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On the quasilinear elliptic problem with a Hardy-Sobolev critical exponent

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Abstract. In this article, we consider a quasilinear elliptic equation involving Hardy-Sobolev critical exponents and superlinear nonlinearity. The right hand side nonlinearity $f(x, u)$ which is $(p-1)$-superlinear nearby 0. However, it does not satisfy the usual Ambrosetti-Rabinowitz condition (AR-condition). Instead we employ a more general condition. Using a variational approach based on the critical point theory and the Ekeland variational principle, we show the existence of two nontrivial positive solutions. Moreover, the obtained results extend some existing ones.

Contents

1. Introduction and main results

We will consider the following problem

$$\begin{cases} -\triangle_p u = \mu \frac{|u|^{p^*(s)-2}}{|x|^s} u + \lambda f(x,u), \ x \in \Omega\backslash\{0\}, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-laplacian differential operator, Ω is an open bounded domain in $R^N (N \geq 3)$ with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $0 \leq$

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$s < p,\ 1 < p < N,\ 0 < \mu < \infty,\ p^*(s) = \frac{p(N-s)}{N-p}$ is the Hardy-Sobolev critical exponent and $p^* = p^*(0) = \frac{Np}{N-p}$ is the Sobolev critical exponent, $\lambda > 0$ is a real parameter. $W_0^{1,p}(\Omega)$ is the Sobolev space with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}},$$

which is equivalent to the usual norm of $W_0^{1,p}(\Omega)$ due to the Poincaré inequality and

$$A_s(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\left(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx\right)^{\frac{p}{p^*(s)}}} \tag{1.2}$$

is the best Hardy-Sobolev constant.

In the case $s = 0$ and $p = 2$, this problem has been widely studied (see [**1, 2, 11**] and the references therein). In the case $s = 0$, Goncalves and Alves in [**8**] have studied Problem (1.1) in R^N involving $f(x,u) = h(x)u^q,\ u \geq 0$ and $u \not\equiv 0$ to obtain existence of positive solutions where $2 \leq p < N,\ 0 < q < p-1$ or $p-1 < q < p^* - 1$ and a suitable h. Ghoussoub and Yuan have studied problem (1.1)(see [**9**]), when $f(x,u) = |u|^{r-2}u,\ p \leq r \leq p^*$. For other relevant papers see [**6, 10, 7, 12**] and the references herein.

A direct extension of these methods to the case $p \neq 2$ is faced with serious difficulties. Such as, the energy functional associated to (1.1) is defined on $W_0^{1,p}(\Omega)$, which is not a Hilbert space for $p \neq 2$. Due to the lack of compactness of the embedding in $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega, |x|^{-s}dx)$, we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale ((PS) for short) condition in $W_0^{1,p}(\Omega)$. However, a local (PS) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and the Mountain Pass Lemma due to A. Ambrosetti and P.H. Rabinowitz (see also [**15**]).

$F(x,t)$ is a primitive function of $f(x,t)$ defined by $F(x,t) := \int_0^t f(x,s)ds$ for $x \in \Omega,\ t \in R$. For problem (1.1) we have the following assumptions:

$$(A_1)\ f \in C(\overline{\Omega} \times R^+, R),\ f(x,0) \equiv 0,\ \lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = +\infty \text{ and } \lim_{t\to\infty} \frac{f(x,t)}{t^{p^*(s)-1}} = 0$$

uniformly for $x \in \overline{\Omega}$.

(A_2) $f : \Omega \times R^+ \to R$ is nondecreasing with respect to the second variable.

$$(A_3)\ N > p \geq \max\left\{2,\ \frac{3N}{N+3-s},\ \frac{s-1+\sqrt{(1-s)^2+4N}}{2}\right\}.$$

In what follows, $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. Now, our main results are as follows:

Theorem 1. *Suppose that* $N \geq 3,\ 0 < \mu < \infty,\ 0 \leq s < p,\ 1 < p < N$. *Assume* (A_1) *holds, then there exists* $\lambda^* > 0$ *such that problem (1.1) has at least one nontrivial positive solution* u_λ *for every* $\lambda \in (0, \lambda^*)$.

Theorem 2. *Suppose that* $N \geq 3,\ 0 < \mu < \infty,\ \max\left\{0, \frac{p^2-N}{p-1}\right\} < s < p$. *If* $(A_1) - (A_3)$ *all hold, then there exists* $\lambda^* > 0$ *such that problem (1.1) has at least two nontrivial positive solutions for every* $\lambda \in (0, \lambda^*)$.

Remark 1. Here we give some examples of the nonlinearity satisfying (A_1) and (A_2).
(1) $f(x,t) = t^{q-1},\ t \geq 0$ with $1 < q < p$.
(2) $f(x,t) = v(x)t^r + h(x)t^\nu,\ t \geq 0$, where $v(x),\ h(x) \in L^\infty(\Omega),\ v(x),\ h(x) > 0,\ 0 \leq r < p-1$ and $p-1 < \nu < p^*(s)-1$.

This paper is organized as follows. In Section 2, we manage to give the proof of Theorem 1. The proof of Theorem 2 is given in Section 3. Throughout the article the letters C or C_i $(i = 1,\ 2,\ 3,\ ...)$ will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem.

2. Proof of Theorem 1

It is obvious that the values of $f(x,t)$ for $t < 0$ are irrelevant in Theorem 1-2, and we may define

$$f(x,t) \equiv 0 \text{ for } x \in \Omega,\ t \leq 0.$$

Let $u^\pm := \max\{\pm u, 0\}$. The functional corresponding to (1.1) is

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}dx - \lambda\int_\Omega F(x,u^+)dx,\ u \in W_0^{1,p}(\Omega).$$

By Hardy-Sobolev inequalities (see [**4, 9**]) and (A_1), $I \in C^1(W_0^{1,p}(\Omega), R)$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of problem (1.1) and the critical points of I on $W_0^{1,p}(\Omega)$. More precisely we say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1.1), if for any $v \in W_0^{1,p}(\Omega)$, there holds

$$\langle I'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx - \mu\int_\Omega \frac{(u^+)^{p^*(s)-1}}{|x|^s}v dx - \lambda\int_\Omega f(x,u^+)v dx = 0.$$

Proof of Theorem 1. Let $X := W_0^{1,p}(\Omega)$. From the Sobolev and Hardy-Sobolev inequalities, we can easily get

$$\|u\|_p^p \leq C\|u\|^p;\ \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}dx \leq C\|u\|^{p^*(s)};\ \|u\|_{p^*}^{p^*} \leq C\|u\|^{p^*},\ \forall\ u \in X. \tag{2.1}$$

It follows from (A_1) that

$$\exists\delta > 0 \text{ such that } |F(x,t)| < \frac{t^{p^*(s)}}{p^*(s)|x^s|} \text{ for } t > \delta,$$

$$\exists M_1 > 0 \text{ such that } |F(x,t)| \leq M_1 \text{ for all } t \in [0,\delta],$$

uniformly for all $x \in \overline{\Omega}\setminus\{0\}$. Therefore, we deduce that

$$|F(x,t)| \leq M_1 + \frac{t^{p^*(s)}}{p^*(s)|x^s|} \tag{2.2}$$

for all $t \in R$ and for $x \in \overline{\Omega}\setminus\{0\}$. By (2.1) and (2.2), we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \frac{\mu}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}dx - \lambda\int_\Omega F(x,u^+)dx \\ &\geq \frac{1}{p}\|u\|^p - C_1\|u\|^{p^*(s)} - \lambda M_1|\Omega| \end{aligned}$$

for all $\lambda \in (0,1]$ and some $C_1 = \frac{C\mu}{p^*(s)}$, so there exist $\rho > 0$ and $\lambda^* \in (0,1]$ such that

$$I(u) > 0 \text{ if } \|u\| = \rho, \text{ and } I(u) \geq -C_2 \text{ if } \|u\| \leq \rho$$

for every $0 < \lambda < \lambda^*$, where $C_2 = C_1\rho^{p^*(s)} + \lambda^* M_1|\Omega|$. Choose $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $u_0^+ \neq 0$. Let $M_2 := \|u_0\|^p/(\lambda\|u_0^+\|_p^p)$. From (A_1), there exists δ_1 such that

$$|F(x,t)| \geq \frac{2M_2}{p}|t|^p, \ 0 < t < \delta_1.$$

Hence we have

$$\begin{aligned} I(ru_0) &= \tfrac{r^p}{p}\|u_0\|^p - \tfrac{\mu r^{p^*(s)}}{p^*(s)}\int_\Omega \tfrac{(u_0^+)^{p^*(s)}}{|x|^s}dx - \lambda\int_\Omega F(x, ru_0^+)dx \\ &\leq \tfrac{r^p}{p}\|u_0\|^p - \tfrac{2r^p}{p}\lambda M_2\|u_0^+\|_p^p \\ &= -\tfrac{r^p}{p}\|u_0\|^p < 0 \end{aligned}$$

for every $0 < \lambda < \lambda^*$ and $0 < r < \min\{\rho, \delta_1/\|u_0^+\|_\infty\}$. Thus there exists u small enough such that $I(u) < 0$. Then we deduce that

$$\inf_{u\in \overline{B_\rho(0)}} I(u) < 0 < \inf_{u\in \partial \overline{B_\rho(0)}} I(u).$$

By applying Ekeland's variational principle (see [**13**], Theorem 4.1) in $\overline{B_\rho(0)}$, there is a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I(u_n) \leq \inf_{u\in \overline{B_\rho(0)}} I(u) + \frac{1}{n}, \quad I(\omega) \geq I(u_n) - \frac{1}{n}\|\omega - u_n\|, \ \omega \in \overline{B_\rho(0)}.$$

Therefore, we have

$$\|I'(u_n)\| \to 0 \text{ and } I(u_n) \to c_\lambda \text{ as } n \to \infty,$$

where c_λ stands for the infimum of $I(u)$ on $\overline{B_\rho(0)}$. Since $\{u_n\}$ is bounded and $\overline{B_\rho(0)}$ is a closed convex set, there exist $u_\lambda \in \overline{B_\rho(0)} \subset W_0^{1,p}(\Omega)$. Going if necessary to a subsequence, one can get that(see [**9**])

$$\begin{cases} u_n \rightharpoonup u_\lambda \text{ weakly in } W_0^{1,p}(\Omega), \\ u_n \to u_\lambda \text{ strongly in } L^\gamma(\Omega), \ 1 < \gamma < p^*, \\ u_n \to u_\lambda \text{ a.e. in } \Omega, \\ \nabla u_n \to \nabla u_\lambda \text{ a.e. in } \Omega, \\ \frac{u_n}{x} \rightharpoonup \frac{u_\lambda}{x} \text{ weakly in } L^p(\Omega), \\ \int_\Omega \frac{|u_n|^{p^*(s)-2}u_n}{|x|^s}vdx \to \int_\Omega \frac{|u_\lambda|^{p^*(s)-2}u_\lambda}{|x|^s}vdx, \ \forall\, v \in W_0^{1,p}(\Omega). \end{cases}$$

Consequently, passing to the limit in $\langle I'(u_n), v\rangle$, as $n \to \infty$, we have

$$\int_\Omega |\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla v dx - \mu\int_\Omega \frac{(u_\lambda^+)^{p^*(s)-1}v}{|x|^s}dx - \lambda\int_\Omega f(x, u_\lambda^+)vdx = 0$$

for all $v \in W_0^{1,p}(\Omega)$. That is, $\langle I'(u_\lambda), v\rangle = 0$. Thus u_λ is a critical point of the functional I . Since $\|u_\lambda^-\|^p = -\langle I'(u_\lambda), u_\lambda^-\rangle = 0$, thus $u_\lambda = u_\lambda^+ \geq 0$. Moreover, we deduce from (A_1) and the boundedness of Ω that

$$\exists\, M_3 > 0 \text{ such that } |f(x,t)| < \frac{\mu}{\lambda}\frac{t^{p^*(s)-1}}{|x|^s} \text{ for } t > M_3,$$

$$\exists\, \delta_2 \in (0, M_3) \text{ such that } |f(x,t)| > 0 \text{ for } 0 < t < \delta_2,$$

$$\exists\ M_4 > 0 \text{ such that } |f(x,t)| \leq M_4 \text{ for all } t \in [\delta_2, M_3],$$

for all $x \in \overline{\Omega} \setminus \{0\}$. Therefore, we deduce that

$$f(x,t) \geq -\frac{\mu}{\lambda}\frac{t^{p^*(s)-1}}{|x|^s} - M_4 t \delta_2^{-1} \tag{2.3}$$

for all $t \in R^+$ and for $x \in \overline{\Omega} \setminus \{0\}$. From (1.1) and (2.3), we have

$$-\triangle_p u_\lambda + \lambda M_4 \delta_2^{-1} u_\lambda \geq 0.$$

From the strong maximum principle, we deduce that $u_\lambda > 0$. So Theorem 1 is proved. □

3. Proof of Theorem 2

The first positive solution u_λ of problem (1.1) have been obtained in previous section, we can look for the second positive solution by a translated functional as in [**1**]. For fixed $\lambda \in (0, \lambda^*)$, we look for the second solution of problem (1.1) of the form $u = u_\lambda + v$, where u_λ is the first positive solution obtained in previous section. The corresponding equation for v is

$$\begin{cases} -\triangle_p v = \mu\frac{(u_\lambda+v)^{p^*(s)-1}}{|x|^s} - \mu\frac{(u_\lambda)^{p^*(s)-1}}{|x|^s} + \lambda f(x, u_\lambda + v) - \lambda f(x, u_\lambda),\ x \in \Omega \setminus \{0\}, \\ v = 0,\ x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let us define

$$g(x,t) = \begin{cases} \mu\frac{(u_\lambda+t)^{p^*(s)-1}}{|x|^s} - \mu\frac{(u_\lambda)^{p^*(s)-1}}{|x|^s} + \lambda f(x, u_\lambda + t) - \lambda f(x, u_\lambda),\ t \geq 0, \\ 0,\ t < 0, \end{cases} \tag{3.2}$$

$$G(x,t) = \int_0^t g(x,s)ds$$

and

$$\begin{aligned} J(v) &= \tfrac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega G(x, v^+)dx \\ &= \tfrac{1}{p}\|v\|^p - \tfrac{\mu}{p^*(s)}\int_\Omega \left(\tfrac{(u_\lambda+v^+)^{p^*(s)}}{|x|^s} - \tfrac{u_\lambda^{p^*(s)}}{|x|^s} - p^*(s)\tfrac{u_\lambda^{p^*(s)-1}v^+}{|x|^s} \right) dx \\ &\quad -\lambda \int_\Omega \left[F(x, u_\lambda + v^+) - F(x, u_\lambda) - f(x, u_\lambda)v^+ \right] dx. \end{aligned}$$

Now we have one-to-one correspondence between critical points of J in $W_0^{1,p}(\Omega)$ and solutions of problem (3.1). That is, if $v \in W_0^{1,p}(\Omega),\ v \not\equiv 0$ is a critical point of J, then v is a solution of (3.1). Since $\|v^-\|^p = -\langle J'(v), v^-\rangle = 0$, thus $v = v^+ \geq 0$. Moreover, by the Maximum Principle, $v > 0$ in Ω. Here $u = u_\lambda + v$ is a positive solution of (1.1) and $u \neq u_\lambda$. We will prove the existence of a second positive solution of (1.1) by contradiction. Assume that $v = 0$ is the only critical point of J in $W_0^{1,p}(\Omega)$.

Lemma 1. $v = 0$ *is a local minimum of* J *in* $W_0^{1,p}(\Omega)$.

Proof. For any $v \in W_0^{1,p}(\Omega)$, write $v = v^+ - v^-$. From the expression of J and direct computation, we obtain that

$$J(v) = \frac{1}{p}\|v^-\|^p + I(u_\lambda + v^+) - I(u_\lambda). \tag{3.3}$$

Since u_λ is a local minimizer of I in $W_0^{1,p}(\Omega)$, we have

$$J(v) \geq \frac{1}{p}\|v^-\|^p$$

as long as $\|v\| \leq \varepsilon$ for ε small enough. □

Lemma 2. ([**9**]) *Suppose $1 < p < N,\ 0 \leq s < p$. Then we have the following:*
(i) $A_s(\Omega)$ is independent of Ω (and will henceforth be denoted by A_s).
(ii) A_s is attained when $\Omega = \mathrm{R}^N$ by the functions

$$l_\varepsilon(x) = \left[\varepsilon(N-s)\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{N-p}{p(p-s)}} \left(\varepsilon + |x|^{\frac{p-s}{p-1}}\right)^{\frac{p-N}{p-s}}$$

for some $\varepsilon > 0$. Moreover the functions $l_\varepsilon(x)$ are the only positive radial solutions of

$$-\triangle_p u = \frac{u^{p^*(s)-1}}{|x|^s}$$

in R^N, and satisfy

$$\int_{R^N} |\nabla l_\varepsilon|^p dx = \int_{R^N} \frac{|l_\varepsilon|^{p^*(s)}}{|x|^s} dx = A_s^{\frac{N-s}{p-s}}.$$

Lemma 3. *Suppose $0 < \mu < \infty$, f satisfies $(A_1)-(A_3)$. Assume that v=0 is the only critical point of J. Let $\{v_n\}$ be a $(PS)_c$ sequence with $0 < c < \frac{p-s}{p(N-s)} A_s^{\frac{N-s}{p-s}} \mu^{\frac{p-N}{p-s}}$. Then we have*

$$v_n \to 0 \textit{ in } W_0^{1,p}(\Omega) \textit{ as } n \to \infty.$$

Proof. Let v_n be a sequence in $W_0^{1,p}(\Omega)$ such that

$$J(v_n) \to c < \frac{p-s}{p(N-s)} A_s^{\frac{N-s}{p-s}} \mu^{\frac{p-N}{p-s}} \text{ and } J'(v_n) \to 0 \text{ in } \left(W_0^{1,p}(\Omega)\right)^*. \tag{3.4}$$

Then from (3.3) and (3.4), we have

$$J(v_n) = \frac{1}{p}\|v_n^-\|^p + I(u_\lambda + v_n^+) - I(u_\lambda) = c + o(1), \tag{3.5}$$

$$\langle J'(v_n), u_\lambda + v_n^+\rangle = \int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^- \nabla u_\lambda dx + \langle I'(u_\lambda + v_n^+), u_\lambda + v_n^+\rangle = o(1)\|u_\lambda + v_n^+\|.$$

It yields that

$$\begin{aligned} &J(v_n) - \tfrac{1}{p}\langle J'(v_n), u_\lambda + v_n^+\rangle \\ &= \tfrac{1}{p}\left(\|v_n^-\|^p - \textstyle\int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^- \nabla u_\lambda dx - \langle I'(u_\lambda + v_n^+), u_\lambda + v_n^+\rangle\right) \\ &+ I(u_\lambda + v_n^+) - I(u_\lambda) \\ &\leq c + 1 + o(1)\|u_\lambda + v_n^+\|. \end{aligned}$$

Therefore, one gets

$$\begin{aligned} &\tfrac{1}{p}\left(\|v_n^-\|^p - \textstyle\int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^- \nabla u_\lambda dx\right) + \mu\left(\tfrac{1}{p} - \tfrac{1}{p^*(s)}\right)\textstyle\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx \\ &+ \lambda \textstyle\int_\Omega \left[\tfrac{1}{p} f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+) - F(x, u_\lambda + v_n^+)\right] dx \\ &\leq I(u_\lambda) + c + 1 + o(1)\|u_\lambda + v_n^+\|. \end{aligned} \tag{3.6}$$

By (A_1) and the boundedness of Ω, for any $\varepsilon > 0$, there exists $M_5 = M_5(\varepsilon) > 0$ such that

$$|f(x,t)t| \le \varepsilon \frac{|t|^{p^*(s)}}{|x|^s},\ x \in \Omega\backslash\{0\},\ |t| > M_5, \quad |f(x,t)t| \le C_3(\varepsilon),\ x \in \Omega,\ |t| \in [0, M_5],$$

$$|F(x,t)| \le \frac{\varepsilon}{p} \frac{|t|^{p^*(s)}}{|x|^s},\ x \in \Omega\backslash\{0\},\ |t| > M_5, \quad |F(x,t)| \le C_4(\varepsilon),\ x \in \Omega,\ |t| \in [0, M_5],$$

where $C_3(\varepsilon),\ C_4(\varepsilon) > 0$. Therefore, we have

$$|f(x,t)t| \le C_3(\varepsilon) + \varepsilon \frac{|t|^{p^*(s)}}{|x|^s},\ (x,t) \in (\Omega \setminus \{0\}) \times R, \tag{3.7}$$

$$|F(x,t)| \le C_4(\varepsilon) + \frac{\varepsilon}{p} \frac{|t|^{p^*(s)}}{|x|^s},\ (x,t) \in (\Omega \setminus \{0\}) \times R. \tag{3.8}$$

Let $C(\varepsilon) := \frac{1}{p} C_3(\varepsilon) + C_4(\varepsilon)$, combining (3.7) and (3.8), one gets

$$F(x,t) - \frac{1}{p} f(x,t)t \le C(\varepsilon) + \frac{2\varepsilon}{p} \frac{|t|^{p^*(s)}}{|x|^s},\ (x,t) \in (\Omega \setminus \{0\}) \times R. \tag{3.9}$$

From (3.6) and (3.9), we deduce that

$$\begin{aligned} &\left(\frac{\mu(p-s)}{p(N-s)} - \frac{2\lambda\varepsilon}{p} \right) \int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx \\ &\le \lambda C(\varepsilon)|\Omega| - \frac{1}{p} \|v_n^-\|^p + C_5 \|v_n^-\|^{p-1} + C_6 + o(1)\|u_\lambda + v_n^+\|, \end{aligned}$$

where $C_5 = \frac{1}{p}\|u_\lambda\|,\ C_6 = I(u_\lambda) + c + 1$. Let $\varepsilon = \frac{\mu(p-s)}{4(N-s)\lambda}$, we have

$$\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx \le C_7 \|v_n^-\|^{p-1} + C_8 + o(1)\|u_\lambda + v_n^+\|,$$

where $C_7 = \frac{2p(N-s)}{\mu(p-s)} C_5,\ C_8 = \frac{2p(N-s)}{\mu(p-s)} (\lambda C(\varepsilon)|\Omega| + C_6)$, which together with (3.3), (3.5) and (3.8) imply that

$$\begin{aligned} &\tfrac{1-\varepsilon}{p} \|v_n^-\|^p + \tfrac{1}{p} \left[(1-\varepsilon)\|v_n^+\|^p - \overline{C_\varepsilon}\|u_\lambda\|^p - (1-\varepsilon)\|v_n^+\|^{p-1} \right] \\ &\le \tfrac{1}{p} \|v_n^-\|^p + \tfrac{1}{p} \left[(1-\varepsilon)\|v_n^+\|^p - \overline{C_\varepsilon}\|u_\lambda\|^p \right] \\ &\le \tfrac{1}{p} \|v_n^-\|^p + \tfrac{1}{p} \left|(\|v_n^+\| - \|u_\lambda\|)\right|^p \\ &\le \tfrac{1}{p} \|v_n^-\|^p + \tfrac{1}{p} \|u_\lambda + v_n^+\|^p \\ &= \tfrac{\mu}{p^*(s)} \int_\Omega \tfrac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s} dx + \lambda \int_\Omega F(x, u_\lambda + v_n^+) dx + J(v_n) + I(u_\lambda) + o(1) \\ &\le C_9 \|v_n^-\|^{p-1} + C_{10} + o(1)\|u_\lambda + v_n^+\|, \end{aligned}$$

where in the second step we used the fact that, the elementary inequality $|a-b|^t \ge (1-\varepsilon)a^t - \overline{C_\varepsilon} b^t$ $(t \ge 1,\ a, b > 0)$ holds. $C_9 = \left(\frac{\mu}{p^*(s)} + \frac{\lambda\varepsilon}{p} \right) C_7,\ C_{10} = \lambda C_4(\varepsilon)|\Omega| + \left(\frac{\mu}{p^*(s)} + \frac{\lambda\varepsilon}{p} \right) C_8 + I(u_\lambda) + c + o(1)$. Since $\|v_n^-\|^{p-1} + \|v_n^+\|^{p-1} = \|v_n\|^{p-1}$, then we deduce

$$\|v_n\|^p - C_{11}\|v_n^+\|^{p-1} - C'_{11}\|v_n^-\|^{p-1} \le C_{12} + o(1)\|u_\lambda\|,$$

where $C_{11} = 1 + o(1)\frac{p}{1-\varepsilon}$, $C'_{11} = \frac{C_9 p}{1-\varepsilon}$, $C_{12} = \frac{\overline{C_\varepsilon}\|u_\lambda\|^p + pC_{10}}{1-\varepsilon}$. So we get

$$\|v_n\|^p - C_{13}\|v_n\|^{p-1} \le C_{12} + o(1)\|u_\lambda\|,$$

where $C_{13} = C_{11} + C'_{11}$. It shows that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, one gets that

$$\begin{cases} v_n \rightharpoonup v_0 \text{ weakly in } W_0^{1,p}(\Omega), \\ v_n \to v_0 \text{ strongly in } L^\gamma(\Omega),\ 1 < \gamma < p^*, \\ v_n \to v_0 \text{ a.e. in } \Omega, \end{cases} \tag{3.10}$$

as $n \to \infty$.

In addition, by the Sobolev embedding theorem, there exists $M' > 0$ such that $\|u_\lambda + v_n^+\|_{p^*(s)}^{p^*(s)} \le M'$, denote by meas$E$ the measure of E. By (A_1), for any $\varepsilon > 0$, there exists $C_{14}(\varepsilon) > 0$ such that

$$|f(x,t)t| \le C_{14}(\varepsilon) + \frac{\varepsilon}{2M'}|t|^{p^*(s)},\ (x,t) \in \overline{\Omega} \times R.$$

Set $\delta := \frac{\varepsilon}{2C_{14}(\varepsilon)} > 0$, when $E \subset \Omega$, meas$E < \delta$, we have

$$\begin{aligned} \left|\int_E f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)dx\right| &\le \int_E |f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)|\, dx \\ &\le \int_E C_{14}(\varepsilon)dx + \frac{\varepsilon}{2M'}\int_E |u_\lambda + v_n^+|^{p^*(s)}dx \\ &\le C_{14}(\varepsilon)\text{meas } E + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

By Vitali's theorem, we prove that

$$\int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)dx \to \int_\Omega f(x, u_\lambda + v_0^+)(u_\lambda + v_0^+)dx \text{ as } n \to \infty.$$

Hence one has

$$\begin{aligned} &\int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n)dx \\ &= \int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)dx - \int_\Omega f(x, u_\lambda)(v_n^-)dx \\ &\to \int_\Omega f(x, u_\lambda + v_0^+)(u_\lambda + v_0)dx \text{ as } n \to \infty. \end{aligned} \tag{3.11}$$

Using the same method, we deduce that

$$\int_\Omega F(x, u_\lambda + v_n^+)dx \to \int_\Omega F(x, u_\lambda + v_0^+)dx, \tag{3.12}$$

$$\int_\Omega f(x, u_\lambda + v_n^+)\omega dx \to \int_\Omega f(x, u_\lambda + v_0^+)\omega dx,$$

as $n \to \infty$ for $\omega \in W_0^{1,p}(\Omega)$. Hence, similar to the proof of Theorem 1, we have

$$0 = \lim_{n\to\infty}\langle J'(v_n), \omega\rangle = \langle J'(v_0), \omega\rangle$$

for $\omega \in W_0^{1,p}(\Omega)$, which implies that $J'(v_0) = 0$. Therefore, v_0 is a critical point of J in $W_0^{1,p}(\Omega)$. From the assumption that $v = 0$ is the only critical point of J, we know that v_0=0. Now we want to prove $v_0 \to 0$ strongly in $W_0^{1,p}(\Omega)$. From (3.10), (3.12) and the Brezis-Leib Lemma (see [**3**]), we have

$$J(v_n) = \frac{1}{p}\|v_n^-\|^p + I(u_\lambda + v_n^+) - I(u_\lambda) = \frac{1}{p}\|v_n\|^p - \frac{\mu}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}dx + o(1).$$

Therefore, we get

$$\langle J'(v_n), v_n\rangle = \|v_n\|^p - \mu\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}dx + o(1) \to 0,$$

then $\|v_n\|^p \to 0$ as $n\to\infty$. Otherwise, there exists a subsequence (still denoted by v_n) such that

$$\lim_{n\to\infty}\|v_n\|^p = k, \quad \lim_{n\to\infty}\mu\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}dx = k,\ k>0.$$

By (1.2), we deduce that

$$\|v_n\|^p \geq A_s\left(\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}dx\right)^{\frac{p}{p^*(s)}}, \quad \text{for all } n\in N.$$

Then, $k \geq A_s\left(\frac{k}{\mu}\right)^{\frac{p}{p^*(s)}}$, that is, $k \geq A_s^{\frac{N-s}{p-s}}\mu^{\frac{p-N}{p-s}}$. Thus we get that

$$\begin{aligned} c = o(1) + J(v_n) &= \tfrac{1}{p}\|v_n\|^p - \tfrac{\mu}{p^*(s)}\int_\Omega \tfrac{(v_n^+)^{p^*(s)}}{|x|^s}dx + o(1) \\ &= \tfrac{p-s}{p(N-s)}k + o(1) \\ &\geq \tfrac{p-s}{p(N-s)}A_s^{\frac{N-s}{p-s}}\mu^{\frac{p-N}{p-s}}. \end{aligned}$$

This is a contradiction. So $v_n \to 0$ strongly in $W_0^{1,p}(\Omega)$ as $n\to\infty$. □

Since $u_\lambda > 0$ is a solution of problem (1.1), in a way similar to the proof of Theorem 1.1 in [**5**], we obtain positive constants R and r_0 such that $B_{2R}(0)\subset\Omega$ and

$$0 < r_0 \leq u_\lambda(x),\ \forall\ x\in B_{2R}(0)\backslash\{0\}. \tag{3.13}$$

In the following, we shall give some estimates for the extremal functions. Let

$$C_\varepsilon := \left[\varepsilon(N-s)\left(\frac{N-p}{p-1}\right)^{p-1}\right]^{\frac{N-p}{p(p-s)}},\ U_\varepsilon(x) := \frac{l_\varepsilon(x)}{C_\varepsilon}.$$

Define a function $\varphi\in C_0^\infty(\Omega),\ 0\leq\varphi(x)\leq 1$ such that

$$\varphi(x) = \begin{cases} 1, & |x|\leq R,\\ 0, & |x|\geq 2R,\end{cases}$$

where $B_{2R}(0)\subset\Omega$. Set

$$u_\varepsilon(x) := \varphi(x)U_\varepsilon(x), \quad v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\left(\int_\Omega \frac{|u_\varepsilon|^{p^*(s)}}{|x|^s}dx\right)^{\frac{1}{p^*(s)}}},$$

so that $\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}dx = 1$. Then, by using the argument as [**9**], we can get the following results:

$$A_s + C_{15}\varepsilon^{\frac{N-p}{p-s}} \leq \|v_\varepsilon\|^p \leq A_s + C_{16}\varepsilon^{\frac{N-p}{p-s}} \tag{3.14}$$

and

$$C_{17}\left(\varepsilon^{\frac{(p-s)(p-1)}{p}}\right) \leq \int_\Omega \frac{|v_\varepsilon|^p}{|x|^s}dx \leq C_{18}\left(\varepsilon^{\frac{(p-s)(p-1)}{p}}\right),\ p > \frac{N-s}{N-p}(p-1). \tag{3.15}$$

Lemma 4. *Suppose that* $N \geq 3,\ 0 < \mu < \infty$ *and* $\max\left\{0, \frac{p^2-N}{p-1}\right\} < s < p$. *Assume* $(A_1) - (A_3)$ *and* $f(x,0) \equiv 0$ *hold. Then there exists* $v_* \in W_0^{1,p}(\Omega),\ v_* \not\equiv 0$, *such that*

$$\sup_{t \geq 0} J(tv_*) < \frac{p-s}{p(N-s)} A_s^{\frac{N-s}{p-s}} \mu^{\frac{p-N}{p-s}}.$$

Proof. By (3.2), (A_2) and an elementary inequality (the proof of this inequality is elementary but not straightforward. We postpone it to Appendix A.)

$(a+b)^\gamma \geq a^\gamma + b^\gamma + Ca^{\gamma-t}b^t,\ \gamma \geq 2,\ 1 \leq t \leq \gamma-1,\ a,b>0$, where C is a positive constant,

we have

$$g(x,l) \geq \mu \frac{l^{p^*(s)-1}}{|x|^s} + C\mu \frac{l^{p-1}u_\lambda^{p^*(s)-p}}{|x|^s},$$

where (A_3) implies $p^*(s) - 1 \geq 2,\ 1 \leq p-1 \leq (p^*(s)-1)-1$. Therefore, we have

$$G(x, tv_\varepsilon) \geq \mu \frac{t^{p^*(s)}}{p^*(s)} \frac{v_\varepsilon^{p^*(s)}}{|x|^s} + \frac{C\mu t^p}{p} \frac{v_\varepsilon^p u_\lambda^{p^*(s)-p}}{|x|^s}.$$

Note that $s > \frac{p^2-N}{p-1}$ implies $p > \frac{N-s}{N-p}(p-1)$, then (3.15) holds. Therefore, from (3.13)-(3.15), we deduce that

$$\begin{aligned} J(tv_\varepsilon) &= \tfrac{t^p}{p}\|v_\varepsilon\|^p - \textstyle\int_\Omega G(x,tv_\varepsilon)dx \\ &\leq \tfrac{t^p}{p}\|v_\varepsilon\|^p - \mu\tfrac{t^{p^*(s)}}{p^*(s)} - C_{19}t^p \textstyle\int_\Omega \frac{v_\varepsilon^p}{|x|^s}dx \\ &\leq \tfrac{t^p}{p}\|v_\varepsilon\|^p - \mu\tfrac{t^{p^*(s)}}{p^*(s)} - C_{20}t^p\varepsilon^{\frac{(p-s)(p-1)}{p}} \\ &\leq \tfrac{A_s}{p}t^p + C_{21}t^p\varepsilon^{\frac{N-p}{p-s}} - \mu\tfrac{t^{p^*(s)}}{p^*(s)} - C_{20}t^p\varepsilon^{\frac{(p-s)(p-1)}{p}}, \end{aligned}$$

where $C_{19} = \frac{C\mu r_0^{p^*(s)-p}}{p},\ C_{20} = C_{17}C_{19}$ and $C_{21} = \frac{C_{16}}{p}$. Let

$$Q(t) := \frac{A_s}{p}t^p + C_{21}t^p\varepsilon^{\frac{N-p}{p-s}} - \mu\frac{t^{p^*(s)}}{p^*(s)} - C_{20}t^p\varepsilon^{\frac{(p-s)(p-1)}{p}}.$$

It is clear that the equation

$$0 = Q'(t) = A_s t^{p-1} + pC_{21}t^{p-1}\varepsilon^{\frac{N-p}{p-s}} - \mu t^{p^*(s)-1} - pC_{20}t^{p-1}\varepsilon^{\frac{(p-s)(p-1)}{p}}$$

has only one positive root

$$t_\varepsilon := \left(\frac{A_s + pC_{21}\varepsilon^{\frac{N-p}{p-s}} - pC_{20}\varepsilon^{\frac{(p-s)(p-1)}{p}}}{\mu}\right)^{\frac{1}{p^*(s)-p}}.$$

We have

$$\begin{aligned} Q(t_\varepsilon) &= \tfrac{1}{p}\left(A_s + pC_{21}\varepsilon^{\frac{N-p}{p-s}} - pC_{20}\varepsilon^{\frac{(p-s)(p-1)}{p}}\right)t_\varepsilon^p - \mu\frac{t_\varepsilon^{p^*(s)}}{p^*(s)} \\ &= \mu\left(\tfrac{1}{p} - \tfrac{1}{p^*(s)}\right)\left(\frac{A_s + pC_{21}\varepsilon^{\frac{N-p}{p-s}} - pC_{20}\varepsilon^{\frac{(p-s)(p-1)}{p}}}{\mu}\right)^{\frac{p^*(s)}{p^*(s)-p}} \\ &= \tfrac{p-s}{p(N-s)}\mu^{\frac{p-N}{p-s}}\left(A_s + pC_{21}\varepsilon^{\frac{N-p}{p-s}} - pC_{20}\varepsilon^{\frac{(p-s)(p-1)}{p}}\right)^{\frac{N-s}{p-s}} \\ &< \tfrac{p-s}{p(N-s)}\mu^{\frac{p-N}{p-s}}A_s^{\frac{N-s}{p-s}}, \end{aligned}$$

for $\varepsilon > 0$ sufficiently small due to the fact that

$$\frac{N-p}{p-s} > \frac{(p-s)(p-1)}{p}, \text{ for } s > \frac{p^2-N}{p-1}.$$

Noting that $Q(0) = 0$ and $\lim_{t\to+\infty} Q(t) = -\infty$, we have

$$\sup_{t\geq 0} Q(t) = Q(t_\varepsilon) < \frac{p-s}{p(N-s)}\mu^{\frac{p-N}{p-s}}A_s^{\frac{N-s}{p-s}},$$

for $\varepsilon > 0$ sufficiently small. Hence we obtain

$$\sup_{t\geq 0} J(tv_\varepsilon) \leq \sup_{t\geq 0} Q(t) < \frac{p-s}{p(N-s)}\mu^{\frac{p-N}{p-s}}A_s^{\frac{N-s}{p-s}},$$

for $\varepsilon > 0$ sufficiently small, which complete the proof by letting $v_* = v_\varepsilon$ for $\varepsilon > 0$ sufficiently small. □

Proof of Theorem 2. By contradiction. Assume that $v = 0$ is the only critical point of J in $W_0^{1,p}(\Omega)$. From Lemma 1, there exists $\alpha > 0$ such that $J(v) > \alpha$ for all $v \in \partial B_\rho = \{v \in W_0^{1,p}(\Omega),\ \|v\| = \rho\}$, where $\rho > 0$ small enough. By Lemma 4 there exists $v_* \in W_0^{1,p}(\Omega),\ v_* \not\equiv 0$, such that

$$\sup_{t\geq 0} J(tv_*) < \frac{p-s}{p(N-s)}A_s^{\frac{N-s}{p-s}}\mu^{\frac{p-N}{p-s}}.$$

From (3.8), we easily note that $\lim_{t\to\infty} J(tv_*) \to -\infty$. Hence we can choose $t_0 > 0$ such that $\|t_0 v_*\| > \rho$ and $J(t_0 v_*) < 0$. Applying the Mountain Pass Lemma (see [**15**] or [**14**]), there is a sequence $\{v_n\} \subset W_0^{1,p}(\Omega)$ satisfying

$$J(v_n) \to c \geq \alpha \text{ and } J'(v_n) \to 0,$$

where

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} J(h(t))$$

and

$$\Gamma = \{h \in C([0,1], X) |\ h(0) = 0,\ h(1) = t_0 v_*\}.$$

Note that

$$\begin{aligned} &0 < \alpha \leq c = \inf_{h\in\Gamma}\max_{t\in[0,1]} J(h(t)) \\ &\leq \max_{t\in[0,1]} J(tt_0 v_*) \leq \sup_{t\geq 0} J(tv_*) < \frac{p-s}{p(N-s)}A_s^{\frac{N-s}{p-s}}\mu^{\frac{p-N}{p-s}}. \end{aligned}$$

Together with Lemma 3, we know that $v_n \to 0$ strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Hence one has $0 = J(0) = \lim_{n\to\infty} J(v_n) = c \geq \alpha > 0$, this is a contradiction. So Theorem 2 holds. □

Appendix A

Here we give the proof of the elementary inequality in Lemma 4, that is,

$$(a+b)^\gamma \geq a^\gamma + b^\gamma + Ca^{\gamma-t}b^t,\ \gamma \geq 2,\ 1 \leq t \leq \gamma-1,\ a,b>0,$$

where C is a positive constant.

Proof. Indeed, by scaling it suffices to show that

$$(1+x)^\gamma \geq 1 + x^\gamma + Cx^t,\ 0 < x < \infty.$$

Let $\gamma = k+\theta,\ t = m+\eta$, where $k \geq 2,\ 1 \leq m \leq k-1$ are integral numbers and $0 \leq \eta \leq \theta < 1$ are real numbers. It is obvious that

$$\begin{aligned}(1+x)^\gamma = (1+x)^{k+\theta} &= (1+x)^k(1+x)^\theta \\ &\geq (1+x^k+Cx^m)(1+x)^\theta \\ &\geq 1 + x^{k+\theta} + Cx^m(1+x)^\theta \\ &\geq 1 + x^{k+\theta} + Cx^m x^\eta = 1 + x^\gamma + Cx^t.\end{aligned}$$

Therefore, this inequality holds. □

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