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On the focusing mass critical problem in six dimensions with splitting spherically symmetric initial data

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Abstract. In this paper, we consider the six-dimensional focusing mass critical NLS: $iu_t+\Delta u = -|u|^{\frac{2}{3}}u$ with splitting-spherical initial data $u_0(x_1,\cdots x_6) = u_0(\sqrt{x_1^2+x_2^2+x_3^2}, \sqrt{x_4^2+x_5^2+x_6^2})$. We prove that any finite mass solution which is almost periodic modulo scaling in both time directions must have Sobolev regularity H_x^{1+}. Moreover, the kinetic energy of the solution is localized around the spatial origin uniformly in time. As important applications of the results, we prove the scattering conjecture for solutions with mass smaller than that of the ground state. We also prove that any two-way non-scattering solution must be global and coincides with the solitary wave up to symmetries. Here the ground state is the unique positive, radial solution of the nonlinear elliptic equation $\Delta Q - Q + Q^{\frac{5}{3}} = 0$. To prove the smoothness of the solution, we use a new local iteration scheme which first appears in [**19**].

Contents

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1. Introduction

1.1. Background and main results. The d-dimensional focusing mass-critical nonlinear Schrödinger equation takes the form

$$iu_t + \Delta u = -|u|^{\frac{4}{d}} u. \tag{1.1}$$

Here $u(t,x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$. The name "mass critical" refers to the fact that the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{d}{2}} u(\lambda^2 t, \lambda x), \ \forall \lambda > 0 \tag{1.2}$$

leaves both the equation and the mass invariant. Here the mass is defined as

$$\text{Mass: } M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0).$$

For the initial value problem of (1.1), the local theory was established by Cazenave and Weissler in [**3**]. To summarize, for any initial data $u_0 \in L_x^2(\mathbb{R}^d)$, they constructed the unique local solution $u(t,x) \in C_t([-T,T]; L_x^2) \cap L_{t,x}^{\frac{2(d+2)}{d}}([-T,T] \times \mathbb{R}^d)$. Moreover, when the mass of the initial data is small enough, the solution is global and satisfies the global spacetime estimate

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leq C(\|u_0\|_{L_x^2}).$$

This estimate implies that the solution scatters in both time directions asymptotically: more precisely, there exist $u_\pm \in L_x^2(\mathbb{R}^d)$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it\Delta}u_+\|_{L_x^2} = 0 \quad \text{and} \quad \lim_{t\to-\infty} \|u(t) - e^{it\Delta}u_-\|_{L_x^2} = 0.$$

When the solution has large mass, blowup may occur at finite time. The existence of finite blowup solutions was proved by Glassey [**8**], based on a virial argument. On the other hand, the equation (1.1) also admits solitary wave solutions of the form $e^{it}R(x)$, where $R = R(x)$ solves the elliptic equation

$$\Delta R - R + |R|^{\frac{4}{d}} R = 0. \tag{1.3}$$

There are infinitely many solutions to this equation, but only one positive solution which is spherically symmetric (up to translation) and has minimal mass among all these $R's$. This solution is usually called the

Definition 1.1 (Ground state). The ground state Q refers to the unique, radial, Schwartz solution to the equation (1.3).

It is believed that the mass of Q serves as the minimal mass among all the non-scattering solutions. The precise statement of this general belief is the following scattering conjecture:

Conjecture 1.2 (Scattering conjecture). *Let $u_0 \in L_x^2(\mathbb{R}^d)$ be such that $M(u_0) < M(Q)$. Then the corresponding solution exists globally and scatters.*

So far, this conjecture has been proved in dimensions $d \geq 2$ when the initial data u_0 is spherically symmetric, see [**15, 16**]. In the proof of all these results, the spherical symmetry is used in an essential way. First of all, the spherical symmetry forces to freeze the center of mass at the origin in both physical and frequency spaces. Secondly, a few crucial technical tools like weighted Strichartz estimates are no longer available in the nonradial setting.

At the level of minimal mass, there are two explicit examples of non-scattering solutions: [1] the solitary wave SW and the pseudo-conformal ground state $Pc(Q)$.

$$SW = e^{it}Q(x),$$
$$Pc(Q) = |t|^{-\frac{d}{2}} e^{\frac{i|x|^2-4}{4t}} Q(\frac{x}{t}).$$

It is conjectured that these are the only two threshold solutions for scattering at the level of minimal mass. Associated with this is the following rigidity conjecture which identifies all solutions with ground state mass as either SW or $Pc(Q)$ if they do not scatter. Since both mass and the equation are invariant under a couple of symmetries, the coincidence of the solutions with the examples only hold modulo these symmetries. Specifically, the symmetries are: translation, phase rotation, scaling and the Galilean boost.

CONJECTURE 1.3 (Rigidity conjecture at the ground state mass). *Let $u_0 \in L_x^2(\mathbb{R}^d)$ satisfy $M(u_0) = M(Q)$. Then only the following cases can occur*

1. The solution u blows up at finite time, then in this case u must coincide with $Pc(Q)$ up to symmetries of the equation.

2. The solution u is a global solution. Then in this case, u either scatters in both time directions or u must coincide with SW up to symmetries of the equation.

In [**20**], Merle considered the first part of the conjecture, where he identified all finite time blowup solutions as $Pc(Q)$ under an additional H_x^1 assumption on the initial data. See also [**32**] for the preliminary result due to Weinstein and [**9**] for a simplified proof of Merle's argument due to Hmidi-Keraani. By Merle's result and pseudoconformal transformation, the second part of the conjecture, which characterizes all global solutions with ground state mass, still holds if we make the strong assumption that the initial data $u_0 \in \Sigma = \{f \in H_x^1, xf \in L_x^2\}$. Finally it is worthwhile to notice that Merle's argument works for all dimensions without any symmetry assumption on the initial data.

Without the Σ assumption on the initial data, it is not clear at all how to deal with the case when u_0 is merely in L_x^2 and the corresponding solution is global. Recently in [**14**] and [**18**], we proved the second part of the conjecture when the initial data $u_0 \in H_x^1(\mathbb{R}^d)$, $d \geq 2$ and is spherically symmetric. In dimension $d \geq 4$, the results hold even under a weaker symmetry assumption, namely, the initial data is only required to be splitting-spherical symmetric (see [**18**] for more details).

As stated, all the results concerning the rigidity conjecture require the H_x^1 regularity on the initial data since it is the minimal regularity to define the energy and to carry out the spectral analysis. Here the energy refers to

$$\text{Energy:} \;\; E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L_x^2}^2 - \frac{d}{2(d+2)}\|u(t)\|_{L_x^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} = E(u_0).$$

The purpose of this paper is two-folded. First of all, we lower down the symmetry assumption on the initial data in the scattering conjecture. Secondly, we prove certain rigidity results for all *finite mass solutions* under this weak symmetry

[1]Here by "non-scattering", we mean that the $L_{t,x}^{\frac{2(d+2)}{d}}$ norm is infinite. Obviously, the "non-scattering" solution may blow up at finite time, or can exist globally.

assumption. The weak symmetry we refer to here is the "splitting spherical symmetry". For the notational simplicities, we choose to work in 6-dimensions and we require the initial data u_0 satisfy

$$u_0(x_1, \cdots, x_6) = u_0(\sqrt{x_1^2+x_2^2+x_3^2}, \sqrt{x_4^2+x_5^2+x_6^2}). \tag{1.4}$$

The extension to arbitrary higher dimensions is also expected after some technical changes. Our main results are the following

THEOREM 1.4 (scattering in 6-dimensions with symmetry (1.4)). *Let* $u_0 \in L_x^2(\mathbb{R}^6)$ *satisfying* $M(u_0) < M(Q)$ *and symmetry* (1.4). *Then the solution to* (1.1) *with this initial data exists globally and satisfies*

$$\|u\|_{L_{t,x}^{\frac{8}{3}}(\mathbb{R}\times\mathbb{R}^6)} \leq C(\|u_0\|_{L_x^2}).$$

In particular, this implies scattering: there exist $u_\pm \in L_x^2$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{L_x^2} = 0.$$

The following theorem proves the rigidity conjecture for merely L_x^2 solutions under certain assumptions.

THEOREM 1.5 (the only two-way non-scattering solution is solitary wave). *Let* $u_0 \in L_x^2(\mathbb{R}^6)$ *satisfy* $M(u_0) = M(Q)$ *and symmetry 1.4. Suppose the maximal-lifespan solution* $u(t,x)$ *on time interval* $(-T_*, T^*)$ *does not scatter in the sense that*

$$\|u\|_{L_{t,x}^{\frac{8}{3}}([0,T^*))} = \|u\|_{L_{t,x}^{\frac{8}{3}}((-T_*,0])} = \infty.$$

Then $T_* = T^* = \infty$ *and* $u = e^{it}Q$ *up to scaling and phase rotation.*

1.2. Outline of the proof. The proof of the above two theorems follows roughly the same general strategy as in previous works [**14, 18, 15, 16**]. The main part of the proof, as we shall see soon, is devoted to upgrading the regularity of the solutions. However to lower the regularity assumption on initial data we have to adopt a new local iteration scheme which is developed in our very recent work [**19**]. The main advantage of this new scheme is that we only need to use the local in time information of the solution. Therefore it works for all solutions with certain compactness. This sets free the procedure of picking up three candidates (or referred to as "three enemies" in [**15, 16**])[2] in the proof of scattering conjecture). At the same time it enables us to deal with the rigidity conjecture in the critical L_x^2 space.

In what follows we outline the proof of Theorem 1.4 in three steps. The proof of Theorem 1.5 is only slightly different at the first step and therefore we shall give the necessary modifications (see Remark 1.8 below).

Step 1. Reduction to *almost periodic modulo scaling* solutions.

This is by now a standard step. We argue by contradiction and suppose the scattering result Theorem 1.4. does not hold. Then the failure of the theorem implies the existence of solutions with certain compactness. More precisely, the argument in [**25, 13, 1**] establishes the following

[2]In other words, we will eliminate all enemies without performing the reduction procedure to three enemies

Proposition 1.6. [**25, 13, 1**] *Suppose Theorem 1.4 fails, then there exists minimal mass $M_c < M(Q)$ and maximal-lifespan solution $u(t,x) : I \times \mathbb{R}^6 \to \mathbb{C}$ such that u satisfies* (1.4) *and*
• *u has the minimal mass: $M(u) = M_c$;*
• *u is* almost periodic modulo scaling *in the following sense: for any $t \in I$, there exists $N(t) > 0$ such that $\{N(t)^{-3} u(t, \frac{x}{N(t)})\}$ is precompact in L_x^2.*

Here $N(t)$ is the frequency scale. In the physical space, it also measures the concentration size of the solution. Basically we do not have any a priori control on $N(t)$[3] which is the heart of the problem. The explicit control on $N(t)$ provides the control of Strichartz norm of the solution on certain time intervals which is crucial to upgrading the regularity of the solution. However, as we shall show in the second step, to gain additional regularity of the solution we will use a local iteration scheme in which we do not need any a priori control on $N(t)$. This is different from the works [**15, 16**].

Remark 1.7. In more detail, the argument in the proof of scattering conjecture for radial solutions in $d \geq 2$ (see [**15, 16**]) relies on performing yet another limiting procedure to pick up the candidates for which we have good control on $N(t)$. This is a crucial step in the proof in [**15, 16**], namely to prove the existence of three enemies and then kill the three enemies by proving additional regularities. The three enemies are: soliton like solution with $N(t) = 1$; high-to-low cascade with $N(t) \leq 1$ and $\liminf_{t \to \pm\infty} N(t) = 0$; self similar solution $N(t) = t^{-\frac{1}{2}}$. However in the present proof, we shall not reduce to three enemies and we will directly kill all enemies.

Remark 1.8. The proof of Theorem 1.5 is based on Theorem 1.4. Namely, knowing that Theorem 1.4 holds true, then $M(Q)$ is the minimal mass for solution to not scatter. As a consequence of this minimality and the compactness argument in [**25**] for example, all the solutions with symmetry (1.4) and ground state mass must be almost periodic modulo scaling. Since by the assumption, the solution does not scatter on both time directions, the almost periodicity holds on both time directions as well. Therefore, similar to Theorem 1.4, all the work is reduced to proving additional regularity of the *almost periodic modulo scaling* solution and localization of kinetic energy. Finally, to exclude all possibilities when the solution does not coincide with SW and $Pc(Q)$, we will use the truncated virial argument. See Step 2 and Step 3 below.

To conclude, after establishing the compactness property of the solution, the proofs of both Theorem 1.4 and Theorem 1.5 are both hinged on showing that the *almost periodic modulo scaling* solution on both time directions must have Sobolev regularity $H^{1+\varepsilon}$. This is the main content in the

Step 2. Additional regularity for *almost periodic modulo scaling* solutions.

More precisely we prove the following

Theorem 1.9 (Additional regularity for *almost periodic modulo scaling* solutions). *Let u be a maximal-lifespan solution of* (1.1) *on I in six dimensions obeying*

[3] One can keep in mind the two examples of *almost periodic modulo scaling solutions*: SW and $Pc(Q)$. For SW, $N(t) = 1$; for $Pc(Q)$, $N(t) = \frac{1}{t}$.

symmetry (1.4)*. Let* u *be almost periodic modulo scaling on* I*. Then there exists* $\varepsilon > 0$ *such that* $\forall t \in I$*,*

$$u(t) \in H_x^{1+\varepsilon}.$$

Moreover, the kinetic energy of u *is uniformly localized: for any* $\eta > 0$*, there exists* $C(\eta) > 0$ *such that*

$$\sup_{t\in I} \|\phi_{>C(\eta)} \nabla u(t)\|_{L_x^2} \le \eta. \tag{1.5}$$

We will give the proof in Section 4, Section 5 and Section 6. Here we describe the main idea of the proof. We will work with each single dyadic frequency of u:

$$\|P_N u(t)\|_{L_x^2}.$$

First by using in/out decomposition and weighted Strichartz inequalities, we prove

$$\|\phi_{>1} P_N u(t)\|_{L_x^2} \lesssim N^{-1-\varepsilon} \tag{1.6}$$

with a uniform in time bound, where $\phi_{>1}$ is a smooth cut-off function supported in the region $|x| > 1$. This then reduces matters to considering the part of the solution near the spatial origin, i.e. $\|\phi_{\le 1} P_N u(t)\|_{L_x^2}$. This piece is trivially bounded by

$$A_N = \|P_N u\|_{S([t,t+\frac{1}{\sqrt{N}}])},$$

i.e. the Strichartz norm of $P_N u$ on a local time interval $[t, t+\frac{1}{\sqrt{N}}]$. It turns out, after some technical manipulations, that this latter quantity is better suited for iteration and bootstrapping. Indeed we shall establish recurrent relation for A_N and we will iterate our estimates only finitely many (but sufficiently many) steps. The crucial point is that during the iteration process, we only need to use the information of the solution on a unit time interval $[t, t + 1]$. Therefore we do not need to use the full control on $N(t)$. We remark that although as a sacrifice the $H_x^{1+\varepsilon}$ norm of $u(t)$ depends on t, the analysis here combined with (1.6) and a further spatial decay estimate (see Section 6) give rise to the uniform localization of kinetic energy (1.5). This property is enough for us to use the truncated virial argument in Step 3.

Step 3. Truncated virial argument.

The contradiction for the scattering result Theorem 1.4 and final coincidence with the examples in the rigidity result Theorem 1.5 follows quickly from the kinetic localization by using a virial type argument. This part of the proof is standard and will be given in Section 6.

We have explained all three steps of the proofs of Theorem 1.4 and Theorem 1.5. A few remarks are in order. First of all, as we will see later in the detailed proof, the additional regularity comes from the compactness *in both time directions.* In other words, only knowing the solution is almost periodic modulo scaling in one time direction is not enough to prove the additional regularity. This fact is not fatal in the proof of scattering result 1.4 since we have the freedom to choose the two-way non-scattering solution to work with (see Proposition 1.6). However, concerning the rigidity result Theorem 1.5, two-way non-scattering has to come in as our assumption. As a matter of fact, it is an interesting problem to deal with the case when the solution does not scatter in one time direction but scatters on the other. Secondly, we want to point out that, due to the anisotropicity of the function, the proof here is more involved than in [**19**] especially in getting the uniform regularity away from the origin. Finally, the extension to higher dimensions is also expected where we need to use the tools developed in [**18**].

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2. Preliminaries

2.1. Some notations. We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \le CY$ for some constant $C > 0$. We use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$.

We use the 'Japanese bracket' convention $\langle x \rangle := (1 + |x|^2)^{1/2}$.

We write $L_t^q L_x^r$ to denote the Banach space with norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} := \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |u(t,x)|^r \, dx\Big)^{q/r} \, dt\Big)^{1/q},$$

with the usual modifications when q or r are equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of spacetime such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L_t^q L_x^q$ as $L_{t,x}^q$. For notational consistence, in this section, d is understood to ≥ 3.

Throughout this paper, we will use $\phi \in C^\infty(\mathbb{R}^d)$ be a radial bump function supported in the ball $\{x \in \mathbb{R}^d : |x| \le \frac{25}{24}\}$ and equal to one on the ball $\{x \in \mathbb{R}^d : |x| \le 1\}$. We also denote $\psi(x) = \phi(x) - \phi(2x)$. For any constant $C > 0$, we denote $\phi_{\le C}(x) := \phi\big(\frac{x}{C}\big)$, $\phi_{>C} := 1 - \phi_{\le C}$ and $\psi_C(x) = \psi(\frac{x}{C})$.

In this paper, we use x^1 to denote the vector (x_1, x_2, x_3) and x^2 to denote (x_4, x_5, x_6). Same explanation applies to y^1, y^2. We use Δ_j, ∇_j to denote the Laplacian and gradient operators restricted in x^j direction. For any two functions $f, g : \mathbb{R}^6 \to \mathbb{R}$, we use $*$ to denote the usual convolution, i.e.

$$f * g(x) = \int_{\mathbb{R}^6} f(x-y)g(y)dy.$$

We use $*_{x^1}$ to denote the convolution in (x_1, x_2, x_3) variable, for example

$$f(\cdot, x^2) *_{x^1} g(\cdot, y^2) = \int_{\mathbb{R}^3} f(x^1 - y^1, x^2) g(y^1, y^2) dy^1.$$

Similar convention also applies to $*_{x^2}$.

2.2. Basic harmonic analysis. For each number $N > 0$, we define the Fourier multipliers

$$\widehat{P_{\le N} f}(\xi) := \phi_{\le N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_{> N} f}(\xi) := \phi_{> N}(\xi)\hat{f}(\xi)$$
$$\widehat{P_N f}(\xi) := \psi_N(\xi)\hat{f}(\xi)$$

and similarly $P_{<N}$ and $P_{\ge N}$. We also define

$$P_{M<\cdot\le N} := P_{\le N} - P_{\le M} = \sum_{M<N'\le N} P_{N'}$$

whenever $M < N$. We will usually use these multipliers when M and N are *dyadic numbers* (that is, of the form 2^n for some integer n); in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. As P_N is not truly a projection, $P_N^2 \neq P_N$, we will occasionally need to use fattened Littlewood-Paley operators:

$$\tilde{P}_N := P_{N/2} + P_N + P_{2N}. \tag{2.1}$$

These obey $P_N \tilde{P}_N = \tilde{P}_N P_N = P_N$.

Like all Fourier multipliers, the Littlewood-Paley operators commute with the propagator $e^{it\Delta}$, as well as with differential operators such as $i\partial_t + \Delta$. We will use basic properties of these operators many times, including

LEMMA 2.1 (Bernstein estimates). *For* $1 \leq p \leq q \leq \infty$,

$$\begin{aligned} \big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^p} &\sim N^{\pm s} \|P_N f\|_{L_x^p}, \\ \|P_{\leq N} f\|_{L_x^q} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq N} f\|_{L_x^p}, \\ \|P_N f\|_{L_x^q} &\lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L_x^p}. \end{aligned}$$

While it is true that spatial cutoffs do not commute with Littlewood-Paley operators, we still have the following:

LEMMA 2.2 (Mismatch estimates in real space). *Let* $R, N > 0$. *Then*

$$\begin{aligned} \big\| \phi_{>R} \nabla P_{\leq N} \phi_{\leq \frac{R}{2}} f \big\|_{L_x^p} &\lesssim_m N^{1-m} R^{-m} \|f\|_{L_x^p} \\ \big\| \phi_{>R} P_{\leq N} \phi_{\leq \frac{R}{2}} f \big\|_{L_x^p} &\lesssim_m N^{-m} R^{-m} \|f\|_{L_x^p} \end{aligned}$$

for any $1 \leq p \leq \infty$ *and* $m \geq 0$.

PROOF. We will only prove the first inequality; the second follows similarly.

It is not hard to obtain kernel estimates for the operator $\phi_{>R} \nabla P_{\leq N} \phi_{\leq \frac{R}{2}}$. Indeed, an exercise in non-stationary phase shows

$$\big| \phi_{>R} \nabla P_{\leq N} \phi_{\leq \frac{R}{2}}(x,y) \big| \lesssim N^{d+1-2k} |x-y|^{-2k} \phi_{|x-y| > \frac{R}{2}}$$

for any $k \geq 0$. An application of Young's inequality yields the claim. □

Similar estimates hold when the roles of the frequency and physical spaces are interchanged. More generally we have the following

LEMMA 2.3 (Mismatch estimates in frequency space). *Let* $1 \leq p \leq 2$, *then for any* $m \geq 1$,

$$\begin{aligned} \big\| P_N \phi_{\leq R} P_{>8N} f \big\|_{L_x^2} &\lesssim_m N^{\frac{d}{p} - \frac{d}{2}} (RN)^{-m} \|f\|_{L_x^p}. \\ \big\| P_N \phi_{\leq R} P_{\leq \frac{N}{8}} f \|_{L_x^2} &\lesssim N^{\frac{d}{p} - \frac{d}{2}} (RN)^{-m} \|f\|_{L_x^p} \end{aligned}$$

Proof. We first prove the second inequality, the treatment for the first one is similar. By Plancherel, we write

$$\begin{aligned}\left\|P_N\phi_{\leq R}P_{<N/8}f\right\|_{L_x^2} &= \left\|\psi(\frac{\xi}{N})R^d\int\hat{\phi}(R(\xi-\eta))\phi(\frac{8\eta}{N})\hat{f}(\eta)d\eta\right\|_{L_\xi^2}\\ &\lesssim R^d\left\|\psi(\frac{\xi}{N})\left(\int\left|\hat{\phi}(R(\xi-\eta))\phi(\frac{8\eta}{N})\right|^p d\eta\right)^{\frac{1}{p}}\right\|_{L_\xi^2}\|\hat{f}\|_{L_\eta^{p'}}\\ &\leq R^dN^{\frac{d}{p}}\left\|\psi(\frac{\xi}{N})\left(\int\left|\hat{\phi}(R(\xi-\frac{N}{8}\eta))\phi(\eta)\right|^p d\eta\right)^{\frac{1}{p}}\right\|_{L_\xi^2}\|f\|_{L_x^p}\\ &\leq R^dN^{\frac{d}{p}+\frac{d}{2}}(RN)^{-m}\|f\|_{L_x^p}\\ &\leq (RN)^{d-m}N^{\frac{d}{p}-\frac{d}{2}}\|f\|_{L_x^p}.\end{aligned}$$

□

Before finishing this subsection, we remark that in this paper, we will use P_N^1 to denote the frequency projection in subspace $\mathbb{R}^3_{x^1}$. Same convention applies to P_N^2 and $P_{\leq N}^i$ for $i=1,2$.

2.3. Some analysis tools. We will need the following fractional chain rule lemma.

Lemma 2.4 (Fractional chain rule for a C^1 function, [**5**][**22**][**27**]). *Let $F\in C^1(\mathbb{C})$, $\sigma\in(0,1)$, and $1<r,r_1,r_2<\infty$ such that $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$. Then we have*

$$\||\nabla|^\sigma F(u)\|_{L_x^r}\lesssim\|F'(u)\|_{L_x^{r_1}}\||\nabla|^\sigma u\|_{L_x^{r_2}}.$$

Proof. See [**5**], [**22**] and [**27**]. □

We also need the following lemma from [**16**].

Lemma 2.5. *Let $0<s<1+\frac{4}{d}$, then*

$$\||\nabla|^sF(u)\|_{L_x^{\frac{2(d+2)}{d+4}}}\lesssim\||\nabla|^su\|_{L_x^{\frac{2(d+2)}{d}}}\|u\|^{\frac{4}{d}}_{L_x^{\frac{2(d+2)}{d}}}.$$

We will need the following sharp Gagliardo-Nirenberg inequality

Lemma 2.6. *Let Q be the ground state in the Definition 1.1. Then for any $f\in H_x^1(\mathbb{R}^d)$, we have*

$$\|f\|^{\frac{2(d+2)}{d}}_{L_x^{\frac{2(d+2)}{d}}}\leq\frac{d+2}{d}\left(\frac{M(f)}{M(Q)}\right)^{\frac{2}{d}}\|\nabla f\|^2_{L_x^2}. \tag{2.2}$$

The equality holds only and if only

$$f=ce^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda(x-x_0)). \tag{2.3}$$

for $(c,\theta,\lambda)\in(\mathbb{R}^+,\mathbb{R},\mathbb{R}^+)$.

We will need the following radial Sobolev embedding

LEMMA 2.7 (Radial Sobolev embedding, [**24**]). *Let dimension $d \geq 3$. Let $s > 0$, $\alpha > 0$, $1 < p, q < \infty$ obeys the scaling restriction: $\alpha + s = d(\frac{1}{q} - \frac{1}{p})$. Then the following holds:*

$$\||x|^\alpha f\|_{L_x^p(\mathbb{R}^d)} \lesssim \||\nabla|^s f\|_{L_x^q(\mathbb{R}^d)},$$

where the implicit constant depends on s, α, p, q.

2.4. Strichartz estimates. The free Schrödinger flow has the explicit expression:

$$e^{it\Delta} f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) dy,$$

from which we can derive the kernel estimate of the frequency localized propagator. We record the following

LEMMA 2.8 (Kernel estimate[**15, 16**]). *For any $m \geq 0$, we have*

$$|(P_N e^{it\Delta}(x,y)| \lesssim_m \begin{cases} |t|^{-d/2}, & : |x-y| \sim Nt; \\ \frac{N^d}{|N^2 t|^m \langle N|x-y|\rangle^m} & : otherwise \end{cases}$$

for $|t| \geq N^{-2}$ and

$$|(P_N e^{it\Delta})(x,y)| \lesssim_m N^d \langle N|x-y|\rangle^{-m}$$

for $|t| \leq N^{-2}$.

We will frequently use the standard Strichartz estimate. Let I be a time interval. We define the Strichartz space on I:

$$S(I) = L_t^\infty L_x^2(I \times \mathbb{R}^d) \cap L_t^2 L_x^{\frac{2d}{d-2}}(I \times \mathbb{R}^d).$$

We also define $N(I)$ to be the dual space of $S(I)$. Then the standard Strichartz estimate reads

LEMMA 2.9 (Strichartz estimate). *Let $d \geq 3$. Let I be an interval, $t_0 \in I$, and let $u_0 \in L_x^2(\mathbb{R}^d)$ and $F \in N(I)$. Then, the function u defined by*

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-t')\Delta} F(t')\, dt'$$

obeys the estimate

$$\|u\|_{S(I)} \lesssim \|u_0\|_{L_x^2} + \|F\|_{N(I)},$$

where all spacetime norms are over $I \times \mathbb{R}^d$.

PROOF. See, for example, [**7, 23**]. For the endpoint see [**10**]. □

We will also need a weighted Strichartz estimate for splitting spherically symmetric functions in 6 dimensions.

LEMMA 2.10. *(Weighted Strichartz estimate in splitting-spherical symmetric case).*

Let I be a time interval, $t_0 \in I$. Let $u_0 \in L_x^2(\mathbb{R}^6)$, $F \in L_{t,x}^{\frac{8}{5}}(I \times \mathbb{R}^6)$ be splitting-spherically symmetric. Then the function $u(t,x)$ defined by

$$u(t) := e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-s)\Delta} F(s) ds$$

is also splitting-spherically symmetric and obeys the estimate

$$\||x^1|^{\frac{1}{2}-}u\|_{L_t^{2+}L_x^{4-}(I\times\mathbb{R}^6)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^6)} + \|F\|_{L_{t,x}^{\frac{8}{5}}(I\times\mathbb{R}^6)}.$$

The same conclusion holds true if we replace x^1 by x^2.

Proof. See [**18**] for the proof. □

2.5. The in-out decomposition. We will need an incoming/outgoing decomposition which was developed in [**15, 16**]. As there, we define operators $P^\pm$ by

$$[P^\pm f](r) := \tfrac{1}{2}f(r) \pm \frac{i}{\pi}\int_0^\infty \frac{r^{2-d}\, f(\rho)\,\rho^{d-1}\,d\rho}{r^2-\rho^2},$$

where the radial function $f:\mathbb{R}^d\to\mathbb{C}$ is written as a function of radius only. We will refer to P^+ is the projection onto outgoing spherical waves; however, it is not a true projection as it is neither idempotent nor self-adjoint. Similarly, P^- plays the role of a projection onto incoming spherical waves; its kernel is the complex conjugate of the kernel of P^+ as required by time-reversal symmetry.

For $N>0$ let $P_N^\pm$ denote the product $P^\pm P_N$ where P_N is the Littlewood-Paley projection. We record the following properties of $P^\pm$ from [**15, 16**]:

Proposition 2.11 (Properties of $P^\pm$, [**15, 16**]).
(*i*) *$P^+ + P^-$ represents the projection from L^2 onto L^2_{rad}.*
(*ii*) *Fix $N>0$. Then*

$$\big\|\chi_{\gtrsim\frac{1}{N}}P^\pm_{\geq N}f\big\|_{L^2(\mathbb{R}^d)} \lesssim \big\|f\big\|_{L^2(\mathbb{R}^d)}$$

with an N-independent constant.
(*iii*) *For $|x|\gtrsim N^{-1}$ and $t\gtrsim N^{-2}$, the integral kernel obeys*

$$\big|[P_N^\pm e^{\mp it\Delta}](x,y)\big| \lesssim \begin{cases} (|x||y|)^{-\frac{d-1}{2}}|t|^{-\frac{1}{2}} & : \ |y|-|x|\sim Nt \\ \frac{N^d}{(N|x|)^{\frac{d-1}{2}}\langle N|y|\rangle^{\frac{d-1}{2}}}\big\langle N^2t+N|x|-N|y|\big\rangle^{-m} & : \ \textit{otherwise} \end{cases}$$

for all $m\geq 0$.
(*iv*) *For $|x|\gtrsim N^{-1}$ and $|t|\lesssim N^{-2}$, the integral kernel obeys*

$$\big|[P_N^\pm e^{\mp it\Delta}](x,y)\big| \lesssim \frac{N^d}{(N|x|)^{\frac{d-1}{2}}\langle N|y|\rangle^{\frac{d-1}{2}}}\big\langle N|x|-N|y|\big\rangle^{-m}$$

for any $m\geq 0$.

3. Proofs of Theorem 1.4 and 1.5

As was already explained in the introduction, proofs of Theorem 1.4 and Theorem 1.5 are reduced to obtaining the additional regularity for the solutions which is almost periodic modulo scaling in both time directions. In this section, we explain in more details the proof of both theorems.

The only property of the *almost periodic modulo scaling* solution we rely on is the following improved Duhamel formula. We record the following result from [**25**]:

Proposition 3.1 (Improved Duhamel formula). *Let $u(t,x)$, $t \in I$ be the maximal-lifespan solution of* (1.1) *and is almost periodic modulo scaling on I, then we have the following*

$$u(t) = w - \lim_{T\to\inf I} -i\int_{\inf I}^{t} e^{i(t-\tau)\Delta}F(u(\tau))d\tau$$
$$= w - \lim_{T\to\sup I} i\int_{t}^{\sup I} e^{i(t-\tau)\Delta}F(u(\tau))d\tau$$

weakly in L_x^2.

Comparing with the usual Duhamel formula, the linear flow, which is not smoothed out with time, vanishes in this improved Duhamel formula. This is where the additional smoothness comes from.

The key step in proving both theorems is to establish the following

Theorem 3.2 (Additional regularity for solutions obeying Proposition 3.1). *Let u be a maximal-lifespan solution of* (1.1) *in six dimensions on time interval I. Let u obey the Duhamel formula Proposition 3.1 and symmetry* (1.4)*. Then there exists $\varepsilon > 0$ such that $\forall\, t \in I$,*

$$u(t) \in H_x^{1+\varepsilon}.$$

Moreover, the kinetic energy of u is uniformly localized: for any $\eta > 0$, there exists $C(\eta) > 0$ such that

$$\sup_{t\in I} \|\phi_{>C(\eta)}\nabla u(t)\|_{L_x^2} \le \eta. \tag{3.1}$$

The proof of this theorem is the main part of the paper and will be presented in the remaining sections. Here we assume this theorem hold momentarily and finish the proof of Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.4:

Proof. Suppose by contradiction that Theorem 1.4 does not hold, then the same argument as in [**15**], [**16**] yields that: there exists minimal mass $M_c < M(Q)$ and maximal-lifespan solution $u(t,x)$ on $I = (-T_*, T^*)$ such that u obeys the symmetry (1.4) and

1. $M(u) = M_c$;
2. $u(t)$ is almost periodic modulo scaling on I;

Applying Proposition 3.1 and Theorem 3.2, we know that $u \in H_x^{1+\varepsilon}$. This combined with sharp Gagliardo-Nirenberg inequality and the fact that $M(u) < M(Q)$ yields that

$$\|u(t)\|_{H_x^1} \lesssim_{M(u)} 1.$$

From this and the standard H_x^1 local theory, we know that u exists globally, i.e.: $T_* = T^* = \infty$. In this situation, the contradiction will come from the truncated virial and the kinetic energy localization as we now explain. Let $\phi_{\le R}$ be the smooth cutoff function, we define truncated virial as

$$V_R(t) = \int \phi_{\le R}(x)|x|^2|u(t,x)|^2dx.$$

Obviously

$$V_R(t) \lesssim R^2,\ \forall\, t \in \mathbb{R}. \tag{3.2}$$

On the other hand, we compute the second derivative of virial with respect to t, this gives

$$\partial_{tt} V_R(t) = 8E(u) + \tag{3.3}$$

$$O\Big(\int_{|x|>R} \Big(|\nabla u(t,x)|^2 + |u(t,x)|^{\frac{8}{3}}\Big)\, dx + \frac{1}{R^2}\int_{|x|>R} |u(t,x)|^2 dx\Big). \tag{3.4}$$

Since $M(u) < M(Q)$ and $u \in H^1_x$, from sharp Gagliardo-Nirenberg inequality (2.2) we have

$$E(u) > 0.$$

Now we can use the kinetic energy localization (3.1) and Gagliardo-Nirenberg inequality to control the error term (3.4) and finally arrives at

$$\partial_{tt} V_R(t) \geq 4E(u) > 0.$$

This obvious contradicts (3.2). The proof of Theorem 1.4 is finished. □

Proof of Theorem 1.5:

Proof. Let u be the solution of (1.1) in six dimensions obeying the symmetry (1.4) and

1. $M(u) = M(Q)$,
2. u does not scatter in both time directions.

Since by Theorem 1.4, $M(Q)$ is the minimal mass, the compactness argument in [**25, 13, 1**] shows that u is almost periodic modulo scaling in both time directions. Now we can apply Theorem 3.2 to deduce that $u \in H^1_x$. Since from Merle's result, the only finite time blowup solution must be $Pc(Q)$ up to symmetries and $Pc(Q)$ scatters in one time direction, we know from condition 2 that u must be a global solution.

From (2.2), this global solution u satisfies $E(u) \geq 0$. Moreover, the same virial argument as in the proof of Theorem 1.4 allows us to preclude the case $E(u) > 0$, therefore we obtain $E(u) = 0$. From this the coincidence of the solution with solitary wave follows immediately, again by the sharp Gagliardo-Nirenberg inequality. The proof is completed.

□

The remaining part of the paper is devoted to proving Theorem 3.2. This is done in several steps: In section 4, we prove additional smoothness away from the origin with uniform in time bound through finite steps of iteration; In Section 5, we prove $u(t) \in H^{1+\varepsilon}_x$ by using a new local iteration scheme. In Section 6, we prove the uniform localization property for kinetic energy.

4. Smoothness away from the origin with uniform in time bound

In this section, we show that for any L^2_x solution which obeys the improved Duhamel formula Proposition 3.1, the H^{1+} norm away from the origin is well defined and has the uniform in time control. This property will be used crucially in the local iteration scheme in the next Section. In the radial case, this result was proved in [**14**] by a two-step iteration. However, in this $3+3$ case, the proof is much more complicated due to the anisotropicity of the function.

PROPOSITION 4.1. *Let $u_0 \in L_x^2(\mathbb{R}^6)$ and obey symmetry (1.4). Let $u(t,x)$ be the maximal-lifespan solution on $I = (-T_*, T^*)$ satisfying the improved Duhamel formula Proposition 3.1. Then for any $N \geq 1$,*

$$\|\phi_{>1} P_N u(t)\|_{L_x^2} \lesssim N^{-1-\frac{1}{10}}, \quad \forall\, t \in (-T_*, T^*) \tag{4.1}$$

In particular,

$$\sup_{t \in (-T_*, T^*)} \|\phi_{>1} \nabla u(t)\|_{L_x^2} \lesssim 1. \tag{4.2}$$

In both of the two estimates, the implicit constant depends only on $M(u)$.

PROOF. We only give the proof of (4.1), since (4.2) follows from (4.1) and simple manipulation involving dyadic decomposition and mismatch estimate Lemma 2.3.

We prove (4.1) in two steps. In the first step, we get a little decay in N, say $N^{-\frac{1}{50}}$. Then in the next step, we improve this decay via finitely many iterations. For the notational convenience and without loss of generality, we assume $T_* = T^* = \infty$.

Step 1. In this step, we prove

$$\sup_{t \in \mathbb{R}} \|\phi_{>1} P_N u(t)\|_{L_x^2} \lesssim N^{-\frac{1}{50}}, \ \forall N \geq 1. \tag{4.3}$$

By time translation invariance, it is enough to prove

$$\|\phi_{>1} P_N u_0\|_{L_x^2} \lesssim N^{-\frac{1}{50}}, \ \forall N \geq 1. \tag{4.4}$$

where the implicit bound depends only on $M(u)$.

Since $P_N \sim \tilde{P}_N^1 P_{\lesssim N}^2 + \tilde{P}_N^2 P_{\lesssim N}^1$, we use triangle inequality to bound

$$\|\phi_{>1} P_N u_0\|_{L_x^2} \lesssim \|\phi_{>1} \tilde{P}_N^1 P_{\lesssim N}^2 u_0\|_{L_x^2} + \|\phi_{>1} \tilde{P}_N^2 P_{\lesssim N}^1 u_0\|_{L_x^2}.$$

By symmetry and without loss of generality, we only need to estimate

$$\|\phi_{>1} P_N^1 P_{\leq N}^2 u_0\|_{L_x^2}, \tag{4.5}$$

here and afterwards, we write P_N^1 instead of $\tilde{P}_N^1$ for notational convenience. This can be trivially bounded by

$$\|\phi_{>1} \phi_{|x^1| \leq \frac{1}{N}} P_N^1 P_{\leq N}^2 u_0\|_{L_x^2} + \|\phi_{>1} \phi_{|x^1| > \frac{1}{N}} P_N^1 P_{\leq N}^2 u_0\|_{L_x^2}. \tag{4.6}$$

We first estimate the term with cutoff $\phi_{|x^1| > \frac{1}{N}}$, for which we use in/out decomposition to get

$$\begin{aligned} \|\phi_{>1} \phi_{|x^1| > \frac{1}{N}} P_N^1 P_{\leq N}^2 u_0\|_{L_x^2} \leq\; & \|\phi_{>1} \phi_{|x^1| > \frac{1}{N}} P_N^{1+} P_{\leq N}^2 u_0\|_{L_x^2} \\ & + \|\phi_{>1} \phi_{|x^1| > \frac{1}{N}} P_N^{1-} P_{\leq N}^2 u_0\|_{L_x^2}. \end{aligned}$$

For the outgoing piece, we use Duhamel formula forward in time; for the incoming piece, we use Duhamel formula backward in time. Since the two give the same contributions, we only estimate the outgoing piece. Let $0 < \sigma < 1$ whose value will be chosen later. We use Duhamel formula, cut the time integral into short, long time pieces, and then introduce spatial cutoff to get

$$\begin{aligned}
&\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2 u_0\|_{L_x^2} \\
&\leq \left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2\int_0^\infty e^{-i\tau\Delta}F(u(\tau))d\tau\right\|_{L_x^2} \\
&\lesssim \left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2\int_{\frac{1}{N^{1+\sigma}}}^\infty e^{-i\tau\Delta}\phi_{>N\tau/4}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.7)\\
&\quad+\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2\int_{\frac{1}{N^{1+\sigma}}}^\infty e^{-i\tau\Delta}\phi_{\leq N\tau/4}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.8)\\
&\quad+\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{>1/4}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.9)\\
&\quad+\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\leq N}^2\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{\leq 1/4}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.10)
\end{aligned}$$

We first estimate the tail terms (4.8) and (4.10). For (4.8), we use the kernel estimate

$$\left|(\phi_{|x^1|>\frac{1}{N}}P_N^{1+}e^{-i\tau\Delta_1}\phi_{|y^1|\leq N\tau/4})(x^1,y^1)\right|\lesssim N^3(N^2\tau)^{-11}\langle N|x^1-y^1|\rangle^{-11}.$$

Using this estimate in the x^1 direction and Bernstein in the other, (4.8) can be estimated as follows:

$$\begin{aligned}
(4.8) &\lesssim \int_{\frac{1}{N^{1+\sigma}}}^\infty \|e^{-i\tau\Delta_2}(\phi_{|x^1|>\frac{1}{N}}P_N^{1+}e^{-i\tau\Delta_1}\phi_{|y^1|\leq N\tau/4}P_{\leq N}^2\phi_{\leq N\tau/4}F(u(\tau))\|_{L_x^2}d\tau \\
&\lesssim \int_{\frac{1}{N^{1+\sigma}}}^\infty \|\phi_{|x^1|>\frac{1}{N}}P_N^{1+}e^{-i\tau\Delta_1}\phi_{|y^1|\leq N\tau/4}P_{\leq N}^2\phi_{\leq N\tau/4}F(u(\tau))\|_{L_x^2}d\tau \\
&\lesssim \int_{\frac{1}{N^{1+\sigma}}}^\infty N^3(N^2\tau)^{-11}\|\|\langle N|\cdot|\rangle *_{x^1} P_{\leq N}^2\phi_{\leq N\tau/4}F(u(\tau))\|_{L_{x^1}^2}\|_{L_{x^2}^2}d\tau \\
&\lesssim N^{3-22}\int_{\frac{1}{N^{1+\sigma}}}^\infty \tau^{-11}d\tau\sup_\tau\|\langle N|\cdot|\rangle\|_{L_{x^1}^{\frac{3}{2}}}\|\|P_{\leq N}^2\phi_{\leq N\tau/4}F(u(\tau))\|_{L_{x^1}^{\frac{6}{5}}}\|_{L_{x^2}^2} \\
&\lesssim N^{-19}N^{10(1+\sigma)}N^{-2}N\sup_\tau\|F(u(\tau))\|_{L_x^{\frac{6}{5}}} \\
&\lesssim N^{-10+10\sigma}.
\end{aligned}$$

For (4.10), we further split the cutoff in x^1 to estimate

$$\begin{aligned}
(4.10) &\lesssim \left\|\phi_{>1}\phi_{\frac{1}{N}<|x^1|\leq\frac{1}{2}}P_N^{1+}P_{\leq N}^2\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{\leq\frac{1}{4}}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.11)\\
&\quad+\left\|\phi_{|x^1|>\frac{1}{2}}P_N^{1+}\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}P_{\leq N}^2\phi_{\leq\frac{1}{4}}F(u(\tau))d\tau\right\|_{L_x^2} \qquad (4.12)
\end{aligned}$$

We first estimate (4.11). Since $\phi_{>1}\phi_{|x^1|\leq\frac{1}{2}}=\phi_{>1}\phi_{|x^1|\leq\frac{1}{2}}\phi_{|x^2|>\frac{1}{2}}$ and

$$P_N^1P_{\leq N}^2=P_N^1P_{\leq N}^2\tilde{P}_N,$$

we first throw away the bounded operator $\phi_{\frac{1}{N}<|x^1|\le\frac12}P_N^{1+}P_{\le N}^2$ then use kernel estimate Lemma 2.8 to get

$$\begin{aligned}(4.11) &\lesssim \left\|\phi_{>\frac12}\tilde P_N\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{\le\frac14}F(u(\tau))d\tau\right\|_{L_x^2}\\ &\lesssim N^{-1-\sigma}\sup_{\tau\in[0,\frac{1}{N^{1+\sigma}}]}\|\phi_{>\frac12}\tilde P_N e^{-i\tau\Delta}\phi_{\le\frac14}F(u(\tau))\|_{L_x^2}\\ &\lesssim N^{-1-\sigma}N^{-4}\|\langle N|\cdot|\rangle^{-10}*F(u)\|_{L_\tau^\infty L_x^2}\\ &\lesssim N^{-5-\sigma}\|\langle N|\cdot|\rangle^{-10}\|_{L_x^{\frac32}}\|F(u)\|_{L_\tau^\infty L_x^{\frac65}}\\ &\lesssim N^{-5}.\end{aligned}$$

For (4.12), we use Proposition 2.11 in the x^1 variable and Bernstein in the other to obtain

$$\begin{aligned}(4.12) &\lesssim N^{-1-\sigma}\|e^{-i\tau\Delta_2}\phi_{|x^1|>\frac12}P_N^{1+}e^{-i\tau\Delta_1}\phi_{|y^1|\le\frac14}P_{\le N}^2\phi_{\le\frac14}F(u)\|_{L_\tau^\infty L_x^2}\\ &\lesssim N^{-1-\sigma}\sup_{\tau\in\mathbb{R}}\|(\phi_{|x^1|>\frac12}P_N^{1+}e^{-i\tau\Delta_1}\phi_{|y^1|\le\frac14})P_{\le N}^2\phi_{\le\frac14}F(u(\tau))\|_{L_x^2}\\ &\lesssim N^{-1-\sigma}\sup_{\tau\in\mathbb{R}}N^3N^{-10}\|\langle N|\cdot|\rangle^{-10}\|_{L_{x^1}^{\frac32}}\|\|P_{\le N}^2\phi_{\le\frac14}F(u)\|_{L_{x^1}^{\frac65}}\|_{L_{x^2}^2}\\ &\lesssim N^{-1-\sigma}N^{-8}\sup_{\tau\in\mathbb{R}}\|F(u(\tau))\|_{L_x^{\frac65}}\\ &\lesssim N^{-5}.\end{aligned}$$

So the contribution from the tail terms is negligible if we take $0<\sigma\le\frac45$:

$$(4.8)+(4.10)\lesssim N^{-2}.$$

Now we look at the main term (4.7), for which we use Strichartz to bound as

$$\begin{aligned}(4.7) &\lesssim \left\|\tilde P_N\int_{\frac{1}{N^{1+\sigma}}}^\infty e^{-i\tau\Delta}\phi_{>N\tau/4}F(\phi_{>N\tau/8}u)(\tau)d\tau\right\|_{L_x^2}\\ &\lesssim \left\|\tilde P_N\int_{\frac{1}{N^{1+\sigma}}}^\infty e^{-i\tau\Delta}\phi_{>N\tau/4}(P_{N/8<\cdot\le 8N}+P_{>8N}+P_{\le N/8})F(\phi_{>N\tau/8}u)(\tau)d\tau\right\|_{L_x^2}\end{aligned}$$

$$(4.13)\quad \lesssim \|\tilde P_N\phi_{>N\tau/4}(P_{>8N}+P_{\le N/8})F(\phi_{>N\tau/8}u)\|_{L_\tau^1L_x^2([\frac{1}{N^{1+\sigma}},\infty))}$$

$$(4.14)\quad +\left\|\int_{\frac{1}{N^{1+\sigma}}}^\infty e^{-i\tau\Delta}\phi_{>\frac{N\tau}{4}}\cdot\tilde P_NF(\phi_{>\frac{N\tau}{8}}u)(\tau)d\tau\right\|_{L_x^2}.$$

We use mismatch estimate Lemma 2.3 to deal with (4.13):

$$\begin{aligned}(4.13) &\lesssim N^2\|(N^2\tau)^{-11}F(\phi_{>N\tau/8}u)\|_{L_\tau^1L_x^{\frac65}([\frac{1}{N^{1+\sigma}},\infty)\times\mathbb{R}^6)}\\ &\lesssim N^{-20}\|\tau^{-11}\|_{L_\tau^1([\frac{1}{N^{1+\sigma}},\infty))}\\ &\lesssim N^{10(\sigma-1)}.\end{aligned}$$

For (4.14), without loss of generality, we assume

$$\phi_{>\frac{N\tau}{4}}=\phi_{>\frac{N\tau}{4}}\phi_{|x^1|>\frac{N\tau}{8}}.$$

Then using weighted Strichartz Lemma 2.10, radial Sobolev embedding in the x^1 variable and Bernstein, we have

$$
\begin{aligned}
(4.14) &\lesssim \||x^1|^{-\frac{1}{2}+}\phi_{|x^1|>N\tau/8}\tilde{P}_N F(\phi_{>N\tau/8}u)\|_{L_\tau^{2-}L_x^{\frac{4}{3}+}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{-\frac{1}{2}+}\|\tau^{-\frac{1}{2}+}\phi_{|x^1|>N\tau/8}\tilde{P}_N F(\phi_{>N\tau/8}u)\|_{L_\tau^{2-}L_x^{\frac{4}{3}+}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{-\frac{1}{2}+}\|\tau^{-\frac{1}{2}+}(N\tau)^{-\frac{1}{8}}\||x^1|^{\frac{1}{8}}\tilde{P}_N F(\phi_{>N\tau/8}u)\|_{L_x^{\frac{4}{3}+}}\|_{L_\tau^{2-}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{-\frac{1}{2}+}\|\tau^{-\frac{1}{2}+}(N\tau)^{-\frac{1}{8}}\|\||\nabla^1|^{\frac{1}{8}-}\tilde{P}_N F(\phi_{>N\tau/8}u)\|_{L_{x^1}^{\frac{6}{5}}}\|_{L_{x^2}^{\frac{4}{3}+}}\|_{L_\tau^{2-}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{-\frac{1}{2}+}\|\tau^{-\frac{1}{2}+}(N\tau)^{-\frac{1}{8}}N^{\frac{1}{8}-}N^{\frac{1}{4}+}\|\tilde{P}_N F(\phi_{>N\tau/8}u)\|_{L_x^{\frac{6}{5}}}\|_{L_\tau^{2-}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{-\frac{1}{4}+}\|\tau^{-\frac{5}{8}+}\|_{L_\tau^{2-}([\frac{1}{N^{1+\sigma}},\infty))} \\
&\lesssim N^{\frac{1}{8}(\sigma-1)+}.
\end{aligned}
$$

Finally, we estimate the other main term (4.9). We have

$$
\begin{aligned}
(4.9) &\lesssim \left\|\tilde{P}_N\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{>\frac{1}{4}}F(\phi_{>\frac{1}{8}}u)(\tau)d\tau\right\|_{L_x^2} \\
&\lesssim \left\|\tilde{P}_N\int_0^{\frac{1}{N^{1+\sigma}}} e^{-i\tau\Delta}\phi_{>\frac{1}{4}}(P_{\frac{N}{8}<\cdot\leq 8N}+P_{\leq N/8}+P_{>8N})F(\phi_{>\frac{1}{8}}u)(\tau)d\tau\right\|_{L_x^2} \\
&\lesssim \|P_{\frac{N}{8}<\cdot\leq 8N}F(\phi_{>\frac{1}{8}}u)\|_{L_\tau^{2-}L_x^{\frac{4}{3}+}([0,\frac{1}{N^{1+\sigma}}])} \\
&+\|\tilde{P}_N\phi_{>\frac{1}{4}}(P_{\leq\frac{N}{8}}+P_{>8N})F(\phi_{>\frac{1}{8}}u)\|_{L_\tau^1L_x^2([0,\frac{1}{N^{1+\sigma}}])} \\
&\lesssim N^{-\frac{1+\sigma}{2}-}\|\tilde{P}_N F(\phi_{>\frac{1}{8}}u)\|_{L_\tau^\infty L_x^{\frac{4}{3}+}}+N^{-5} \\
&\lesssim N^{-\frac{1+\sigma}{2}+\frac{1}{2}+}\|F(\phi_{>\frac{1}{8}}u)\|_{L_\tau^\infty L_x^{\frac{6}{5}}}+N^{-5} \\
&\lesssim N^{-\frac{\sigma}{2}+}.
\end{aligned}
$$

Collecting all the pieces together and taking $\sigma=\frac{4}{5}$, we get the desired $N^{-\frac{1}{50}}$ bound for the second term in (4.6). We still have to estimate the first term involving the spatial cutoff $\phi_{|x^1|\leq\frac{1}{N}}$. The estimate will be similar, so here we only sketch the proof. We first use improved Duhamel formula to bound it by

$$
\left\|\phi_{>1}\phi_{|x^1|\leq\frac{1}{N}}P_N^1P_{\leq N}^2\int_0^\infty e^{-i\tau\Delta}F(u(\tau))d\tau\right\|_{L_x^2}
$$

Then we split the integral into long time piece from $\frac{1}{N^{1+\sigma}}$ to ∞ and short time piece from 0 to $\frac{1}{N^{1+\sigma}}$.

For the long time piece, we insert the spatial cutoff $\phi_{>N\tau/4}$ and $\phi_{<N\tau/4}$ in front of $F(u)$. The term with cutoff $\phi_{>N\tau/4}$ can be treated in exact the same way as (4.7). The term with the cutoff $\phi_{\leq N\tau/4}$ is still a tail term, since the desired decay in N comes from the decay of kernel

$$
\phi_{|x^1|\leq\frac{1}{N}}P_N^1e^{-i\tau\Delta_1}\phi_{|y^1|<N\tau/4}(x^1,y^1)
$$

in the x^1 variable and Bernstein in the other.

For the short time piece, we insert the cutoff $\phi_{>\frac{1}{4}} + \phi_{\le\frac{1}{4}}$. Again the term with cutoff $\phi_{>\frac{1}{4}}$ can be treated in the way as (4.9). The term with cutoff $\phi_{\le\frac{1}{4}}$ is a tail term and will give us the bound, say N^{-5}.

Therefore finally we establish

$$\|\phi_{>1}P_N u\|_{L_t^\infty L_x^2} \lesssim N^{-\frac{1}{50}}. \tag{4.15}$$

Moreover, it is not hard to find that after minor notational change, the above proof also gives that for any $0 < c \le 1$

$$\|\phi_{>c}P_N u(t)\|_{L_x^2} \lesssim_c N^{-\frac{1}{50}}. \tag{4.16}$$

Step 2. This step we prove the following: Let $0 < s \le 1$ and $0 < c \le 1$. If

$$\sup_{t\in\mathbb{R}} \|\phi_{>c/8}P_N u(t)\|_{L_x^2} \lesssim_c N^{-s}. \tag{4.17}$$

Then

$$\sup_{t\in\mathbb{R}} \|\phi_{>c}P_N u(t)\|_{L_x^2} \lesssim_c N^{-s-\min(\frac{s}{3},\frac{1}{10})}. \tag{4.18}$$

Without loss of generality, we assume $c = 1$. By time translation invariance, it suffices to prove

$$\|\phi_{>1}P_N u_0\|_{L_x^2} \lesssim N^{-s-\min(\frac{s}{3},\frac{1}{10})} \tag{4.19}$$

under the assumption (4.17). Similar to the estimate (4.4), it is reduced to estimating the following two terms:

$$\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^1 P_{\le N}^2 u_0\|_{L_x^2}, \tag{4.20}$$

$$\|\phi_{>1}\phi_{|x^1|\le\frac{1}{N}}P_N^1 P_{\le N}^2 u_0\|_{L_x^2}. \tag{4.21}$$

To estimate (4.20) we use the in-out decomposition and improved Duhamel formula thus reduce to obtaining the following

$$\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\le N}^2 \int_0^\infty e^{-i\tau\Delta}F(u(\tau))d\tau\right\|_{L_x^2} \lesssim N^{-s-\min(\frac{s}{3},\frac{1}{10})}. \tag{4.22}$$

Now we adopt a slightly different strategy to treat the LHS of (4.22). This time we split the time integral into $[0, \frac{1}{N}]$ and $[\frac{1}{N}, \infty)$, thus reduce matters to proving

$$\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\le N}^2 \int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}F(u(\tau))d\tau\right\|_{L_x^2} \lesssim N^{-s-\min(\frac{s}{3},\frac{1}{10})}, \tag{4.23}$$

$$\left\|\phi_{>1}\phi_{|x^1|>\frac{1}{N}}P_N^{1+}P_{\le N}^2 \int_0^{\frac{1}{N}} e^{-i\tau\Delta}F(u(\tau))d\tau\right\|_{L_x^2} \lesssim N^{-s-\min(\frac{s}{3},\frac{1}{10})}. \tag{4.24}$$

To obtain (4.23), we still insert the spatial cutoff $\phi_{\le N\tau/4} + \phi_{>N\tau/4}$ in front of $F(u)$. The term with cutoff $\phi_{\le N\tau/4}$ is a tail term, the same calculation as in estimating (4.8) gives us the bound N^{-5}.

Now we turn to the term with cutoff $\phi_{>N\tau/4}$. This is the main regime where we need to use weighted Strichartz. After certain mismatch estimates and throwing away the bounded operator, it suffices to consider the piece

$$\left\|\int_{\frac{1}{N}}^\infty e^{-i\tau\Delta}\phi_{>N\tau/4}\tilde{P}_N F(\phi_{>N\tau/8}u)(\tau)d\tau\right\|_{L_x^2}. \tag{4.25}$$

Without loss of generality, we may assume

$$\phi_{>N\tau/4} = \phi_{>N\tau/4}\phi_{|x^1|>N\tau/8}.$$

The other case $\phi_{>N\tau/4}\phi_{|x^2|>N\tau/8}$ is similar. By weighted Strichartz, radial Sobolev embedding, Bernstein inequality and the inductive assumption (4.17), we compute

$$\begin{aligned}
(4.25) &\lesssim N^{-\frac12+}\|\tau^{-\frac12+}\phi_{|x^1|>N\tau/8}\tilde P_N F(\phi_{>N\tau/8}u)\|_{L_\tau^{2-}L_x^{\frac43+}([\frac1N,\infty))}\\
&\lesssim N^{-\frac12+}\|\tau^{-\frac12+}(N\tau)^{-\frac18}\|_{L_\tau^{2-}([\frac1N,\infty))}\cdot\sup_{\tau>\frac1N}\||x^1|^{\frac18}\tilde P_N F(\phi_{>N\tau/8}u)(\tau)\|_{L_x^{\frac43+}}\\
&\lesssim N^{-\frac12+}\|\tau^{-\frac12+}(N\tau)^{-\frac18}\|_{L_\tau^{2-}([\frac1N,\infty))}\cdot\sup_{\tau>\frac1N}\Big\|\||\nabla^1|^{\frac18-}\tilde P_N F(\phi_{>N\tau/8}u)(\tau)\|_{L_{x^1}^{\frac65}}\Big\|_{L_{x^2}^{\frac43+}}\\
&\lesssim N^{-\frac14+}\|\tau^{-\frac12-\frac18+}\|_{L_\tau^{2-}([\frac1N,\infty))}\sup_{\tau>\frac1N}\|\tilde P_N F(\phi_{>N\tau/8}u)(\tau)\|_{L_x^{\frac65}}\\
&\lesssim N^{-\frac18+}\sup_{\tau>\frac1N}N^{-s+}\||\nabla|^{s-}F(\phi_{>N\tau/8}u)\|_{L_x^{\frac65}}\\
&\lesssim N^{-s-\frac{1}{10}}.
\end{aligned}$$

Adding the two pieces together, we establish (4.23).

Now we establish (4.24). Again we split the term on the RHS into two terms by inserting spatial cutoffs $\phi_{>1/4}$ and $\phi_{\le 1/4}$ in front of $F(u)$. The term with cutoff $\phi_{\le1/4}$ gives us the contribution N^{-5} which is negligible. Applying the mismatch estimate and throwing away the bounded operator, the estimate of the term with cutoff $\phi_{>1/4}$ is essentially reduced to estimating the following

$$\left\|\int_0^{\frac1N} e^{-i\tau\Delta}\phi_{>1/4}\tilde P_N F(\phi_{>1/8}u)(\tau)d\tau\right\|_{L_x^2}. \tag{4.26}$$

We estimate (4.26) by using the weighted Strichartz Lemma 2.10, radial Sobolev embedding, Bernstein and the inductive assumption(4.17). Without loss of generality, assume $\phi_{>1/4} = \phi_{>1/4}\phi_{|x^1|>1/8}$. Let $s_0 = \min(\frac12, \frac45 s)$, we have

$$\begin{aligned}
(4.26) &\lesssim \||x^1|^{-\frac12+}\phi_{|x^1|>1/8}\tilde P_N F(\phi_{>1/8}u)\|_{L_\tau^{2-}L_x^{\frac43+}([0,\frac1N])}\\
&\lesssim N^{-\frac12-}\sup_{\tau\in\mathbb{R}}\|\tilde P_N F(\phi_{>1/8}u(\tau))\|_{L_x^{\frac43+}}\\
&\lesssim N^{-\frac12-}N^{6(\frac{15-2s_0}{18}-\frac34)+}N^{-s+}\sup_{\tau\in\mathbb{R}}\||\nabla|^{s-}F(\phi_{>1/8}u)(\tau)\|_{L_x^{\frac{18}{15-2s_0}}}\\
&\lesssim N^{-\frac12-}N^{\frac12-\frac23 s_0+}N^{-s+}\sup_{\tau\in\mathbb{R}}\|\phi_{>1/8}u\|_{L_x^{\frac{6}{3-s_0}}}^{\frac23}\||\nabla|^{s-}(\phi_{>1/8}u)\|_{L_x^2}\\
&\lesssim N^{-\frac23 s_0-s+}\sup_{\tau\in\mathbb{R}}\||\nabla|^{s-}(\phi_{>1/8}u)\|_{L_x^2}\||\nabla|^{s_0}(\phi_{>1/8}u)\|_{L_x^2}^{\frac23}\\
&\lesssim N^{-\frac23 s_0-s+}.
\end{aligned}$$

Collecting the estimates, we obtain

$$(4.20) \lesssim N^{-s-\min(\frac s3,\frac1{10})}.$$

The contribution due to (4.21) can be estimated similarly, so we have

$$(4.21) \lesssim N^{-s-\min(\frac s3,\frac1{10})}.$$

Therefore we obtain (4.19) and finish the discussion of step 2. The proof of Proposition 4.1 is then completed.

□

5. Local iteration to prove $u(t) \in H_x^{1+\frac{1}{200}}$

LEMMA 5.1. *For any $N \geq 1$ we have*

$$\|P_{\geq N} u_0\|_{L_x^2} \lesssim N^{-1-\frac{1}{10}} + \|P_{\geq N} F(u)\|_{L_{t,x}^{\frac{8}{5}}([0,\frac{1}{\sqrt{N}}])}. \tag{5.1}$$

PROOF. Since by Proposition 4.1, $\|\phi_{>1} P_{\geq N} u_0\|_{L_x^2} \lesssim N^{-1-\frac{1}{10}}$, we only need to estimate the near-origin piece $\|\phi_{\leq 1} P_{\geq N} u_0\|_{L_x^2}$. By the improved Duhamel formula Proposition 3.1, we get

$$\begin{aligned}
\|\phi_{\leq 1} P_{\geq N} u_0\|_{L_x^2} &= \left\| \phi_{\leq 1} P_{\geq N} \int_0^\infty e^{-i\tau\Delta} F(u(\tau)) d\tau \right\|_{L_x^2} \\
&\leq \left\| \phi_{\leq 1} P_{\geq N} \int_0^{\frac{1}{\sqrt{N}}} e^{-i\tau\Delta} F(u(\tau)) d\tau \right\|_{L_x^2} &(5.2)\\
&\quad + \left\| \phi_{\leq 1} P_{\geq N} \int_{\frac{1}{\sqrt{N}}}^\infty e^{-i\tau\Delta} \phi_{\leq N\tau/4} F(u(\tau)) d\tau \right\|_{L_x^2} &(5.3)\\
&\quad + \left\| \phi_{\leq 1} P_{\geq N} \int_{\frac{1}{\sqrt{N}}}^\infty e^{-i\tau\Delta} \phi_{> N\tau/4} F(u(\tau)) d\tau \right\|_{L_x^2}. &(5.4)
\end{aligned}$$

For (5.2), we use Strichartz to bound it by

$$\|P_{\geq N} F(u)\|_{L_{t,x}^{\frac{8}{5}}([0,\frac{1}{\sqrt{N}}])}.$$

For (5.3), using kernel estimate Lemma 2.8 with $m = 20$, we have

$$\begin{aligned}
(5.3) &\leq \sum_{M \geq N} \left\| \phi_{\leq 1} P_M \int_{\frac{1}{\sqrt{N}}}^\infty e^{-i\tau\Delta} \phi_{\leq N\tau/4} F(u(\tau)) d\tau \right\|_{L_x^2} \\
&\lesssim \sum_{M \geq N} M^{6-40} \int_{\frac{1}{\sqrt{N}}}^\infty \tau^{-20} \|\langle M|\cdot|\rangle^{-20} * F(u(\tau))\|_{L_x^2} d\tau \\
&\lesssim \sum_{M \geq N} M^{-34} M^{\frac{19}{2}} \|F(u)\|_{L_\tau^\infty L_x^{\frac{6}{5}}} \|\langle M|\cdot|\rangle^{-20}\|_{L_x^{\frac{3}{2}}} \\
&\lesssim \sum_{M \geq N} M^{-10} \\
&\lesssim N^{-5}.
\end{aligned}$$

The estimate of (5.4) follows the same way as the estimate of (4.7) or (4.25). We have

$$\begin{aligned}
(5.4) &\lesssim N^{-\frac{3}{16}+} \sup_{\tau \in [\frac{1}{\sqrt{N}},\infty)} \|\tilde{P}_N F(\phi_{>N\tau/8} u)\|_{L_x^{\frac{6}{5}}} + N^{-5} \\
&\lesssim N^{-1-\frac{3}{16}+} \sup_{\tau \in [\frac{1}{\sqrt{N}},\infty)} \|\nabla F(\phi_{>N\tau/8} u)\|_{L_x^{\frac{6}{5}}} + N^{-5} \\
&\lesssim N^{-1-\frac{1}{10}}.
\end{aligned}$$

Here in the last line, we have used the uniform boundedness estimate (4.2).

This finishes the proof of Lemma 5.1.

□

LEMMA 5.2 (Dual Strichartz norm control). *Let $\beta > 0$, $N_0 \geq 1$, $N > \frac{1}{\beta} N_0$. Then for any $0 < s < 1 + \frac{2}{3}$, we have*

$$\|P_{\geq N} F(u)\|_{L^{\frac{8}{5}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u\|^{\frac{2}{3}}_{S([0,\frac{1}{\sqrt{N}}])} \sum_{M \leq \beta N} \left(\frac{M}{N}\right)^s \|P_M u\|_{S([0,\frac{1}{\sqrt{N}}])}$$
$$+ \|u_{>\beta N}\|_{S([0,\frac{1}{\sqrt{N}}])} \left(N_0^{\frac{1}{2}} N^{-\frac{1}{8}} + \|u_{>N_0}\|^{\frac{1}{6}}_{L^\infty_\tau L^2_x} \|u_{>N_0}\|^{\frac{1}{2}}_{S([0,\frac{1}{\sqrt{N}}])} \right). \tag{5.5}$$

PROOF. By splitting u into low, medium and high frequencies:

$$u = u_{\leq N_0} + u_{N_0 < \cdot \leq \beta N} + u_{>\beta N},$$

we write

$$F(u) = F(u_{\leq \beta N}) + O(u_{>\beta N} |u_{\leq N_0}|^{\frac{2}{3}}) + O(u_{>\beta N} |u_{>N_0}|^{\frac{2}{3}}). \tag{5.6}$$

The contribution due to the first term can be estimated as follows. By using Lemma 2.5 and Bernstein, we have

$$\|P_{\geq N} F(u_{\leq \beta N})\|_{L^{\frac{8}{5}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim N^{-s} \| |\nabla|^s P_{\geq N} F(u_{<\beta N})\|_{L^{\frac{8}{5}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim N^{-s} \| |\nabla|^s u_{\leq \beta N}\|_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])} \|u_{\leq \beta N}\|^{\frac{2}{3}}_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u\|^{\frac{2}{3}}_{S([0,\frac{1}{\sqrt{N}}])} \sum_{M \leq \beta N} \left(\frac{M}{N}\right)^s \|P_M u\|_{S([0,\frac{1}{\sqrt{N}}])}.$$

For the contribution due to the second part of (5.6), we use Bernstein to get

$$\|u_{>\beta N} |u_{\leq N_0}|^{\frac{2}{3}}\|_{L^{\frac{8}{5}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u_{>\beta N}\|_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])} \|u_{\leq N_0}\|^{\frac{2}{3}}_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u_{>\beta N}\|_{S([0,\frac{1}{\sqrt{N}}])} N_0^{\frac{1}{2}} N^{-\frac{1}{8}} \|u_{\leq N_0}\|^{\frac{2}{3}}_{L^\infty_\tau L^2_x}$$
$$\lesssim \|u_{>\beta N}\|_{S([0,\frac{1}{\sqrt{N}}])} N_0^{\frac{1}{2}} N^{-\frac{1}{8}}.$$

To estimate the third term in (5.6), we have

$$\|u_{>\beta N} |u_{>N_0}|^{\frac{2}{3}}\|_{L^{\frac{8}{5}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u_{>\beta N}\|_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])} \|u_{>N_0}\|^{\frac{2}{3}}_{L^{\frac{8}{3}}_{t,x}([0,\frac{1}{\sqrt{N}}])}$$
$$\lesssim \|u_{>\beta N}\|_{S([0,\frac{1}{\sqrt{N}}])} \|u_{>N_0}\|^{\frac{1}{6}}_{L^\infty_t L^2_x([0,\frac{1}{\sqrt{N}}])} \|u_{>N_0}\|^{\frac{1}{2}}_{S([0,\frac{1}{\sqrt{N}}])}.$$

Collecting the three pieces together, we get (5.5). □

Now we use the above two lemmas to establish the recurrent relation for the local Strichartz norm. The precise quantity we consider is

$$A_N = \|P_{\geq N} u\|_{S([0,\frac{1}{\sqrt{N}}])}.$$

Our purpose is to prove that

$$A_N \leq C(\|u\|_{S([0,1])}) N^{-1-\frac{1}{100}}, \ \forall N \geq 1. \tag{5.7}$$

Here the constant depends on the Strichartz norm of u on unit time interval $[0,1]$ which is certainly bounded. This automatically gives

$$\|P_N u_0\|_{L_x^2} \lesssim N^{-1-\frac{1}{100}}. \tag{5.8}$$

In particular,

$$\|u_0\|_{H_x^{1+\frac{1}{200}}} \lesssim C(\|u\|_{S([0,1])}). \tag{5.9}$$

Now we establish the recurrent relation for A_N through which the bound (5.7) can be deduced. Let

$$A := \|u\|_{S([0,1])} + 1 < \infty.$$

We use Strichartz inequality, Lemma 5.1, Lemma 5.2 to estimate A_N. Fixing $s = \frac{4}{3}$, we obtain

$$\begin{aligned}
A_N &\lesssim \|P_{\geq N} u_0\|_{L_x^2} + \|P_{\geq N} F(u)\|_{L_{t,x}^{\frac{8}{5}}([0,\frac{1}{\sqrt{N}}])} \\
&\lesssim N^{-1-\frac{1}{10}} + \|P_{\geq N} F(u)\|_{L_{t,x}^{\frac{8}{5}}([0,\frac{1}{\sqrt{N}}])} \\
&\lesssim N^{-1-\frac{1}{10}} +
\end{aligned}$$

$$A^{\frac{2}{3}} \sum_{M \leq \beta N} \left(\frac{M}{N}\right)^{\frac{4}{3}} \|P_M u\|_{S([0,\frac{1}{\sqrt{N}}])} + \tag{5.10}$$

$$\|P_{\geq \beta N} u\|_{S([0,\frac{1}{\sqrt{N}}])} \left(N_0^{\frac{1}{2}} N^{-\frac{1}{8}} + A^{\frac{1}{2}} \|u_{\geq N_0}\|_{L_t^\infty L_x^2([0,\frac{1}{\sqrt{N}}])}^{\frac{1}{6}}\right). \tag{5.11}$$

For (5.10), we make a little modification. Noting $P_M = P_M P_{\geq M/2}$, we have

$$\begin{aligned}
(5.10) &\leq A^{\frac{2}{3}} \sum_{M \leq \beta N} \left(\frac{M}{N}\right)^{\frac{4}{3}} \|P_{\geq M/2} u\|_{S([0,\frac{1}{\sqrt{N}}])} \\
&\leq A^{\frac{2}{3}} \sum_{M \leq 2\beta N} \left(\frac{M}{N}\right)^{\frac{4}{3}} \|P_{\geq M} u\|_{S([0,\frac{1}{\sqrt{N}}])}.
\end{aligned}$$

Now we absorb (5.11) into (5.10) through taking suitable parameters. First we take $N_0 = N_0(\beta, A)$ such that

$$A^{\frac{1}{2}} \|u_{>N_0}\|_{L_t^\infty L_x^2([0,1])}^{\frac{1}{6}} \leq \frac{1}{100} \beta^{\frac{4}{3}}.$$

This is certainly possible since u is *almost periodic modula scaling* and $[0,1]$ is a compact interval. Then we assume $N \geq M_0$ where M_0 satisfies

$$M_0^{-\frac{1}{8}} N_0^{\frac{1}{2}} \leq \frac{1}{100} \beta^{\frac{4}{3}}. \tag{5.12}$$

Under these conditions we have

$$(5.11) \leq \frac{1}{2}\beta^{\frac{4}{3}}\|P_{\geq \beta N}u\|_{S([0,\frac{1}{\sqrt{N}}])}. \tag{5.13}$$

Therefore we get for all $N \geq M_0$ that

$$A_N \lesssim N^{-1-\frac{1}{10}} + A^{\frac{2}{3}} \sum_{M \leq 2\beta N} \left(\frac{M}{N}\right)^{\frac{4}{3}} \|P_{\geq M}u\|_{S([0,\frac{1}{\sqrt{N}}])}.$$

By taking β sufficiently small then taking M_0 large enough, we can kill the implicit constant above and get

$$A_N \leq N^{-1-\frac{1}{20}} + \sum_{M \leq 2\beta N} \left(\frac{M}{N}\right)^{\frac{7}{6}} \|P_{\geq M}u\|_{S([0,\frac{1}{\sqrt{N}}])}, \quad \forall\, N \geq M_0. \tag{5.14}$$

Now, we split the summation into $M \leq M_0$ and $M > M_0$. For large M, we trivially bound the summand by

$$\left(\frac{M}{N}\right)^{\frac{7}{6}} A_M.$$

Then we sum all the pieces for small M, this gives that

$$\sum_{M \leq M_0} \left(\frac{M}{N}\right)^{\frac{7}{6}} \|P_{\geq M}u\|_{S([0,\frac{1}{\sqrt{N}}])} \leq CAM_0^{\frac{7}{6}} N^{-\frac{7}{6}}.$$

Here the constant C is produced in dyadic summation. Finally we establish the following recurrent relation for A_N: There exists an absolute constant $C > 0$ such that for all $N \geq M_0$,

$$A_N \leq N^{-1-\frac{1}{20}} + CAM_0^{\frac{7}{6}} N^{-\frac{7}{6}} + \sum_{M_0 < N \leq 2\beta N} \left(\frac{M}{N}\right)^{\frac{7}{6}} A_M \tag{5.15}$$

We need the following lemma to get the final bound for A_N. We have

Lemma 5.3 (recursive control). *Let $s > 1$, $\gamma > 0$ and $s - \gamma > 1$. Let $C_1 > 0$ be such that for all $N \geq M_0$,*

$$A_N \leq C_1 M_0^s N^{-s} + \sum_{M_0 \leq N \leq \beta' N} \left(\frac{M}{N}\right)^{s} A_M, \tag{5.16}$$

$$A_N \leq A. \tag{5.17}$$

Then there exists a constant $c(s,\gamma,A) > 0$ such that for all $0 < \beta' < c(s,\gamma,A)$, we have

$$A_N \leq 2C_1 M_0^s N^{-s+\gamma},\ \forall N \geq M_0. \tag{5.18}$$

Proof. We will inductively prove

$$A_N \leq 2C_1 M_0^s N^{-s+\gamma} + (\beta')^j. \tag{5.19}$$

First, plugging the bound (5.17) into (5.16), we get

$$A_N \leq C_1 M_0 N^{-s} + C(s)A(\beta')^s \leq 2C_1 M_0 N^{-s+\gamma} + \beta'.$$

by requiring $(\beta')^{s-1} < \frac{1}{100C(s)A}$. This establishes (5.19) for $j = 1$.

Now assume (5.19) hold for j-th step, we plug this bound into (5.16) to compute

$$\begin{aligned} A_N &\le C_1 M_0^s N^{-s} + 2C(s)(\beta')^\gamma \cdot C_1 M_0^s N^{-s+\gamma} + C(s)(\beta')^{s-1} \cdot (\beta')^{j+1} \\ &\le 2C_1 M_0^s N^{-s+\gamma} + (\beta')^{j+1}. \end{aligned}$$

by requiring $(\beta')^\gamma < \frac{1}{100C(s)}$. This establishes (5.19) for $j+1$.

Finally, (5.18) follows by taking $j \to \infty$ in (5.19). □

Now we apply this lemma with $s = \frac{21}{20}$, $\gamma = \frac{1}{50}$. By taking β sufficiently small depending on A which is certainly possible, we obtain

$$A_N \le 4CAM_0^{\frac{21}{20}} N^{-1-\frac{1}{100}}, \tag{5.20}$$

for all sufficiently large frequencies. This together with the estimate on finite frequencies establishes (5.7)-(5.9). The $H_x^{1+\frac{1}{200}}$ regularity for u_0 is then established. This implies that $u(t) \in H_x^{1+\frac{1}{200}}$ for any t in the maximal lifespan, although the bound for this Sobolev norm bound may not be uniform in time.

6. Uniform localization of kinetic energy

In this section, we prove the uniform localization for kinetic energy (3.1) in Theorem 3.2. To this end, we will use Proposition 4.1, the fact $u \in H_x^{1+}$ which was proved in Section 5 and the following spatial decay estimate

Proposition 6.1. *Let u be the maximal-lifespan solution of* (1.1) *on I and obey Duhamel formula Proposition 3.1 and symmetry* (1.4). *Let $N_0 \le N_1$ be two arbitrary dyadic numbers. Then there exists $R_0 = R_0(N_0, N_1)$ such that for all $N \in [N_0, N_1]$ and $R \ge R_0$, we have*

$$\sup_{t\in I} \|\phi_{>R} P_N u(t)\|_{L_x^2} \le R^{-\frac{1}{10}}. \tag{6.1}$$

Proof. The proof follows the same argument as Proposition 4.2 in our previous paper [**18**]. Here we give a self-contained proof for the sake of completeness.

Without loss of generality we assume $t = 0$ and u is a global solution. The general case can be treated after minor notational changes. Now by symmetry and without loss of generality, it suffices to estimate

$$\|\phi_{>R} P_N^1 P_{\le N}^2 u_0\|_{L_x^2}. \tag{6.2}$$

This can be controlled by

$$\|\phi_{>R} \phi_{|x^1| \le \frac{1}{N}} P_N^1 P_{\le N}^2 u_0\|_{L_x^2} \tag{6.3}$$

$$+ \|\phi_{>R} \phi_{|x^1| > \frac{1}{N}} P_N^{1+} P_{\le N}^2 u_0\|_{L_x^2} \tag{6.4}$$

$$+ \|\phi_{>R} \phi_{|x^1| > \frac{1}{N}} P_N^{1-} P_{\le N}^2 u_0\|_{L_x^2} \tag{6.5}$$

To estimate (6.3), we first use Duhamel forward in time (or backward in time), then put spatial cutoffs in front of $F(u)$ to get the bound

$$(6.3) \le \left\| \phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 \int_0^\infty e^{-i\tau\Delta} F(u(\tau))d\tau \right\|_{L_x^2} \tag{6.6}$$

$$\le \left\| \phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 \int_0^{\frac{R}{100N}} e^{-i\tau\Delta} \phi_{\le R/4} F(u(\tau))d\tau \right\|_{L_x^2} \tag{6.7}$$

$$+ \left\| \phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 \int_0^{\frac{R}{100N}} e^{-i\tau\Delta} \phi_{>R/4} F(u(\tau))d\tau \right\|_{L_x^2} \tag{6.8}$$

$$+ \left\| \phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 \int_{\frac{R}{100N}}^\infty e^{-i\tau\Delta} \phi_{\le N\tau/4} F(u(\tau))d\tau \right\|_{L_x^2} \tag{6.9}$$

$$+ \left\| \phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 \int_{\frac{R}{100N}}^\infty e^{-i\tau\Delta} \phi_{>N\tau/4} F(u(\tau))d\tau \right\|_{L_x^2}. \tag{6.10}$$

Since we will take R large enough depending on N_0, N_1 and N varies on the interval $[N_0, N_1]$, we will not keep track of the precise power of N. We first deal with the tail terms (6.7) and (6.9). By using the kernel estimate, we have

$$\begin{aligned}
(6.7) &\lesssim \int_0^{\frac{R}{100N}} \|\phi_{>R} P_N^1 P_{\le N}^2 e^{-i\tau\Delta}\phi_{\le R/4} F(u(\tau))\|_{L_x^2} d\tau \\
&\lesssim N^C \int_0^{\frac{R}{100N}} (NR)^{-10} \|\langle|\cdot|\rangle^{-10} * F(u(\tau))\|_{L_x^2} d\tau \\
&\lesssim N^C R^{-9} \sup_\tau \|F(u(\tau))\|_{L_x^{\frac{6}{5}}} \\
&\lesssim R^{-5}.
\end{aligned}$$

Here N^C denotes a power of N whose precise value is not important. Similarly

$$\begin{aligned}
(6.9) \lesssim &\int_{\frac{R}{100N}}^\infty \|\phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 e^{-i\tau\Delta}\phi_{\le N\tau/4} P_{\le 8N} F(u(\tau))\|_{L_x^2} d\tau \\
+ &\int_{\frac{R}{100N}}^\infty \|\phi_{>R}\phi_{|x^1|\le\frac{1}{N}} P_N^1 P_{\le N}^2 e^{-i\tau\Delta}\phi_{\le N\tau/4} P_{>8N} F(u(\tau))\|_{L_x^2} d\tau.
\end{aligned}$$

The second term has arbitrary decay in R by using the mismatch estimate of the operator

$$P_N^1 P_{\le N}^2 \phi_{\le N\tau/4} P_{>8N}.$$

Therefore

$$(6.9) \lesssim \int_{\frac{R}{100N}}^\infty \|\phi_{|x^1|\le\frac{1}{N}} P_N^1 e^{-i\tau\Delta_1}\phi_{\le N\tau/4} P_{\le 8N} F(u(\tau))\|_{L_x^2} d\tau + R^{-5}.$$

Now the desired decay comes from the decay estimate of the kernel

$$\phi_{|x^1|\le\frac{1}{N}} P_N^1 e^{-i\tau\Delta_1}\phi_{\le N\tau/4}$$

in x^1 variable and Bernstein in the other. So finally we obtain for $R_0 = R_0(N_0, N_1)$ sufficiently large and $R \ge R_0$,

$$(6.9) \lesssim R^{-5}.$$

We now estimate (6.8) by weighted Strichartz estimate Lemma 2.10. Without loss of generality assume $\phi_{>R/4} = \phi_{>R/4}\phi_{|x^1|>R/8}$, we have

$$\begin{aligned}
(6.8) &\lesssim \| \int_0^{\frac{R}{100N}} e^{-i\tau\Delta}\phi_{>R/4}F(\phi_{>R/8}u)(\tau)d\tau\|_{L_x^2} \\
&\lesssim \||x^1|^{-\frac12+}\phi_{|x^1|>R/8}F(\phi_{>R/8}u)\|_{L_\tau^{2-}L_x^{\frac43+}([0,\frac{R}{100N}])} \\
&\lesssim R^{-\frac12+}(\frac{R}{100N})^{\frac12+}\|F(\phi_{>R/8}u)\|_{L_\tau^\infty L_x^{\frac43+}} \\
&\lesssim R^{0+}N^{-\frac12-}R^{-\frac18}\||x^1|^{\frac18}F(\phi_{>R/8}u)\|_{L_\tau^\infty L_x^{\frac43+}} \\
&\lesssim R^{-\frac19}\||\nabla^1|^{\frac18-}F(\phi_{>R/8}u)\|_{L_\tau^\infty L_{x^2}^{\frac43+}L_{x^1}^{\frac65}} \\
&\lesssim R^{-\frac19}\||\nabla^1|^{\frac18-}|\nabla^2|^{\frac14+}F(\phi_{>R/8}u)\|_{L_\tau^\infty L_x^{\frac65}} \\
&\lesssim R^{-\frac19}\|\phi_{>R/8}u\|_{L_\tau^\infty H_x^1}\|\phi_{>R/8}u\|^{\frac23}_{L_\tau^\infty L_x^2} \\
&\lesssim R^{-\frac19}.
\end{aligned}$$

The estimate of (6.10) follows the same way as this. So finally we obtain

$$(6.3) \lesssim R^{-\frac{1}{10}} \tag{6.11}$$

for all $R > R_0(N_0, N_1)$.

To obtain (6.1), we still have to deal with (6.4), (6.5). Since these two terms can be estimated similarly as before, we omit the details. The proof of the Proposition is then completed.

□

We have collected enough material to deduce the following uniform localization for the kinetic energy. We write as

PROPOSITION 6.2. *Let u be an H_x^1 maximal-lifespan solution on I. Assume there exist $\varepsilon > 0, \delta > 0$ such that*

$$\|\phi_{>1}P_N u(t)\|_{L_x^2} \lesssim N^{-1-\varepsilon},\ \forall\, t \in I,\ N \geq 1. \tag{6.12}$$

$$\|\phi_{>R}P_N u(t)\|_{L_x^2} \lesssim R^{-\delta},\ \forall\, t \in I, N \in [N_0', N_1'],\ R > R(N_0', N_1'). \tag{6.13}$$

Then the kinetic energy of u is uniformly localized in the following sense: for any $\eta > 0$, there exists $C(\eta) > 0$ such that

$$\sup_{t\in I}\|\phi_{>C(\eta)}\nabla u(t)\|_{L_x^2} \leq \eta. \tag{6.14}$$

PROOF. The proof of this proposition is essentially contained in [**18**]. Here we sketch the proof for the sake of completeness.

Let $N_1(\eta), N_2(\eta)$ be dyadic numbers and $C(\eta)$ a large constant to be specified later. We estimate the LHS of (6.14) by splitting it into low, medium and high frequencies:

$$\begin{aligned}
\|\phi_{>C(\eta)}\nabla u(t)\|_{L_x^2} \lesssim\ &\|\phi_{>C(\eta)}\nabla P_{\leq N_1(\eta)}u(t)\|_{L_x^2} \\
&+ \|\phi_{>C(\eta)}\nabla P_{N_1(\eta)<\cdot\leq N_2(\eta)}u(t)\|_{L_x^2} \\
&+ \|\phi_{>C(\eta)}\nabla P_{>N_2(\eta)}u(t)\|_{L_x^2}
\end{aligned}$$

For the low frequencies, we simply discard the cutoff and use Bernstein,

$$\|\phi_{>C(\eta)}\nabla P_{\le N_1(\eta)}u(t)\|_{L_x^2} \lesssim N_1(\eta)\|u(t)\|_{L_x^2} \lesssim N_1(\eta). \tag{6.15}$$

To estimate the medium frequencies, we use Bernstein, mismatch estimate Lemma 2.2 and (6.13) to obtain

$$\begin{aligned}
&\|\phi_{>C(\eta)}\nabla P_{N_1(\eta)<\cdot\le N_2(\eta)}u(t)\|_{L_x^2} \\
\le& \sum_{N_1(\eta)<N\le N_2(\eta)} \|\phi_{>C(\eta)}\nabla P_N u(t)\|_{L_x^2} \\
\le& C(N_1(\eta),N_2(\eta)) \max_{N_1(\eta)<N\le N_2(\eta)} \|\phi_{>C(\eta)}\nabla P_N u(t)\|_{L_x^2} \\
\le& C(N_1(\eta),N_2(\eta)) \max_{N_1(\eta)<N\le N_2(\eta)} \Big(\|\phi_{>C(\eta)}\nabla P_N \phi_{\le \frac{C(\eta)}{2}} \tilde{P}_N u(t)\|_{L_x^2} \\
&\quad + \|\phi_{>C(\eta)}\nabla P_N \phi_{> \frac{C(\eta)}{2}} \tilde{P}_N u(t)\|_{L_x^2} \Big) \\
\le& C(N_1(\eta),N_2(\eta)) \Big(\max_{N_1(\eta)<N\le N_2(\eta)} N^{-1}C(\eta)^{-2} + NC(\eta)^{-\delta} \Big) \\
\le& C(N_1(\eta),N_2(\eta)) \Big(C(\eta)^{-2} + C(\eta)^{-\delta} \Big) \\
\le& C(N_1(\eta),N_2(\eta))C(\eta)^{-\delta}.
\end{aligned}$$

For the high frequencies, we use mismatch estimate Lemma 2.2, Lemma 2.3 and (6.12) to get

$$\begin{aligned}
&\|\phi_{>C(\eta)}\nabla P_{>N_2(\eta)}u(t)\|_{L_x^2} \\
\le& \sum_{M>N_2(\eta)} \|\phi_{>C(\eta)}\nabla P_M u(t)\|_{L_x^2} \\
\le& \sum_{M\ge N_2(\eta)} \|\phi_{>C(\eta)}\nabla P_M \phi_{<C(\eta)/2}u(t)\|_{L_x^2} + \sum_{M\ge N_2(\eta)} M\|P_M\phi_{>C(\eta)/2}u(t)\|_{L_x^2} \\
\le& \sum_{M\ge N_2(\eta)} M^{-9}\cdot C(\eta)^{-10} + \sum_{M\ge N_2(\eta)} M\|P_M\phi_{>C(\eta)/2}\tilde{P}_M u(t)\|_{L_x^2} \\
&\quad + \sum_{M\ge N_2(\eta)} M\cdot M^{-10}\cdot C(\eta)^{-10} \\
\le& N_2(\eta)^{-2\varepsilon} + C(\eta)^{-10}N_2(\eta)^{-9}.
\end{aligned}$$

Therefore, the high frequencies give

$$\|\phi_{>C(\eta)}\nabla P_{>N_2(\eta)}u(t)\|_{L_x^2} \lesssim N_2(\eta)^{-\varepsilon} + C(\eta)^{-1}.$$

Adding the estimates of three pieces together we obtain

$$\begin{aligned}
\|\phi_{>C(\eta)}\nabla u(t)\|_{L_x^2} \lesssim& N_1(\eta) + C(N_1(\eta),N_2(\eta))C(\eta)^{-\delta} \\
&+ N_2(\eta)^{-\varepsilon} + C(\eta)^{-1}.
\end{aligned}$$

Now first taking $N_1(\eta)$ sufficiently small, $N_2(\eta)$ sufficiently large depending on η, then choosing $C(\eta)$ sufficiently large depending on (η, $N_1(\eta)$, $N_2(\eta)$), we obtain

$$\|\phi_{>C(\eta)}\nabla u(t)\|_{L_x^2} \le \eta,$$

as desired.

□

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