Dynamics of PDE, Vol.8, No.3, 239-260, 2011

Sharp blow-up criteria for the Davey-Stewartson system in $\mathbb{R}^3$

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Communicated by Y. Charles Li, received October 27, 2010.

Abstract. In this paper, we study the blow-up solutions for the Davey-Stewartson system

$$iu_t + \triangle u + |u|^{p-1}u + E(|u|^2)u = 0,\ t \geq 0,\ x \in \mathbb{R}^3. \qquad (DS)$$

Using the profile decomposition of the bounded sequences in $H^1(\mathbb{R}^3)$, we give some new variational characteristics for the ground states and generalized Gagliardo-Nirenberg inequalities. Then, we obtain the precise expressions on the sharp blow-up criteria to (DS) for $1 + \frac{4}{3} \leq p < 5$.

Contents

1. Introduction

This paper is concerned with the Cauchy problem of the following Davey-Stewartson system

$$iu_t + \triangle u + |u|^{p-1}u + E(|u|^2)u = 0,\ t \geq 0,\ x \in \mathbb{R}^N, \tag{1.1}$$

$$u(0, x) = u_0, \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35B44; 35Q55.

Key words and phrases. Davey-Stewartson system, profile decomposition, blow-up solution, sharp criteria.

Supported by National Natural Science Foundation of P.R.China (No. 11071177).

where $i=\sqrt{-1}$; $\triangle=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_N^2}$ is the Laplace operator on $\mathbb{R}^N$; $u=u(t,x)$: $[0,T)\times\mathbb{R}^N\to\mathbb{C}$ is the complex valued function and $0<T\le+\infty$; the parameter $1<p<2^*-1$ (where $2^*=+\infty$ for $N=2$; $2^*=\frac{2N}{N-2}$ for $N\ge3$); E is the singular integral operator with symbol $\sigma_1(\xi)=\frac{\xi_1^2}{|\xi|^2},\xi\in\mathbb{R}^N$, $E(\psi)=\mathcal{F}^{-1}\frac{\xi_1^2}{|\xi|^2}\mathcal{F}\psi$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and Fourier inverse transform on $\mathbb{R}^N$, respectively.

For the Cauchy problem (1.1)-(1.2), Ghidaglia and Saut [**8**], Guo and Wang [**9**] established the local well-posedness in the energy space $H^1(\mathbb{R}^N)$ for $N=2$ and $N=3$ respectively (see [**1, 22**] for a review). Cipolatti [**2**] showed the existence of the standing waves. Cipolatti [**3**], Ohta [**15, 16**], Gan and Zhang [**6**] showed the stability and instability of the standing waves. Ghidaglia and Saut [**8**], Guo and Wang [**9**] showed the existence of the blow-up solutions. Ozawa [**17**] gave the exact blow-up solutions. Wang and Guo [**23**] studied the scattering of solutions. Richards [**19**], Papanicolaou etal [**18**], Gan and Zhang [**5, 6**], Shu and Zhang [**20**] studied the sharp conditions of blow-up and global existence. Li etal [**13**], Richards [**19**] obtained the mass-concentration properties of the blow-up solutions in L^2-critical case in $\mathbb{R}^2$.

We note that in $\mathbb{R}^2$, Richards [**19**] and Papanicolaou etal [**18**] gave the precise expression on the sharp blow-up criteria in L^2-critical case that is $N=2$ and $p=3$. But in $\mathbb{R}^3$ Equation (1.1) has not any L^2-critical case because of influence of the nonlocal term $E(|u|^2)u$. Although in [**5, 6, 20**] some sharp thresholds of blow-up and global existence are gotten, we also note that the upper bound d of the energy functional $I(u)$ is not determined. This motivates us to investigate the precise expression on the sharp blow-up criteria in $\mathbb{R}^3$.

In the present paper, at first, we study the sharp blow-up criteria for the Cauchy problem (1.1)-(1.2) in $\mathbb{R}^3$ for $1+\frac{4}{3}\le p<5$. Motivated by Holmer and Roudenko's studies [**11**] for the classical L^2-super nonlinear Schrödinger equation, we consider the following elliptic equations

$$\frac{3}{2}\triangle Q-\frac{1}{2}Q+|Q|^2Q+E(|Q|^2)Q=0,\ Q\in H^1(\mathbb{R}^3) \tag{1.3}$$

and

$$\frac{3}{2}\triangle R-\frac{1}{2}R+E(|R|^2)R=0,\quad R\in H^1(\mathbb{R}^3). \tag{1.4}$$

Applying the profile decomposition of the bounded sequences in $H^1(\mathbb{R}^3)$, we obtain the following generalized Gagliardo-Nirenberg inequalities

$$\int_{\mathbb{R}^3}|f|^4+E(|f|^2)|f|^2dx\le\frac{2}{\|Q\|_2^2}\|\nabla f\|_{L^2}^3\|f\|_{L^2},\quad\forall\ f\in H^1(\mathbb{R}^3) \tag{1.5}$$

and

$$\int_{\mathbb{R}^3}E(|f|^2)|f|^2dx\le\frac{2}{\|R\|_{L^2}^2}\|\nabla f\|_{L^2}^3\|f\|_{L^2},\quad\forall\ f\in H^1(\mathbb{R}^3) \tag{1.6}$$

where Q is the solution of Equation (1.3) and R is the solution of Equation(1.4). Using the above Gagliardo-Nirenberg inequalities, we obtain the sharp blow-up criteria to the Cauchy problem (1.1)-(1.2) for $1+\frac{4}{3}\le p<5$ by overcoming the loss of scaling invariance. We remark that we get a clear bound value of energy functional, which corresponds to d in [**6**]. Furthermore, we prove that there is no

L^3 strong limit of the blow-up solutions to the Cauchy problem (1.1)-(1.2) provided $1 < p \leq 3$.

There are two major difficulties in the analysis of blow-up solutions to the Davey-Stewartson system (1.1)-(1.2) in $H^1(\mathbb{R}^3)$: One is the nonlinearity containing the singular integral operator E; The other is the loss of scaling invariance to Equation (1.1) for $p \neq 3$, which destroys the balance between $|u|^{p-1}u$ and $E(|u|^2)u$. Due to the singular integral operator E, we have to establish the corresponding generalized Gagliardo-Nirenberg inequalities and variational structures. Since there is no scaling invariance for $p \neq 3$, improving Holmer and Roudenko's method [**11**] we use the ground state of the classical nonlinear Schrödinger equation to describe the sharp blow-up criteria to the Davey-Stewartson system (1.1)-(1.2). Finally, for the time being, as we have mentioned, the results in the present paper are new for the Davey-Stewartson system (1.1)-(1.2). In particular, the sharp blow-up criteria are different from [**6**], and the sharp blow-up criteria obtained in this paper are more precisely, which is very useful from the viewpoint of physics.

We conclude this section with several notations. We abbreviate $L^q(\mathbb{R}^3)$, $\|\cdot\|_{L^q(\mathbb{R}^3)}$, $H^s(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \cdot dx$ by L^q, $\|\cdot\|_q$, H^s and $\int \cdot dx$. The various positive constants will be simply denoted by C.

2. Preliminary

For the Cauchy problem (1.1)-(1.2), the energy space is defined by

$$H^1 := \{u \in L^2 \ ; \ \nabla u \in L^2\},$$

which is a Hilbert space. The norm of H^1 is denoted by$\|\cdot\|_{H^1}$. Moreover, we define the energy functional $H(u)$ in H^1 by

$$H(u) := \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{1}{p+1}\int |u(t,x)|^{p+1} dx - \frac{1}{4}\int E(|u|^2)|u|^2 dx.$$

The functional H is well-defined according to the Sobolev embedding theorem and the properties of the singular operator E.

Guo and Wang [**9**] established the local well-posedness of the Cauchy problem (1.1)-(1.2) in energy space H^1.

Proposition 2.1. *Let $u_0 \in H^1$. There exists an unique solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) on the maximal time $[0,T)$ such that $u(t,x) \in C([0,T); H^1)$ and either $T = +\infty$ (global existence), or $T < +\infty$ and $\lim_{t\to T} \|u(t,x)\|_{H^1} = +\infty$ (blow-up). Furthermore, for all $t \in [0,T)$, $u(t,x)$ satisfies the following conservation laws*

(i) *Conservation of mass*

$$\|u(t,x)\|_2 = \|u_0\|_2, \tag{2.1}$$

(ii) *Conservation of energy*

$$H(u(t,x)) = H(u_0). \tag{2.2}$$

For more specific results concerning the Cauchy problem (1.1)-(1.2), we refer the reader to [**8**, **22**]. In addition, by some basic calculations, we have the following proposition(see also Ohta [**16**]).

Proposition 2.2. *Assume that $u_0 \in H^1$, $|x|u_0 \in L^2$ and the corresponding solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) on the interval $[0,T)$. Then, for all $t \in [0,T)$ we have $|x|u(t,x) \in L^2$. Moreover, let $J(t) := \int |x|^2|u(t,x)|^2 dx$, we have*

$$J^{'}(t) = -4\Im \int xu\nabla\overline{u}dx \tag{2.3}$$

and

$$J^{''}(t) = 8\int |\nabla u|^2 dx - 12\frac{p-1}{p+1}\int |u|^{p+1}dx - 6\int E(|u|^2)|u|^2 dx. \tag{2.4}$$

We give some known facts of the singular integral operator E (see Cipolatti [**2**, **3**]), as follows.

Lemma 2.3. *Let E be the singular integral operator defined in Fourier variables by*

$$\mathcal{F}[E(\psi)](\xi) = \sigma_1(\xi)\mathcal{F}[\psi](\xi),$$

where $\sigma_1(\xi) = \frac{\xi_1^2}{|\xi|^2}, \xi \in \mathbb{R}^3$ and $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^3$. For $1 < p < +\infty$, E satisfies the following properties:

(i) *$E \in \mathcal{L}(L^p, L^p)$, where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded linear operators from L^p to L^p.*
(ii) *If $\psi \in H^s(\mathbb{R}^3)$, then $E(\psi) \in H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$.*
(iii) *If $\psi \in W^{m,p}$, then $E(\psi) \in W^{m,p}$ and*

$$\partial_k E(\psi) = E(\partial_k \psi), \quad k = 1, 2.$$

(iv) *E preserves the following operations:*
   - *translation: $E(\psi(\cdot + y))(x) = E(\psi)(x + y)$, $y \in \mathbb{R}^3$;*
   - *dilatation: $E(\psi(\lambda\cdot))(x) = E(\psi)(\lambda x)$, $\lambda > 0$;*
   - *conjugation: $\overline{E(\psi)} = E(\overline{\psi})$,*

   where $\overline{\psi}$ is the complex conjugate of ψ.

At the end of this section, we shall give the profile decomposition of bounded sequences in H^1 proposed by Gérard [**7**], Hmidi and Keraani [**10**], which is important to study the variational characteristic of the ground state.

Proposition 2.4. *Let $\{v_n\}_{n=1}^\infty$ be a bounded sequence in H^1. Then there is a subsequence of $\{v_n\}_{n=1}^\infty$ (still denoted by $\{v_n\}_{n=1}^\infty$) and a sequence $\{V^j\}_{j=1}^\infty$ in H^1 and a family of $\{x_n^j\}_{j=1}^\infty \subset \mathbb{R}^3$ such that*

(i) *for every $j \neq k$, $|x_n^j - x_n^k| \overset{n\to\infty}{\to} +\infty$;*
(ii) *for every $l \geq 1$ and every $x \in \mathbb{R}^3$*

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x) \tag{2.5}$$

with

$$\lim_{n\to\infty} \sup \|v_n^l\|_r \overset{l\to\infty}{\to} 0, \tag{2.6}$$

for every $r \in (2,6)$.

Moreover, we have, as $n \to \infty$,

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1) \tag{2.7}$$

and

$$\|\nabla v_n\|_2^2 = \sum_{j=1}^{l} \|\nabla V^j\|_2^2 + \|\nabla v_n^l\|_2^2 + o(1). \tag{2.8}$$

3. Sharp Gagliardo-Nirenberg Inequalities

In order to study the sharp blow-up criteria to the Cauchy problem (1.1)-(1.2), we have to establish the generalized Gagliardo-Nirenberg inequalities corresponding to the Davey-Stewartson system. For $p = 3$, we consider the standing wave solutions of Equation (1.1) in the form $u(t,x) = Q(\sqrt{\frac{3}{2}}x)e^{-\frac{i}{2}t}$. It is easy to check that $Q(x)$ satisfies

$$\frac{3}{2}\triangle Q - \frac{1}{2}Q + |Q|^2 Q + E(|Q|^2)Q = 0,\ Q \in H^1. \tag{3.1}$$

We say that $Q \in H^1$ is a ground state solution of (3.1), if it satisfies

$$\tilde{S}(v) = \inf\{\tilde{S}(v)|\ v \in H^1 \setminus \{0\} \text{ is a solution of (3.1)}\},$$

where $\tilde{S}(v)$ is defined for $v \in H^1$ by

$$\tilde{S}(v) = \frac{1}{2}\int |\nabla v|^2 dx - \frac{1}{4}\int (|v|^4 + E(|v|^2)|v|^2) dx + \frac{1}{2}\int |v|^2 dx.$$

For any solution Q of Equation (3.1), we claim the following Pohozhaev identities:

$$-\frac{1}{2}\int |Q|^2 dx - \frac{3}{2}\int |\nabla Q|^2 dx + \int |Q|^4 + E(|Q|^2)|Q|^2 dx = 0 \tag{3.2}$$

and

$$\int |Q|^2 dx + \int |\nabla Q|^2 dx - \int |Q|^4 + E(|Q|^2)|Q|^2 dx = 0. \tag{3.3}$$

Indeed, multiplying (3.1) by Q and integrating by parts, we have that (3.2) is true. On the other hand, multiplying (3.1) by $x \cdot \nabla Q$ and integrating by parts, we have

$$\frac{3}{2}\int \triangle Q x \cdot \nabla Q dx - \frac{1}{2}\int Q x \cdot \nabla Q dx + \int Q^2 Q x \cdot \nabla Q dx + \int E(|Q|^2) Q x \cdot \nabla Q dx = 0.$$

It follows from some basic calculations that

$$-\int \triangle Q x \cdot \nabla Q dx = \int |\nabla Q|^2 dx + \int x \cdot \nabla(\frac{|\nabla Q|^2}{2}) dx = -\frac{1}{2}\int |\nabla Q|^2 dx,$$

$$\int Q x \cdot \nabla Q dx = -\frac{3}{2}\int |Q|^2 dx,$$

$$\int Q^2 Q x \cdot \nabla Q dx = -\frac{3}{4}\int |Q|^4 dx$$

and

$$\int E(|Q|^2) Q x \cdot \nabla Q dx = -\frac{3}{4}\int E(|Q|^2) Q^2 dx.$$

Collecting the above identities, we have that (3.3) is true.

In this section, using the profile decomposition of bounded sequence in H^1, we shall give a new and simple proof of the existence of the ground state of (3.1). Moreover, we compute the best constant of a generalized Gagliardo-Nirenberg inequality in dimension three, which is corresponding to Equation (1.1). More precisely, we have the following theorem.

Theorem 3.1. *Let $f \in H^1$, then*

$$\int |f|^4 + E(|f|^2)|f|^2 dx \leq \frac{2}{\|Q\|_2^2}\|\nabla f\|_2^3 \|f\|_2, \tag{3.4}$$

where Q is the solution of Equation (3.1).

We shall give the proof of Theorem 3.1 by three steps. First, from the definition of E and the classical Gagliardo-Nirenberg inequality, we observe that

$$\begin{aligned} \textstyle\int_{\mathbb{R}^3}(|v|^4 + E(|v|^2)|v|^2)dx &\leq C \textstyle\int_{\mathbb{R}^3}(|v|^4 + |v|^4)dx \\ &\leq C\|v\|_2 \|\nabla v\|_2^3, \end{aligned}$$

which motivates us to investigate the best constant C of the above inequality. Thus, we consider the variational problem

$$J := \inf\{J(u) \mid u \in H^1\} \quad \text{where} \quad J(u) := \frac{(\int |u|^2 dx)^{\frac{1}{2}} (\int |\nabla u|^2 dx)^{\frac{3}{2}}}{\int (|u|^4 + E(|u|^2)|u|^2) dx}. \tag{3.5}$$

It is obvious that if W is the minimizer of $J(u)$, then $|W|$ is also a minimizer. Hence we can assume that W is an real positive function. Indeed, $W = |W|e^{i\theta(x)}$ we have

$$|\nabla |W|| \leq |\nabla W|$$

in the sense of distribution. On the other hand, if $W \in H^1$, then$|W| \in H^1$ and $J(|W|) \leq J(W)$.

Second, by some basic calculations, if W is the minimizer of $J(u)$, we have the following lemma.

Lemma 3.2. *If W is the minimizer of $J(u)$, then W satisfies*

$$\frac{3}{2}\|W\|_2 \|\nabla W\|_2 \triangle W - \frac{1}{2}\frac{\|\nabla W\|_2^3}{\|W\|_2} W + 2J(|W|^2 + E(|W|^2))W = 0. \tag{3.6}$$

Proof. Since W is a minimizing function of $J(u)$ in H^1, and we have $\forall\ v \in C_0^\infty(\mathbb{R}^3)$

$$\frac{d}{d\varepsilon} J(W + \varepsilon v) \mid_{\varepsilon=0} = 0. \tag{3.7}$$

By some basic calculations, we have

$$\begin{aligned} &\frac{d}{d\varepsilon}\{\|(W+\varepsilon v)\|_2 \|\nabla(W+\varepsilon v)\|_2^3\} \mid_{\varepsilon=0} \\ &= \frac{1}{2}\frac{\|\nabla W\|_2^3}{\|W\|_2}\int 2\Re W\overline{v}dx - \frac{3}{2}\|W\|_2\|\nabla W\|_2 \int 2\Re \triangle W \overline{v} dx \end{aligned} \tag{3.8}$$

and

$$\frac{d}{d\varepsilon}\{\int |(W+\varepsilon v)|^4 + E(|W+\varepsilon v|^2)|W+\varepsilon v|^2 dx\} \mid_{\varepsilon=0} = 4\int (|W|^2 + E(|W|^2)\Re W \overline{v} dx. \tag{3.9}$$

By (3.7)-(3.9), we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\|\nabla W\|_2^3}{\|W\|_2}\int 2\Re W\overline{v}dx-\tfrac{3}{2}\|W\|_2\|\nabla W\|_2\int 2\Re\triangle W\overline{v}dx \\
= \quad & 2\tfrac{\|\nabla W\|_2^3\|W\|_2}{\int |W|^4+E(|W|^2)|W|^2dx}\int(|W|^2+E(|W|^2))2\Re W\overline{v}dx,
\end{aligned}
\tag{3.10}
$$

which implies that (3.6) is true.

The Euler-Lagrange equation (Lemma 3.2) shows that any minimizer of $J(u)$ is a solution of (3.1). Since any smaller mass solution would yield a lower value of $J(u)$, the Pohozhaev identities show that it is in fact a minimal mass solution of (3.1). Therefore, to prove the existence of a ground state it suffices to prove the existence of a minimizer for $J(u)$.

Thirdly, we use the profile decomposition of bounded sequences in H^1 to prove the following proposition, which give a proof of the existence of a minimizer for $J(u)$. Moreover, Theorem 3.1 is a direct conclusion of the following proposition.

Proposition 3.3. *J is attained at a function $U(x)$ with the following properties:*

$$U(x)=aQ(\lambda x+b) \quad \text{for some} \quad a\in\mathbb{C}^*,\ \lambda>0 \text{ and } \ b\in\mathbb{R}^3 \tag{3.11}$$

where Q is the solution of Equation(3.1). Moreover, we have

$$J=\frac{\|Q\|_2^2}{2}. \tag{3.12}$$

Proof. If we set $u^{\lambda,\mu}=\mu u(\lambda x)$, where $\lambda=\frac{\|u\|_2}{\|\nabla u\|_2}$, $\mu=\frac{\|u\|_2^{\frac{1}{2}}}{\|\nabla u\|_2^{\frac{3}{2}}}$, we have

$$\|u^{\lambda,\mu}\|_2=1,\quad \|\nabla u^{\lambda,\mu}\|_2=1 \quad\text{and}\quad J(u^{\lambda,\mu})=J(u).$$

Now, choosing a minimizing sequence $\{u_n\}_{n=1}^{\infty}\subset H^1$ such that $J(u_n)\to J$ as $n\to\infty$, after scaling, we may assume

$$\|u_n\|_2=1 \quad\text{and}\quad \|\nabla u_n\|_2=1, \tag{3.13}$$

and we have

$$J(u_n)=\frac{1}{\int |u_n|^4+E(|u_n|^2)|u_n|^2dx}\to J,\quad\text{as}\quad n\to\infty. \tag{3.14}$$

Note that $\{u_n\}_{n=1}^{\infty}$ is bounded in H^1. It follows form the profile decomposition (Proposition 2.4) that

$$u_n(x)=\sum_{j=1}^{l}U^j(x-x_n^j)+r_n^l(x),$$

$$\sum_{j=1}^{l}\|U_n^j\|_2^2\le 1 \quad\text{and}\quad \sum_{j=1}^{l}\|\nabla U_n^j\|_2^2\le 1, \tag{3.15}$$

where $U_n^j=U^j(x-x_n^j)$. Moreover, using the Hölder's inequality for r_n^l and the properties of E, we have

$$\begin{aligned} \int(|r_n^l|^4+E(|r_n^l|^2)|r_n^l|^2)dx &\le C(\|r_n^l\|_4^4+\|E(|r_n^l|^2)\|_2\|r_n^l\|_4^2) \\ &\le C(\|r_n^l\|_4^4+\|r_n^l\|_4^4)\to 0, \quad \text{as} \quad l\to\infty. \end{aligned} \tag{3.16}$$

Applying the orthogonal conditions and the properties of E, we have the following claims:

(i)

$$\int|\sum_{j=1}^{l}U^j(x-x_n^j)|^4dx\to\sum_{j=1}^{l}\int|U^j|^4dx, \quad \text{as} \quad n\to\infty. \tag{3.17}$$

(ii)

$$\int E(|\sum_{j=1}^{l}U^j(x-x_n^j)|^2)|\sum_{j=1}^{l}U^j(x-x_n^j)|^2dx\to\sum_{j=1}^{l}\int E(|U^j|^2)|U^j|^2dx, \ \text{as} \ n\to\infty. \tag{3.18}$$

Indeed, for (3.17), it suffices to show that

$$I_n=\int U_n^{j_1}U_n^{j_2}U_n^{j_3}U_n^{j_4}dx\to 0, \quad \text{as} \quad n\to\infty \tag{3.19}$$

for $1\le j_k\le l$ and at least two j_k are different. Assuming for example $j_1\ne j_2$, by the Hölder's inequality, we can estimate

$$|I_n|^2\le C^2\|U_n^{j_1}U_n^{j_2}\|_2^2,$$

where $C=\Pi_{k=3}^4\|U_n^{j_k}\|_4$. Without loss of generality, we can assume that both U^{j_1} and U^{j_2} are continuous and compactly supported. Now, we use the pairwise orthogonal conditions, and we have the following estimation

$$|I_n|^2\le C^2\int|U^{j_1}(y)U^{j_2}(y-(x_n^{j_2}-x_n^{j_1}))|^2dy\to 0, \ \text{as} \ n\to\infty. \tag{3.20}$$

This completes the proof of Claim (i).

For (3.18), we have

$$\begin{aligned} &\int E(|\sum_{j=1}^{l}U^j(x-x_n^j)|^2)|\sum_{j=1}^{l}U^j(x-x_n^j)|^2dx \\ \le C \quad &(\int[\sum_{j=1}^{l}E(|U_n^j|^2)+\sum_{1\le j,k\le l,j\ne k}E(|U_n^jU_n^k|)][\sum_{j=1}^{l}|U_n^j|^2+\sum_{1\le j,k\le l,j\ne k}|U_n^jU_n^k|]dx), \end{aligned}$$

which implies that

(3.21)
$$
\begin{aligned}
&\int E(|\sum_{j=1}^{l} U^j(x-x_n^j)|^2)|\sum_{j=1}^{l} U^j(x-x_n^j)|^2dx - \sum_{j=1}^{l}\int E(|U^j|^2)|U^j|^2dx\\
\leq\;& C\sum_{j=1}^{l}\int E(|U_n^j|^2)|U_n^j|^2dx + \sum_{1\leq j,k\leq l, j\neq k}\int E(|U_n^j|^2)|U_n^k|^2dx\\
&+C\sum_{1\leq i,j,k\leq l, i\neq j}\int E(|U_n^iU_n^j|)|U_n^k|^2dx + \sum_{1\leq i,j,k\leq l, j\neq k}\int E(|U_n^i|^2)|U_n^jU_n^k|dx\\
&\quad +C\sum_{1\leq i,j,k,m\leq l, i\neq j, k\neq m}\int E(|U_n^iU_n^j|)|U_n^kU_n^m|dx\\
=\;& I+II+III.
\end{aligned}
$$

Without loss of generality, we can assume that U^i, U^j, U^k and U^m are continuous and compactly supported. Using the orthogonal conditions and the properties of the singular operator $E(u)$, we have

$$
\begin{aligned}
I &= C\sum_{1\leq j,k\leq l, j\neq k}\int E(|U_n^j|^2)|U_n^k|^2dx\\
&= \sum_{1\leq j,k\leq l, j\neq k}\int E(|U^j|^2)(x-x_n^j)|U^k(x-x_n^k)|^2dx\\
&= C\sum_{1\leq j,k\leq l, j\neq k}\int E(|U^j|^2)(x)|U^k(x-(x_n^k-x_n^j))|^2dx\\
&\to 0,\ \text{as } n\to\infty,
\end{aligned}
$$

$$
\begin{aligned}
II\quad &\leq C\sum_{1\leq i,j,k\leq l, i\neq j}[\|E(|U_n^iU_n^j|)\|_2\|U_n^k\|_4^2 + \|E(|U_n^k|^2)\|_{L^2}\|U_n^iU_n^j\|_2]\\
&\leq C\sum_{1\leq i,j,k\leq l, i\neq j}\|U_n^iU_n^j\|_2\|U_n^k\|_4^2\\
&\to 0,\ \text{as } n\to\infty
\end{aligned}
$$

and

$$
III \leq C\sum_{1\leq i,j,k,m\leq l, i\neq j, k\neq m}\|E(|U_n^iU_n^j|)\|_2\|U_n^kU_n^m\|_2 \to 0, \quad \text{as} \quad n\to\infty.
$$

The last step estimations of I, II and III follows from the proof of Claim (i) and this completes the proof of Claim (ii).

Therefore, by (3.14) and (3.16)- (3.18), we have

$$
\sum_{j=1}^{l}\int(|U^j|^4 + E(|U^j|^2)|U^j|^2)dx \to \frac{1}{J}, \quad \text{as} \quad n\to\infty. \tag{3.22}
$$

On the other hand, by the definition of J, we have

$$
J\int(|U^j|^4 + E(|U^j|^2)|U^j|^2)dx \leq \|U^j\|_2\|\nabla U^j\|_2^3. \tag{3.23}
$$

Since the series $\sum_j \|U^j\|_2^2$ is convergent, there exists a $j_0 \geq 1$ such that

$$
\|U^{j_0}\|_2 = \sup\{\|U^j\|_2 \mid \ j\geq 1\}. \tag{3.24}
$$

It follows from (3.22)-(3.24) that

(3.25)

$$1 \le J\left(\sum_{j=1}^{l}\int(|U^j|^4+E(|U^j|^2)|U^j|^2)dx\right) \le \sup\{\|U^j\|_2 \mid \ j\ge 1\}\left(\sum_{j=1}^{l}\|\nabla U^j\|_2^3\right)$$
$$\le \|U^{j_0}\|_2\left(\sum_{j=1}^{l}\|\nabla U^j\|_2^2\right) \le \|U^{j_0}\|_2.$$

It follows from (3.15) that $\|U^{j_0}\|_2 = 1$, which implies that there exists only one term $U^{j_0} \neq 0$ such that

$$\|U^{j_0}\|_2 = 1,\ \|\nabla U^{j_0}\|_2 = 1 \ \text{ and } \int(|U^{j_0}|^4+E(|U^{j_0}|^2)|U^{j_0}|^2)dx = \frac{1}{J}. \tag{3.26}$$

Therefore, we show that U^{j_0} is the minimizer of $J(u)$. It follows from Lemma 3.2 that

$$\frac{3}{2}\triangle U^{j_0} - \frac{1}{2}U^{j_0} + 2J(|U^{j_0}|^2 + E(|U^{j_0}|^2))U^{j_0} = 0. \tag{3.27}$$

We may assume that U^{j_0} is an real positive function by the definition of $J(u)$.

Now, we take $U^{j_0} = aQ(\lambda x + b)$ with Q is the positive solution of (3.1). By some computations, we have that $\|U^{j_0}\|_2^2 = \frac{a^2}{\lambda^3}\|Q\|_2^2 = 1$, $\|\nabla U^{j_0}\|_2^2 = \frac{a^2}{\lambda}\|\nabla Q\|_2^2 = 1$ and $\int(|U^{j_0}|^4+E(|U^{j_0}|^2)|U^{j_0}|^2)dx = \frac{a^4}{\lambda^3}\int(|Q|^4+E(|Q|^2)|Q|^2)dx = \frac{1}{J}$. Applying Claim (3.2) and (3.3), we have

$$\int(|Q|^4+E(|Q|^2)|Q|^2)dx = 2\int Q^2dx = 2\int|\nabla Q|^2dx,$$

which implies that

$$J = \frac{\lambda^3}{a^4}\frac{1}{\int(|Q|^4+E(|Q|^2)|Q|^2)dx} = \frac{1}{2a^2} = \frac{\|Q\|_2^2}{2}. \tag{3.28}$$

This completes the proof Proposition 3.3.

Next, we consider the following elliptic equation

$$\frac{3}{2}\triangle R - \frac{1}{2}R + E(|R|^2)R = 0, \quad R \in H^1. \tag{3.29}$$

By the same argument in Theorem 3.1, we have the following theorem.

Theorem 3.4. *Let $f \in H^1$, then*

$$\int E(|f|^2)|f|^2dx \le \frac{2}{\|R\|_2^2}\|\nabla f\|_2^3\|f\|_2, \tag{3.30}$$

where R is the solution of Equation(3.29).

Remark 3.5. *(i) To our knowledge, the uniqueness of the ground state $Q(x)$ and $R(x)$ is still an open problem. Indeed, the known proofs of the uniqueness rely on radiality (see [**12**]). Since the E operator does not commute with rotation, one cannot deduce the radiality of minimizers to $J(u)$ by symmetric rearrangement.*

*(ii) The best constants of the generalized Gagliardo-Nirenberg inequalities (3.4) and (3.30) are dependent on the space dimension N, but they are independent of the choices of the ground state solution $Q(x)$ and $R(x)$. Cipolatti [**2**] showed the*

*existence of a ground state solution of (3.1) in dimension two and three. Papanicolaou etal [**18**] computed the best constant of the generalized Gagliardo-Nirenberg inequality in dimension two. We point out that the method used in this paper is different from [**2, 18**]. Moreover, our method can also be applied in the dimension* $N = 2$*.*

In the end, we collect Weinstein's results [**24**], and we consider the following elliptic equation

$$\frac{3(p-1)}{4}\triangle P - \frac{5-p}{4}P + |P|^{p-1}P = 0,\ P \in H^1. \tag{3.31}$$

Strauss [**21**] showed the existence of equation (3.31). Weinstein [**24**] showed the best constant of the Gagliardo-Nirenberg inequality, as follows

Proposition 3.6. *Let* $f \in H^1$ *and* $1 < p < 5$*, then*

$$\|f\|_{p+1}^{p+1} \leq \frac{p+1}{2\|P\|_2^{p-1}} \|\nabla f\|_2^{\frac{3(p-1)}{2}} \|f\|_2^{\frac{5-p}{2}}, \tag{3.32}$$

where P *is the solution of Equation (3.31).*

4. Sharp Blow-up Criteria

In this section, using the sharp Gagliardo-Nirenberg inequalities obtained in Section 3, we obtain the sharp blow-up criteria to the Cauchy problem (1.1)-(1.2). More precisely, establishing four classes of invariant evolution flows according to the value of p, we obtain the sharp blow-up criteria to the Cauchy problem (1.1)-(1.2) for all $1 + \frac{3}{4} \leq p < 5$.

• **Sharp Criteria for** $p = 3$

Theorem 4.1. *Let* $p = 3$, $u_0 \in H^1$ *and satisfy*

$$H(u_0)\|u_0\|_2^2 < \frac{2}{27}\|Q\|_2^4. \tag{4.1}$$

Then, we have that

(i) *If*

$$\|\nabla u_0\|_2 \|u_0\|_2 < \frac{2}{3}\|Q\|_2^2, \tag{4.2}$$

then the solution $u(t,x)$ *of the Cauchy problem (1.1)-(1.2) exists globally. Moreover,* $u(t,x)$ *satisfies*

$$\|\nabla u(t,x)\|_2 \|u(t,x)\|_2 < \frac{2}{3}\|Q\|_2^2. \tag{4.3}$$

(ii) *If*

$$\|\nabla u_0\|_2 \|u_0\|_2 > \frac{2}{3}\|Q\|_2^2, \tag{4.4}$$

and $|x|u_0 \in L^2$*, then the solution* $u(t,x)$ *of the Cauchy problem (1.1)-(1.2) blows up in finite time* $T < +\infty$*,*

where Q *is the solution of Equation (3.1).*

Proof. Applying the generalized Gagliardo-Nirenberg inequality (Theorem 3.1), we have

$$H(u) = \frac{1}{2}\int |\nabla u|^2 dx - \frac{1}{4}\int |u|^4 + E(|u|^2)|u|^2 dx \\ \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{\|u\|_2}{2\|Q\|_2^2}\|\nabla u\|_2^3. \tag{4.5}$$

Now, we define a function $f(y)$ on $[0, +\infty)$ by

$$f(y) = \frac{1}{2}y^2 - \frac{\|u_0\|_2}{2\|Q\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0, +\infty)$ and

$$f^{'}(y) = y - \frac{3\|u_0\|_2}{2\|Q\|_2^2}y^2 = y(1 - \frac{3\|u_0\|_2}{2\|Q\|_2^2}y). \tag{4.6}$$

It is obvious that there are two roots for equation $f^{'}(y) = 0$: $y_1 = 0$, $y_2 = \frac{2\|Q\|_2^2}{3\|u_0\|_2}$. Hence, we have that y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_2)$ and decreasing on the interval $[y_2, +\infty)$.

Note that $f(0) = 0$ and $f_{max} = f(y_2) = \frac{2\|Q\|_2^4}{27\|u_0\|_2^2}$. By the conservation of energy and assumption (4.1), we have

$$f(\|\nabla u\|_2) \leq H(u) = H(u_0) < \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2} = f(y_2). \tag{4.7}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (1.1)-(1.2), as follows.

$$K_1 := \{u \in H^1 \mid 0 < \|\nabla u\|_2\|u\|_2 < \frac{2}{3}\|Q\|_2^2,\ 0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}\}$$

and

$$K_2 := \{u \in H^1 \mid \|\nabla u\|_2\|u\|_2 > \frac{2}{3}\|Q\|_2^2,\ 0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}\}.$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_1$, we have $0 < H(u)\|u\|_2^2 < \frac{2\|Q\|_2^4}{27}$ and $\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\|Q\|_2^2$, which implies that $\|\nabla u_0\|_2 < y_2$. Since $f(y)$ is continuous and increasing on $[0, y_2)$ and $f(y) < f_{max} = \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t, x)\|_2 < y_2,$$

which implies that K_1 is invariant.

If $u_0 \in K_2$, we have

$$\|\nabla u_0\|_2\|u_0\|_2 > \frac{2}{3}\|Q\|_2^2,$$

which implies that $\|\nabla u_0\|_2 > y_2$. Since $f(y)$ is continuous and decreasing on $[y_2, +\infty)$ and

$$f(y) < f_{max} = \frac{2\|\|Q\|_2^4}{27\|u_0\|_2^2},$$

we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t, x)\|_2 > y_2 \quad \text{and} \quad \|\nabla u(t, x)\|_2\|u(t, x)\|_2 > \frac{2}{3}\|Q\|_2^2, \tag{4.8}$$

which implies that K_2 is invariant.

Now, we return to the proof the Theorem 4.1. By (4.1) and (4.2), we have $u_0 \in K_1$. Applying the invariant of K_1, we have that (4.3) is true and the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally. This completes the part (i) of the proof.

By (4.1) and (4.4), we have $u_0 \in K_2$. Applying the invariant of K_2, we have (4.8) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$\begin{aligned} J^{''}(t) &= \tfrac{d^2}{dt^2}\int |x|^2|u(t,x)|^2dx \\ &= 8\int |\nabla u|^2dx - 6\int |u|^4 + E(|u|^2)|u|^2dx \\ &= 24H(u_0) - 4\|\nabla u\|_2^2. \end{aligned} \tag{4.9}$$

Multiplying both side of (4.9) by $\|u_0\|_2^2$, applying the conservation laws, (4.1) and (4.8), we have

$$\begin{aligned} \|u_0\|_2^2\tfrac{d^2}{dt^2}\int |x|^2|u(t,x)|^2dx &= 24H(u_0)\|u_0\|_2^2 - 4\|\nabla u\|_2^2\|u_0\|_2^2 \\ &< \tfrac{16}{9}\|Q\|_2^4 - \tfrac{16}{9}\|Q\|_2^4 = 0. \end{aligned} \tag{4.10}$$

By the classical analysis identity

$$J(t) = J(0) + J^{'}(0)t + \int_0^t J^{''}(s)(t-s)ds, \tag{4.11}$$

we have that the maximal existence interval I of $u(t,x)$ must be finite, which implies that the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$. This completes the proof.

- **Sharp Criteria for** $p = 1 + \frac{4}{3}$

Theorem 4.2. *Let* $p = 1 + \frac{4}{3}$, $u_0 \in H^1$ *and satisfy*

$$\|u_0\|_2 < \|P\|_2 \quad \text{and} \quad H(u_0) < \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}. \tag{4.12}$$

Then, we have

(i) *If*

$$\|\nabla u_0\|_2\|u_0\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{4.13}$$

then the solution $u(t,x)$ *of the Cauchy problem (1.1)-(1.2) exists globally. Moreover, for all time* t, $u(t,x)$ *satisfies*

$$\|\nabla u(t,x)\|_2\|u(t,x)\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}. \tag{4.14}$$

(ii) *If*

$$\|\nabla u_0\|_2\|u_0\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{4.15}$$

and $|x|u_0 \in L^2$, then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$,

where P is the solution of Equation (3.31) and R is the solution of Equation (3.29).

Proof. Applying Theorem 3.4 and Proposition 3.6, we have

$$\begin{aligned} H(u) &= \tfrac{1}{2}\int |\nabla u|^2 dx - \tfrac{1}{2+\frac{4}{3}}\int |u|^{2+\frac{4}{3}} dx - \tfrac{1}{4}\int E(|u|^2)|u|^2 dx \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{\|u\|_2^{\frac{4}{3}}}{2\|P\|_2^{\frac{4}{3}}}\|\nabla u\|_2^2 - \tfrac{\|u\|_2}{2\|R\|_2^2}\|\nabla u\|_2^3. \end{aligned} \tag{4.16}$$

Now, we define a function $f(y)$ on $[0,+\infty)$ by

$$f(y) = (\frac{1}{2} - \frac{\|u_0\|_2^{\frac{4}{3}}}{2\|P\|_2^{\frac{4}{3}}})y^2 - \frac{\|u_0\|_2}{2\|R\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0,+\infty)$ and

$$f'(y) = (1 - \frac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}})y - \frac{3\|u_0\|_2}{2\|R\|_2^2}y^2. \tag{4.17}$$

It is obvious that there are two roots for equation $f'(y) = 0$: $y_1 = 0$, $y_2 = \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2\|P\|_2^{\frac{4}{3}}} > 0$. Hence, we have that y_1 and y_2 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_2)$ and decreasing on the interval $[y_2, +\infty)$.

Note that $f(y_1) = 0$ and

$$f_{max} = f(y_2) = \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}.$$

By the conservation of energy and the assumption (4.12), we have

$$f(\|\nabla u\|_2) \leq H(u) = H(u_0) < f(y_2). \tag{4.18}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (1.1)-(1.2), as follows.

$$\begin{aligned} K_3 := \{u \in H^1 \mid 0 < \|\nabla u\|_2\|u\|_2 \\ < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}},\ \|u\|_2 < \|P\|_2,\ 0 < H(u) < D\} \end{aligned}$$

and

$$\begin{aligned} K_4 := \{u \in H^1 \mid \|\nabla u\|_2\|u\|_2 > \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}},\ \|u\|_2 \\ < \|P\|_2,\ 0 < H(u) < D\}, \end{aligned}$$

where $D = \frac{2\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u\|_2^{\frac{4}{3}})^3}{27\|u_0\|_2^2\|P\|_2^4}$. Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_3$, we have $\|u\|_2 < \|P\|_2$, $0 < H(u) < D$ and $\|\nabla u_0\|_2 < \frac{2}{3}\frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2\|P\|_2^{\frac{4}{3}}}$, which implies that $\|\nabla u_0\|_2 <$

y_2. Since $f(y)$ is increasing on $[0, y_2)$ and $0 < f(y) < D$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 \|u(t,x)\|_2 < \frac{2}{3} \frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}},$$

which implies that K_3 is invariant.

If $u_0 \in K_4$, we have

$$\|u\|_2 < \|P\|_2, 0 < H(u) < D$$

and

$$\|\nabla u_0\|_2 > \frac{2}{3} \frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|u_0\|_2 \|P\|_2^{\frac{4}{3}}},$$

which implies $\|\nabla u_0\|_2 > y_2$. Since $f(y)$ is decreasing on $[y_2, \infty)$ and $0 < f(y) < D$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 > y_2 \quad \text{and} \quad \|\nabla u(t,x)\|_2 \|u(t,x)\|_2 > \frac{2}{3} \frac{\|R\|_2^2(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})}{\|P\|_2^{\frac{4}{3}}}, \tag{4.19}$$

which implies that K_4 is invariant.

Now, we return to the proof the Theorem 4.2. By (4.12) and (4.13), we have $u_0 \in K_3$. Applying the invariant of K_3, we have that (4.14) is true and the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally. This completes the part (i) of the proof.

By (4.12) and (4.15), we have $u_0 \in K_4$. Applying the invariant of K_4, we have (4.19) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$\begin{aligned} J''(t) &= \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \\ &= 8 \int |\nabla u|^2 dx - \tfrac{16}{2+\frac{4}{3}} \int |u|^{2+\frac{4}{3}} - 6 \int E(|u|^2)|u|^2 dx \\ &= 24 H(u_0) - 4 \int |\nabla u|^2 dx + \tfrac{8}{2+\frac{4}{3}} \int |u|^{2+\frac{4}{3}} dx \\ &\le 24 H(u_0) - 4[1 - \tfrac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}] \|\nabla u\|_2^2. \end{aligned} \tag{4.20}$$

By the assumption (4.12), it follows from (4.19) and (4.20) that

(4.21)

$$\begin{aligned} \|u_0\|_2^2 \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx &\le 24 H(u_0) \|u_0\|_2^2 - 4[1 - \tfrac{\|u_0\|_2^{\frac{4}{3}}}{\|P\|_2^{\frac{4}{3}}}] \|\nabla u\|_2^2 \|u_0\|_2^2 \\ &< \tfrac{16\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{9\|P\|_2^4} - \tfrac{16\|R\|_2^4(\|P\|_2^{\frac{4}{3}} - \|u_0\|_2^{\frac{4}{3}})^3}{9\|P\|_2^4} = 0, \end{aligned}$$

which implies that the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$. This completes the proof.

In order to study the sharp thresholds of blow-up and global existence for the Cauchy problem (1.1)-(1.2) for $1+\frac{4}{3}<p<3$ and $3<p<5$, we need the following preparations.

Let us define a function $g(y)$ on $[0,+\infty)$

$$g(y)=1-\frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}-2}-\frac{3\|u_0\|_2}{2\|P\|_2^2}y, \tag{4.22}$$

where P is the solution of Equation (3.31). We claim that there exists an unique positive solution y_0 for the equation $g(y)=0$. Indeed, by some computations, we have for $y>0$

$$g^{'}(y)=-\frac{3(p-1)(3p-7)\|u_0\|_2^{\frac{5-p}{2}}}{8\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}-3}-\frac{3\|u_0\|_2}{2\|P\|_2^2}<0, \tag{4.23}$$

which implies that $g(y)$ is decreasing on $[0,+\infty)$. Notice that

$$g(0)=1>0,$$

and

$$g(\frac{2\|P\|_2^2}{3\|u_0\|_2})=-\frac{3(p-1)}{4}(\frac{2}{3})^{\frac{3p-7}{2}}\frac{\|u_0\|_2^{6-2p}}{\|P\|_2^{6-2p}}<0.$$

Since $g(y)$ is continuous on $[0,+\infty)$, there exists an unique positive $y_0\in[0,\frac{2\|P\|_2^2}{3\|u_0\|_2}]$ such that $g(y_0)=0$.

- **Sharp Criteria for** $1+\frac{4}{3}<p<3$

Theorem 4.3. *Let* $1+\frac{4}{3}<p<3$, $u_0\in H^1$ *and satisfy*

$$0<H(u_0)<\frac{3p-7}{6(p-1)}y_0^2. \tag{4.24}$$

Then, we have

(i) *If*

$$\|\nabla u_0\|_2<y_0, \tag{4.25}$$

then the solution $u(t,x)$ *of the Cauchy problem (1.1)-(1.2) exists globally. Moreover, for all time* t, $u(t,x)$ *satisfies*

$$\|\nabla u(t,x)\|_2<y_0. \tag{4.26}$$

(ii) *If*

$$\|\nabla u_0\|_2>y_0, \tag{4.27}$$

and $|x|u_0\in L^2$, *then the solution* $u(t,x)$ *of the Cauchy problem (1.1)-(1.2) blows up in finite time* $T<+\infty$,

where y_0 *is the unique positive solution of the equation* $g(y)=0$ *and* $g(y)$ *is defined in (4.22).*

Proof. Applying the Gagliardo-Nirenberg inequality (Proposition 3.6), we have

$$\begin{aligned} H(u) &= \tfrac{1}{2}\int|\nabla u|^2dx - \tfrac{1}{1+p}\int|u|^{p+1}dx - \tfrac{1}{4}\int E(|u|^2)|u|^2dx \\ &\geq \tfrac{1}{2}\int|\nabla u|^2dx - \tfrac{1}{1+p}\int|u|^{p+1}dx - \tfrac{1}{4}\int|u|^4dx \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{\|u\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}\|\nabla u\|_2^{\frac{3(p-1)}{2}} - \tfrac{\|u\|_2}{2\|P\|_2^2}\|\nabla u\|_2^3. \end{aligned} \tag{4.28}$$

Now, we define a function $f(y)$ on $[0,+\infty)$ by

$$f(y) = \frac{1}{2}y^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0,+\infty)$ and

$$f^{'}(y) = y[1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}-2} - \frac{3\|u_0\|_2}{2\|P\|_2^2}y] = yg(y). \tag{4.29}$$

By the properties of $g(y)$, we have

$$f^{'}(0) = f^{'}(y_0) = y_0g(y_0) = 0 \text{ and } f^{''}(y_0) = g(y_0) + y_0g^{'}(y_0) < 0, \tag{4.30}$$

which implies that 0 and y_0 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_0)$ and decreasing on the interval $[y_0,+\infty)$.

Note that $f_{max} = f(y_0)$ and $f(0) = 0$. Since $g(y_0) = 0$, we have

$$\begin{aligned} f_{max} = f(y_0) &= \tfrac{1}{2}y_0^2 - \tfrac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{2}} - \tfrac{\|u_0\|_2}{2\|P\|_2^2}y_0^3 \\ &= [\tfrac{1}{2} - \tfrac{2}{3(p-1)}]y_0^2 + [\tfrac{1}{p-1} - \tfrac{1}{2}]\tfrac{\|u_0\|_2}{\|P\|_2^2}y_0^3 \\ &\geq \tfrac{3p-7}{6(p-1)}y_0^2. \end{aligned} \tag{4.31}$$

By the conservation of energy and the assumption (4.24), we have

$$H(u) = H(u_0) < \frac{3p-7}{6(p-1)}y_0^2 < f_{max}. \tag{4.32}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (1.1)-(1.2), as follows. We set

$$K_5 := \{u \in H^1 \mid 0 < \|\nabla u\|_2 < y_0,\ 0 < H(u) < \frac{3p-7}{6(p-1)}y_0^2\}$$

and

$$K_6 := \{u \in H^1 \mid \|\nabla u\|_2 > y_0,\ 0 < H(u) < \frac{3p-7}{6(p-1)}y_0^2\}.$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_5$, we have $0 < H(u) < \frac{3p-7}{6(p-1)}y_0^2$ and $\|\nabla u_0\|_2 < y_0$. Since $f(y)$ is continuous and increasing on $[0, y_0)$ and $f(y) < \frac{3p-7}{6(p-1)}y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 < y_0,$$

which implies that K_5 is invariant.

If $u_0 \in K_6$, we have $0 < H(u) < \frac{3p-7}{6(p-1)} y_0^2$ and $\|\nabla u_0\|_2 > y_0$. Since $f(y)$ is continuous and decreasing on $[y_0, +\infty)$ and $f(y) < \frac{3p-7}{6(p-1)} y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 > y_0, \tag{4.33}$$

which implies that K_6 is invariant.

Now, we return to the proof the Theorem 4.3. By (4.24) and (4.25), we have $u_0 \in K_5$. Applying the invariant of K_5, we have that (4.26) is true and the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally . This completes the part (i) of the proof.

By (4.24) and (4.27), we have $u_0 \in K_6$. Applying the invariant of K_6, we have (4.33) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

(4.34)
$$\begin{aligned} J''(t) &= \tfrac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx \\ &= 8 \int |\nabla u|^2 dx - \tfrac{12(p-1)}{p+1} \int |u|^{p+1} dx - \tfrac{1}{4} \int E(|u|^2)|u|^2 dx \\ &= 12(p-1)H(u_0) - [6(p-1)-8]\|\nabla u\|_2^2 + [3(p-1)-6] \int E(|u|^2)|u|^2 dx \\ &\le 2[3(p-1)-4]y_0^2 - [6(p-1)-8]y_0^2 = 0, \end{aligned}$$

for $1 + \frac{4}{3} < p < 3$, which implies that the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$. This completes the proof.

- **Sharp Criteria for** $3 < p < 5$

Theorem 4.4. *Let $3 < p < 5$, $u_0 \in H^1$ and satisfy*

$$0 < H(u_0) < \frac{1}{6} y_0^2. \tag{4.35}$$

Then, we have that

(i) *If*

$$\|\nabla u_0\|_2 < y_0, \tag{4.36}$$

then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally . Moreover, for all time t, $u(t,x)$ satisfies

$$\|\nabla u(t,x)\|_2 < y_0. \tag{4.37}$$

(ii) *If*

$$\|\nabla u_0\|_2 > y_0, \tag{4.38}$$

and $|x|u_0 \in L^2$, then the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$,

where y_0 is the unique positive solution of the equation $g(y) = 0$ and $g(y)$ is defined in (4.22).

Proof. Applying the Gagliardo-Nirenberg inequality (Proposition 3.6), we have

$$\begin{aligned} H(u) &= \tfrac{1}{2}\int |\nabla u|^2dx - \tfrac{1}{1+p}\int |u|^{p+1} - \tfrac{1}{4}\int E(|u|^2)|u|^2dx \\ &\geq \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{\|u\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}\|\nabla u\|_2^{\frac{3(p-1)}{2}} - \tfrac{\|u\|_2}{2\|P\|_2^2}\|\nabla u\|_2^3. \end{aligned} \tag{4.39}$$

Now, we define a function $f(y)$ on $[0,+\infty)$

$$f(y) = \frac{1}{2}y^2 - \frac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}} - \frac{\|u_0\|_2}{2\|P\|_2^2}y^3,$$

then we have $f(y)$ is continuous on $[0,+\infty)$ and

$$f^{'}(y) = y[1 - \frac{3(p-1)\|u_0\|_2^{\frac{5-p}{2}}}{4\|P\|_2^{p-1}}y^{\frac{3(p-1)}{2}-2} - \frac{3\|u_0\|_2}{2\|P\|_2^2}y] = yg(y), \tag{4.40}$$

where $g(y)$ is defined in (4.22). By the properties of $g(y)$, we have

$$f^{'}(0) = f^{'}(y_0) = y_0g(y_0) = 0 \;\text{ and }\; f^{''}(y_0) = g(y_0) + y_0g^{'}(y_0) < 0, \tag{4.41}$$

which implies that 0 and y_0 are two minimizers of $f(y)$, and $f(y)$ is increasing on the interval $[0, y_0)$ and decreasing on the interval $[y_0,+\infty)$.

Note that $f(0) = 0$ and $f_{max} = f(y_0)$. Since $g(y_0) = 0$, we have

$$\begin{aligned} f_{max} = f(y_0) &= \tfrac{1}{2}y_0^2 - \tfrac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{2}} - \tfrac{\|u_0\|_2}{2\|P\|_2^2}y_0^3 \\ &= [\tfrac{1}{2} - \tfrac{1}{3}]y_0^2 + [\tfrac{p-1}{4} - \tfrac{1}{2}]\tfrac{\|u_0\|_2^{\frac{5-p}{2}}}{2\|P\|_2^{p-1}}y_0^{\frac{3(p-1)}{2}} \\ &\geq \tfrac{1}{6}y_0^2. \end{aligned} \tag{4.42}$$

By the conservation of energy and the assumption (4.35), we have

$$0 < H(u) = H(u_0) < \frac{1}{6}y_0^2 < f_{max}. \tag{4.43}$$

Therefore, using the convexity and monotony of $f(y)$ and the conservation laws, we obtain two invariant evolution flows generated by the Cauchy problem (1.1)-(1.2), as follows. We set

$$K_7 := \{u \in H^1 \mid 0 < \|\nabla u\|_2 < y_0,\ 0 < H(u) < \frac{1}{6}y_0^2\}$$

and

$$K_8 := \{u \in H^1 \mid \|\nabla u\|_2 > y_0,\ 0 < H(u) < \frac{1}{6}y_0^2\}.$$

Indeed, by the conservation of mass and energy, we have $\|u\|_2 = \|u_0\|_2$ and $H(u) = H(u_0)$. If $u_0 \in K_7$, we have $\|\nabla u_0\|_2 < y_0$. Since $f(y)$ is continuous and increasing on $[0, y_0)$ and $\forall\, y \in [0,+\infty)$, $f(y) < \frac{1}{6}y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t, x)\|_2 < y_0,$$

which implies that K_7 is invariant.

If $u_0 \in K_8$, we have $\|\nabla u_0\|_2 > y_0$. Since $f(y)$ is continuous and decreasing on $[y_0, +\infty)$ and $\forall\, y \in [0, +\infty)$, $f(y) < \frac{1}{6}y_0^2 < f_{max}$, we have that for all $t \in I$(maximal existence interval)

$$\|\nabla u(t,x)\|_2 > y_0, \tag{4.44}$$

which implies that K_8 is invariant.

Now, we return to the proof the Theorem 4.4. By (4.35) and (4.36), we have $u_0 \in K_7$. Applying the invariant of K_7, we have that (4.37) is true and the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) exists globally . This completes the part (i) of the proof.

By (4.35) and (4.38), we have $u_0 \in K_8$. Applying the invariant of K_8, we have (4.44) is true. If we assume $|x|u_0 \in L^2$, then we have $|x|u(t,x) \in L^2$ by the local well-posedness. Thus, we recall the virial identity and the conservation of energy $H(u(t)) = H(u_0)$, and we have

$$\begin{aligned} J^{''}(t) &= \tfrac{d^2}{dt^2}\int |x|^2|u(t,x)|^2dx \\ &= 8\int |\nabla u|^2dx - \tfrac{12(p-1)}{p+1}\int |u|^{p+1}dx - \tfrac{1}{4}\int E(|u|^2)|u|^2dx \\ &= 24H(u_0) - 4\|\nabla u\|_2^2 + \tfrac{24-12(p-1)}{p+1}\int |u|^{p+1}dx \\ &< 4y_0^2 - 4y_0^2 = 0, \end{aligned} \tag{4.45}$$

for $3 < p < 5$, which implies that the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T < +\infty$. This completes the proof.

5. Properties of Blow-up Solutions

In this section, we shall investigate the blow-up properties of the solutions to the Cauchy problem (1.1)-(1.2). We prove the nonexistence of the L^3 strong limit to the blow-up solutions of the Cauchy problem (1.1)-(1.2) for $1 < p \leq 3$, as follows.

Theorem 5.1. *Let $1 < p \leq 3$ and the initial data $u_0 \in H^1$. If the solution of the Cauchy problem (1.1)-(1.2) $u(t,x)$ blows up in finite time $T < +\infty$, Then for any sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to T$ as $n \to \infty$, $\{u(t_n,x)\}_{n=1}^{\infty}$ does not have any strong limit in L^3 as $n \to \infty$.*

Proof. We prove this result by contradiction. Suppose that $\{u(t_n,x)\}_{n=1}^{\infty}$ has a strong limit in L^3 along a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to T$ as $n \to \infty$. Since the solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) blows up at finite time T in H^1, we have $\|\nabla u(t_n)\|_2 \to +\infty$ as $n \to +\infty$. By the conservation of energy

$$H(u) := \frac{1}{2}\int |\nabla u(t,x)|^2dx - \frac{1}{p+1}\int |u(t,x)|^{p+1}dx - \frac{1}{4}\int E(|u|^2)|u|^2dx = H(u_0),$$

for $1 < p \leq 3$, we claim $\forall n \neq m$,

$$\|\nabla u(t_n)\|_2^2 \leq C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C. \tag{5.1}$$

Indeed, if $p = 3$, by the conservation of energy, we have

$$\begin{aligned} \|\nabla u(t_n)\|_{L^2}^2 &\leq 2H(u_0) + \tfrac{1}{2}\|u(t_n)\|_4^4 + \tfrac{1}{2}\int E(|u(t_n)|^2)|u(t_n)|^2dx \\ &\leq 2H(u_0) + C\|u(t_n)\|_4^4 + C \\ &\leq C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C. \end{aligned}$$

If $1 < p < 3$, using the Gagliardo-Nirenberg inequality and Hölder inequality, we have $\forall\ \varepsilon > 0$

$$\|u(t_n)\|_{p+1}^{p+1} \leq C\|\nabla u(t_n)\|_2^{\frac{3(p-1)}{2}}\|u_0\|_2^{\frac{5-p}{2}} \leq \varepsilon\|\nabla u(t_n)\|_2^2 + C(\varepsilon).$$

By the conservation of energy, we have

$$\begin{aligned}\|\nabla u(t_n)\|_2^2 &\leq 2H(u_0) + \varepsilon\|\nabla u(t_n)\|_2^2 + C\|u(t_n)\|_4^4 + C(\varepsilon)\\ &\leq \varepsilon\|\nabla u(t_n)\|_2^2 + C\|u(t_n) - u(t_m)\|_4^4 + C\|u(t_m)\|_4^4 + C,\end{aligned}$$

for $\varepsilon < 1$, which implies that Claim (5.1) is true.

Since $3 < 4 < 6$, applying the Hölder's inequality for $\frac{1}{4} = \frac{\theta}{3} + \frac{1-\theta}{6}$, $\theta \in (0,1)$, we have

$$\begin{aligned}\|u(t_n) - u(t_m)\|_4^4 &\leq C\|u(t_n) - u(t_m)\|_3^{4\theta}\|u(t_n) - u(t_m)\|_6^{4(1-\theta)}\\ &\leq C\|u(t_n) - u(t_m)\|_3^2\|\nabla(u(t_n) - u(t_m))\|_2^2.\end{aligned} \tag{5.2}$$

It follows from (5.1) and (5.2) that for $m \neq n$ large enough

$$\|\nabla u(t_n)\|_2^2 \leq C\|u(t_n) - u(t_m)\|_3^2\|\nabla(u(t_n)\|_2^2 + C_m, \tag{5.3}$$

where C_m depends on m.

On the other hand, since the sequence $\{u(t_n)\}_{n=1}^{\infty}$ converges strongly in L^3, there is a positive integer k such that for all $n \geq k, m \geq k$

$$C\|u(t_n) - u(t_m)\|_3^2 \leq \frac{1}{2}.$$

Therefore, choosing $m = k$ in the inequality (5.3), we obtain that for all $n \geq n_k$

$$\|\nabla u(t_n)\|_2^2 \leq \frac{1}{2}\|\nabla u(t_n)\|_2^2 + C_k, \tag{5.4}$$

which implies that the sequence $\{\nabla u(t_n)\}_{n=1}^{\infty}$ is bounded in L^2. This is contradictory to that $u(t,x)$ blows up in finite time $T < +\infty$. This completes the proof.

Acknowledgement

We thank the referee for pointing out some misprints and helpful suggestions.

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