

Hausdorff dimension of random attractor for stochastic Navier-Stokes-Voight equations and primitive equations

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Communicated by Yuncheng You, received October 7, 2009.

ABSTRACT. We study the three dimensional stochastic Navier-Stokes-Voight model of viscoelastic incompressible fluid and three dimensional stochastic Primitive Equations with additive noise. In [12] and [11], we showed that the long time dynamics is captured by a random attractor, along this line, in this paper, we show that the Hausdorff dimension of attractor is an invariant random variable, and it is bounded by d , provided the random dynamics system contracts d -dimensional volumes exponentially fast.

CONTENTS

1. Introduction	307
2. Technical Preliminaries	308
3. Finite Hausdorff Dimension for SNSV	309
4. Finite Dimension of the Attractor for SPEs	317
References	325

1. Introduction

The need to take stochastic effects into account for modeling complex systems has now become widely recognized. Stochastic partial differential equations arise naturally as mathematical models for nonlinear macroscopic dynamics under random influences. It is thus desirable to understand the impact of such random influences on the system evolution [7], [21] and [23]. The theory of random attractors is becoming very useful for the study of the asymptotic behavior of dissipative

2000 *Mathematics Subject Classification.* Primary 35Q99, 60H15; Secondary 76M35, 86A05, 86A10.

Key words and phrases. Hausdorff dimension, Navier-Stokes-Voight equations, primitive equations.

Partially supported by a China NSF Grant No. 10871097 and National Basic Research Program of China (973 Program) No. 2007CB814800.

random dynamical systems, Crauel and Flandoli [5] and Crauel, Debussche and Flandoli [4] developed a theory for the existence of random attractors for stochastic systems that closely parallels the deterministic theory. A random attractor is a random invariant compact set which attracts every trajectory as time goes to infinite. Moreover, it can be proved that, in some cases, the random attractor has finite Hausdorff dimension. It seems that the asymptotic behavior of the random dynamical systems is governed by a finite number of degrees of freedom. Crauel and Flandoli [6] developed a method for bounding the Hausdorff dimension of attractors for certain systems, and their techniques required the noise to be bounded; Debussche [8] used a random squeezing property (cf. [10]) to bound the Hausdorff dimension without the assumption of bounded noise, a technique generalized to treat the fractal dimension by Langa [19]. However, the best bounds in the deterministic theory come not from a use of the squeezing property, but from the method involving Lyapunov exponents developed by Constantin, Foias and Temam [3]. It is this method that was adapted to the stochastic case by Debussche [9] to obtain an upper bound on the Hausdorff dimension. In his paper he remarks that the same arguments could be used to obtain a bound on the fractal dimension of such sets.

In this paper, we study the three dimensional Stochastic Navier-Stokes-Voigt model of viscoelastic incompressible fluid (SNSV) and three dimensional Stochastic Primitive Equations (SPEs) with additive noise. In [12] and [11], we showed that the long time dynamics is captured by a random attractor. Using the method in [6], we show that the Hausdorff dimension is an invariant random variable, and it is bounded by d , provided the random dynamics system contracts d -dimensional volumes exponentially fast.

This paper is organized as follows. Some technical preliminaries to obtain the finite Hausdorff dimension is in §2. Finite Hausdorff dimension of random attractor for SNSV is showed in §3. Finally, finite Hausdorff dimension of random attractor for SPEs is showed in §4.

2. Technical Preliminaries

In order to obtain the finite Hausdorff dimension, we should need some technical preliminaries.

DEFINITION 2.1. *The Hausdorff dimension $\dim_H(Y)$ of a subset Y of a metric space is*

$$\dim_H(Y) = \inf\{s \geq 0 : \mu_H(Y, s) = 0\} = \sup\{s \geq 0 : \mu_H(Y, s) > 0\},$$

where $\mu_H(Y, s)$ is the s -dimensional Hausdorff measure of Y (see Definition in Temam [22]).

We next show that volume contraction of an random dynamical system (RDS) on a random compact invariant set implies that the Hausdorff of this set is finite.

Suppose S is a continuous RDS on a separable Hilbert space H with norm $|\cdot|$. L is a bounded linear operator on H . $\wedge^d L$ denotes the d -fold exterior product of L , and $\|\cdot\|$ denotes the operator norm defined by

$$\|\wedge^d L\|_{\mathcal{L}(H^d, \wedge^d H)} = \sup_{\varphi_1, \dots, \varphi_d \in H; |\varphi_i|_H \leq 1, \forall i} |(\wedge^d L)(\varphi_1, \dots, \varphi_d)|_{\wedge^d H}$$

where

$$(\wedge^d L)(\varphi_1, \dots, \varphi_d) = L\varphi_1 \wedge \dots \wedge L\varphi_d.$$

Here we briefly recall the definitions of the exterior product of spaces and vectors in the case of Hilbert spaces, for further details the reader is referred to R. Temam [22].

Suppose $w \mapsto X(w)$ is a compact strictly invariant set for S . We assume that S is weakly differentiable on X in the sense that (for \mathbb{P} -almost all w) for every $u \in X(w)$ and $t > 0$ there exists a linear map $D_u S(t, w) : H \rightarrow H$ such that

$$(2.1) \quad g_\delta(t, w) = \sup \left\{ \frac{|S(t, w)u - S(t, w)v - D_u S(t, w)(u - v)|}{|u - v|} : u, v \in X(w), |u - v| \leq \delta \right\}$$

is finite for any $\delta > 0$, and converges to zero \mathbb{P} -a.s. as $\delta \rightarrow 0$ for any $t > 0$ fixed. In case S is Frécher differentiable in x , the map $D_x S(t, w)$ is the differential. Put

$$\gamma_1(t, w) = \sup_{u \in X(w)} \|D_u S(t, w)\|, \quad \Gamma_d(t, w) = \sup_{u \in X(w)} \|\wedge^d D_u S(t, w)\|.$$

The following is the main theorem to obtain the finite Hausdorff dimension.

THEOREM 2.2. [6] *Suppose S is an RDS on a separable Hilbert space H . Let $w \mapsto X(w)$ be a random compact set, strictly invariant for S , and suppose S is weakly differentiable on X in the sense of (2.1). Suppose that there exist d and $t_0 > 0$ such that \mathbb{P} -a.s.*

$$\sup_{u \in X(w)} \|\wedge^d D_u S(t_0, w)\| = \Gamma_d(t_0, w) < 1.$$

Suppose further that

- $\gamma_1(t, w) \in L^\infty(\Omega; \mathbb{P})$ for every $t \geq 0$.
- $g_\delta(t, w) \in L^\infty(\Omega; \mathbb{P})$ for every $t \geq 0$.
- $g_\delta(t, w)$ converges to zero in $L^\infty(\Omega; \mathbb{P})$ for every $t > 0$.

Then the Hausdorff dimension of X satisfies $\dim_H(X) \leq d$.

3. Finite Hausdorff Dimension for SNSV

3.1. Existence of the Attractor. We consider the three-dimensional Stochastic Navier-Stokes-Voigt (SNSV) system of equations

$$(3.1) \quad v_t - \nu \Delta v - \alpha^2 \Delta v_t + (v \cdot \nabla)v + \nabla p = f(x) + n(t), \quad x \in D, \quad t \in \mathbb{R},$$

$$(3.2) \quad \operatorname{div} v = 0, x \in D; \quad v(x, t) = 0, \quad x \in \partial D,$$

$$(3.3) \quad v(x, 0) = v_0(x), \quad x \in D,$$

where $D \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary ∂D , $v = v(x, t)$ is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity, α is a length scale parameter characterizing the elasticity of the fluid, f is a given force field, $n(t)$ are noise forcing terms and are taken as $\frac{\partial}{\partial t} W(t)$, here we assume that $W(t)$ is an H -value infinite dimensional Brownian motion of the form

$$W(t) = \sum_{j=1}^{\infty} \sigma_j \beta_j(t) e_j,$$

where β_1, β_2, \dots is a sequence of independent standard Brownian motions on a complete probability space (Ω, \mathcal{F}, P) (with expectation denoted by \mathbb{E}) and $(e_j)_{j \in \mathbb{N}}$ is an orthonormal basis of a function space H which we'll give below.

The system (1.1)-(1.3) models the stochastic dynamical of a Kelvin-Voight viscoelastic incompressible fluid. The determining model (for $n(t) = 0$) was introduced in [20] as a model of motion of linear viscoelastic fluids. The authors of [1] proposed the inviscid simplified Bardina model (or equivalently the inviscid NSV equations) as a non-viscous (inviscid) regularization of the 3D Euler equations, subject to periodic boundary conditions. Motivated by this observation the system (1.1)-(1.3) was also proposed in [1] as a regularization, for small values of α , of the 3D Navier-Stokes (NS) equations for the purpose of direct numerical simulations for both the periodic and the no-slip Dirichlet boundary conditions. In [20] it is shown that the initial boundary value problem (1.1)-(1.3) has a unique weak solution. In [15] and [16], it is shown that the semigroup generated by the problem (1.1)-(1.3) has a finite dimensional global attractor. Recently Cao, Lunasin, Titi [1] have obtained the global regularity of the inviscid version of 3D Kelvin-Voight model. The Gevrey regularity of the global attractor and determining modes for the 3D Navier-Stokes-Voight equations (Kelvin-Voight equation) are given by [17]-[18].

In this section, as in [12], we use the following notations in the mathematical theory for SNSV equations:

- $L^p(D)$, $1 \leq p \leq \infty$, and $H^s(D)$ are the usual Lebesgue and Sobolev spaces, respectively.
- For $v = (v_1, v_2, v_3)$ and $u = (u_1, u_2, u_3)$, we denote by

$$(u, v) = \sum_{j=1}^3 (u_j, v_j)_{L_2(D)}, \quad \|v\|^2 = \sum_{j=1}^3 \|v_j\|_{L_2(D)}^2, \quad \|\nabla v\|^2 = \sum_{i,j=1}^3 \|\partial_i v_j\|_{L_2(D)}^2.$$

- We set

$$\mathcal{V} := \{v \in (C_0^\infty(D))^3; \nabla \cdot v = 0\}.$$

- H is the closure of the set \mathcal{V} in $(L_2(D))^3$ topology.
- P is the Helmholtz-Leray orthogonal projection in $(L_2(D))^3$ onto the space H .
- $A := -P\Delta$ is the Stokes operator subject to the no-slip homogeneous Dirichlet boundary condition with the domain $(H_2(D))^3 \cap V$. The operator A is a selfadjoint positively definite operator in H , whose inverse A^{-1} is a compact operator from H into H . Thus it has an orthonormal system of eigenfunctions $\{e_j\}_{j=1}^\infty$ of A .
- We denote by $\{\lambda_j\}_{j=1}^\infty$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$, the eigenvalues of the Stokes operator A corresponding to eigenfunctions $\{e_j\}_{j=1}^\infty$, repeated according to their multiplicities.
- $V_s := D(A^{s/2})$, $\|v\|_s = \|A^{s/2}v\|$, $s \in \mathbb{R}$. $V = V_1 = (H_0^1(D))^3 \cap H$ is the Hilbert space with the norm $\|\cdot\|_1$, thanks to the Poincaré inequality. Clearly $V_0 = H$.
- For $u, v, w \in \mathcal{V}$ we define the following bilinear form

$$B(u, v) := P((u \cdot \nabla)v) \quad \text{and the trilinear form } b(u, v, w) = (B(u, v), w).$$

The bilinear form $B(\cdot, \cdot)$ can be extended as a continuous operator $B : V \times V \rightarrow V'$, where V' is the dual of V (see, e.g., [2]).

- For each $u, v, w \in V$,

$$(3.4) \quad b(u, v, v) = 0 \quad \text{and} \quad b(u, v, w) = -b(u, w, v).$$

In [12], for convenience, we noted $\alpha = 1$ and considered the auxiliary equation

$$(3.5) \quad (I + A)h_t + (\nu + \tau)(I + A)h = \sum_{j=1}^{\infty} \sigma_j \frac{d\beta_j(t)}{dt} e_j,$$

since $(I + A)$ is positive self-adjoint operators, $(I + A)^{-1}$ exists and is compact, and $(I + A)^{-1}e_j = (1 + \lambda_j)^{-1}e_j$. Equation (3.5) equivalently the following Ornstein-Uhlenbeck equation

$$(3.6) \quad h_t + (\nu + \tau)h = \frac{dn_1(t)}{dt},$$

where $n_1(t) = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-1} \sigma_j \beta_j(t) e_j$. Since $W(t)$ is an H -valued Wiener process, we obtained that the process

$$(3.7) \quad h(t) = \int_{-\infty}^t e^{-(t-s)(\nu+\tau)} dn_1(s)$$

is a stationary ergodic solution with continuous trajectories, take values in $D(A)$. In particular,

$$\mathbb{E}\|Ah(t)\|^2 = \sum_{j=1}^{\infty} \frac{\lambda_j^2 \sigma_j^2}{2(\nu + \tau)(1 + \lambda_j)^2} < \infty.$$

In a similar way, we have $\mathbb{E}\|h\|^2, \mathbb{E}\|A^{1/2}h\|^2 < \infty$. However, when $\alpha \neq 1$, above results are also correct.

Then we can transform the SNSV to a random partial differential equation. The main advantage of this method is that we can now solve our equation ω -wise, because of the absence of stochastic integrals. In this way we avoid the issue of exceptional sets of the solution depending on the initial condition.

Using the change of variable $u(t) = v(t) - h(t)$, then $u(t, x)$ satisfies the equation (which depends on a random parameter):

$$(3.8) \quad u_t + \nu Au + Au_t + B(u + h, u + h) = f(x) + (\nu + \tau)h + \tau Ah,$$

$$(3.9) \quad u(0, x) = v_0 - h_0.$$

Now we give the well-posedness of the equation (3.8)-(3.9).

THEOREM 3.1. (see [12]) *Assume that $t_0 \in \mathbb{R}$ and $u_0 \in V$, for \mathbb{P} -almost every $\omega \in \Omega$ the following holds:*

(i) *there exist a unique solution of (3.8)-(3.9) on the time interval $[t_0, \infty)$ and satisfies $u \in C([t_0, \infty), V)$.*

(ii) *denoting such solution by $u(t, \omega; t_0, u_0)$, the mapping $u_0 \mapsto u(t, \omega; t_0, u_0)$ is continuous for all $t \geq t_0$.*

Then having the mapping $v_0 \mapsto u(t, w; 0, u_0) + h(t, w)$, one can define a stochastic flow $S(t, w)$ by

$$(3.10) \quad S(t, w)v_0 = u(t, w; 0, u_0) + h(t, w).$$

Also in [12], using uniform estimates and asymptotic compactness, we showed that the long time dynamics is captured by a random attractor.

THEOREM 3.2. (see [12]) Let $K(w)$ be the ball in V of radius $R_0(w) = r_0(w)^{\frac{1}{2}} + \|h(0, w)\|_1$, where

$$\begin{aligned} r_0(w) &= C \int_{-\infty}^0 (g(\sigma) + C\|f\|^2) \exp\left(-\int_{\sigma}^0 [d_0 - C(\|h(\varsigma)\| + \|h(\varsigma)\|_1)] d\varsigma\right) d\sigma \\ &\quad + C \sup_{s \in (-\infty, 0]} \left([\|h(s)\|^2 + \|h(s)\|_1^2] \exp\left(s \frac{d_0}{2}\right) \right) + 1, \\ d_0 &= \frac{\nu}{2} \min\{1, \lambda_1\}, \\ g(t) &= C(\|h\|^2 + \|h\|_1^2 + \|h\|_1^3 \|h\|), \end{aligned}$$

such that for any B bounded in V , there exists $s_1(w, B)$, for $s < s_1(w, B)$, we have

$$(3.11) \quad S(0, s; w)B \subset K(w), \quad \mathbb{P} - a.s.$$

and the stochastic flow S is asymptotically compact in V for \mathbb{P} -a.s., thus we obtain a random attractor $\mathcal{A}(\omega)$ which attracts every bounded nonrandom set $B \subset V$.

3.2. Volume Contraction. Consider the differential equation of (3.1) on V

$$(3.12) \quad U_t(t, w) + \alpha^2 AU_t(t, w) = F'(v(t))U(t),$$

$$(3.13) \quad U(0, w) = \xi \in V,$$

where

$$F'(v(t))U = -\nu AU(t, w) - B(S(t, w)v_0, U(t, w)) - B(U(t, w), S(t, w)v_0).$$

The existence of solution $U \in C([0, T]; V)$ for all $T > 0$ can be established, path by path, as in the deterministic case in [18]. One can show that the flow $S(t, w)$ associated with (3.12)-(3.13) has the property that $v \mapsto S(t, w)v$ is differentiable and for each v_0 and $\xi \in V$ the differential $D_{v_0}S(t, w)\xi$ solves (3.12).

Before we give the contraction of sufficiently high dimensional volumes, the following two lemma is crucial.

LEMMA 3.3. For every family of elements U_1, \dots, U_d of V , we have

$$\begin{aligned} &\left(F'(v(t))U_1 \wedge (I - \alpha^2 \Delta)U_2 \wedge \dots \wedge (I - \alpha^2 \Delta)U_d, U_1 \wedge \dots \wedge U_d \right)_{\wedge^d H} + \dots \\ &\quad + \left((I - \alpha^2 \Delta)U_1 \wedge (I - \alpha^2 \Delta)U_2 \wedge \dots \wedge F'(v(t))U_d, U_1 \wedge \dots \wedge U_d \right)_{\wedge^d H} \\ (3.14) \quad &= |U_1 \wedge \dots \wedge U_d|_{\wedge^d V}^2 \sum_{j=1}^d \langle F'(v(t))\vartheta_j, \vartheta_j \rangle_H, \end{aligned}$$

where ϑ_j $j = 1, \dots, d$, be an orthonormal basis of $\text{Span}[U_1, \dots, U_d]$ in V .

PROOF. As in [22], we consider the case where U_1, \dots, U_d are linearly independent, $U_1 \wedge \dots \wedge U_d \neq 0$ and, without loss of generality, we can assume that U_1, \dots, U_d is an orthogonal family of V . If U_1, \dots, U_d is not orthogonal we can consider the orthogonal family ψ_1, \dots, ψ_d associated with U_1, \dots, U_d , by the usual Gram-Schmidt process. Since

$$\psi_j \in U_j + \text{Span}\{U_1, \dots, U_{j-1}\}, \quad j = 1, \dots, d,$$

we have clear

$$(3.15) \quad \psi_1 \wedge \dots \wedge \psi_d = U_1 \wedge \dots \wedge U_d$$

and

$$(3.16) \quad |\psi_1 \wedge \cdots \wedge \psi_d|_{\wedge^d V}^2 = \|\psi_1\|_V^2 \cdots \|\psi_d\|_V^2.$$

Furthermore, since linearity of $F'(v(t))$ and $I - \alpha^2 \Delta$, by the fact that ψ_1, \dots, ψ_d is the orthogonal family in V and (3.16), we obtain

$$\begin{aligned} & \left(F'(v(t))U_1 \wedge (I - \alpha^2 \Delta)U_2 \wedge \cdots \wedge (I - \alpha^2 \Delta)U_d, U_1 \wedge \cdots \wedge U_d \right)_{\wedge^d H} + \cdots \\ & + \left((I - \alpha^2 \Delta)U_1 \wedge (I - \alpha^2 \Delta)U_2 \wedge \cdots \wedge F'(v(t))U_d, U_1 \wedge \cdots \wedge U_d \right)_{\wedge^d H} \\ & = \left(F'(v(t))\psi_1 \wedge (I - \alpha^2 \Delta)\psi_2 \wedge \cdots \wedge (I - \alpha^2 \Delta)\psi_d, \psi_1 \wedge \cdots \wedge \psi_d \right)_{\wedge^d H} + \cdots \\ & + \left((I - \alpha^2 \Delta)\psi_1 \wedge (I - \alpha^2 \Delta)\psi_2 \wedge \cdots \wedge F'(v(t))\psi_d, \psi_1 \wedge \cdots \wedge \psi_d \right)_{\wedge^d H} \\ & = \langle F'(v(t))\psi_1, \psi_1 \rangle_H \|\psi_2\|_V^2 \cdots \|\psi_d\|_V^2 + \cdots \\ & + \|\psi_1\|_V^2 \cdots \|\psi_{d-1}\|_V^2 \langle F'(v(t))\psi_d, \psi_d \rangle_H \\ & = \|\psi_1\|_V^2 \cdots \|\psi_d\|_V^2 \sum_{j=1}^d \left\langle \frac{F'(v(t))\psi_j}{\|\psi_j\|_V}, \frac{\psi_j}{\|\psi_j\|_V} \right\rangle_H \\ & = |\psi_1 \wedge \cdots \wedge \psi_d|_{\wedge^d V}^2 \sum_{j=1}^d \left\langle \frac{F'(v(t))\psi_j}{\|\psi_j\|_V}, \frac{\psi_j}{\|\psi_j\|_V} \right\rangle_H \\ & = |U_1 \wedge \cdots \wedge U_d|_{\wedge^d V}^2 \sum_{j=1}^d \left\langle \frac{F'(v(t))\psi_j}{\|\psi_j\|_V}, \frac{\psi_j}{\|\psi_j\|_V} \right\rangle_H \\ (3.17) \quad & = |U_1 \wedge \cdots \wedge U_d|_{\wedge^d V}^2 \sum_{j=1}^d \langle F'(v(t))\vartheta_j, \vartheta_j \rangle_H. \end{aligned}$$

□

LEMMA 3.4. *Suppose $\xi_1, \dots, \xi_d \in V$, and denote $D_{v_0}S(t, w)\xi_k$ by U_k , $k = 1, \dots, d$. Then U_k solves (3.12)-(3.13), $k = 1, \dots, d$, and*

$$(3.18) \quad \begin{aligned} & |U_1(t) \wedge \cdots \wedge U_d(t)|_{\wedge^d V}^2 \\ & = |\xi_1 \wedge \cdots \wedge \xi_d|_{\wedge^d V}^2 \exp\left(\int_0^t \sum_{j=1}^d \langle F'(v(\tau))\vartheta_j, \vartheta_j \rangle_H d\tau \right). \end{aligned}$$

PROOF. We rewrite the norm

$$(3.19) \quad \|\phi\|_V^2 = \|\phi\|^2 + \alpha^2 \|\phi\|_1^2$$

for all $\phi \in V$, thanks to the Poincaré inequality, we have this norm is equivalent to the former norm $\|\cdot\|_1$.

By time differentiation, applying (3.19) and Lemma 3.3, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} |U_1(t) \wedge \cdots \wedge U_d(t)|_{\wedge^d V}^2 \\
&= \left(\frac{d}{dt} (U_1(t) \wedge \cdots \wedge U_d(t)), U_1(t) \wedge \cdots \wedge U_d(t) \right)_{\wedge^d V} \\
&= (U'_1(t) \wedge \cdots \wedge U_d(t), U_1(t) \wedge \cdots \wedge U_d(t))_{\wedge^d V} \\
&\quad + \cdots + (U_1 \wedge \cdots \wedge U'_d(t), U_1(t) \wedge \cdots \wedge U_d(t))_{\wedge^d V} \\
&= \left| \begin{array}{cccc} \langle U'_1, U_1 \rangle_V & \langle U'_1, U_2 \rangle_V & \cdots & \langle U'_1, U_d \rangle_V \\ \langle U_2, U_1 \rangle_V & \langle U_2, U_2 \rangle_V & \cdots & \langle U_2, U_d \rangle_V \\ \cdots & \cdots & \cdots & \cdots \\ \langle U_d, U_1 \rangle_V & \langle U_d, U_2 \rangle_V & \cdots & \langle U_d, U_d \rangle_V \end{array} \right| + \cdots \\
&\quad + \left| \begin{array}{cccc} \langle U_1, U_1 \rangle_V & \langle U_1, U_2 \rangle_V & \cdots & \langle U_1, U_d \rangle_V \\ \langle U_2, U_1 \rangle_V & \langle U_2, U_2 \rangle_V & \cdots & \langle U_2, U_d \rangle_V \\ \cdots & \cdots & \cdots & \cdots \\ \langle U'_d, U_1 \rangle_V & \langle U'_d, U_2 \rangle_V & \cdots & \langle U'_d, U_d \rangle_V \end{array} \right| \\
&= \left| \begin{array}{cccc} \langle (I - \alpha^2 \Delta) U'_1, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U'_1, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U'_1, U_d \rangle_H \\ \langle (I - \alpha^2 \Delta) U_2, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U_2, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U_2, U_d \rangle_H \\ \cdots & \cdots & \cdots & \cdots \\ \langle (I - \alpha^2 \Delta) U_d, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U_d, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U_d, U_d \rangle_H \end{array} \right| + \cdots \\
&\quad + \left| \begin{array}{cccc} \langle (I - \alpha^2 \Delta) U_1, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U_1, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U_1, U_d \rangle_H \\ \langle (I - \alpha^2 \Delta) U_2, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U_2, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U_2, U_d \rangle_H \\ \cdots & \cdots & \cdots & \cdots \\ \langle (I - \alpha^2 \Delta) U'_d, U_1 \rangle_H & \langle (I - \alpha^2 \Delta) U'_d, U_2 \rangle_H & \cdots & \langle (I - \alpha^2 \Delta) U'_d, U_d \rangle_H \end{array} \right| \\
&= \left(F'(v(t)) U_1 \wedge (I - \alpha^2 \Delta) U_2 \wedge \cdots \wedge (I - \alpha^2 \Delta) U_d, U_1 \wedge \cdots \wedge U_d \right)_{\wedge^d H} + \cdots \\
&\quad + \left((I - \alpha^2 \Delta) U_1 \wedge (I - \alpha^2 \Delta) U_2 \wedge \cdots \wedge F'(v(t)) U_d, U_1 \wedge \cdots \wedge U_d \right)_{\wedge^d H} \\
(3.20) \quad &= |U_1 \wedge \cdots \wedge U_d|_{\wedge^d V}^2 \sum_{j=1}^d \langle F'(v(t)) \vartheta_j, \vartheta_j \rangle_H.
\end{aligned}$$

Therefore

$$\begin{aligned}
& |U_1(t) \wedge \cdots \wedge U_d(t)|_{\wedge^d V} \\
&= |\xi_1 \wedge \cdots \wedge \xi_d|_{\wedge^d V} \exp \left(\int_0^t \sum_{j=1}^d \langle F'(v(\tau)) \vartheta_j, \vartheta_j \rangle_H d\tau \right).
\end{aligned}$$

□

REMARK 3.5. *It is worth stressing that since adding the regularizing term $(-\alpha \Delta v_t)$ to the Navier Stokes equations, the linear initial-value problem (3.12)-(3.13) is different to the one in Section V.2.3 in [22], in which for every $d \in \mathbb{N}$,*

$$\begin{aligned}
(3.21) \quad & |U_1(t) \wedge \cdots \wedge U_d(t)|_{\wedge^d V}^2 \\
&= |\xi_1 \wedge \cdots \wedge \xi_d|_{\wedge^d V}^2 \exp \left(\int_0^t \text{Tr} F'(S(\tau) \chi_0) \circ Q_d(\tau) d\tau \right),
\end{aligned}$$

where $S(\tau)\chi_0 = \chi(\tau) = (v(\tau), \theta(\tau))$ and Q_d is the orthogonal projector in V onto the space spanned by $U_1(\tau), \dots, U_d(\tau)$. However, applying Lemma 3.3 and Lemma 3.4, we have (3.18) here.

Volume contraction is shown as follows.

THEOREM 3.6. *Let S be the flow associated with (3.10) and let $w \mapsto \mathcal{A}(w)$ be the global attractor given by Theorem 3.2, then there exist a positive integer d and a time $t > 0$, such that*

$$(3.22) \quad \|\Gamma_d(t, \cdot)\|_{L^\infty} < 1,$$

where $\Gamma_d(t, \cdot) = \sup_{v \in \mathcal{A}(w)} \|\wedge^d D_v S(t, w)\|$.

PROOF. Suppose $\xi_1, \dots, \xi_d \in V$, $v_0 \in V$, and denote $D_{v_0} S(t, w)\xi_k$ by U_k , $k = 1, \dots, d$. Then according to Lemma 3.4, we have

$$(3.23) \quad \begin{aligned} & |U_1(t) \wedge \dots \wedge U_d(t)|_{\wedge^d V}^2 \\ &= |\xi_1 \wedge \dots \wedge \xi_d|_{\wedge^d V}^2 \exp\left(\int_0^t \sum_{j=1}^d \langle F'(v(\tau))\vartheta_j, \vartheta_j \rangle_H d\tau\right). \end{aligned}$$

By the Gagliardo-Nirenberg inequality, Poincaré inequality and the Young's inequality, we deduce

$$(3.24) \quad \begin{aligned} & \sum_{j=1}^d \langle F'(v(t))\vartheta_j, \vartheta_j \rangle_H \\ &= \sum_{j=1}^d \langle -\nu A\vartheta_j + B(S(t, w)v_0, \vartheta_j) + B(\vartheta_j, S(t, w)v_0), \vartheta_j \rangle_H \\ &\leq -\nu \sum_{j=1}^d \|\vartheta_j\|_1^2 + \left| \sum_{j=1}^d b(\vartheta_j, v, \vartheta_j) \right| \\ &\leq -\nu \sum_{j=1}^d \|\vartheta_j\|_1^2 + \sum_{j=1}^d \|\vartheta_j\|_{L^4}^2 \|v\|_1 \\ &\leq -\nu \sum_{j=1}^d \|\vartheta_j\|_1^2 + \|v\|_1 \sum_{j=1}^d \|\vartheta_j\|_1^{\frac{3}{2}} \|\vartheta_j\|_1^{\frac{1}{2}} \\ &\leq -\nu d + \|v\|_1 \sum_{j=1}^d \|\vartheta_j\|_1^{\frac{1}{2}} \\ &\leq -\nu d + \frac{\nu}{2} \sum_{j=1}^d \|\vartheta_j\|_1 + c\|v\|_1^2 \\ &\leq -\frac{\nu}{2}d + c\|v\|_1^2. \end{aligned}$$

Therefore

$$(3.25) \quad \begin{aligned} & |U_1(t) \wedge \dots \wedge U_d(t)|_{\wedge^d V}^2 \\ &\leq |\xi_1 \wedge \dots \wedge \xi_d|_{\wedge^d V}^2 \exp\left(-\frac{\nu dt}{2} + c \int_0^t \|v(\tau)\|_1^2 d\tau\right), \end{aligned}$$

for every fixed $t > 0$.

Since $v(\tau) = S(\tau, w)v_0 \in \mathcal{A}(\theta_\tau w)$ and Theorem 3.2, moreover, by ergodicity of the shift θ_τ , we obtain, for \mathbb{P} -a.s.,

$$(3.26) \quad \begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|v(\tau)\|_1^2 d\tau &\leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t R_0(\theta_\tau w)^2 d\tau \\ &\leq \mathbb{E}R_0(w)^2, \end{aligned}$$

then for $\forall t > 0$, we have $\int_0^t \|v(\tau)\|_1^2 d\tau \leq c_1(w) + c_2(w)t$, where the constant c_1 and c_2 are random and finite \mathbb{P} -a.s..

Recalling the notations for $\Gamma_d(t, \cdot)$, we get

$$(3.27) \quad \Gamma_d(t, w) \leq e^{(-\frac{\nu d}{2} + c_2)t + c_1}, \quad \mathbb{P}\text{-a.s.}$$

for every $t > 0$, let d sufficiently large, we obtain $\|\Gamma_d(t, \cdot)\|_{L^\infty} < 1$ for \mathbb{P} -a.s.. \square

3.3. Finite Dimension of the Attractor.

THEOREM 3.7. *The Hausdorff dimension of the global attractor for the SNSV is finite.*

PROOF. From Theorem 3.6 we get existence of some d such that

$$\|\Gamma_d(t, \cdot)\|_{L^\infty} < 1.$$

This is the first basic estimate we need to apply Theorem 2.2.

From the proof of Theorem 3.6, we get for the particular case $d = 1$

$$\|U_1\|_V \leq \|\xi_1\|_V e^{c_1 + c_2 t}.$$

Thus the assumption $\gamma_1(t) \in L^\infty(\Omega, \mathbb{P})$ is fulfilled. Finally, we have to prove that $g_\delta(t)$ convergence to 0 in $L^\infty(\Omega, \mathbb{P})$ for each $t > 0$.

Let $x \in V$ and $x_0 \in \mathcal{A}(w)$ be given, and put $u(t) = S(t, w)x$, $v(t) = S(t, w)x_0$, $z(t) = D_{x_0}S(t, w)(x - x_0)$ and $\eta(t) = u(t) - v(t) - z(t)$. Then

$$(3.28) \quad (u - v)_t + \alpha^2 A(u - v)_t = -\nu A(u - v) - B(u, u) + B(v, v)$$

and

$$z_t + \alpha^2 z_t = -\nu A z_t - B(v, z) - B(z, v),$$

hence

$$\eta_t + \alpha^2 \eta_t = -\nu A \eta - B(v, \eta) - B(\eta, v) - B(u - v, u - v).$$

Therefore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\eta\|^2 + \alpha^2 \|\eta\|_1^2) &\leq -\nu \|\eta\|_1^2 + \langle B(\eta, v), \eta \rangle_H + \langle B(u - v, u - v), \eta \rangle_H \\ &\leq -\nu \|\eta\|_1^2 + c_3 \|\eta\|_1^2 \|v\|_1 + c_4 \|u - v\|^{\frac{1}{2}} \|u - v\|_1^{\frac{3}{2}} \|\eta\|_1 \\ &\leq -\frac{\nu}{2} \|\eta\|_1^2 + \frac{c_3}{\alpha^2} \|v\|_1 (\|\eta\|^2 + \alpha^2 \|\eta\|_1^2) + \frac{c_4}{2\nu} \|u - v\| \|u - v\|_1^3. \end{aligned}$$

By the Gronwall lemma, for \mathbb{P} -a.s., we get

$$(3.29) \quad \begin{aligned} \|\eta\|^2 + \alpha^2 \|\eta\|_1^2 &\leq \frac{c_4}{\nu} \int_0^t e^{\int_s^t \frac{2c_3}{\alpha^2} \|v(\sigma)\|_1 d\sigma} \|u - v\| \|u - v\|_1^3 ds \\ &\leq \frac{c_4}{\nu} e^{\frac{2c_3}{\alpha^2} (c_1 + c_2 t)} \int_0^t \|u - v\| \|u - v\|_1^3 ds. \end{aligned}$$

Taking the inner product of (3.28) with $u - v$, one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u - v\|^2 + \alpha^2 \|u - v\|_1^2) &\leq -\nu \|u - v\|_1^2 - \langle B(u - v, v), u - v \rangle_H \\ &\leq \frac{c_5}{\alpha^2} (\|u - v\|^2 + \alpha^2 \|u - v\|_1^2) \|v\|_1, \end{aligned}$$

then we have

$$\begin{aligned} \|u - v\|^2 + \alpha^2 \|u - v\|_1^2 &\leq e^{\frac{2c_5}{\alpha^2} \int_0^t \|v\|_1 ds} (\|x - x_0\|^2 + \alpha^2 \|x - x_0\|_1^2) \\ (3.30) \quad &\leq e^{\frac{2c_5}{\alpha^2} (c_1 + c_2 t)} \|x - x_0\|_1^2. \end{aligned}$$

By (3.29)-(3.30), one has

$$\|\eta\|_1^2 \leq c(t) \|x - x_0\|_1^4,$$

where $c(t) > 0$ depending on $c_1, c_2, c_3, c_4, c_5, \alpha, t$, for \mathbb{P} -a.s.. Recalling the definition of $g_\delta(t)$, we see that it converges to 0 in $L^\infty(\Omega, \mathbb{P})$ for each $t > 0$. Thus Theorem 2.2 applies. \square

4. Finite Dimension of the Attractor for SPEs

4.1. Existence of Attractor. Given a smooth bounded domain $M \subset \mathbb{R}^2$ and the cylindrical domain $\mathcal{U} = M \times (-h, 0) \subset \mathbb{R}^3$, our aim is to study the following 3D viscous SPEs with additive noise in \mathcal{U} .

$$(4.1) \quad (SPEs) \quad \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v + \eta \frac{\partial v}{\partial z} + \nabla p + f v^\perp + L_1 v = \sum_{j=1}^m \phi_j \frac{dw_j(t)}{dt}; \\ \partial_z p + \theta = 0; \\ \nabla \cdot v + \partial_z \eta = 0; \\ \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta + \eta \frac{\partial \theta}{\partial z} + L_2 \theta = Q, \end{cases}$$

with fluid velocity field $(u, w) = (u_1, u_2, w) \in \mathbb{R}^3$ with $u = (u_1, u_2)$ and $u^\perp = (-u_2, u_1)$ being horizontal, the temperature θ , $\phi = (u, \theta)$ and the pressure p ; $f = f_0(\beta + y)$ is the given Coriolis rotation frequency with β -plane approximation and Q is a given heat source. The viscosity and the heat diffusion operators L_1 and L_2 are given respectively as the following:

$$L_i = -\nu_i \Delta - \mu_i \frac{\partial^2}{\partial z^2}, \quad i = 1, 2;$$

here the positive constants ν_1, μ_1 are the horizontal and vertical viscous coefficients; while the positive constants ν_2, μ_2 are the horizontal and vertical heat diffusivity coefficients, without lost of generality in this paper, we take $\nu_{1,2}, \mu_{1,2}$ to be 1. In the above equations and in this article, we denote by ∇ the horizontal gradient operator and by Δ the horizontal Laplacian:

$$\nabla \equiv (\partial_x, \partial_y), \quad \Delta \equiv \partial_x^2 + \partial_y^2,$$

and

$$\nabla_3 \equiv (\partial_x, \partial_y, \partial_z), \quad \Delta_3 \equiv \partial_x^2 + \partial_y^2 + \partial_z^2.$$

The function $\phi_j = \phi_j(x, y)$, $j = 1, \dots, m$, are time independent and specified below. The random function w_j , $j = 1, \dots, m$, are independent two-sided realvalue Wiener processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. More precisely, let $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}^m) \mid$

$\omega(0) = 0\}$, with \mathbb{P} being a product measure of two Wiener measure on the negative and the positive time parts of Ω ; then we have

$$(w_1(t, \omega), w_2(t, \omega), \dots, w_m(t, \omega)) = \omega(t), \quad t \in \mathbb{R},$$

the time shift is simply defined by

$$\vartheta_s \omega(t) = \omega(t + s) - \omega(s), \quad t, s \in \mathbb{R},$$

it is a family of ergodic transformations.

The boundary of \mathcal{U} is partitioned into three parts: $\partial\mathcal{U} = \Gamma_u \cup \Gamma_b \cup \Gamma_s$, where

$$\Gamma_u = \{(x, y, z) \in \overline{\mathcal{U}} : z = 0\},$$

$$\Gamma_b = \{(x, y, z) \in \overline{\mathcal{U}} : z = -h\},$$

$$\Gamma_s = \{(x, y, z) \in \overline{\mathcal{U}} : (x, y) \in \partial M, -h \leq z \leq 0\},$$

here $h > 0$ is a given constant. We consider the following boundary conditions:

$$\begin{aligned} \Gamma_u : \frac{\partial v}{\partial z} &= 0, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = 0; \\ \Gamma_b : \frac{\partial v}{\partial z} &= 0, \quad w = 0, \quad \frac{\partial \theta}{\partial z} = 0; \\ \Gamma_s : v \cdot n &= 0, \quad \frac{\partial v}{\partial n} \times n = 0, \quad \frac{\partial \theta}{\partial n} = 0. \end{aligned}$$

We define the following Hilbert spaces as in [11]:

$$(4.2) \quad H = H_1 \times H_2, \quad V = V_1 \times V_2,$$

$$(4.3) \quad H_1 = \{v \in L_2(\mathcal{U})^2 | \nabla \cdot \bar{v} = 0, \quad v \cdot n |_{\Gamma_s} = 0\}, \quad H_2 = L_2(\mathcal{U}),$$

$$(4.4) \quad V_1 = \{v \in H_1(\mathcal{U})^2 | \nabla \cdot \bar{v} = 0, \quad v \cdot n |_{\Gamma_s} = 0\}, \quad V_2 = H_1(\mathcal{U}).$$

These spaces are endowed with the L_2 and H_1 norms which we respectively denote by $|\cdot|_2$ and $\|\cdot\|$. Let V' be the dual space of V . We have the dense and continuous embeddings $V \hookrightarrow H = H' \hookrightarrow V'$ and denote by $\langle \phi, \psi \rangle$ the duality between $\phi \in V$ (resp. V_i) and $\psi \in V'$ (resp. V'_i).

Consider an unbounded linear operator $A = (L_1, L_2) : H \rightarrow H$ with $D(A) = D(L_1) \times D(L_2)$ where $D(L_1) = V_1 \cap H^2(\mathcal{U})$, $D(L_2) = V_2 \cap H^2(\mathcal{U})$ and define

$$\langle L_1 u, v \rangle = ((u, v)), \quad \langle L_2 \theta, \eta \rangle = ((\theta, \eta)), \quad \forall u, v \in D(L_1), \quad \forall \theta, \eta \in D(L_2).$$

The operator L_1 and L_2 are self-adjoint, positive, with compact self-adjoint inverses. They map V to V' .

Let $\delta > 0$ be given, we shall impose condition below on δ . For each $k = 1, \dots, m$, let ς_k be the stationary solution of the Ornstein-Uhlenbeck process

$$d\varsigma_k = -\delta\varsigma_k dt + dw_k(t),$$

so

$$\varsigma_k(t) = \int_{-\infty}^t e^{-\delta(t-s)} dw_k(s).$$

Putting $\varsigma(t) = \sum_{k=1}^m \phi_k \varsigma_k(t)$ we have

$$d\varsigma = -\delta\varsigma dt + dw(t),$$

where $w(t) = \sum_{k=1}^m \phi_k w_k(t)$. Since the trajectories of ς_k are \mathbb{P} -a.s. continuous and $\phi_k \in D(A_1)$, we have

$$(4.5) \quad \sup_{-n \leq t \leq n} \{ \|\varsigma(t)\|_{H_1}^2 + \|\varsigma(t)\|_{V_1}^2 + \|A_1 \varsigma(t)\|_{H_1}^2 \} \leq M, \quad \mathbb{P} - a.s.$$

Letting $u(t) = v(t) - \varsigma(t)$, then u satisfies the equations (which depends on a random parameter):

$$(4.6) \quad \begin{aligned} & \frac{\partial u}{\partial t} + L_1 u + ((u + \varsigma) \cdot \nabla)(u + \varsigma) - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \frac{\partial u}{\partial z} \\ & + \nabla p_s(x, y, t) - \int_{-h}^z \nabla \theta(x, y, \xi, t) d\xi + f(u + \varsigma)^\perp = \delta \varsigma - L_1 \varsigma, \end{aligned}$$

$$(4.7) \quad \frac{\partial \theta}{\partial t} + L_2 \theta + (u + \varsigma) \cdot \nabla \theta - \left(\int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \right) \frac{\partial \theta}{\partial z} = Q,$$

$$(4.8) \quad \frac{\partial u}{\partial z} |_{\Gamma_u} = \frac{\partial u}{\partial z} |_{\Gamma_b} = 0, \quad u \cdot n |_{\Gamma_s} = 0, \quad \frac{\partial u}{\partial n} \times n |_{\Gamma_s} = 0,$$

$$(4.9) \quad \left(\frac{\partial \theta}{\partial z} + \alpha \theta \right) |_{\Gamma_u} = \frac{\partial \theta}{\partial z} |_{\Gamma_b} = 0, \quad \frac{\partial \theta}{\partial n} |_{\Gamma_s} = 0,$$

$$(4.10) \quad u(x, y, z, t_0, \omega) = v_0(x, y, z, \omega) - \varsigma_0(x, y, \omega), \theta(x, y, z, t_0, \omega) = \theta_0(x, y, z, \omega).$$

In [11], we have already obtained the following wellposedness for the problems (4.6)-(4.10).

THEOREM 4.1. *Assume that $t_0 \in \mathbb{R}$ and $(u_0, \theta_0) \in V_1 \times V_2$, for \mathbb{P} -almost every $\omega \in \Omega$ the following holds:*

(i) *there exist a unique strong solution of (4.6)-(4.10) on the time interval $[t_0, \infty)$ and satisfies $u \in C([t_0, \infty), V_1) \cap L^2(t_0, \infty; D(A_1))$, $\theta \in C([t_0, \infty), V_2) \cap L^2(t_0, \infty; D(A_2))$.*

(ii) *denoting such solution by $u(t, \omega; t_0, u_0)$, the mapping $u_0 \mapsto u(t, \omega; t_0, u_0)$ is continuous for all $t \geq t_0$.*

In [11], by uniform estimates and Aubin-Lions compactness lemma combined with the use of the Riesz lemma and a continuity argument, we showed that the long time dynamics is captured by a random attractor. V-weak topology attractor was given in [13].

THEOREM 4.2. *(see [11]) Let $K(w)$ be the ball in V of radius $R_0(w)$ such that for any B bounded in $V = V_1 \times V_2$, there exists $s_1(w, B)$, for $s < s_1(w, B)$, we have*

$$(4.11) \quad S(0, s; w)B \subset K(w), \quad \mathbb{P} - a.s.$$

and the stochastic flow S is asymptotically compact in V for \mathbb{P} -a.s., thus we obtain a random attractor $\mathcal{A}(\omega)$ which attracts every bounded nonrandom set $B \subset V$.

4.2. Volume Contraction for SPEs. Consider the differential equation on $V = V_1 \times V_2$, here U will denote a pair (ϱ, ψ) .

$$(4.12) \quad \begin{aligned} & \frac{\partial \varrho}{\partial t} + L_1 \varrho + (\varrho \cdot \nabla)v + (v \cdot \nabla)\varrho - \left(\int_{-h}^z \nabla \cdot \varrho(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z} \\ & - \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \varrho}{\partial z} - \int_{-h}^z \nabla \psi(x, y, \xi, t) d\xi + f \varrho^\perp = 0, \end{aligned}$$

$$(4.13) \quad \begin{aligned} & \frac{\partial \psi}{\partial t} + L_2 \psi + v \cdot \nabla \psi + \varrho \cdot \nabla \theta - \left(\int_{-h}^z \nabla \cdot \varrho(x, y, \xi, t) d\xi \right) \frac{\partial \theta}{\partial z} \\ & - \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \psi}{\partial z} = 0, \end{aligned}$$

$$(4.14) \quad U(0, w) = (\varrho(0, w), \psi(0, w)) = \xi \in V.$$

The existence of a solution $U \in C([t_0, \infty), V) \cap L^2(t_0, \infty; D(A))$ for all $T > 0$, of (4.12)-(4.14) can be established, path by path, as in the deterministic case. For $\xi = \xi_1, \dots, \xi_d \in V$, let U_1, \dots, U_d denote the corresponding solutions of (4.12)-(4.14), hence following the procedure in Section V.2.3 in [22], we consider, for every $d \in \mathbb{N}$,

$$(4.15) \quad \begin{aligned} & |U_1(t) \wedge \dots \wedge U_d(t)|_{\wedge^d V}^2 \\ & = |\xi_1 \wedge \dots \wedge \xi_d|_{\wedge^d V}^2 \exp\left(\int_0^t \text{Tr} F'(S(\tau)\chi_0) \circ Q_d(\tau) d\tau \right), \end{aligned}$$

where $S(\tau)\chi_0 = \chi(\tau) = (v(\tau), \theta(\tau))$ and Q_d is the orthogonal projector in V onto the space spanned by $U_1(\tau), \dots, U_d(\tau)$.

At a given time τ , let $\varphi_j(\tau) = (\varrho_j(\tau), \psi_j(\tau))$, $j \in \mathbb{N}$, be an orthonormal basis of V , hence the trace of $\text{Tr} F'(S(\tau)\chi_0) \circ Q_d(\tau)$ is given by

$$(4.16) \quad \begin{aligned} \text{Tr} F'(S(\tau)\chi_0) \circ Q_d(\tau) & \leq \sum_{j=1}^d -|L_1 \varrho_j|^2 - |L_2 \psi_j|^2 + |b_1(\varrho_j, v, \Delta_3 \varrho_j)| + |b_1(v, \varrho_j, \Delta_3 \varrho_j)| \\ & + \left| \int_{\Omega} \left(\int_{-h}^z \nabla \cdot \varrho_j(x, y, \xi, t) d\xi \right) \frac{\partial v}{\partial z} \Delta_3 \varrho_j d\Omega \right| \\ & + \left| \int_{\Omega} \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \varrho_j}{\partial z} \Delta_3 \varrho_j d\Omega \right| \\ & + \left| \int_{\Omega} \left(\int_{-h}^z \nabla \psi_j(x, y, \xi, t) d\xi \right) \Delta_3 \varrho_j d\Omega \right| \\ & + |b_2(\varrho_j, \theta, \Delta_3 \psi_j)| + |b_2(v, \psi_j, \Delta_3 \psi_j)| \\ & + \left| \int_{\Omega} \left(\int_{-h}^z \nabla \cdot \varrho_j(x, y, \xi, t) d\xi \right) \frac{\partial \theta}{\partial z} \Delta_3 \psi_j \right| \\ & + \left| \int_{\Omega} \left(\int_{-h}^z \nabla \cdot v(x, y, \xi, t) d\xi \right) \frac{\partial \psi_j}{\partial z} \Delta_3 \psi_j \right|, \end{aligned}$$

by the Hölder, Sobolev inequality and Lemma 4.2 in [14], we get

$$\begin{aligned} \text{Tr}F'(S(\tau)\chi_0) \circ Q_d(\tau) &\leq \sum_{j=1}^d \left(-|\Delta_3\varphi_j|_{L^2}^2 + |\Delta_3\varrho_j|_{L^2}\|v\|_{V_1}^{\frac{1}{2}}|\Delta_3v|_{L^2}^{\frac{1}{2}}\|\varrho_j\|_{V_1} \right. \\ &\quad + |\Delta_3\varrho_j|_{L^2}^{\frac{3}{2}}\|\varrho_j\|_{V_1}^{\frac{1}{2}}\|v\|_{V_1} + |\Delta_3\varrho_j|_{L^2}^{\frac{3}{2}}\|\varrho_j\|_{V_1}^{\frac{1}{2}}|v_z|_{L^2}^{\frac{1}{2}}|v_z|_{V_1}^{\frac{1}{2}} \\ &\quad + |\Delta_3\varrho_j|_{L^2}^{\frac{3}{2}}\|\varrho_j\|_{V_1}^{\frac{1}{2}}|v|_{V_1}^{\frac{1}{2}}|\Delta_3v|_{L^2}^{\frac{1}{2}} + |\Delta_3\psi_j|_{L^2}\|\varrho_j\|_{V_1}|\theta|_{V_2}^{\frac{1}{2}}|\Delta_3\theta|_{L^2}^{\frac{1}{2}} \\ &\quad + |\Delta_3\psi_j|_{L^2}^{\frac{3}{2}}\|\psi_j\|_{V_1}^{\frac{1}{2}}\|v\|_{V_1} + |\Delta_3\varphi_j|_{L^2}^{\frac{3}{2}}\|\varrho_j\|_{V_1}^{\frac{1}{2}}|\theta_z|_{L^2}^{\frac{1}{2}}|\theta_z|_{V_2}^{\frac{1}{2}} \\ &\quad \left. + |\Delta_3\psi_j|_{L^2}^{\frac{3}{2}}\|\psi_j\|_{V_1}^{\frac{1}{2}}|v|_{V_1}^{\frac{1}{2}}|\Delta_3v|_{L^2}^{\frac{1}{2}} \right). \end{aligned}$$

Since $\{\varphi_j\}_j$ being orthonormal in $V = V_1 \times V_2$, thus by Young's inequality, we have

$$\begin{aligned} \text{Tr}F'(S(\tau)\chi_0) \circ Q_d(\tau) &\leq \left(\sum_{j=1}^d -\frac{1}{2}|\Delta_3\varphi_j|_{L^2}^2 \right) + |v|_{V_1}^2|\Delta_3v|_{L^2}^2 \\ (4.17) \quad &\quad + |\theta|_{V_2}^2|\Delta_3\theta|_{L^2}^2 + \|v\|_{V_1}^4 + 1, \end{aligned}$$

applying Generalization of the Sobolev-Lieb-Thirring Inequality in [22], we have

$$\begin{aligned} \sum_{j=1}^d \int_{\Omega} |\Delta_3\varphi_j|^2 d\Omega &\geq \frac{1}{\kappa_2} \left(\int_{\Omega} \left(\sum_{j=1}^d |\nabla\varphi_j|_{L^2}^2 \right)^2 d\Omega \right)^{\frac{2}{3}} - \frac{1}{|\Omega|^{\frac{2}{3}}} \int_{\Omega} \sum_{j=1}^d |\nabla\varphi_j|_{L^2}^2 d\Omega \\ (4.18) \quad &\geq d^{\frac{4}{3}}|\Omega|^{\frac{2}{3}} \frac{1}{\kappa_2} - d|\Omega|^{\frac{1}{3}}. \end{aligned}$$

We deduce that

$$\begin{aligned} &|U_1(t) \wedge \cdots \wedge U_d(t)|_{\wedge^d V}^2 \\ &\leq |\xi_1 \wedge \cdots \wedge \xi_d|_{\wedge^d V}^2 \exp\left(-d^{\frac{4}{3}}|\Omega|^{\frac{2}{3}} \frac{1}{2\kappa_2} t + d(|\Omega|^{\frac{1}{3}} + 1)t \right) \\ (4.19) \quad &+ \int_0^t |v|_{V_1}^2 |\Delta_3v|_{L^2}^2 + |\theta|_{V_2}^2 |\Delta_3\theta|_{L^2}^2 + \|v\|_{V_1}^4 d\tau. \end{aligned}$$

Then we left the estimates of $\int_0^t |v|_{V_1}^2 |\Delta_3v|_{L^2}^2 + |\theta|_{V_2}^2 |\Delta_3\theta|_{L^2}^2 + \|v\|_{V_1}^4 d\tau$.

LEMMA 4.3. *For any $\chi_0 = (v_0, \theta_0) \in \mathcal{A}(w)$, we have $\chi(t) = S(t, w)\chi_0 = (v(t), \theta(t))$ satisfy*

$$(4.20) \quad \int_0^t |v|_{V_1}^2 |\Delta_3v|_{L^2}^2 + |\theta|_{V_2}^2 |\Delta_3\theta|_{L^2}^2 + \|v\|_{V_1}^4 d\tau \leq C_6 + C_7 t$$

for \mathbb{P} -a.s.

PROOF. In [11], we get

$$(4.21) \quad \|\theta(t)\|_2 \leq e^{-ct}\|\theta_0\|_2 + \frac{1}{c}\|Q\|_2,$$

$$(4.22) \quad \int_t^{t+1} \|\theta\|_{V_2}^2 ds \leq 2e^{-2ct}\|\theta_0\|_2^2 + \frac{3}{c^2}\|Q\|_2^2$$

and

$$(4.23) \quad \frac{d\|u\|_2^2}{dt} + \frac{1}{2}\|u\|_{V_1}^2 + \left(\frac{\lambda}{2} - c \sum_{k=1}^m |\varsigma_k(t)|^2\right)\|u\|_2^2 \leq cA,$$

with

$$(4.24) \quad A = \|\nabla \varsigma\|_3^2 \|\varsigma\|_2^2 + \|\theta\|_2^2 + \|\varsigma\|_2^2 + \|A_1 \varsigma\|_2^2,$$

by Gronwall's lemma for (4.23),

$$(4.25) \quad \begin{aligned} \|u(t)\|_2^2 &\leq \|u(0)\|_2^2 \exp\left(-\int_0^t \left(\frac{\lambda}{2} - c \sum_{k=1}^m |\varsigma_k(\sigma)|^2\right) d\sigma\right) \\ &\quad + \exp\left(-\int_0^t \left(\frac{\lambda}{2} - c \sum_{k=1}^m |\varsigma_k(\tau)|^2\right) d\tau\right) \int_0^t A(\sigma) d\sigma. \end{aligned}$$

Since the process ς_k , $k = 1, \dots, m$ is stationary and ergodic, thus we know from the ergodic theorem that

$$\frac{1}{t} \int_0^t \sum_{j=1}^m |\varsigma_j(\sigma)|^2 d\sigma \rightarrow E\left(\sum_{j=1}^m |\varsigma_j(0)|^2\right), \quad t \rightarrow \infty,$$

thus there exists $t_1(\omega)$ such that any $t > t_1(\omega)$,

$$\begin{aligned} \frac{1}{t} \int_0^t \sum_{j=1}^m |\varsigma_j(\sigma)|^2 d\sigma &\leq 2E\left(\sum_{j=1}^m |\varsigma_j(0)|^2\right) \\ &\leq \frac{m}{\delta}. \end{aligned}$$

Takeing δ large enough so that

$$c \int_0^t \sum_{j=1}^m |\varsigma_j(\sigma)|^2 d\sigma \leq \frac{\lambda}{4} t.$$

As in [11], $A(t)$ grow at most polynomially, thus we deduce

$$\begin{aligned} &\lim_{t \rightarrow \infty} \exp\left(-\int_0^t \left(\frac{\lambda}{2} - c \sum_{k=1}^m |\varsigma_k(\tau)|^2\right) d\tau\right) \int_0^t A(\sigma) d\sigma \\ &\leq \lim_{t \rightarrow \infty} \exp\left(-\frac{\lambda}{4} t\right) \int_0^t A(\sigma) d\sigma = 0, \end{aligned}$$

thus $\exists t_2(w)$, for $t > t_2(w)$, we have

$$\exp\left(-\int_0^t \left(\frac{\lambda}{2} - c \sum_{k=1}^m |\varsigma_k(\tau)|^2\right) d\tau\right) \int_0^t A(\sigma) d\sigma \leq 1.$$

Taking $t_0(w) = \max(t_1, t_2)$, for $t > t_0(w)$ and $u_0 \in \mathcal{A}(w)$, we have

$$(4.26) \quad \|u(t)\|_2^2 \leq \|u(0)\|_{V_1}^2 + 1 \leq R_0(w) + 1$$

Applying again Gronwall's lemma for (4.23) with $0 \leq t \leq t_0$ and noticing (4.26), we get, for $\forall t > 0$

$$(4.27) \quad \|u(t)\|_2^2 \leq C_0(w).$$

Moreover, we integrate (4.23) on $[t, t+1]$ and applying (4.5),

$$(4.28) \quad \begin{aligned} \int_t^{t+1} \|u(s)\|_{V_1}^2 ds &\leq 2\|u(t)\|_2^2 + C \int_t^{t+1} \sum_{j=1}^m |\varsigma_j(\sigma)|^2 \|u(\sigma)\|_2^2 d\sigma + C \int_t^{t+1} A(s) ds \\ &\leq C_1(R_0(w), M, Q). \end{aligned}$$

Then we will use the uniform Gronwall lemma to obtain the time-uniform estimates for $\|\tilde{u}\|_6^6$, in [11], we have got

$$(4.29) \quad \frac{d}{dt} \|\tilde{u}\|_6^2 \leq H(t) \|\tilde{u}\|_6^2 + c \|\theta\|_6^2$$

with

$$(4.30) \quad H(t) = c(1 + \|u\|_2^2 \|u\|_{V_1}^2 + \|\varsigma\|_2^2 \|\varsigma\|_{V_1}^2 + \|u\|_{V_1}^2).$$

Since (4.27), (4.28), we get

$$\begin{aligned} \int_t^{t+1} \|\tilde{u}\|_6^2 ds &\leq \int_t^{t+1} \|u\|_{V_1}^2 ds \leq C_2(w), \\ \int_t^{t+1} H(s) ds &\leq C_3(R_0(w), M, Q), \\ \int_t^{t+1} \|\theta\|_6^2 &\leq \int_t^{t+1} \|\theta\|_{V_2}^2 \leq C_4(R_0(w), Q). \end{aligned}$$

By uniform Gronwall lemma we obtain $\|\tilde{u}\|_6^2 \leq C_5(w)$. Then we estimate steps one steps as in [11] and by the uniform gronwall lemma, we are not difficulty to obtain the time-uniform estimates for $\|u\|_{V_1}^2, \|\theta\|_{V_2}^2 \leq C_6(w)$ and $\int_0^t |\Delta_3 v(\tau)|_{L^2}^2 + |\Delta_3 \theta(\tau)|_{L^2}^2 d\tau \leq C_7(w)t$, where C_6, C_7 are random and finite \mathbb{P} -a.s.. \square

Recalling the notations for $\Gamma_d(t, \cdot)$, and (4.19), (4.20), we get,

$$(4.31) \quad \begin{aligned} \Gamma_d(t, w) &\leq \exp\left(-d^{\frac{4}{3}} |\Omega|^{\frac{2}{3}} \frac{1}{2\kappa_2} t + d(|\Omega|^{\frac{1}{3}} + 1)t + C_6 + C_7 t\right) \\ &\leq \exp\left((-C_8(w)d^{\frac{4}{3}} + C_9(w))t + C_{10}(w)\right). \end{aligned}$$

Let d sufficiently large, we have $\|\Gamma_d(t, w)\|_{L^\infty(\Omega)} < 1$ for \mathbb{P} -a.s..

4.3. Finite Dimension of the Attractor.

THEOREM 4.4. *The Hausdorff dimension of the global attractor for the SPEs is finite.*

PROOF. From (4.31) we get existence of some d such that

$$\|\Gamma_d(t, \cdot)\|_{L^\infty} < 1.$$

This is the first basic estimate we need to apply Theorem 2.2. From the proof of (4.31), we get for the particular case $d = 1$,

$$\|U_1\|_V \leq \|\xi_1\|_V e^{C_9 t + C_{10}}.$$

Thus also the assumption $\gamma_1(t) \in L^\infty(\Omega, \mathbb{P})$ is fulfilled. Finally, we have to prove that $g_\delta(t)$ convergence to 0 in $L^\infty(\Omega, \mathbb{P})$ for each $t > 0$.

Let $x \in V = V_1 \times V_2$ and $x_0 \in \mathcal{A}(w)$ be given, and put $\varphi_1(t) = (v_1, \theta_1) = S(t, w)x$, $\varphi_2(t) = (v_2, \theta_2) = S(t, w)x_0$, $\varphi_3(t) =$

$(\varrho_1, \psi_1) = D_{x_0} S(t, w)(x - x_0)$, and $\eta(t) = (\varrho_2, \psi_2) = \varphi_1(t) - \varphi_2(t) - \varphi_3(t)$, then

$$(4.32) \quad \begin{aligned} & \frac{\partial(v_1 - v_2)}{\partial t} + L_1(v_1 - v_2) + \mathcal{B}_1(v_1, v_1 - v_2) + \mathcal{B}_1(v_1 - v_2, v_2) \\ & - \int_{-h}^z \nabla(\theta_1 - \theta_2) d\xi + f(v_1 - v_2)^\perp = 0, \end{aligned}$$

$$(4.33) \quad \frac{\partial(\theta_1 - \theta_2)}{\partial t} + L_2(\theta_1 - \theta_2) + \mathcal{B}_2(v_1, \theta_1 - \theta_2) + \mathcal{B}_2(v_1 - v_2, \theta_2) = 0,$$

and

$$(4.34) \quad \frac{\partial \varrho_1}{\partial t} + L_1 \varrho_1 + \mathcal{B}_1(\varrho_1, v_2) + \mathcal{B}_1(v_2, \varrho_1) - \int_{-h}^z \nabla \psi_1 d\xi + f \varrho_1^\perp = 0,$$

$$(4.35) \quad \frac{\partial \psi_1}{\partial t} + L_2 \psi_1 + \mathcal{B}_2(v_2, \psi_1) + \mathcal{B}_2(\varrho_1, \theta_2) = 0.$$

Hence

$$(4.36) \quad \begin{aligned} & \frac{\partial \varrho_2}{\partial t} + L_1 \varrho_2 + \mathcal{B}_1(\varrho_2, v_2) + \mathcal{B}_1(v_2, \varrho_2) \\ & + \mathcal{B}_1(v_1 - v_2, v_1 - v_2) - \int_{-h}^z \nabla \psi_2 d\xi + f \varrho_2 = 0, \end{aligned}$$

$$(4.37) \quad \frac{\partial \psi_2}{\partial t} + L_2 \psi_2 + \mathcal{B}_2(\varrho_2, \theta_2) + \mathcal{B}_2(v_2, \psi_2) + \mathcal{B}_2(v_1 - v_2, \theta_1 - \theta_2) = 0,$$

where

$$(4.38) \quad \mathcal{B}_1(u, v) := (u \cdot \nabla)v - \int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \frac{\partial v}{\partial z},$$

$$(4.39) \quad \mathcal{B}_2(u, \theta) := u \cdot \nabla \theta - \int_{-h}^z \nabla \cdot u(x, y, \xi, t) d\xi \frac{\partial \theta}{\partial z},$$

and

$$\begin{aligned} \langle \mathcal{B}_1(u, v), w \rangle &= \mathfrak{b}_1(u, v, w), \\ \langle \mathcal{B}_2(u, \theta), \eta \rangle &= \mathfrak{b}_2(u, \theta, \eta). \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{2} \frac{\partial \|\varrho_2\|_{V_1}^2}{\partial t} + \|L_1 \varrho_2\|_2^2 + \mathfrak{b}_1(\varrho_2, v_2, \Delta_3 \varrho_2) + \mathfrak{b}_1(v_2, \varrho_2, \Delta_3 \varrho_2) \\ & + \mathfrak{b}_1(v_1 - v_2, v_1 - v_2, \Delta_3 \varrho_2) + \int_{\Omega} \left(\int_{-h}^z \nabla \psi_2 d\xi \right) \Delta_3 \varrho_2 d\Omega = 0, \\ & \frac{1}{2} \frac{\partial \|\psi_2\|_{V_2}^2}{\partial t} + \|L_2 \psi_2\|_2^2 + \mathfrak{b}_2(\varrho_2, \theta_2, \Delta_3 \psi_2) + \mathfrak{b}_2(v_2, \psi_2, \Delta_3 \psi_2) \\ & + \mathfrak{b}_2(v_1 - v_2, \theta_1 - \theta_2, \Delta_3 \psi_2) = 0, \end{aligned}$$

by the Hölder, Sobolev inequality and Lemma 4.2 in [14], we get

$$\begin{aligned} & \frac{\partial(\|\varrho_2\|_{V_1}^2 + \|\psi_2\|_{V_2}^2)}{\partial t} + \frac{1}{2}(\|\Delta_3 \varrho_2\|_2^2 + \|\psi_2\|_2^2) \\ & \leq C \left(1 + \|v_2\|_{V_1}^4 + \|v_2\|_{V_1}^2 \|\Delta_3 v_2\|_2^2 + \|\theta_2\|_{V_2}^2 \|\Delta_3 \theta_2\|_2^2 \right) (\|\varrho_2\|_{V_1}^2 + \|\psi_2\|_{V_2}^2) \\ & + C \left(\|v_1 - v_2\|_{V_1}^3 \|\Delta_3(v_1 - v_2)\|_2 + \|v_1 - v_2\|_{V_1}^2 \|\theta_1 - \theta_2\|_{V_2} \|\Delta_3(\theta_1 - \theta_2)\|_2 \right). \end{aligned}$$

By the Gronwall lemma and (4.20),

$$(4.40) \quad \begin{aligned} \|\eta(t)\|_{\mathbb{V}}^2 &\leq C_{11}(t, w) \int_0^t \left(\|v_1 - v_2\|_{\mathbb{V}_1}^3 \|\Delta_3(v_1 - v_2)\|_2 \right. \\ &\quad \left. + \|v_1 - v_2\|_{\mathbb{V}_1}^2 \|\theta_1 - \theta_2\|_{\mathbb{V}_2} \|\Delta_3(\theta_1 - \theta_2)\|_2 \right) ds. \end{aligned}$$

As we obtain (4.40), concerning $\varphi_1 - \varphi_2$, denoting

$$g(t) = 1 + \|v_1\|_{\mathbb{V}_1}^2 \|\Delta_3 v_1\|_{\mathbb{V}_1}^2 + \|v_2\|_{\mathbb{V}_1}^2 \|\Delta_3 v_2\|_{\mathbb{V}_1}^2 + \|\theta_1\|_{\mathbb{V}_2}^2 \|\Delta_3 \theta_1\|_{\mathbb{V}_2}^2 + \|\theta_2\|_{\mathbb{V}_2}^2 \|\Delta_3 \theta_2\|_{\mathbb{V}_2}^2,$$

we finally obtain,

$$(4.41) \quad \begin{aligned} \|\varphi_1(t) - \varphi_2(t)\|_{\mathbb{V}}^2 &\leq \exp \left[C \int_0^t \left(1 + \|v_1\|_{\mathbb{V}_1}^2 \|\Delta_3 v_1\|_{\mathbb{V}_1}^2 + \|v_2\|_{\mathbb{V}_1}^2 \|\Delta_3 v_2\|_{\mathbb{V}_1}^2 \right. \right. \\ &\quad \left. \left. + \|\theta_1\|_{\mathbb{V}_2}^2 \|\Delta_3 \theta_1\|_{\mathbb{V}_2}^2 + \|\theta_2\|_{\mathbb{V}_2}^2 \|\Delta_3 \theta_2\|_{\mathbb{V}_2}^2 \right) ds \right] \|x - x_0\|_{\mathbb{V}}^2 \\ &\leq C_{12}(t, w) \|x - x_0\|_{\mathbb{V}}^2. \end{aligned}$$

Consequently, we have

$$(4.42) \quad \begin{aligned} \int_0^t \|\Delta_3(\varphi_1 - \varphi_2)\|_{\mathbb{V}}^2 ds &\leq \|x - x_0\|_{\mathbb{V}}^2 + \int_0^t g(s) \|\varphi_1(s) - \varphi_2(s)\|_{\mathbb{V}}^2 ds \\ &\leq C_{13}(t, w) \|x - x_0\|_{\mathbb{V}}^2. \end{aligned}$$

Inserting these estimates in (4.40), we finally obtain

$$(4.43) \quad \|\eta(t)\|_{\mathbb{V}}^2 \leq C_{14}(t, w) \|x - x_0\|_{\mathbb{V}}^4,$$

where $C_{14}(t, w)$ depending on t is random and finite \mathbb{P} -a.s.. Recalling the definition of $g_\delta(t)$, we see that it converges to 0 in $L^\infty(\Omega, \mathbb{P})$ for each $t > 0$. Thus Theorem 2.2 applies. □

Acknowledgements. The authors are indebted to the referee for giving some important suggestions which improved the presentations of this paper.

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