Global dynamics of a PDE model for Aedes aegypti mosquitoe incorporating female sexual preference

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ABSTRACT. In this paper we study the long time dynamics of a reaction diffusion system, describing the spread of Aedes aegypti mosquitoes, which are the primary cause of dengue infection. The system incorporates a control attempt via the sterile insect technique. The model incorporates female mosquitoes sexual preference for wild males over sterile males. We show global existence of strong solution for the system. We then derive uniform estimates to prove the existence of a global attractor in $L^2(\Omega)$, for the system. The attractor is shown to be $L^\infty(\Omega)$ regular and posess state of extinction, if the injection of sterile males is large enough. We also provide upper bounds on the Hausdorff and fractal dimensions of the attractor.

CONTENTS

1. Introduction

Dengue infection is a leading illness in the tropics and subtropics causing as many as 100 million infections yearly [**29**]. Dengue is caused by any one of four

related viruses transmitted by Aedes female mosquitoes. There are no available vaccines yet to prevent infection with dengue virus and avoiding mosquito bites is the most effective protective measures against infection. Thus all efforts must be geared against proliferation of the mosquito population.

The sterile insect technique (SIT) is a biological control which disrupts the natural reproductive process of insects. Male insects are first made sterile by gamma radiation before releasing them in large numbers into the environment to mate with the native wild insects. The native wild females that mates with these sterile male will produce unviable eggs which will not hatch. This causes the native insects populatin to decrease over time, resulting in the increase in the ratio of sterile to normal insects increases, which then drives the native population to extinction [**33**].

In vector borne disease control it is important to include analyzing mating behavior, which is an aspect of mosquito biology that is not fully understood [**26**]. The buzz of a flying female mosquito acts as a mating signal, for attracting males [**26**]. Mate assessment interactions in swarming insects occurs when these insects enter swarms and this happens very quickly in flight [**27**]. In mosquito mating swarms, it is important to converge quickly to a mate harmonic signal before others, this is to ensure that a desirable mate locks faster into its signal, than the signal from a swarming competitor [**31**]. This is as indication that harmonic convergence may be used in mate assessment. The female mosquito have the ability to react to variation in flight tone, which further suggests that this signal may be used to assess the males during precopulatory interactions [**27**]. Single females fly into the swarm and are detected by their lower wing-beat frequency [**24, 30**]. It has been observed that several males may arrive near the female, which departs with one of them from the swarm in copula. Reports have it that larger males were more successful in mating than smaller ones [**35, 36**]. Thus there is evidence that sexual selection operates when these insects enter swarms, and hence should be incorporated into any reasonable model.

The sterile insect technique, has been tried in a number of scenarios, as an attempt to combat malaria and dengue with little success. In the current work, we extend the model derived in [**33**], to include both spatial spread of the insect, and sexual preference of the female mosquito for the wild males against modified sterile males. To the best of our knowledge, this is a novel feature and has not been incorporated earlier in reaction diffusion equations modelling mosquitoe dynamics and/or dengue control. Thus the question we ask is this, is it possible to achieve finite time extinction in a reaction diffusion system describing the evolution of a mosquito species, where both the SIT and female sexual preference are incorporated?

The answer lies in investigating the long time dynamics of our derived model. It is well known that usually, under the action of diffusion, dynamical systems tend to smooth out. This mechanism is commonly referred to as "dissipation" [**1**]. Thus we are lead to believe that the diffusive system that we consider, should posess a global attractor, that supports states of extinction. Our goal in the current manuscript is to show this precisely. The global attractor which is the object that encompasses the long-time dynamics, is by definition a compact invariant set in the phase space, to which almost all trajectories eventually evolve. To study this object in the PDE case often involves making detailed estimates of various functional norms. Heuristically, the goal behind these is to show the existence of a bounded absorbing set in the

phase space, and then to establish asymptotic compactness of the semigroup for the system of equations. The opposite signs on the nonlinearities in (2.1) pose a problem to show asymptotic compactness, and thus the uniform gronwall lemma has to be resorted to.

There is a large literature on the global and asymptotic dynamics of PDE's arising in ecological modelling. Results for general cross diffusion systems were reported in [18]. More recent work presenting general theory has been done by Shim [19], [20]. Also of much interest have been systems with time delays [15], [9],[10], [11]. More specific cross diffusion systems have been explored in [16], and systems with stage structure in [14],[13]. [21] has explored diffusion in tri-trophic food models. Zhang et al also investigated a cross diffusion PDE model with Holling type III functional response. They investigated the long time dynamics via construction of appropriate Lyapunov functions. Ko and Ryu have also investigated predator prey models with Holling type II responses, including the long time dynamics. Here there is scope of extensive prey refuge [17]. Questions of persistence and existence in two species models have been explored in [8]. Much has also been done on questions regarding stability and boundedness of solutions in long time of some of these model systems (see [12]). Pao has investigated in some detail the global dynamics of diffusive competition systems [6], [7]. Recently there is also work by You on predator prey systems with Holling type response, where the dissipation condition is not met, [22].

Our approach in the current manuscript is as follows. We begin by demonstarting global existence of weak and strong solution to the system under consideration. This is done via theorems 3.2, 4.2. We derive the existence of a bounded absorbing sets in $L^2(\Omega)$. We also derive the uniform estimates by means of which we tackle the question of asymptotic compactness of the semigroup for the model in $L^2(\Omega)$. With these estimates at hand, we will demonstrate the existence of a global attractor for the model in $L^2(\Omega)$. To this end we provide our main result, theorem 5.6. We next show that this global attractor is actually $L^\infty(\Omega)$ regular via theorem 5.7.

We lastly show that the global attractor is finite dimensional, and derive upper bounds on both it's Hausdorff and fractal dimensions. Thus entailing our result, theorem 6.4. We also make some concluding remarks.

2. Model Formulation

Our model is an extension of the model in [33] to included both spatial spread, and preferential selection of the female mosquito for the wild males, against modified sterile males. The mosquito population is divided into male and female classes. The female classes are further divided into immature and adult depending on the insect sexual preference. The class A_I represents the immature phase of the mosquitoes (eggs, larvae and pupae) from adults without sexual preference, while A_P represents the immature phase from adults with sexual preference. For the adult form, I corresponds to mating singles without sexual preference, P corresponds to mating singles with sexual preference. The class F_I represents mating fertilized females without sexual preference and F_P represents mating fertilized females with sexual preference. The class M correspond to the wild male insects population, while M_S corresponds to the sterile (irradiated or transgenic) insects. The parameter ϕ is the oviposition rate per female mosquito which is proportional to female density, but it is also regulated by a carrying capacity effect, C, which is related to the

amount of available nutrients and space. The aquatic population becomes winged mosquitoes at a rate γ, and a proportion r transforms into female, and $(1-r)$, into male. The parameters μ_A, μ, μ_F, μ_M and μ_{M_S}, denotes the mortality rates of the immature form, unmating females, mating fertilized females, mating unfertilized females, with and without sexual preference as well as wild males and sterile male insects, respectively.

A female mosquito mates once in its life, and oviposits its eggs in different places during its entire life [23]. The per capita mating rate of a unmating female with a natural male mosquito are given by $\frac{\beta_I M}{M+M_S}$ and $\frac{\beta_P M}{M+M_S}$, where $\beta_P < \beta_I$. Since irradiated insects are placed artificially, and the effective mating rate could be diminished due to the sterilization, the per capita mating rate of a female with an irradiated male is given by $\frac{\beta_S M}{M+M_S}$, where $\beta_S = pq\beta$, and $0 \leq p; q \leq 1$. The parameter p is related to the effectiveness of sterile male introduction regarded to the spatial distribution of female insects, and q can be thought of as physiological modifications induced by the sterilization technique. Since the mating rate of the wild females with sexual preference is lower than mating rate of females without sexual preference, we assume that the population of females with sexual preference is maintained by mutation from female without sexual preference at the rate δ. Finally the parameter α is the rate at which sterile males are released and sprayed.

It is a modelling assumption that the females with preference do not mate with the sterile male (even in small probability). This we feel is a natural way to incorporate selection in the model, and helps us clearly differentiate between the classes I and P. Note, there is no conclusive evidence, that sexual selection exists in mosquitoe mating, however there are a number of studies, [26, 27, 24, 30, 35, 36] that suggest, this might be so.

Concerning the mutation, we assume what is known as a mutation-selection balance, [25]. Essentially, we assume that females that display preference in mate selection have a lower mating rate overall (e.g. due to time wasted searching for a male). In the absence of a high number of fit males, these females are at a fitness disadvantage, because they do not reproduce as quickly. As a result, over time, they would eventually be driven extinct by competitive exclusion with the females without preference. Thus preference can be thought of as a deletrious allele. In many population biology models, deletrious alleles are assumed maintained in the population by mutation. The deletrious alleles keep appearing as a result of mutation, but because they are selected against, they are only maintained at low levels. If preference/selection is not a deletrious allele maintained at low populations, then only a few other options are possible.
(i) Preference confers a fitness advantage, even in the absence of SIT. In this case, the population would consist almost entirely of females with preference and treatment would not work even from the beginning. We have no evidence for this.
(ii) There is some alternate trade-off maintaining preference in the population. While this may be possible, it would require a more complicated model with additional assumptions regarding the maintenance of preference in the population. In the absence of any evidence for such a trade-off, we prefer to assume that preference is maintained through mutation instead, since this will always be possible.
The model is described by the following system of partial differential equations:

$$\frac{\partial A_I}{\partial t} = \Delta A_I + \phi(1 - \frac{A_I + A_P}{C})F_I - (\gamma + \mu_A + \delta)A_I$$

$$\frac{\partial A_P}{\partial t} = \Delta A_P + \phi(1 - \frac{A_P + A_I}{C})F_P - (\gamma + \mu_A)A_P + \delta A_I$$

$$\frac{\partial I}{\partial t} = \Delta I + r\gamma A_I - \frac{\beta_I M I}{M + M_S} - \frac{\beta_S M_S I}{M + M_S} - \mu I$$

$$\frac{\partial P}{\partial t} = \Delta P + r\gamma A_P - \frac{\beta_P M P}{M + M_S} - \mu P$$

(2.1)

$$\frac{\partial F_I}{\partial t} = \Delta F_I + \frac{\beta_I M I}{M + M_S} - \mu_F F_I$$

$$\frac{\partial F_P}{\partial t} = \Delta F_P + \frac{\beta_P M P}{M + M_S} - \mu_F F_P$$

$$\frac{\partial M}{\partial t} = \Delta M + (1 - r)\gamma(A_I + A_P) - \mu_M M$$

$$\frac{\partial M_S}{\partial t} = \Delta M_S + \alpha - \mu_{M_S} M_S,$$

The problem is posed on $\Omega \subset \mathbb{R}^3$. Ω is bounded, and $\partial\Omega$ is assumed to be smooth. We consider Neumann boundary conditions

(2.2)
$$\nabla A_I \cdot \mathbf{n} = \nabla A_P \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = \nabla P \cdot \mathbf{n} = \nabla F_I \cdot \mathbf{n} = \nabla F_P \cdot \mathbf{n} = \nabla M \cdot \mathbf{n} = \nabla M_S \cdot \mathbf{n} = 0$$

on $\partial\Omega$ We also impose suitable initial conditions

(2.3) $$A_I(x, 0) = A_{I0}, \ A_P(x, 0) = A_{P0}, \ I(x, 0) = I_0, \ P(x, 0) = P_0,$$

(2.4) $$F_I(x, 0) = F_{I0}, \ F_P(x, 0) = F_{P0}, \ M(x, 0) = M_0, \ M_S(x, 0) = M_{S0}$$

In all the estimates made henceforth C, C_1, C_3, C_4 are generic constants that can change in their value from line to line, and sometimes within the same line if so required.

3. Existence of weak solution

In this section we show there exists a weak solution to the system (2.1).

DEFINITION 3.1 (Weak solution). A given compartment u_1 of system (2.1) is said to posess a weak solution if the following equation is satisfied in the distributional sense for $\forall v \in H_0^1(\Omega)$.

(3.1) $$\frac{d}{dt}(u_1, v) + (d_1 \nabla u_1, \nabla v) + C_1 \langle u_1, v \rangle = \langle G(u_i), v \rangle$$

Variable	Description
$A(t)$	Immature phase of insect
$I(t)$	Single females with male preference
$P(t)$	Single females with wild male preference
$F(t)$	Mating fertilized females
$M(t)$	Wild males
$M_S(t)$	Sterile males

Parameter	Description
C	Carrying capacity related to the amount of available nutrients and space
ϕ	Intrinsic oviposition rate
β	Mating rate of natural insects
$\beta_T = qp\beta$	Mating rate of SIT
q	Percentage of reduction of mating capacity
p	Ability of dispersion
γ	Mature rate to adulthood
$r\gamma$	Female mature rate to adulthood
$(1-r)\gamma$	Male mature rate to adulthood
$\mu_A, \mu, \mu_F, \mu_M, \mu_{M_S}$	Natural mortality rate
α	Release rates of sterile male mosquitoes

TABLE 1. Description of the Variables and Parameters of the Model (2.1).

Here $G(u_i)$ is the nonlinear functional response which can depend on other compartments. Also, $\langle .,. \rangle = \langle .,. \rangle_2$ is the standard inner product in $L^2(\Omega)$. $G(u_i)$ is a nonlinear term, possibly depending on other compartments. We introduce the following spaces

$$(3.2) \qquad H = [L^2(\Omega)]^8, \ V = [H^1(\Omega)]^8, \ V^* = [H^1(\Omega)]'.$$

We demonstrate with A_I. For this compartment the Galerkin truncation take the form,

$$(3.3) \qquad A_{In}(t) = \sum_{j=1}^{n} A_{In_j}(t) w_j.$$

Here w_j are the eigenfunctions of the negative Laplacian, i.e., $-\Delta w_i = \lambda_i w_i$. The goal is to then derive uniform estimates on these truncations and then use the standard functional analysis theory to extract subsequences, weakly convergent to a limiting function. This limiting function will satisfy (3.1), making it a weak solution. The truncation solves the following equation,

$$(3.4) \qquad \frac{\partial A_{In}}{\partial t} = \Delta A_{In} + \phi \left(1 - \frac{A_{In} + A_{Pn}}{C}\right) F_{In} - (\gamma + \mu_A + \delta)A_{In},$$

$$(3.5) \qquad A_{In}(0) = A_{In0}.$$

In order to proceed we note $F_{In} \leq A_{In}$. Hence we can write the equation for A_{In} as

$$(3.6) \qquad \frac{\partial A_{In}}{\partial t} = \Delta A_{In} + \phi \left(1 - \frac{A_{In} + A_{Pn}}{C} \right) (A_{In} - g) - (\gamma + \mu_A + \delta) A_{In}$$

where g is a positive function, representing combination of eggs that hatched into males and single females. Now we need only consider region where $A_{In} + A_{Pn} \leq C$, as when $A_{In} + A_{Pn} > C$, the logistic term is negative, hence $\phi \left(1 - \frac{A_{In} + A_{Pn}}{C} \right) F_{In} < 0$, hence the estimates are made easily. We multiply (3.6) by A_{In} and integrate by parts to obtain

$$(3.7) \qquad |A_{In}|_{L^\infty(0,T;L^2(\Omega))} \leq C$$

$$(3.8) \qquad |A_{In}|_{L^2(0,T;H^1(\Omega))} \leq C$$

We can now use the uniform bounds in (3.7) and (3.8) to extract a subsequence A_{In_j} such that

$$(3.9) \qquad A_{In_j} \overset{*}{\rightharpoonup} A_I^* \text{ in } L^\infty(0,T;L^2(\Omega))$$

$$(3.10) \qquad A_{In_j} \rightharpoonup A_I^* \text{ in } L^2(0,T;H^1(\Omega))$$

We need estimates on the time derivatives of solutions. To this end

$$\left| \frac{\partial A_n}{\partial t} \right|_{V^*}$$
$$\leq \quad |\Delta A_n + \phi F_n|_{V^*}$$
$$\leq \quad |\nabla A_n|_2 + \phi |F_n|_2$$
$$\leq \quad C(|A_n|_{H^1(\Omega)} + |F_n|_2)$$

(3.11)

Integrating the above in the time interval $[0, T]$ yields

$$\int_0^T \left| \frac{\partial A_n}{\partial t} \right|_{V^*} dt$$
$$\leq \quad C|A_n|_{L^2(0,T;H^1(\Omega))} + C|F_n|_{L^\infty(0,T;L^2(\Omega))}$$
$$(3.12) \qquad \leq \quad C$$

The estimates on F_n are derived next. This yields the following estimate

$$(3.13) \qquad A_n \in H^1(0,T;L^2(\Omega)).$$

But via the embedding of

$$(3.14) \qquad H^1(0,T) \hookrightarrow C(0,T),$$

it follows

$$(3.15) \qquad A_{In} \in C(0,T;L^2(\Omega)).$$

Thus we have

(3.16)
$$A_{In_j} \to A_* \text{ in } L^2(0, T; L^2(\Omega))$$

(3.17)
$$\frac{\partial A_{In_j}}{\partial t} \rightharpoonup \frac{\partial A_*}{\partial t} \text{ in } L^2(0, T; L^2(\Omega))$$

We multiply the equation for F_{In} by F_{In} and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}|F_{In}|_2^2 + |\nabla F_{In}|_2^2 + \mu_{F_I}|F_{In}|_2^2 = \int_\Omega \left(\frac{\beta M_n I_n F_{In}}{M_n + M_{S_n}} \right) d\mathbf{x}$$

Using the $L^2(\Omega)$ bounds on I_n, P_n, (which are easily obtainable after we have the estimates on A_I, A_P) we obtain

(3.18)
$$\frac{1}{2}\frac{d}{dt}|F_n|_2^2 + C_1|F_n|_2^2 \le C_2$$

we obtain via Gronwalls inequality

(3.19)
$$|F_n(t)|_2^2 \le e^{-(C_1)t}|F_n(0)|_2^2| + \frac{C_2}{C_1}$$

furthermore integration of (3.18) in the time interval $[0, T]$ yields

(3.20)
$$\int_0^T |\nabla F_n|_2^2 dt \le C_2 + |F_n(0)|_2^2$$

These estimates via the standard functional analysis theory, [3] as earlier yield

(3.21)
$$F_{In_j} \to F_* \text{ in } L^2(0, T; L^2(\Omega))$$

(3.22)
$$\frac{\partial F_{In_j}}{\partial t} \rightharpoonup \frac{\partial F_*}{\partial t} \text{ in } L^2(0, T; L^2(\Omega))$$

The main technical difficulty in proving convergence is to show convergence of the nonlinear terms in the equations. To this end note for two distinct nonlinear terms as they appear in model (2.1)

(3.23)
$$G_1(A_{In}, A_{Pn}, F_{In}) = \phi \left(1 - \frac{A_{In} + A_{Pn}}{C} \right) F_{In},$$

we have the following estimate holds

$$|G_1(A_{In}, A_{Pn}, F_n) - G_2(A_{In}^*, A_{Pn}^*, F_n^*)|_2$$
$$\le \quad C_1|A_{In} - A_{In}^*|_2 + C_2|F_n - F_n^*|_2 + C_3|A_{Pn} - A_{Pn}^*|_2$$
(3.24)

With all the above a priori estimates at hand, we will pass to the weak limit. We focus on the equation for A_{In} for demonstration.

$$\left(\frac{d\,A_{In_j}}{dt}, \phi(t)w_j\right) + \left(\nabla A_{In_j}, \nabla w_j\phi(t)\right) + (\gamma + \mu_A + \delta)\left\langle A_{In_j}, \phi(t)w_j\right\rangle$$
$$= \left\langle G_1(F_{In_j}, A_{In_j}, A_{Pn_j}), \phi(t)w_j\right\rangle,$$

(3.25)

Upon passing to the weak limit of (3.25) we will obtain

$$\lim_{j\to\infty} \int_0^T (A_{In_j}, \phi^{'}(t)w_j)dt + \int_0^T (\nabla A_{In_j}, \nabla w_j\phi)dt$$
$$+(\gamma + \mu_A + \delta)\int_0^T (A_{In_j}, \phi w_j)dt - \int_0^T (G_1(F_{In_j}, A_{In_j}, A_{Pn_j}), \phi w_j)dt$$
$$= \int_0^T (A_I^*, \phi^{'}(t)w_j)dt + \int_0^T (\nabla A_I^*, \nabla w_j\phi)dt + (\gamma + \mu_A + \delta)\int_0^T (A_I^*, \phi w_j)dt$$
$$- \int_0^T (G_1(F_*, A_I^*, A_P^*), \phi w_j)dt$$
$$= 0$$

(3.26)

This implies that we have continuity with respect to w_j. Thus we obtain that for any $v \in H_0^1(\Omega)$ we have the existence of A_I^*, A_P^*, F^* such that the following is true

(3.27) $\qquad \dfrac{d}{dt}(A_I^*, v) + (\nabla A_I^*, \nabla v) - (\gamma + \mu_A + \delta)(A_I^*, v) = (G_1(F^*, A_I^*, A_P^*), v)$

A similar analysis for the remaining compartments tells us that $(A_I^*, \cdots, M_T^*)$ solves the system (2.1). Thus we can state the following theorem

THEOREM 3.2. *Consider the reaction diffusion system as described via* (2.1). *For initial data in $L^2(\Omega)$, and any time $T > 0$, there exists a unique weak solution $(A_I^*, \cdots, M_T^*)$ to the system such that*

(3.28) $\qquad (A_I^*, .. M_{T_*}) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap C([0, T]; L^2(\Omega))$

and

(3.29) $\qquad \left(\dfrac{\partial A_I^*}{\partial t}, \cdots, \dfrac{\partial M_T^*}{\partial t}\right) \in L^2(0, T; V^*),$

If the initial data is in $L^\infty(\Omega) \cap H^1(\Omega)$ then we have further regularity, that is,

(3.30) $\quad (A_I^*, \cdots, M_{T_*}) \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$

Furthermore $(A_I^*, \cdots, M_{T_*})$ are continuous with respect to initial data.

We postpone the proof of further regularity till after global existence has been proved. This is done in the following section. Note continuity with respect to initial data can also be shown.

4. Global existence of strong solution

In this section we prove global existence of strong solution to the model (2.1). For the general theory of strong solutions the reader is referred to [**4**]. Consider the operator A

$$(4.1) \qquad\qquad A : D(A) \subset [L^2(\Omega)]^8 \mapsto [L^2(\Omega)]^8,$$

with domain $D(A)$,

$$(4.2) \qquad D(A) = \left\{ (A_I, ..M_S) \in H^2(\Omega), \nabla A_I \cdot \mathbf{n} =, ...\nabla M_S \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}.$$

Now (2.1) can be recast in abstract form by the following equation

$$(4.3) \qquad u^{'}(t) = Au(t) + f(t, u(t)), \ t \in [0, T], \ u(0) = u_0 \in D(A).$$

in the real Hilbert space $H = [L^2(\Omega)]^8$. Here
Here $u(t) = \left\{ u^1, ...u^8 \right\} = \{A_I, ...M_T\}$, and $f(t, u(t))$ is the nonlinear map in (2.1).

We first recall the following theorem from [**28**], In order to prove existence of solution to the above abstract problem

THEOREM 4.1. *Let A be the infinitesimal generator of a C_0 semigroup of linear operators $T(t)$, $t \geq 0$, on a reflexive Banach space X. If $f : [0, T] \times X \to X$ is Lipschitz continuous in (t, u) on $[0, T] \times X$ and $u_0 \in D(A)$, then the problem (4.3) admits a unique global strong solution $u \in W^{1,2}(0, T; X)$, with $u \in L^2(0, T; D(A))$. If $f : [0, T] \times X \to X$ is only locally Lipschitz continuous on X, uniformly w.r.t $t \in [0, T]$, then for every $u_0 \in D(A)$, the initial value problem (4.3) has a strong solution u defined on a maximal interval $[0, \delta)$. Moreover, if $\delta < T$ then $\lim_{t \nearrow \delta} ||u(t)|| = \infty$*

We first state the following result concerning the reaction diffusion system we have considered

THEOREM 4.2. *Consider the system (2.1). For initial data in $H^1(\Omega) \cap L^\infty(\Omega)$ there exists a unique global strong solution to the system.*

PROOF. Note that the opposite signs on the holling type responses in the reaction diffusion system prevent the nonlinearities from being Lipschitz continuous in u uniformly w.r.t t. In order to circumvent this difficulty we consider the truncated problem

$$(4.4) \qquad u^{'}_N(t) = Au_N(t) + f_N(t, u_N(t)), \ t \in [0, T], \ u_N(0) = u_0 \in D(A).$$

where N satisfies

$$(4.5) \qquad\qquad N \geq C_{max} \left\{ ||u_0^i||_\infty, \ 1 \leq i \leq 8 \right\}.$$

Where we define $\left\{ f_N^1(t, u_N(t))..., f_N^8(t, u_N(t)) \right\}$, which stand for the nonlinearities in (2.1) in the following way, if $u_N^1(t)...u_N^8(t)$, are greater than N or less than -N, then we replace $u_N^i(t)$ by N or -N. Via this method $\left\{ f_N^1(t, u_N(t))..., f_N^8(t, u_N(t)) \right\}$, become well defined on $H \times [0, T]$, furthermore they are bounded and Lipschitz continuous in $u \in H$, uniformly w.r.t $t \in [0, T]$. This way it can be shown that (4.4)

posesses a strong solution u_N. We will next focus on showing that $u_N^1(t)...u_N^8(t)$ are in $L^\infty(\Omega)$. We will demonstrate this for two critical components in our reaction diffusion system, A_N and F_N.

denote

(4.6)
$$K_N = max\left\{\left|f_N^i\right|_\infty, \left|u_0^i\right|_\infty, \ 1 \le i \le 8\right\},$$

We now construct the following function

(4.7) $Z_N^i(t) = u_N^i(t) - K_N t + \left|u_0^i\right|_\infty, \ t \in [0,T], \ Z_N^i(0) = u_0^i - \left|u_0^i\right|_\infty.$

By the classical theory there is a strong solution to this problem explicitly given as

(4.8) $$Z_N^i(t) = S(t)(u_0^i - \left|u_0^i\right|_\infty) + \int_0^t S(t-s)(f_N^i - K_N)ds$$

where $S(t)$ is the C_0 semi group generated by the laplacian operator. However $u_0^i - \left|u_0^i\right|_\infty \le 0$ and $f_N^i - K_N \le 0$. Therefore $Z_N^i \le 0$. Similarly setting $V_N^i(t) = u_N^i(t) + K_N t + \left|u_0^i\right|_\infty$, we can show $V_N^i \ge 0$. Thus

(4.9) $$|u_N^i(t)| \le K_N t + \left|u_0^i\right|_\infty, \ \text{in } Q$$

Where $Q = \Omega \times [0,T]$. Therefore

(4.10) $$|u_N^i(t)|_{L^\infty(Q)} \le C$$

We now prove positivity of the compartments. Consider the equation for A_{IN}

(4.11) $$\frac{\partial A_{IN}}{\partial t} = \Delta A_{IN} + \phi\left(1 - \frac{A_{IN} + A_{PN}}{C}\right)F_{IN} - (\gamma + \mu_A + \delta)A_{IN},$$

Recall the positive and negative parts of a function.

(4.12) $$(A_{IN})^+ = \sup(A_{IN}, 0),$$

(4.13) $$(A_{IN})^- = -\inf(A_{IN}, 0),$$

we multiply the above by $(A_{IN})^-$ and integrate by parts over Ω to obtain

$$\frac{d}{dt}|(A_{IN})^-|_2^2 + (\gamma + \mu_{A_{IN}} + \delta)|(A_{IN})^-|_2^2 + |\nabla(A_{IN})^-|_2^2$$
$$= \phi\int_\Omega \left(1 - \frac{A_{IN} + A_{PN}}{C}\right)F_{IN}(A_{IN})^- d\mathbf{x}$$

(4.14)

Now using the uniform bounds on $f_N^1 = \phi\left(1 - \frac{A_{IN}+A_{PN}}{C}\right)F_{IN} - (\gamma + \mu_A + \delta)A_{IN})$

$$\left| \phi \left(1 - \frac{A_{IN} + A_{PN}}{C} \right) F_{IN} - (\gamma + \mu_A + \delta) A_{IN} \right|_\infty$$

(4.15) $$\leq \quad C_1 |A_{IN}|_\infty |F_{IN}|_\infty + C_2 |A_{IN}|_\infty$$

we obtain

(4.16) $$\frac{d}{dt} |(A_{IN})^-|_2^2 \leq C |(A_{IN})^-|_2^2$$

Integrating the above in the time interval $(0, t)$ yields,

(4.17) $$|(A_{IN})^-(t)|_2^2 \leq C \int_0^t |(A_{IN})^-(s)|_2^2 ds,$$

Gronwalls inequality yields

(4.18) $$|(A_{IN})^-(t)|_2^2 \leq 0,$$

   or

(4.19) $$(A_{IN})^-(t) = 0,$$

Thus $A_{IN} \geq 0$.

The analysis for F_N is similar, we multiply the above by $(F_N)^-$ and integrate by parts over Ω to obtain

$$\frac{1}{2} \frac{d}{dt} |(F_N)^-|_2^2 + |\nabla (F_N)^-|_2^2 + (\mu_F + v_1) |(F_N)^-|_2^2$$
$$= \quad \int_\Omega \left(\frac{\beta M_n I_N (F_N)^-}{M_n + M_{T_n}} \right) d\mathbf{x} + \int_\Omega \left(\frac{\beta_T M_N P_N F_N}{M_n + M_{T_n}} \right) d\mathbf{x}$$

(4.20)

therefore we obtain

(4.21) $$\frac{d}{dt} |(F_N)^-|_2^2 \leq \left| \frac{\beta M_n I_N}{M_n + M_{S_n}} \right|_\infty |(F_N)^-|_2^2 + \left| \frac{\beta M_n P_N}{M_n + M_{S_n}} \right|_\infty |(F_N)^-|_2^2$$

Now using the uniform bounds on compartments in

(4.22) $$f_N^4 = \frac{\beta M_N I_N}{M_N + M_{S_N}} + \frac{\beta M_N P_N}{M_N + M_{T_N}} - (\mu_{F_N}) F_N$$

to obtain

$$\left| \frac{\beta M_N I_N}{M_N + M_{S_N}} + \frac{\beta M_N P_N}{M_N + M_{T_N}} - (\mu_{F_N} + v_1) F_N \right|_\infty$$
$$\leq \quad \left| \frac{\beta M_N I_N}{M_N + M_{S_N}} \right|_\infty + \left| \frac{\beta M_N P_N}{M_N + M_{T_N}} - (\mu_{F_N} + v_1) F_N \right|_\infty$$
$$\leq \quad C_1 |I_N|_\infty + C_2 |P_N|_\infty + C_3 |F_N|_\infty$$

(4.23)

We integrate (4.21) in the time interval $(0, t)$ to yield,

$$(4.24) \qquad |(F_N)^-(t)|_2^2 \le C \int_0^t |(F_N)^-(s)|_2^2 ds,$$

Gronwalls inequality yields

$$(4.25) \qquad |(F_N)^-(t)|_2^2 \le 0,$$

or

$$(4.26) \qquad (F_N)^-(t) = 0,$$

Thus $F_N \ge 0$. The estimates for the other compartments follow similarly. If N is chosen as in (4.5) then we obtain

$$(4.27) \qquad A_{IN}, F_N \in [-N, N],$$

This follows via the way A_{IN}, F_N have been constructed. Thus for $t \in (0, s)$ $A_{IN} = A_I$ and $F_N = F$. Thus A_N, F_N are a local solution to (2.1). Next we show

$$(4.28) \qquad |A_{IN}|_{L^\infty(\Omega \times (0, s))} < K,$$

$$(4.29) \qquad |F_N|_{L^\infty(\Omega \times (0, s))} < K.$$

we compare our solution to the solution of the following problem

$$(4.30) \qquad \frac{\partial W}{\partial t} = \Delta W + rW, \ (t, x) \in Q$$

$$(4.31) \qquad \nabla W \cdot \mathbf{n} = 0, \text{on} \partial \Omega,$$

$$(4.32) \qquad W(x, 0) = F_0,$$

where

$$(4.33) \qquad r > K_N \max\left(\left|f_N^i\right|_\infty\right), 1 \le i \le 8.$$

We see that from the above

$$(4.34) \qquad F \le W(x, t), \ \forall \ (t, x) \in Q$$

thus

$$(4.35) \qquad |F|_{L^\infty(Q)} \le |W|_{L^\infty(Q)} \le C, \ \forall \ (t, x) \in Q$$

The similar method follows for A_I.

$\square$

We now show that the strong solution to the system achieves further regularity via the following lemma

LEMMA 4.3. *Consider the system* (2.1). *For initial data* $\in H^1(\Omega) \cap L^\infty(\Omega)$ *the strong solution to the system lies in*

$$\tag{4.36} H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$$

PROOF. We consider the equation for A_I

$$\tag{4.37} \frac{\partial A_I}{\partial t} - \Delta A_I = \phi\left(1 - \frac{A_I + A_P}{C}\right) F_I - (\gamma + \mu_A + \delta) A_I,$$

We square both sides to yield

(4.38)
$$\left|\frac{\partial A_I}{\partial t}\right|^2 + |\Delta A_I|^2 - 2\frac{\partial A_I}{\partial t}\Delta A_I = \left[\phi\left(1 - \frac{A_I + A_P}{C}\right) F_I - (\gamma + \mu_A + \delta) A_I\right]^2,$$

We now integrate the left hand side by parts, and use the supremum bounds on the right hand side to yield

$$\tag{4.39} \left|\frac{\partial A_I}{\partial t}\right|_2^2 + |\Delta A_I|_2^2 + 2\frac{d}{dt}|\nabla A_I|_2^2 \le C\left[\left|\left(1 - \frac{A_I + A_P}{C}\right) F_I\right|_\infty + |A_I|_\infty^2\right]$$

We now integrate the above in the time interval $(0, T)$ to obtain

$$\tag{4.40} \int_0^T \left|\frac{\partial A_I}{\partial t}\right|_2^2 dt + \int_0^T |\Delta A_I|_2^2 dt + |\nabla A_I(T)|_2^2 \le Ct + |\nabla A_I(0)|_2^2$$

These estimates yield

$$\tag{4.41} A_I \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega))$$

$\square$

5. Existence of Global Attractor

In this section we will prove the existence of a global attractor for the reaction diffusion system (2.1). Recall the following definition

DEFINITION 5.1. Consider a semi group $S(t)$ acting on a reflexive Banach space H, then the global attractor $\mathcal{A} \subset H$ for this semigroup is an object that has the following properties,
i) $\mathcal{A}$ is compact in H.
ii) $\mathcal{A}$ is invariant, i.e, $S(t)\mathcal{A} = \mathcal{A}$, $t \ge 0$
iii) If B is bounded in M then

$$dist_M(S(t)B, \mathcal{A}) \to 0, \ t \to \infty.$$

Next various preliminaries are presented, detailing the phase spaces of interest and recalling certain standard theory. Let us define our phase spaces of interest.

$$H = [L^2(\Omega)]^8, \ V = [H^1(\Omega)]^8$$

In order to prove the existence of a global attractor we are required to show:
i) There exists a bounded absorbing set in the phase space.
ii) The asymptotic compactness property of the semigroup in question, [2], [4].
These are defined next

DEFINITION 5.2 (Bounded absorbing set). A bounded set $\mathcal{B}$ in a reflexive Banach space H is called a bounded absorbing set if for each bounded subset U of H, there is a time $T = T(U)$, such that $S(t)U \subset \mathcal{B}$ for all $t > T$. The number $T = T(U)$ is referred to as the **compactification time** for $S(t)U$. This is essentially the time after which the semigroup compactifies.

DEFINITION 5.3 (Asymptotic compactness). The semigroup $\{S(t)\}_{t \geq 0} : H \to H$ associated with a dynamical system is said to be asymptotically compact in H if for any $\{u_{i0,n}\}_{n=1}^{\infty}$ bounded in H, and a sequence of times $\{t_n \to \infty\}$, $S(t_n)u_{i0,n}$ possesses a convergent subsequence in H.

5.1. Bounded absorbing set in $L^2(\Omega)$. In order to proceed we note $F_I \leq A_I$. Hence we can write the equation for A_I as

$$(5.1) \qquad \frac{\partial A_I}{\partial t} = \Delta A_I + \phi \left(1 - \frac{A_I + A_P}{C} \right) (A_I - g) - (\gamma + \mu_A + \delta)A_I$$

where g is a positive function, representing the combination of eggs that hatch into males and single females. Now we need only consider region where $A_I \leq C$, as when $A_I > C$, the logistic term is negative, hence $\phi \left(1 - \frac{A_I + A_P}{C} \right) F_I < 0$, hence the estimates are made easily. We multiply (5.1) by A_I and integrate by parts to obtain

$$\frac{1}{2} \frac{d}{dt} |A_I|_2^2 + |\nabla A_I|_2^2 + \frac{\phi}{C} |A_I|_3^3$$
$$+ (\gamma + \mu_A + \delta)|A_I|_2^2 = \phi |A_I|_2^2$$

(5.2)

Using Young's inequality, Cauchy with epsilon, along with Cauchy Schwartz inequality yields

$$(5.3) \qquad \frac{1}{2} \frac{d}{dt} |A_I|_2^2 + \frac{\phi}{C} |A_I|_3^3 + (\gamma + \mu_A + \delta)|A_I|_2^2 \leq \frac{(\epsilon)^2}{4} |A_I|_3^3 + |\Omega|$$

Now using the appropriate choice of epsilon, $\epsilon = \sqrt{\frac{4\phi}{C}}$ yields

$$(5.4) \qquad \frac{d}{dt} |A_I|_2^2 + (\gamma + \mu_A + \delta)|A_I|_2^2 \leq |\Omega|$$

Gronwalls inequality yields

$$(5.5) \qquad |A_I|_2^2 \leq e^{-(\gamma + \mu_A + \delta)t} |A_I(0)|_2^2 + |\Omega|$$

Thus there exists a time

$$(5.6) \qquad t_1 = \max \left(0, \frac{\ln \left(|A_I(0)|_2^2 \right)}{(\gamma + \mu_A + \delta)} \right),$$

such that for times $t > t_1$ the following uniform estimate holds

$$(5.7) \qquad\qquad |A_I|_2^2 \leq 1 + |\Omega| \leq C$$

Here C is independent of time and initial data.

We can integrate (5.2) in the time interval $[t_1, t_1 + 1]$ to obtain

$$(5.8) \qquad\qquad \int_{t_1}^{t_1+1} |\nabla A_I|_2^2 dt \leq |A_I(t_1)|_2^2 + \int_{t_1}^{t_1+1} |\Omega| dt$$

Now using the estimate on $|A_I|_2^2$, for $t > t_1$ yields

$$(5.9) \qquad\qquad \int_{t_1}^{t_1+1} |\nabla A_I|_2^2 dt \leq C$$

Via the mean value theorem, there exists a time $t_2 \in [t_1, t_1 + 1]$ such that the following estimate holds,

$$(5.10) \qquad\qquad |\nabla A_I(t_2)|_2^2 \leq C$$

We now multiply the equation for the I compartment by I and integrate by parts over Ω to obtain

$$(5.11) \qquad\qquad \frac{1}{2}\frac{d}{dt}|I|_2^2 + |\nabla I|_2^2 + (\min(\beta, \beta_S) + \mu)|I|_2^2 \leq \phi \int_\Omega I A_I d\mathbf{x}$$

Now Cauchy with epsilon, Holders and Youngs inequalities, and the embedding of $H^1(\Omega) \hookrightarrow L^2(\Omega)$ yield

$$(5.12) \qquad\qquad \frac{1}{2}\frac{d}{dt}|I|_2^2 + C|I|_2^2 + (\min(\beta, \beta_S) + \mu)|I|_2^2 \leq C|I|_2^2 + \frac{\phi^2}{C}|A_I|_2^2,$$

The application of Gronwalls lemma, and the prior estimate on $|A_I|_2^2$ yields

$$(5.13) \qquad\qquad |I|_2^2 \leq e^{-(\min(\beta, \beta_S) + \mu)t}|I(0)|_2^2 + \frac{\phi^2}{C}(|\Omega| + 1)$$

Thus there exists a time

$$(5.14) \qquad\qquad t_3 = \max\left(0, \frac{\ln\left(|A_I(0)|_2^2\right)}{\frac{\phi^2}{C}(|\Omega| + 1)}, t_2\right),$$

such that for times $t > t_3$ the following uniform estimate holds

$$(5.15) \qquad\qquad |I|_2^2 \leq 1 + (1 + |\Omega|) \leq C$$

Here C is independent of time and initial data.

We can integrate (5.11) in the time interval $[t_3, t_3 + 1]$ to obtain

$$(5.16) \qquad\qquad \int_{t_3}^{t_3+1} |\nabla I|_2^2 dt \leq C \int_{t_3}^{t_3+1} |A_I|_2^2 dt$$

Now using the estimate on $|A_I|_2^2$ this yields

$$(5.17) \qquad \int_{t_3}^{t_3+1} |\nabla I|_2^2 dt \leq C$$

Via the mean value theorem, there exists a time $t_4 \in [t_3, t_3 + 1]$ such that the following estimate holds,

$$(5.18) \qquad |\nabla I(t_4)|_2^2 dt \leq C$$

Simialrly the estimates on the compartment P are made, and using these the estimates on F_I, F_P follow. Thus we have

$$(5.19) \qquad |F_I|_2^2 \leq C, \ \forall t > t_5$$

$$(5.20) \qquad \int_{t_5}^{t_5+1} |\nabla F_I|_2^2 dt \leq C$$

$$(5.21) \qquad |\nabla F_I(t_6)|_2^2 dt \leq C$$

These estimates enable us to state the following lemma concerning bounded absorbing sets

LEMMA 5.4. *Consider the reaction diffusion system described via* (2.1) *with initial data in* $L^2(\Omega)$. *There exists a time* t_{**} , *and a constant* C *independent of time and initial data, and dependent only on the parameters in the system, such that for any* $t > t_{**}$ *the following uniform estimates hold:*

$$|A_I|_2^2 \leq C, \ |A_P|_2^2 \leq C, \ |I|_2^2 \leq C, \ |P|_2^2 \leq C, \ |F_I|_2^2 \leq C, \ |F_P|_2^2 \leq C,$$

$$|M|_2^2 \leq C, \ |M_S|_2^2 \leq C,$$

REMARK 1. The t_{**} in lemma 5.4 is chosen such that $t_{**} = \max t_i$, where t_i range through all the explicit compactification times, for all the compartments, calculated in the estimates of the bounded absorbing sets.

We demonstrate some further estimates on F_I, A_I for arbitary large times, which will be used subsequently. Multiplying the equation for the F_I compartment by $-\Delta F_I$ and integarting by parts yields,

$$\frac{1}{2}\frac{d}{dt}|\nabla F_I|_2^2 + |\Delta F_I|_2^2 + \mu_F|\nabla F_I|_2^2$$

$$= \beta \int_{\Omega} I \frac{M}{M + M_S}(-\Delta F_I) d\mathbf{x}$$

$$\leq \beta \left|\frac{M}{M + M_S}\right|_{\infty} \int_{\Omega} |I||\Delta F_I| d\mathbf{x}$$

$$\leq C_1|I|_2^2 + \frac{1}{2}|\Delta F_I|_2^2$$

$$(5.22)$$

These follow by using Youngs inequality and Cauchy with epsilon. Thus we have

$$(5.23) \qquad \frac{d}{dt}|\nabla F_I|_2^2 + |\Delta F_I|_2^2 \leq C_1 |I|_2^2$$

Integrating from $[t, t+1]$, where $t > t_{**}$ that appears in 5.4 we obtain,

$$(5.24) \qquad \int_t^{t+1} |\Delta F_I|_2^2 dt \leq |\nabla F_I(t)|_2^2 + C_1 \int_t^{t+1} |I|_2^2 dt \leq C$$

This follows via readjustment for compactification times in (5.21) and (5.15), as these can be chosen arbitarily large.

Furthermore via mean value theorems there exists a time $t_* \in [t, t+1]$ such that

$$(5.25) \qquad |\Delta F_I(t_*)|_2^2 \leq C$$

Multiplying the equation for the A_I compartment by $-\Delta A_I$, integarting by parts, and using product rule for differentiation yields,

$$\frac{1}{2}\frac{d}{dt}|\nabla A_I|_2^2 + |\Delta A_I|_2^2 + (\gamma + \mu_A + \delta)|\nabla A_I|_2^2$$
$$\leq \int_\Omega |\nabla F_I||\nabla A_I|d\mathbf{x} + \int_\Omega |A_I||\nabla F_I||\nabla A_I|d\mathbf{x}$$
$$+ \int_\Omega |F_I||\nabla A_I||\nabla A_P|d\mathbf{x} + \int_\Omega |A_P||\nabla F_I||\nabla A_I|d\mathbf{x}$$

(5.26)

Now integrating the above in time interval from $[t, t+1]$, where $t > t_{**}$ that appears in lemma 5.4 yields

$$\int_t^{t+1} |\Delta A_I(s)|_2^2 ds$$
$$\leq |\nabla A_I(t)|_2^2 + \int_t^{t+1} |\nabla F_I(s)|_2 |\nabla A_I(s)|_2 ds + C \int_t^{t+1} |\nabla F_I(s)|_2 |\nabla A_I(s)|_2 ds$$
$$+ \int_t^{t+1} |F_I(s)||\nabla A_I(s)||\nabla A_P(s)|ds + \int_t^{t+1} |A_P(s)||\nabla F_I(s)||\nabla A_I(s)|ds$$
$$\leq C_1 + C_2 \left(\int_t^{t+1} |\nabla F_I(s)|_2^2 ds\right)^{\frac{1}{2}} \left(\int_t^{t+1} |\nabla A_I(s)|_2^2 ds\right)^{\frac{1}{2}}$$
$$+ C_3 \left(\int_t^{t+1} |\nabla A_P(s)|_2^2 ds\right)^{\frac{1}{2}} \left(\int_t^{t+1} |\nabla A_I(s)|_2^2 ds\right)^{\frac{1}{2}}$$
$$\leq C$$

(5.27)

This follows via readjusting the estimate in (5.10), the estimate on $|A_I|_2$ and the estimates in (5.20), (5.9), with readjustment for the compactification times, as these can be chosen arbitarily large. The estimates on $\int_t^{t+1} |\nabla A_P(s)|_2^2 ds$ are derived similarly to the estimates derived in (5.9). Furthermore via mean value theorems there exists a time $t_* \in [t, t+1]$ such that

$$(5.28) \qquad |\Delta A_I(t_*)|_2^2 \leq C$$

5.2. Asymptotic compactness. We first recap the following lemma,

LEMMA 5.5 (Uniform Gronwall Lemma). *Let β, ζ and h be nonnegative functions in $L^1_{loc}[0,\infty;\mathbb{R})$. Assume that β is absolutely continuous on $(0,\infty)$ and the following differential inequality is satisfied*

$$(5.29) \qquad \frac{d\beta}{dt} \leq \zeta\beta + h, \ \text{for } t > 0.$$

If there exists a finite time $t_1 > 0$ and some $r > 0$ such that

$$(5.30) \qquad \int_t^{t+r} \zeta(\tau)d\tau \leq A, \ \int_t^{t+r} \beta(\tau)d\tau \leq B, \ \text{and} \ \int_t^{t+r} h(\tau)d\tau \leq C,$$

for any $t > t_1$, where A, B and C are some positive constants, then

$$(5.31) \qquad \beta(t) \leq \left(\frac{B}{r} + C\right) e^A, \ \text{for} \ \text{any } t > t_1 + r.$$

We begin by demonstarting the estimate for A_I. We multiply the equation for the A_I compartment by $-\Delta A_I$ to obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla A_I|_2^2 + |\Delta A_I|_2^2 + (\gamma + \mu_A + \delta)|\nabla A_I|_2^2$$
$$= \phi\int_\Omega \nabla F_I \cdot \nabla A_I d\mathbf{x} - \phi\int_\Omega \left(\frac{A_I + A_P}{C}\right) F_I(-\Delta A_I)d\mathbf{x}$$

$$(5.32)$$

Differentiation yields

$$\frac{1}{2}\frac{d}{dt}|\nabla A_I|_2^2 + |\Delta A_I|_2^2 + (\gamma + \mu_A + \delta)|\nabla A_I|_2^2$$
$$= \int_\Omega \nabla F_I \cdot \nabla A_I d\mathbf{x} - \int_\Omega |\nabla A_I|^2 F_I d\mathbf{x} - \int_\Omega A_I \nabla F_I \cdot \nabla A_I d\mathbf{x}$$
$$- \int_\Omega F_I \nabla A_P \cdot \nabla A_I d\mathbf{x} - \int_\Omega A_P \nabla F_I \cdot \nabla A_I d\mathbf{x}$$

$$(5.33)$$

Using Holders inequality this yields

$$\frac{1}{2}\frac{d}{dt}|\nabla A_I|_2^2 + |\Delta A_I|_2^2 + (\gamma + \mu_A + \delta)|\nabla A_I|_2^2$$

$$\leq \quad |\nabla A_I|_2|\nabla F_I|_2 + |\nabla A_I|_4^2|F_I|_2|\nabla A_I|_4 + |A_I|_4|\nabla F_I|_4|\nabla A_I|_2$$
$$+|F_I|_2|\nabla A_P|_4|\nabla A_I|_4 + |\nabla F_I|_4|A_P|_2|\nabla A_I|_4$$

$$\leq \quad C_1|\nabla F_I|_2|\nabla A_I|_2^2 + |\nabla A_I|_4^2|F_I|_2|\nabla A_I|_4 + |A_I|_4|\nabla F_I|_4|\nabla A_I|_2$$

$$\leq \quad |F_I|_2|\nabla A_P|_4|\nabla A_I|_4 + |\nabla F_I|_4|A_P|_2|\nabla A_I|_4$$

(5.34)

Therefore we obtain

$$\frac{1}{2}\frac{d}{dt}|\nabla A_I|_2^2$$

$$\leq \quad C_1|\nabla F_I|_2|\nabla A_I|_2^2 + |\nabla A_I|_4^2|F_I|_2|\nabla A_I|_4 + |A_I|_4|\nabla F_I|_4|\nabla A_I|_2$$
$$+|F_I|_2|\nabla A_P|_4|\nabla A_I|_4 + |\nabla F_I|_4|A_P|_2|\nabla A_I|_4$$

(5.35)

we now set

(5.36) $$\zeta(t) = C_1|\nabla F_I|_2,$$

$$h(t)$$
$$= \quad |\nabla A_I|_4^2|F_I|_2|\nabla A_I|_4 + |A_I|_4|\nabla F_I|_4|\nabla A_I|_2$$
$$+|F_I|_2|\nabla A_P|_4|\nabla A_I|_4 + |\nabla F_I|_4|A_P|_2|\nabla A_I|_4$$

(5.37)

Note via (5.28) we obtain that for $t > t_{**}$, integrating the above in $[t, t+1]$

$$\int_t^{t+1} |\nabla A_I|_4^2|F_I|_2|\nabla A_I|_4 ds$$

$$\leq \quad C\int_t^{t+1} |\nabla A_I|_4^3 ds$$

$$\leq \quad C\int_t^{t+1} |\Delta A_I|_2^2|\Delta A_I|_2 ds$$

$$\leq \quad C|\Delta A_I(t_*)|_2 \int_t^{t+1} |\Delta A_I|_2^2$$

$$\leq \quad C$$

(5.38)

Also note for $t > t_{**}$, integrating the above in $[t, t+1]$

$$\int_t^{t+1} |A_I|_4 |\nabla F_I|_4 |\nabla A_I|_2 ds$$

$$\leq C \left(\int_t^{t+1} |A_I|_4^4 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\nabla F_I|_4^4 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\nabla A_I|_2^2 ds\right)^{\frac{1}{2}}$$

$$\leq \left(\int_t^{t+1} |\nabla A_I|_2^4 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\Delta F_I|_2^4 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\nabla A_I|_2^2 ds\right)^{\frac{1}{2}}$$

$$\leq C \left(|\nabla A_I(t_*)|_2^2 \int_t^{t+1} |\nabla A_I|_2^2 ds\right)^{\frac{1}{4}}$$

$$\left(|\Delta F_I(t_*)|_2^2 \int_t^{t+1} |\Delta F_I|_2^2 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\nabla A_I|_2^2 ds\right)^{\frac{1}{2}}$$

$$\leq C$$

(5.39)

This follows via (5.24), (5.25), (5.27) and (5.28), and the Sobolev embedding of

(5.40)
$$H^2(\Omega) \hookrightarrow W^{1,4}(\Omega).$$

Furthermore for $t > t_{**}$, integrating the above in $[t, t+1]$

$$\int_t^{t+1} \left(|F_I|_2 |\nabla A_P|_4 |\nabla A_I|_4 + |\nabla F_I|_4 |A_P|_2 |\nabla A_I|_4\right) ds$$

$$\leq C \int_t^{t+1} |\nabla A_P|_4 |\nabla A_I|_4 ds$$

$$\leq C \left(\int_t^{t+1} |\nabla A_P|_4^4 ds\right)^{\frac{1}{4}} \left(\int_t^{t+1} |\nabla A_I|_4^4 ds\right)^{\frac{1}{4}}$$

$$\leq C \left(|\Delta A_I(t_*)|_2^2 \int_t^{t+1} |\Delta A_I|_2^2 ds\right)^{\frac{1}{4}} \left(|\Delta A_P(t_*)|_2^2 \int_t^{t+1} |\Delta A_P|_2^2 ds\right)^{\frac{1}{4}}$$

$$\leq C$$

(5.41)

This follows via (5.27) and (5.28), and the Sobolev embedding of

(5.42)
$$H^2(\Omega) \hookrightarrow W^{1,4}(\Omega).$$

The estimates for $\int_t^{t+1} |\Delta A_P|_2^2 ds$ are made easily, similarly to (5.27).

Thus via lemma 5.5 it follows that for $t > t_{**} + 1$, we obtain,

(5.43)
$$|\nabla A_I|_2^2 \leq C.$$

The similar estimates can be made on the other compartments.

We now state the following Theorem,

THEOREM 5.6. *Consider the reaction diffusion equation described via* (2.1). *There exists a* (H, H) *global attractor* $\mathcal{A}$ *for the system. This is compact and invariant in* H, *and it attracts all bounded subsets of* H *in the* H *metric.*

PROOF. We have shown that the system is well posed via theorems 3.2, 4.2. Thus there exists a well defined semigroup $\{S(t)\}_{t \geq 0} : H \to H$. The estimates derived in Lemma 5.4 demonstrate the existence of bounded absorbing sets in H. Thus given a sequence $\{u_{i0,n}\}_{n=1}^{\infty}$, for $1 \leq i \leq 8$ that is bounded in $L^2(\Omega)$, we know that for $t > t_{**}$,

$$(5.44) \qquad\qquad S(t)(u_{i0,n}) \subset B \subset H^1(\Omega).$$

Here B is the bounded absorbing set in $H^1(\Omega)$. Now for n large enough $t_n > t_{**}$, thus for such t_n we have

$$(5.45) \qquad\qquad S(t_n)(u_{i0,n}) \subset B \subset H^1(\Omega).$$

This implies that we have the following uniform bound,

$$(5.46) \qquad\qquad |S(t_n)(u_{i0,n})|_{H^1(\Omega)} \leq C_1,$$

For explicit computations on A_I compartment see (5.43).

This implies via standard functional analysis theory, see [**3**], [**5**], the existence of a subsequence still labelled $S(t_n)(u_{i0,n})$ such that

$$(5.47) \qquad\qquad S(t_n)(u_{i0,n}) \rightharpoonup u_i \text{ in } H^1(\Omega),$$

Which implies via the compact Sobolev embedding of

$$(5.48) \qquad\qquad V \hookrightarrow H,$$

that

$$(5.49) \qquad\qquad S(t_n)(u_{i0,n}) \to u_i \text{ in } L^2(\Omega).$$

This yields the asymptotic compactness of the semigroup $\{S(t)\}_{t \geq 0}$ in H. The theorem is now proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

REMARK 2. It is interesting to note that the attractor posesses states of extinction, despite female selection, if the influx of sterile males is large enough. We can demonstarte this heuristically via the equation for F_P.

Multiplying by F_P and integrating by parts yields,

$$\frac{d}{dt}|F_P|_2^2 + C_1|F_P|_2^2$$
$$\leq \quad \beta \int_{\Omega} \left(\frac{M}{M + M_S}\right) P F_P d\mathbf{x}$$
$$\leq \quad C\left(\frac{|P|_{\infty}|M|_{\infty}}{|M_S|_{\infty}}\right) |F_P|_2^2$$

(5.50)

Clearly from the form of the equation for M_S if α is chosen large enough we can cause $|M_S|_\infty$ to be large enough, so as to entail,

$$(5.51) \qquad C_1 - C\left(\frac{|P|_\infty |M|_\infty}{|M_S|_\infty}\right) > 0.$$

Thus a simple application of Gronwall's inequality leads to finite time extinction of the female with preference. Similar analysis can be performed on the F_I compartment. Clearly eradication of the females in finite time leads to subsequent extinction of the population.

We next show that the global attractor $\mathcal{A}$ is more regular than $[L^2(\Omega)]^8$. We state the following theorem

THEOREM 5.7. *The global attractor $\mathcal{A}$ for the reaction diffusion system* (2.1) *is a bounded subset in $[L^\infty(\Omega)]^8$.*

PROOF. We demonstarte with the A_I compartment. Consider the analytic semigroup $\{e^{A\ t}\}_{t \geq 0}$, for the equation for A_I in (2.1). By the $(L^2(\Omega), L^\infty(\Omega))$ regularity of the semigroup $e^{A\ t} : L^2(\Omega) \mapsto L^\infty(\Omega)$, we have via theorem 38.10 in [4], the following estimate

$$(5.52) \qquad ||e^{A\ t}||_{\mathcal{L}(L^2, L^\infty)} \leq C(2)\ t^{-\frac{3}{4}},\ t > 0.$$

There exists a constant steady state, $(A_I^*, ...M_T^*)$ to (2.1). This is easily found via use of MATHEMATICA. Thus

$$(5.53) \qquad G(A_I^*, A_P^*, F_I^*) = \phi(1 - \frac{A_I^* + A_P^*}{C})F_I^* - (\gamma + \mu_A + \delta)A_I^* = 0$$

We can use the variation of constant formula, which works for any strong solution associated with $(A_I, A_P, F_I) \in \mathcal{A}$, to obtain

$$
\begin{aligned}
&||S(t)A_I||_{\mathcal{L}(L^2, L^\infty)} \\
=\ &||e^{A\ t}A_I + \int_0^t e^{A(t-s)}G(S(s)A_I)ds||_{\mathcal{L}(L^2, L^\infty)} \\
=\ &||e^{A\ t}A_I + \int_0^t e^{A(t-s)}(G(S(s)A_I) - G(A_I^*))ds||_{\mathcal{L}(L^2, L^\infty)} \\
\leq\ &||e^{A\ t}||_{\mathcal{L}(L^2, L^\infty)}||A_I|| + \int_0^t ||e^{A(t-s)}||_{\mathcal{L}(L^2, L^\infty)}||G(S(s)A_I) - G(A_I^*)||ds \\
\leq\ &Ct^{-\frac{3}{4}}||g|| \\
&+ \int_0^t C(t-s)^{-\frac{3}{4}}L(\sqrt{C_1})(|A_I - A_I^*|_{H^1(\Omega)} \\
&+|F_I - F_I^*|_{H^1(\Omega)} + |A_P - A_P^*|_{H^1(\Omega)})ds
\end{aligned}
$$
$$(5.54)$$

Here $L(\sqrt{C_1})$ is the lipschitz constant of the nonlinear map in the equation for A_I in (2.1), on the closed bounded ball in $H^1(\Omega)$ centered at the origin with radius $\sqrt{C_1}$. Since the attractor $\mathcal{A}$ is invariant, we have

(5.55) $$\{S(t)\mathcal{A} : t \geq 0\} = \mathcal{A} \subset \left\{u_i \in L^2(\Omega) : |u_i|_2^2 \leq C\right\} \subset H$$

(5.56) $$\{S(t)\mathcal{A} : t \geq 0\} = \mathcal{A} \subset \left\{u_i \in H^1(\Omega) : |u_i|_{H^1(\Omega)}^2 \leq C_1\right\} \subset V$$

Thus we have

$$||S(t)A_I||_\infty \leq Ct^{-\frac{3}{4}}||A_I|| + \int_0^t C(t-s)^{-\frac{3}{4}} L(\sqrt{C_1}) 2\sqrt{C_1} ds$$

which yields

(5.57) $$||S(t)A_I||_\infty \leq C(2)(Ct^{-\frac{3}{4}} + C_2 L(\sqrt{C_1})\sqrt{C_1} t^{\frac{1}{4}}),$$

here C_1 is given in (5.46), and C in lemma 5.4.

Now we can use the invariance of the attractor $S(t)\mathcal{A} = \mathcal{A}$, and $t = 1$ specifically to obtain for any $(A_I, F_I) \in \mathcal{A}$

(5.58) $$||A_I||_\infty \leq C(2)(C + C_2 L(\sqrt{C_1})\sqrt{C_1}) \leq C.$$

This analysis can be carried out for all the compartments. Thus the global attractor $\mathcal{A}$ is a bounded set in $[L^\infty(\Omega)]^8$.

$\square$

6. Finite Dimensionality of the Global Attractor

In this section we show that the Hausdorff and fractal dimensions of the global attractor for the reaction diffusion system (2.1), is finite. Recall

DEFINITION 6.1 (Fractal dimension). Consider a subset X of a Banach space H. If $\bar{X}$ is compact, the fractal dimension of X, denoted $d_f(X)$, is given by

(6.1) $$d_f(X) = \limsup_{\epsilon \to 0} \frac{log N(X, \epsilon)}{log(\frac{1}{\epsilon})}.$$

Here $N(X, \epsilon)$ denotes the minimum number of closed balls of radius ϵ, required to cover X. Note $d_f(X)$ can take the value $+\infty$.

DEFINITION 6.2 (Hausdorff dimension). Consider a subset X of a Banach space H. If $\bar{X}$ is compact, the Hausdorff dimension of X, denoted $d_H(X)$, is given by

(6.2) $$d_H(X) = \inf_{d>0} \left\{d : \mathcal{H}^d(X) = 0\right\}.$$

Here

(6.3) $$\mathcal{H}^d(X) = \lim_{\epsilon \to 0} \mu(X, d, \epsilon).$$

Where

(6.4) $$\mu(X, d, \epsilon) = \inf \left\{\sum_i r_i^d : r_i \leq \epsilon \text{ and } X \subseteq \cup_i B(x_i, r_i)\right\}.$$

and $B(x_i, r_i)$ are balls with radius r_i.

We will provide upper bounds on these dimensions in terms of parameters in the model. There is a standard methodology to derive these estimates. We consider a volume element in the phase space, and try and derive conditions that will cause it to decay, as time goes forward. If $\mathcal{A}$ is the global attractor of the semigroup $\{S(t)\}_{t \geq 0}$ in H associated with (2.1) , we can define

$$q_n(t) = \sup_{u_0 \in A} \sup_{g_i \in H, ||g_i||=1, 1 \leq i \leq n} \frac{1}{t} \int_0^t Tr(\Delta U(\tau) + \delta U(\tau) + F^{'}(S(\tau)u_0) \circ Q_n(\tau) d\tau$$

where

$$q_n = \limsup_{t \to \infty} q_n(t)$$

Here F is the nonlinear map in (2.1), and δ the linear map. also Q_n is the orthogonal projection of the phase space H onto the subspace spanned by

$$U_1(t), U_2(t), \cdots, U_n(t),$$

with

$$U_i(t) = L(S(t)u_0)u_i, i = 1, 2, ..n.$$

$L(S(t)u_0)$ is the Frechet derivative of the map $S(t)$ at u_0. Also for this model, $L(S(t)u_0)U_0 = U(t) = (U1(t)...U8(t))$, where $u = (u_1...u_8)$ is a solution to (2.1), $\phi_j = (\phi_j^1...\phi_j^8,)$ are an orthonormal basis for the subspace $Q_n(\tau)H$ and $(U1(t)...U8(t))$ are strong solutions to the variational equations for the reaction diffusion system (2.1). These have been worked out explicitly in the appendix.

We recall the following Lemma from [2], which will be useful to derive the requisite estimates.

LEMMA 6.3. *If there is an integer n such that $q_n < 0$ then the Hausdorff and fractal dimensions of $\mathcal{A}$, denoted $d_H(\mathcal{A})$ and $d_F(\mathcal{A})$, satisfy the following estimates*

$$d_H(\mathcal{A}) \leq n$$

$$d_F(\mathcal{A}) \leq 2n$$

Our aim is thus clear cut. We will derive exactly which conditions enforce that $q_n < 0$ for the diffusive model. We begin our estimates.

$$Tr(\Delta U(\tau) + F^{'}(S(\tau)u_0) \circ Q_n(\tau)$$
$$+(2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)U(\tau)$$

$$= \sum_{j=1}^{n} 8 \langle \Delta\phi_j(\tau), \phi_j(\tau) \rangle$$

$$+(2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)\sum_{j=1}^{n} \langle \phi_j(\tau), \phi_j(\tau) \rangle$$

$$+ \left\langle F^{'}(S(\tau)u_0)\phi_j(\tau), \phi_j(\tau) \right\rangle$$

$$= -8\sum_{j=1}^{n} |\nabla\phi_j(\tau)|^2$$

$$+(2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)\sum_{j=1}^{n} |\phi_j(\tau)|^2$$

$$+J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8$$

Here

$$J_1$$
$$\leq \sum_{j=1}^{n} \int_{\Omega} \phi(1 - \frac{u1 + u2}{C})\phi_j^1(\tau)\phi_j^5(\tau) - \phi\left(|\phi_j^1(\tau)|^2 + \phi_j^1(\tau)\phi_j^2(\tau)\right)\frac{u_5}{C}d\mathbf{x}$$
$$\leq \sum_{j=1}^{n} C_1(|\phi_j^1(\tau)|_2^2 + |\phi_j^5(\tau)|_2^2) + C_2|\phi_j^1(\tau)|_4^2 + C_3\left(|\phi_j^1(\tau)|_4^2 + |\phi_j^2(\tau)|_4^2\right)$$

$$J_2$$
$$\leq \sum_{j=1}^{n} \int_{\Omega} \phi(1 - \frac{u2 + u1}{C})\phi_j^2(\tau)\phi_j^6(\tau) - \phi\left(|\phi_j^2(\tau)|^2 + \phi_j^1(\tau)\phi_j^2(\tau)\right)\frac{u_6}{C}d\mathbf{x}$$
$$\leq \sum_{j=1}^{n} C_1(|\phi_j^2(\tau)|_2^2 + |\phi_j^6(\tau)|_2^2) + C_2|\phi_j^2(\tau)|_4^2 + C_3\left(|\phi_j^1(\tau)|_4^2 + |\phi_j^2(\tau)|_4^2\right)$$

J_3

$$= \sum_{j=1}^{n} \int_{\Omega} -(\mu)|\phi_j^3|^2 + r\gamma\phi_j^1\phi_j^3$$

$$-\beta \left[\left(\frac{u_7}{u_7 + u_8} \right) u_3 + \left(\frac{u_8\phi_j^7 - u_7\phi_j^8}{(u_7 + u_8)^2} \right) |\phi_j^3|^2 \right]$$

$$-\beta \left[\left(\frac{u_8}{u_7 + u_8} \right) u_3 + \left(\frac{u_7\phi_j^8 - u_8\phi_j^7}{(u_7 + u_8)^2} \right) |\phi_j^3|^2 \right] d\mathbf{x}$$

$$\leq \sum_{j=1}^{n} C_1(|\phi_j^3(\tau)|_2^2 + |\phi_j^1(\tau)|_2^2) + C_2|\phi_j^3(\tau)|_4^2$$

J_4

$$= \sum_{j=1}^{n} \int_{\Omega} -(\mu)|\phi_j^4|^2 + r\gamma\phi_j^2\phi_j^4$$

$$-\beta \left[\left(\frac{u_7}{u_7 + u_8} \right) u_4\phi_j^4 + \left(\frac{u_8\phi_j^7 - u_7\phi_j^8}{(u_7 + u_8)^2} |\phi_j^4|^2 \right) \right] d\mathbf{x}$$

$$\leq \sum_{j=1}^{n} C_1(|\phi_j^4(\tau)|_2^2 + |\phi_j^2(\tau)|_2^2) + C_2|\phi_j^4(\tau)|_4^2$$

J_5

$$= \sum_{j=1}^{n} \int_{\Omega} -(\mu_F)|\phi_j^5|^2$$

$$-\beta \left[\left(\frac{u_7}{u_7 + u_8} \right) u_3\phi_j^5 + \left(\frac{u_8\phi_j^7 - u_7\phi_j^8}{(u_7 + u_8)^2} \phi_j^3\phi_j^5 \right) \right] d\mathbf{x}$$

$$\leq \sum_{j=1}^{n} C_1(|\phi_j^5(\tau)|_2^2 + C_2 \left(|\phi_j^4(\tau)|_4^2 + |\phi_j^3(\tau)|_4^2 \right)$$

$$J_6$$

$$= \sum_{j=1}^{n} \int_{\Omega} -(\mu_F)|\phi_j^6|^2$$

$$- \beta \left[\left(\frac{u_7}{u_7 + u_8} \right) u_3 \phi_j^6 + \left(\frac{u_8 \phi_j^7 - u_7 \phi_j^8}{(u_7 + u_8)^2} \phi_j^4 \phi_j^6 \right) \right] d\mathbf{x}$$

$$\leq \sum_{j=1}^{n} C_1(|\phi_j^4(\tau)|_2^2 + C_2 \left(|\phi_j^4(\tau)|_4^2 + |\phi_j^6(\tau)|_4^2 \right)$$

$$J_7 = \sum_{j=1}^{n} \int_{\Omega} -(\mu_M)|\phi_j^7|^2 + (1-r)\gamma(\phi_j^1 + \phi_j^2)\phi_j^7 d\mathbf{x}$$

$$\leq \sum_{j=1}^{n} C_1 \left(|\phi_j^1(\tau)|_2^2 + |\phi_j^2(\tau)|_2^2 + |\phi_j^7(\tau)|_2^2 \right)$$

$$J_8 \leq \sum_{j=1}^{n} \int_{\Omega} C_1 |\phi_j^8|^2 d\mathbf{x}$$

Recall the Gagliardo-Nirenberg interpolation inequality [4],

(6.5) $$|\phi|_{W^{k,p}(\Omega)} \leq C|\phi|_{W^{k,p}(\Omega)}^{\theta} |\phi|_{L^r(\Omega)}^{1-\theta}, \text{ for } \phi \in W^{m,q}(\Omega)$$

provided $p, q, r \geq 1, 0 < \theta < 1$, and

(6.6) $$k - \frac{n}{p} \leq \theta \left(m - \frac{n}{q} \right) - (1-\theta)\frac{n}{r}, \text{ where } n = \Omega.$$

Now we consider exponents such that,

(6.7) $$W^{k,p}(\Omega) = L^4(\Omega),$$

(6.8) $$W^{m,q}(\Omega) = H^1(\Omega),$$

(6.9) $$L^r(\Omega) = L^2(\Omega),$$

and

(6.10) $$\theta = \frac{3}{4}.$$

Thus we obtain

(6.11) $$|\phi_j(\tau)|_4^2 \leq |\nabla \phi_j(\tau)|_2^{\frac{3}{4}} |\phi_j(\tau)|_2^{\frac{1}{4}} \leq C|\nabla \phi_j(\tau)|_2^{\frac{3}{4}}.$$

Using Young's inequality on the above we obtain,

(6.12) $$C|\nabla\phi_j(\tau)|_2^{\frac{3}{4}} \leq 7|\nabla\phi_j(\tau)|_2^2 + Cn.$$

Where C depends on The various parameters above. Thus we obtain the estimate,

$$Tr(\Delta U(\tau) + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)U(\tau)$$
$$+F'(S(\tau)u_0) \circ Q_n(\tau)$$
$$\leq \quad -8\sum_{j=1}^{n}|\nabla\phi_j(\tau)|_2^2 + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)|\phi_j(\tau)|_2^2$$
$$+ \quad 7|\nabla\phi_j(\tau)|_2^2 + Cm$$

Now via the generalized Sobolev-Lieb-Thirring inequalities [**2**] we obtain

$$\sum_{j=1}^{n}|\nabla\phi_j(\tau)|_2^2 \geq K_1\frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}}$$

Here K_1 depends only on the shape and dimension of Ω. Thus we obtain

(6.13) $$Tr(\Delta U(\tau) + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)U(\tau)$$
$$+F'(S(\tau)u_0) \circ Q_n(\tau)$$
$$\leq \quad -K_1\frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (\delta U(\tau))n + Cn,$$

(6.14)

for $\tau > 0$, $u_0 \in \mathcal{A}$.

Where $C = C(\phi, C, \gamma, r, \mu_i)$ hence forth. Where μ_i are the various μ's for all the compartments.

We now obtain

$$q_n(t) = \sup_{u_0 \in A}\sup_{g_i \in H, ||g_i||=1, 1 \leq i \leq n} \frac{1}{t}\int_0^t Tr(\Delta U(\tau) - (2\mu_A$$
$$+2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)U(\tau)$$
$$+F'(S(\tau)u_0) \circ Q_n(\tau)d\tau$$
$$\leq \quad -K_1\frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (C(\phi, C, \gamma, r, \mu_i) +$$
$$(2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)n, \ \forall t > 0.$$

This yields

(6.15) $$q_n = \limsup_{t \to \infty} \leq -K_1\frac{n^{\frac{5}{3}}}{|\Omega|^{\frac{2}{3}}} + (C(\phi, C, \gamma, r, \mu_i))$$
$$+(2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma)n < 0$$

If the integer n satisfies
(6.16)

$$n - 1 < \left(\frac{(C(\phi, C, \gamma, r, \mu_i) + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma))}{K_1} \right)^{\frac{3}{2}} |\Omega| < n$$

We can now state the following result

THEOREM 6.4. *Consider the reaction diffusion equation described via, (2.1). The global attractor $\mathcal{A}$ of the system is of finite dimension. Furthermore, explicit upper bounds for its Hausdorff and fractal dimensions are given as follows*

$$d_H(A)$$

$$\leq \left(\frac{(C(\phi, C, \gamma, r, \mu_i) + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma))}{K_1} \right)^{\frac{3}{2}} |\Omega| + 1$$

$$d_F(A)$$

$$\leq 2 \left(\frac{(C(\phi, C, \gamma, r, \mu_i) + (2\mu_A + 2\mu + 2\mu_F + \mu_M + \mu_{M_S} + (1-r)\gamma))}{K_1} \right)^{\frac{3}{2}} |\Omega| + 2$$

PROOF. The earlier derived estimates via (6.15), (6.16) along with Lemma 6.3 allow us to obtain the desired result   □

7. Conclusion

In conclusion, we present a new deterministic model,(2.1), for Aedes aegypti mosquitoes, (the vector that causes dengue) incorporating both spatial spread, and preferential selection of the female mosquito for the wild males, against modified sterile males. Some of the main theoretical findings of this study are summarized below:
(i) The system posses strong solution, which is global in time.
(ii) For the system there exists a global attractor in $[L^2(\Omega)]^8$.
(iii) Upper bounds on the Hausdorff and fractal dimensions of the attractor were provided, in terms of parameters in the system.

These estimates become quite handy from the point of view of numerical computation, specially if the object of interest is primarily long time dynamics. Essentially our results show that if sexual selection is incorporated into a reaction diffusion system, modelling the spread of Aedes Aegypti mosquitoes, the sterile insect technique can still be a sucessfull control measure, if the injection of sterile males is large enough. We agree, that the previous ODE models we cite, did obtain eradication if injection of sterile males is large enough. However, we feel that the reason for considering both diffusion, and selection in the current work, is not to mimic the results of the ODE models considered earlier, but in fact to improve them. Also we would like to comment that various modelling/biological assumptions we make seem natural to us as a first attempt to incorporate sexual selection in modelling mosquito dynamics under SIT. However, these can be improved upon in future works. For example it may be possible for the females with preference to mate with sterile males in small probability. These and related questions are all

under current investigation. All in all, we hope this work is a small step, in the further understanding, and control of vector borne disease, such as dengue.

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8. Appendix

Recall $u = (u_1...u_8)$ is a solution to (2.1) and $(U1(t)...U8(t))$ are strong solutions to the variational equations for the reaction diffusion system (2.1). These have been worked out explicitly next.

$$\frac{\partial U1}{\partial t} = \Delta U1 - (\gamma + \mu_A + \delta)U1 + \phi\left(1 - \frac{u_1 + u_2}{C}\right)U_5 - \phi\left(\frac{U_1 + U_2}{C}\right)u_5$$

$$\frac{\partial U2}{\partial t} = \Delta U2 - (\gamma + \mu_A)U2 + \delta U1 + \phi\left(1 - \frac{u_2 + u_1}{C}\right)U_6 - \phi\left(\frac{U_2 + U_1}{C}\right)u_6$$

$$\frac{\partial U3}{\partial t} = \Delta U3 - (\mu)U3 + r\gamma U1$$

$$-\beta\left[\left(\frac{u_7}{u_7+u_8}\right)u_3 + \left(\frac{u_8U7-u_7U8}{(u_7+u_8)^2}\right)U3\right]$$

$$-\beta\left[\left(\frac{u_8}{u_7+u_8}\right)u_3 + \left(\frac{u_7U8-u_8U7}{(u_7+u_8)^2}\right)U3\right]$$

$$\frac{\partial U4}{\partial t} = \Delta U4 - (\mu)U4 + r\gamma U2$$

$$-\beta\left[\left(\frac{u_7}{u_7+u_8}\right)u_4 + \left(\frac{u_8U7-u_7U8}{(u_7+u_8)^2}\right)U4\right]$$

$$\frac{\partial U5}{\partial t} = \Delta U5 - (\mu_F)U5$$

$$+\beta\left[\left(\frac{u_7}{u_7+u_8}\right)u_3 + \left(\frac{u_8U7-u_7U8}{(u_7+u_8)^2}\right)U3\right]$$

$$\frac{\partial U6}{\partial t} = \Delta U6 - (\mu_F)U6$$

$$+\beta\left[\left(\frac{u_7}{u_7+u_8}\right)u_4 + \left(\frac{u_8U7-u_7U8}{(u_7+u_8)^2}\right)U4\right]$$

$$\frac{\partial U7}{\partial t} = \Delta U7 + (1-r)\gamma(U1+U2) - (\mu_M)U7$$

$$\frac{\partial U8}{\partial t} = \Delta U8 + v_2 - (\mu_{M_S})U8$$

$$U1(0) = U1_0, \ U2(0) = U2_0, \ U3(0) = U3_0, \ U4(0) = U4_0, \ U5(0) = U5_0,$$

$$U6(0) = U6_0, \ U7(0) = U7_0, \ U8(0) = U8_0,$$

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