{N}_2 x)(t) + (\mathcal{N}_3 x)(t)$$

where

$$(\mathcal{N}_1 x)(t) = \mathscr{T}(t)[x_0 + g(x)], \ t \in J,$$

$$(\mathcal{N}_2 x)(t) = \int_0^t (t-s)^{\alpha-1} \mathscr{S}(t-s) f(s, x(s))\, ds, \ t \in J, \tag{5.3}$$

and $\mathcal{N}_3$ is the same as the operator Q_2 defined in Theorem 4.2.

It suffices to proceed exactly steps of the proof in Theorem 4.2 while replacing B_r by B_σ to obtain that $\mathcal{N}_2 + \mathcal{N}_3$ are continuous and compact. We want to use the Krasnoselkii's fixed point theorem. Thus, to complete the rest proof of this theorem, it suffices to show that $\mathcal{N}_1$ is a contraction mapping and that if $x, y \in B_\sigma$ then $\mathcal{N}_1 x + (\mathcal{N}_2 + \mathcal{N}_3)y \in B_\sigma$. Indeed, for any $x \in B_\sigma$, we have

$$\begin{aligned} & \|\mathcal{N}_1 x\|_{PC} + \|\mathcal{N}_2 y\|_{PC} + \|\mathcal{N}_3 y\|_{PC} \\ \leq \ & M(\|x_0\| + \|g(0)\| + L_g\sigma) + M\sum_{k=1}^{\delta}\|y_k\| + \frac{b^\alpha M}{\Gamma(1+\alpha)}\|m\|_{L^\infty(J,R^+)}. \end{aligned}$$

Since $ML_g < \frac{1}{2}$, we can deduce that

$$\|\mathcal{N}_1 x + (\mathcal{N}_2 + \mathcal{N}_3)y\|_{PC} \quad \leq \quad \sigma.$$

Next, for any $t \in (t_k, t_{k+1}]$, $x, y \in C((t_k, t_{k+1}], X)$,

$$\|\mathcal{N}_1 x - \mathcal{N}_1 y\|_{C((t_k,t_{k+1}],X)} \quad \leq \quad ML_g\|x - y\|_{C((t_k,t_{k+1}],X)}.$$

Therefore, we can deduce that $\mathcal{N}_1$ is contraction from $ML_g < 1$. Moreover, $\mathcal{N}_2 + \mathcal{N}_3$ is compact and continuous. Hence, by the well known Krasnoselskii's fixed point theorem, we can conclude that system (1.2) has at least one PC-mild solution on J. □

Case 2. g **is not Lipschitz**

[Hg2]: g: $PC(J, X) \to X$ and maps bounded sets into bounded sets.

[C2′]: For each $x_0 \in X$, there exists a constant $r' > 0$ such that

$$M\left[\|x_0\| + \sup_{\phi\in Y'_\Gamma}\|g(\phi)\|\right] + \sum_{k=1}^{\delta}\|y_k\| + \frac{b^\alpha M}{\Gamma(1+\alpha)}\sup_{s\in J,\phi\in Y'_\Gamma}\|f(s,\phi(s))\| \leq r',$$

where

$$Y'_\Gamma = \left\{\phi \in PC(J, X) \mid \|\phi\| \leq r' \text{ for } t \in J\right\}.$$

THEOREM 5.4. *Suppose that [HA], [C1], [C2′] and [Hg2] are satisfied. Then for every $x_0 \in X$, the system (1.2) has at least a PC-mild solution on J.*

Proof. Define an operator $\mathcal{F}$ on $PC(J, X)$ by

$$(\mathcal{F}v)(t) = (\mathcal{F}_1 v)(t) + (\mathcal{F}_2 v)(t)$$

where

$$(\mathcal{F}_1 v)(t) = \mathscr{T}(t)[x_0 + g(x)] + \int_0^t (t-s)^{\alpha-1}\mathscr{S}(t-s)f(s, v(s))\,ds,\ t \in J.$$

and $\mathcal{F}_2$ is the same as Q_2 defined in Theorem 4.2. Thus, we need to check that $\mathcal{F}_1$ is compact. Observing the expression of the $\mathcal{F}_1$, we only check that, for each $t \in J$, the set $\{\mathscr{T}(t)[x_0 + g(v)] \mid v \in Y'_\Gamma\}$ is precompact in X since $\mathscr{T}(t)$, $t > 0$ is compact and [Hg2]. On the other hand, the equicontinuity of $\left\{\mathscr{T}(t)[x_0 + g(v)] \mid \ t \in J,\ v \in Y'_\Gamma\right\}$ can be shown using the same idea.

Therefore, $\mathcal{F}$ is also a compact operator. By Schauder's fixed point theorem again, $\mathcal{F}$ has a fixed point, which gives rise to a PC-mild solution. □

6. Applications

In this section, some interesting examples are presented to illustrate the theory.

Consider the following impulsive fractional differential equations with nonlocal conditions

(6.1)
$$\begin{cases} {}^C D_t^\alpha x(t,y) = \frac{\partial^2}{\partial y^2} x(t,y) + f(t, x(t,y)),\ \alpha \in (0,1),\ y \in (0,\pi),\ t \in [0,t_1) \cup (t_1, 1], \\ x(t,0) = x(t,\pi) = 0, \\ \Delta x(t_1^+) = x(t_1^-) + z,\ t_1 = \frac{1}{2},\ y \in (0,\pi), \\ x(0,y) = x_0(y) + g(x(t,y)),\ t \in [0,1],\ y \in (0,\pi). \end{cases}$$

Let $X = L^2(0,\pi)$. Define

(E1) $Ax = -\frac{\partial^2}{\partial y^2} x$ for $x \in D(A)$ where $D(A) = \{x \in X \mid \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X$ and $x(0) = x(\pi) = 0\}$. Then, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ in $L^2(0,\pi)$. Moreover, $T(\cdot)$ is also compact and $\|T(t)\| \leq e^{-t} \leq 1 = M_1$, $t \geq 0$.

Case 1. Define

(E2) $f(t,x(t))(y) = \frac{e^{-t}|x(t,y)|}{(\rho+e^t)(1+|x(t,y)|)}$, $t \in [0,t_1) \cup (t_1,1]$, $\rho > -1$, $x \in X$, $y \in (0,\pi)$.

(E3) $g(x(t))(y) = \sum_{j=1}^{2} \lambda_j |x(s_j, y)|$, $0 < \lambda_1, \lambda_2$, $0 < s_1 < s_2 < 1$, $s_1, s_2 \neq t_1$, $x \in PC([0,1], X)$, $y \in (0,\pi)$.

Clearly, $f : [0,1] \times X \to X$ is continuous functions,

$$\|f(t,x) - f(t,y)\| \leq L_f \|x - y\|, \text{ with } L_f = \frac{1}{\rho+1} \in L^{\frac{1}{q_1}}([0,1], R^+),\ q_1 \in (0,\alpha).$$

It is obvious that g: $PC([0,1],X) \to X$ satisfies $\|g(x) - g(y)\| \leq L_g \|x - y\|_{PC}$ with $L_g = \sum_{j=1}^{2} \lambda_j$.

• (E1)+(E2)+(E3) makes the assumptions in Theorem 5.2 satisfied. Therefore, the equations (6.1) has a unique PC-mild solution on $[0,1]$ provided that

$$\sum_{j=1}^{2} \lambda_j + \frac{\alpha}{\Gamma(1+\alpha)} \left[\left(\frac{1-q_1}{\alpha - q_1} \right) \right]^{1-q_1} \frac{1}{\rho+1} < 1.$$

Case 2. Define

(E4) $f(t,x(t))(y) = \frac{e^{-t} \sin(x(t,y))}{(1+t)(e^t + e^{-t})} + e^{-t}$, $t \in [0,t_1) \cup (t_1,1]$, $x \in X$, $y \in (0,\pi)$.

Clearly,

$$\|f(t,x)\| \leq \frac{e^{-t}}{e^t + e^{-t}} + e^{-t} = m(t), \text{ with } m(t) \in L^\infty([0,1], R^+).$$

• (E1)+(E3)+(E4) makes the assumptions in Theorem 5.3 satisfied. Therefore, the equations (6.1) has at least one PC-mild solution on $[0,1]$ provided that $\sum_{j=1}^{2} \lambda_j < \frac{1}{2}$.

Case 3. Define

(E5) $f(t,x(t))(y) = c_1 |\sin(x(t,y))|$, $c_1 > 0$, $t \in [0,t_1) \cup (t_1,1]$, $x \in X$, $y \in (0,\pi)$.

(E6) $g(x(t))(y) = \int_0^1 l(s) \ln(1 + |x(s,y)|^{\frac{1}{2}}) ds$, $l \in L^1([0,1], R)$, $x \in PC([0,1], X)$, $y \in (0,\pi)$.

Clearly, f and g are continuous and map a bounded set into a bounded set.

• (E1)+(E5)+(E6) makes the assumptions in Theorem 5.4 satisfied for large $r' > 0$. Therefore, the equations (6.1) has at least one PC-mild solution.

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