

Global existence for nonlinear wave equation

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ABSTRACT. In this work we propose a new approach for investigating the local and global existence of solutions of nonlinear wave equations. This approach gives new results.

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1. Introduction

Nonlinear wave equations are introduced as models of physical phenomena in general relativity, mechanics and relativistic field theories. The main tools for studying these models are the classical representation formula of solutions using the fundamental solution, due to d’Alembert and energy estimates. These estimates were used in the works [3], [5], [6] (and references therein) to establish local existence and uniqueness of solutions of these phenomena. Global existence for these equations with Lipschitz nonlinearities are based on energy estimates (see [3], [5], [6] and references therein). For nonlinearities with restrictive growth are used Strichartz - type estimates for local existence of strong finite energy solutions (see [5], [6] and references therein). Global existence for subcritical problems is obtained using energy conservation laws. Global existence for critical problems follows from a nonconcentration lemma derived from the scaling identity (see [6] and references therein).

In this work we propose a new approach for investigating of these problems. This approach gives new results. It is based on a new integral representation of the

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solutions. We focus our attention on the Cauchy problem

$$(1.1) \quad u_{tt} - \frac{n-1}{r}u_r - u_{rr} = u_t^p, \quad t \geq 0, r \geq r_0,$$

$$(1.2) \quad u(0, r) = u_0(r), \quad u_t(0, r) = u_1(r), \quad r \geq r_0,$$

where $r_0 > 0$ is fixed, $n \geq 2$ is fixed, $r = |x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, $p \geq 0$ is fixed.

Let $Q > 0$ is arbitrary chosen and fixed. Our main assumption for the initial data $u_0(r)$ and $u_1(r)$ are

(H1) $u_0(r) \in \mathcal{C}^2([r_0, \infty))$, $u_1(r) \in \mathcal{C}^1([r_0, \infty))$, $|u_0(r)| \leq Q$, $|u_1(r)| \leq Q$ for every $r \geq r_0$.

Our main result is

THEOREM 1.1. *Let $n \geq 2$ be fixed, $p \geq 0$ be fixed, $Q > 0$ be fixed, $r_0 > 0$ be fixed, the initial data u_0, u_1 satisfy the hypothesis (H1). Then the Cauchy problem (1.1), (1.2) has a solution u for which $u \in \mathcal{C}^2([0, \infty), \mathcal{C}^2([r_0, \infty)))$.*

We note that if u_0 or u_1 is not identically equal to zero then since $u(t, r) \in \mathcal{C}^2([0, \infty), \mathcal{C}^2([r_0, \infty)))$ we have that $u(t, r)$ is nontrivial.

Since we search a solution $u(t, r) \in \mathcal{C}^2([0, \infty), \mathcal{C}^2([r_0, \infty)))$ of the problem (1.1), (1.2) we have not a need of bounded conditions in the formulation of the considered problem because if $u(t, r) \in \mathcal{C}^2([0, \infty), \mathcal{C}^2([r_0, \infty)))$ we have $u(t, r)|_{r=r_0} = u(t, r_0)$, $u_t(t, r)|_{r=r_0} = u_t(t, r_0)$, $u_{tt}(t, r)|_{r=r_0} = u_{tt}(t, r_0)$, $u_r(t, r)|_{r=r_0} = u_r(t, r_0)$, $u_{rr}(t, r)|_{r=r_0} = u_{rr}(t, r_0)$ for every $t \in [0, \infty)$.

Obviously the problems for local existence, global existence, blow up for the equation (1.1) are connected with the values of p and n . This connection is due to the integral representations which are used for their investigation. Here we use new integral representation which show us that these problems are connected with the integral representation and they are not connected with the values of p and n . We note that our condition for p is $p \geq 0$ and for n : $n \geq 2$.

We will compare our result with well known result. Of course, there are many other articles which are connected with the problem (1.1), (1.2), our aim is to give one example with a well known result which illustrates the power of our approach. When $p = \text{const} > 2$, $n = 3$, u_0 and u_1 have compact support, in [7] is obtained the global in time existence in weighted L^∞ - spaces of a classical solution of the problem (1.1), (1.2).

This new nice result is due to our new integral representation.

In the next section we will prove our main result.

2. Proof of Main Result

Let $\epsilon = \frac{1}{100}$. Let also $A > 0$ is constant for which

$$\frac{1}{2}A^2Q^{p-1} + \frac{1}{2} + \frac{1}{2}A^2 + \frac{1}{2}\frac{n-1}{r_0^2}A^2 + 2A^2 + \frac{1}{2}A\frac{n-1}{r_0} \leq (1 - \epsilon),$$

in the case $p > 0$ and

$$\frac{1}{2} + \frac{1}{2}A^2 + \frac{1}{2}\frac{n-1}{r_0^2}A^2 + 2A^2 + \frac{1}{2}A\frac{n-1}{r_0} \leq (1 - \epsilon),$$

in the case $p = 0$. Such choice is possible since we can choose $A < 1$ enough small.

Firstly we will prove that the Cauchy problem

$$(2.1) \quad u_{tt} - \frac{n-1}{r}u_r - u_{rr} = u_t^p, \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0, r_0 + A],$$

$$(2.2) \quad u(0, r) = u_0(r), \quad u_t(0, r) = u_1(r), \quad r \in [r_0, r_0 + A]$$

has a solution which belongs to the space $\mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$. For this purpose we will use fixed point arguments. Therefore we have a need of suitable operator whose fixed point is a solution to the Cauchy problem (2.1), (2.2) and belongs to the space $\mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$.

Let us consider the integral operator

$$(2.3) \quad \begin{aligned} L_{11}(u) = & \int_0^t \int_0^s \int_r^A \int_y^A u_t^p dz dy d\tau ds \\ & + \int_0^t \int_0^s u(\tau, r) d\tau ds + \int_0^t \int_0^s \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + t \int_r^A \int_s^A u_1(y) dy ds + \\ & + \int_r^A \int_s^A u_0(y) dy ds - \int_r^A \int_s^A u(t, y) dy ds - \int_0^t \int_0^s \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau ds + u(t, r). \end{aligned}$$

Let $u \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$ is a fixed point to the operator L_{11} , i.e. $L_{11}(u) = u$. Then, from (2.3), we have

$$(2.4) \quad \begin{aligned} & \int_0^t \int_0^s \int_r^A \int_y^A u_t^p dz dy d\tau ds + \int_0^t \int_0^s u(\tau, r) d\tau ds + \\ & + \int_0^t \int_0^s \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + t \int_r^A \int_s^A u_1(y) dy ds + \\ & + \int_r^A \int_s^A u_0(y) dy ds - \int_r^A \int_s^A u(t, y) dy ds - \int_0^t \int_0^s \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau ds = 0. \end{aligned}$$

Now we differentiate in t the last equality and we obtain

$$(2.5) \quad \begin{aligned} & \int_0^t \int_r^A \int_y^A u_t^p dz dy ds + \int_0^t u(\tau, r) d\tau + \\ & + \int_0^t \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau + \int_r^A \int_s^A u_1(y) dy ds - \\ & - \int_r^A \int_s^A u_t(t, y) dy ds - \int_0^t \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau = 0. \end{aligned}$$

Again we differentiate in t and we obtain

$$\begin{aligned} & \int_r^A \int_y^A u_t^p dz dy + u(t, r) + \int_r^A \int_y^A \frac{n-1}{z^2} u(t, z) dz dy - \\ & - \int_r^A \int_s^A u_{tt}(t, y) dy ds - \int_r^A \frac{n-1}{y} u(t, y) dy = 0. \end{aligned}$$

Now we differentiate in r

$$\begin{aligned} & - \int_r^A u_t^p dz + u_r - \int_r^A \frac{n-1}{y^2} u(t, y) dy + \\ & + \int_r^A u_{tt}(t, y) dy + \frac{n-1}{r} u(t, r) = 0, \end{aligned}$$

again we differentiate in r

$$u_t^p = -u_{rr} + u_{tt}(t, r) - \frac{n-1}{r} u_r(t, r).$$

Now we put $t = 0$ in (2.4) and we obtain

$$\int_r^A \int_s^A u_0(y) dy ds - \int_r^A \int_s^A u(0, y) dy ds = 0.$$

After we differentiate the last equality twice in r we obtain

$$u_0(r) = u(0, r).$$

Now we put $t = 0$ in the equality (2.5) and we obtain

$$\int_r^A \int_s^A u_1(y) dy ds - \int_r^A \int_s^A u_t(0, y) dy ds = 0,$$

after we differentiate twice in r we obtain

$$u_1(r) = u_t(0, r).$$

Consequently, if $u(t, r)$ is a fixed point to the operator L_{11} , we have that $u(t, r)$ is a solution to the Cauchy problem (2.1), (2.2).

To prove that the operator L_{11} has a fixed point we will use the following fixed point theorem

THEOREM 2.1. (see [8], Corrolary 2.4, pp. 3231) *Let X be a nonempty closed convex subset of a Banach space Y . Suppose that T and S map X into Y such that*

(i) *S is continuous, $S(X)$ resides in a compact subset of Y ;*

(ii) *$T : X \rightarrow Y$ is expansive and onto.*

Then there exists a point $x^ \in X$ with $Sx^* + Tx^* = x^*$.*

Here we will use the following definition for expansive operator.

Definition. (see [8], pp. 3230) Let (X, d) be a metric space and M be a subset of X . The mapping $T : M \rightarrow X$ is said to be expansive, if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq hd(x, y) \quad \forall x, y \in M.$$

We define the sets M_{11} and N_{11} as equicontinuous subsets of

$$\tilde{M}_{11} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right), |u(t, r)| \leq Q, |u_t(t, r)| \leq Q, \right.$$

$$\left. \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0, r_0 + A] \right\},$$

$$\tilde{N}_{11} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right), |u(t, r)| \leq (1 + \epsilon)Q, \right.$$

$$\left. |u_t(t, r)| \leq (1 + \epsilon)Q, \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0, r_0 + A] \right\},$$

respectively, in these sets we define a norm as follows

$$\|u\|_2 = \sup \left\{ |u(t, r)|, |u_t(t, r)| \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0, r_0 + A] \right\}.$$

The sets M_{11} , N_{11} are completed normed spaces with this norm, also the set M_{11} is a convex subset of N_{11} .

For the operator L_{11} we will use the following representation

$$L_{11}(u) = T_{11}(u) + S_{11}(u),$$

where

$$\begin{aligned} T_{11}(u) &= (1 + \epsilon)u, \\ S_{11}(u) &= -\epsilon u + \int_0^t \int_0^s \int_r^A \int_y^A u_t^p dz dy d\tau ds \\ &+ \int_0^t \int_0^s u(\tau, r) d\tau ds + \int_0^t \int_0^s \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + t \int_r^A \int_s^A u_1(y) dy ds + \\ &+ \int_r^A \int_s^A u_0(y) dy ds - \int_r^A \int_s^A u(t, y) dy ds - \\ &- \int_0^t \int_0^s \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau ds. \end{aligned}$$

LEMMA 2.2. *The operator $T_{11} : M_{11} \longrightarrow N_{11}$ is an expansive operator and onto.*

PROOF. Firstly we will prove that

$$T_{11} : M_{11} \longrightarrow N_{11}.$$

Really, let $u \in M_{11}$ is arbitrary chosen and fixed. Then $u \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$, from here $(1 + \epsilon)u \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$, therefore $T_{11}(u) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$. Also, from $u \in M_{11}$ follows that $|u(t, r)| \leq Q$, $|u_t(t, r)| \leq Q$ for every $t \in \left[0, \frac{1}{2}\right]$, $r \in [r_0, r_0 + A]$. From here follows that $|T_{11}(u)(t, r)| \leq (1 + \epsilon)Q$, $|(T_{11}(u))_t(t, r)| \leq (1 + \epsilon)Q$ for every $t \in \left[0, \frac{1}{2}\right]$ and every $r \in [r_0, r_0 + A]$. Consequently $T_{11}(u) \in N_{11}$ from where we conclude that $T_{11} : M_{11} \longrightarrow N_{11}$.

Let now $u, v \in M_{11}$ are arbitrary chosen and fixed. Then, from the definition for norm in N_{11} , M_{11} we obtain

$$\|T_{11}(u) - T_{11}(v)\|_2 = (1 + \epsilon)\|u - v\|_2,$$

then $T_{11} : M_{11} \longrightarrow N_{11}$ is an expansive operator with constant $h = 1 + \epsilon$.

Now we will see that the operator $T_{11} : M_{11} \longrightarrow N_{11}$ is onto. Let $v \in N_{11}$ is arbitrary chosen and fixed. Then $|v(t, r)| \leq (1 + \epsilon)Q$, $|v_t(t, r)| \leq (1 + \epsilon)Q$ for every $t \in \left[0, \frac{1}{2}\right]$ and every $r \in [r_0, r_0 + A]$. Let $u = \frac{v}{1 + \epsilon}$. Then $u \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0, r_0 + A])\right)$ and $|u(t, r)| \leq Q$, $|u_t(t, r)| \leq Q$ for every $t \in \left[0, \frac{1}{2}\right]$ and every $r \in [r_0, r_0 + A]$. Consequently $u \in M_{11}$ and the operator $T_{11} : M_{11} \longrightarrow N_{11}$ is onto. \square

LEMMA 2.3. *We have*

$$S_{11} : M_{11} \longrightarrow M_{11}$$

is continuous and M_{11} is a compact subset of N_{11} .

PROOF. Firstly we will prove that

$$S_{11} : M_{11} \longrightarrow M_{11}.$$

Let $u \in M_{11}$ is arbitrary chosen and fixed. Then, using that $|u|, |u_t| \leq Q$, $r \in [r_0, r_0 + A]$, $t \in \left[0, \frac{1}{2}\right]$, and the conditions for the constant A (see the beginning of

the proof), we have

$$\begin{aligned}
|S_{11}(u)| &\leq \epsilon|u| + \int_0^t \int_0^s \int_r^A \int_y^A |u_t^p| dz dy d\tau ds \\
&+ \int_0^t \int_0^s |u(\tau, r)| d\tau ds + \int_0^t \int_0^s \int_r^A \int_y^A \frac{n-1}{z^2} |u(\tau, z)| dz dy d\tau ds + t \int_r^A \int_s^A |u_1(y)| dy ds \\
&+ \int_r^A \int_s^A |u_0(y)| dy ds + \int_r^A \int_s^A |u(t, y)| dy ds \\
&+ \int_0^t \int_0^s \int_r^A \frac{n-1}{y} |u(\tau, y)| dy d\tau ds \\
&\leq \epsilon Q + \frac{1}{4} Q^p A^2 + \frac{1}{4} Q + \frac{1}{4} \frac{n-1}{r_0^2} A^2 Q + \frac{1}{2} A^2 Q + 2A^2 Q + \frac{1}{4} \frac{n-1}{r_0} A Q \\
&= \epsilon Q + \left(\frac{1}{4} Q^{p-1} A^2 + \frac{1}{4} + \frac{1}{4} \frac{n-1}{r_0^2} A^2 + \frac{1}{2} A^2 + 2A^2 + \frac{1}{4} \frac{n-1}{r_0} A \right) Q \\
&\leq \epsilon Q + (1 - \epsilon) Q = Q,
\end{aligned}$$

Now we will estimate $(S_{11}(u))_t$ for it we have

$$\begin{aligned}
(S_{11}(u))_t &= -\epsilon u_t + \int_0^t \int_r^A \int_y^A u_t^p dz dy d\tau ds \\
&+ \int_0^t u(\tau, r) d\tau ds + \int_0^t \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + \int_r^A \int_s^A u_1(y) dy ds \\
&- \int_r^A \int_s^A u_t(t, y) dy ds - \int_0^t \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau ds,
\end{aligned}$$

from here and from $|u| \leq Q$, $|u_t| \leq Q$, $r \in [r_0, r_0 + A]$, $t \in \left[0, \frac{1}{2}\right]$, the conditions for the constant A , we obtain

$$\begin{aligned}
|(S_{11}(u))_t| &\leq \epsilon|u_t| + \int_0^t \int_r^A \int_y^A |u_t^p| dz dy d\tau ds \\
&+ \int_0^t |u(\tau, r)| d\tau ds + \int_0^t \int_r^A \int_y^A \frac{n-1}{z^2} |u(\tau, z)| dz dy d\tau ds + \int_r^A \int_s^A |u_1(y)| dy ds \\
&+ \int_r^A \int_s^A |u_t(t, y)| dy ds + \int_0^t \int_r^A \frac{n-1}{y} |u(\tau, y)| dy d\tau ds \\
&\leq \epsilon Q + \left(\frac{1}{2} A^2 Q^{p-1} + \frac{1}{2} + \frac{1}{2} \frac{n-1}{r_0^2} A^2 + 2A^2 + \frac{1}{2} A \frac{n-1}{r_0} \right) Q \leq \epsilon Q + (1 - \epsilon) Q = Q.
\end{aligned}$$

Consequently

$$S_{11} : M_{11} \longrightarrow N_{11}.$$

From the estimates of $|S_{11}(u)|$ and $|(S_{11}(u))_t|$, follows that if $u \longrightarrow 0$, $u \in M_{11}$, in the sense of topology of the space M_{11} , then $S_{11}(u) \longrightarrow S_{11}(0)$ in the sense of the topology of the space M_{11} . Then the operator $S_{11} : M_{11} \longrightarrow M_{11}$ is continuous operator. The set M_{11} is compact space since every sequence of its elements is bounded. Also the set M_{11} is convex. \square

From Lemma 2.2, Lemma 2.3 and Theorem 2.1 follows that there exists $u^{11} \in M_{11}$ so that $L_{11}(u^{11}) = S_{11}(u^{11}) + T_{11}(u^{11})$, from here u^{11} is a solution to the problem (2.1), (2.2) and belongs to the space $\mathcal{C}^2([0, 1], \mathcal{C}^2([r_0, r_0 + A]))$.

Now we consider the Cauchy problem

$$(2.6) \quad u_{tt} - \frac{n-1}{r}u_r - u_{rr} = u_t^p, \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0 + A, r_0 + 2A],$$

$$(2.7) \quad u(0, r) = u^{11}(0, r), u_t(0, r) = u_t^{11}(0, r), \quad r \in [r_0 + A, r_0 + 2A],$$

We define the sets M_{12} and N_{12} as equicontinuous subsets of

$$\tilde{M}_{12} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0 + A, r_0 + 2A])\right), |u(t, r)| \leq Q, |u_t(t, r)| \leq Q, \right.$$

$$\left. \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0 + A, r_0 + 2A] \right\},$$

$$\tilde{N}_{12} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0 + A, r_0 + 2A])\right), |u(t, r)| \leq (1 + \epsilon)Q, \right. \\ \left. |u_t(t, r)| \leq (1 + \epsilon)Q, \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0 + A, r_0 + 2A] \right\},$$

respectively, in these sets we define a norm as follows

$$\|u\|_2 = \sup \left\{ |u(t, r)|, |u_t(t, r)| \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0 + A, r_0 + 2A] \right\}.$$

The sets M_{12} , N_{12} are completed normed spaces with this norm.

The operator L_{12} we define with the representation

$$L_{12}(u) = T_{12}(u) + S_{12}(u),$$

where

$$T_{12}(u) = (1 + \epsilon)u,$$

$$S_{12}(u) = -\epsilon u + \int_0^t \int_0^s \int_r^{2A} \int_y^{2A} u_t^p dz dy d\tau ds \\ + \int_0^t \int_0^s u(\tau, r) d\tau ds + \int_0^t \int_0^s \int_r^{2A} \int_y^{2A} \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + t \int_r^{2A} \int_s^{2A} u_1(y) dy ds + \\ + \int_r^{2A} \int_s^{2A} u_0(y) dy ds - \int_r^{2A} \int_s^{2A} u(t, y) dy ds - \\ - \int_0^t \int_0^s \int_r^{2A} \frac{n-1}{y} u(\tau, y) dy d\tau ds.$$

As in above we conclude that the Cauchy problem (2.6), (2.7) has a solution $u^{12}(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0 + A, r_0 + 2A])\right)$.

Now we consider the Cauchy problem

$$(2.8) \quad u_{tt} - \frac{n-1}{r}u_r - u_{rr} = u_t^p, \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0 + 2A, r_0 + 3A],$$

$$(2.9) \quad u(0, r) = u^{12}(0, r), u_t(0, r) = u_t^{12}(0, r), \quad r \in [r_0 + 2A, r_0 + 3A],$$

We define the sets M_{13} and N_{13} as equicontinuous subsets of

$$\tilde{M}_{13} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0 + 2A, r_0 + 3A])\right), |u(t, r)| \leq Q, |u_t(t, r)| \leq Q, \right.$$

$$\left. \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0 + 2A, r_0 + 3A] \right\},$$

$$\tilde{N}_{13} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[0, \frac{1}{2}\right], \mathcal{C}^2([r_0 + 2A, r_0 + 3A])\right), |u(t, r)| \leq (1 + \epsilon)Q, \right. \\ \left. |u_t(t, r)| \leq (1 + \epsilon)Q, \forall t \in \left[0, \frac{1}{2}\right], r \in [r_0 + 2A, r_0 + 3A] \right\},$$

respectively, in these sets we define a norm as follows

$$\|u\|_2 = \sup\{|u(t, r)|, |u_t(t, r)| \quad t \in \left[0, \frac{1}{2}\right], r \in [r_0 + 2A, r_0 + 3A]\}.$$

The sets M_{13} , N_{13} are completed normed spaces with this norm.

The operator L_{13} we define with the representation

$$L_{13}(u) = T_{13}(u) + S_{13}(u),$$

where

$$T_{13}(u) = (1 + \epsilon)u,$$

$$\begin{aligned} S_{13}(u) &= -\epsilon u + \int_0^t \int_0^s \int_r^{3A} \int_y^{3A} u_t^p dz dy d\tau ds \\ &+ \int_0^t \int_0^s u(\tau, r) d\tau ds + \int_0^t \int_0^s \int_r^{3A} \int_y^{3A} \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds + t \int_r^{3A} \int_s^{3A} u_1(y) dy ds + \\ &+ \int_r^{3A} \int_s^{3A} u_0(y) dy ds - \int_r^{3A} \int_s^{3A} u(t, y) dy ds - \\ &- \int_0^t \int_0^s \int_r^{3A} \frac{n-1}{y} u(\tau, y) dy d\tau ds. \end{aligned}$$

As in above we conclude that the Cauchy problem (2.8), (2.9) has a solution $u^{13}(t, r) \in \mathcal{C}^2([0, \frac{1}{2}], \mathcal{C}^2([r_0 + 2A, r_0 + 3A]))$ and etc. Then the function

$$u^1 = \begin{cases} u^{11} & t \in \left[0, \frac{1}{2}\right], r \in [r_0, r_0 + A], \\ u^{12} & t \in \left[0, \frac{1}{2}\right], r \in [r_0 + A, r_0 + 2A], \\ \dots & \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - \frac{n-1}{r} u_r - u_{rr} = u_t^p, \quad t \in \left[0, \frac{1}{2}\right], r \geq r_0,$$

$$u(0, r) = u_0(r), u_t(0, r) = u_1(r), \quad r \geq r_0,$$

which belongs to the space $\mathcal{C}^2([0, \frac{1}{2}], \mathcal{C}^2([r_0, \infty)))$. We note that in above definition of the solution u^1 we agglutinate continuous - differentiable u^{11} and u^{12} in $r = r_0 + A$, u^{12} and u^{13} in $r = r_0 + 2A$ and etc. if it is necessary.

Now we consider the Cauchy problem

$$(2.10) \quad u_{tt} - \frac{n-1}{r} u_r - u_{rr} = u_t^p, \quad t \in \left[\frac{1}{2}, 1\right], r \in [r_0, r_0 + A],$$

$$(2.11) \quad u\left(\frac{1}{2}, r\right) = u^1\left(\frac{1}{2}, r\right), u_t\left(\frac{1}{2}, r\right) = u_t^1\left(\frac{1}{2}, r\right), \quad r \in [r_0, r_0 + A],$$

We define the sets M_{21} and N_{21} as equicontinuous subsets of

$$\tilde{M}_{21} = \left\{u(t, r) \in \mathcal{C}^2\left(\left[\frac{1}{2}, 1\right], \mathcal{C}^2([r_0, r_0 + A])\right), |u(t, r)| \leq Q, |u_t(t, r)| \leq Q,\right.$$

$$\left. \forall t \in \left[\frac{1}{2}, 1\right], r \in [r_0, r_0 + A]\right\},$$

$$\tilde{N}_{21} = \left\{ u(t, r) \in \mathcal{C}^2\left(\left[\frac{1}{2}, 1\right], \mathcal{C}^2([r_0, r_0 + A])\right), |u(t, r)| \leq (1 + \epsilon)Q, \right. \\ \left. |u_t(t, r)| \leq (1 + \epsilon)Q, \forall t \in \left[\frac{1}{2}, 1\right], r \in [r_0, r_0 + A] \right\},$$

respectively in these sets we define a norm as follows

$$\|u\|_2 = \sup \left\{ |u(t, r)|, |u_t(t, r)| \quad t \in \left[\frac{1}{2}, 1\right], r \in [r_0, r_0 + A] \right\}.$$

The sets M_{21}, N_{21} are completed normed spaces with this norm.

The operator L_{21} we define with the representation

$$L_{21}(u) = T_{21}(u) + S_{21}(u),$$

where

$$T_{21}(u) = (1 + \epsilon)u,$$

$$S_{21}(u) = -\epsilon u + \int_{\frac{1}{2}}^t \int_{\frac{1}{2}}^s \int_r^A \int_y^A u_t^p dz dy d\tau ds \\ + \int_{\frac{1}{2}}^t \int_{\frac{1}{2}}^s u(\tau, r) d\tau ds + \int_{\frac{1}{2}}^t \int_{\frac{1}{2}}^s \int_r^A \int_y^A \frac{n-1}{z^2} u(\tau, z) dz dy d\tau ds \\ + \left(t - \frac{1}{2}\right) \int_r^A \int_s^A u_1(y) dy ds + \\ + \int_r^A \int_s^A u_0(y) dy ds - \int_r^A \int_s^A u(t, y) dy ds - \\ - \int_{\frac{1}{2}}^t \int_{\frac{1}{2}}^s \int_r^A \frac{n-1}{y} u(\tau, y) dy d\tau ds.$$

As in above we conclude that the Cauchy problem (2.10), (2.11) has a solution $u^{21}(t, r) \in \mathcal{C}^2\left(\left[\frac{1}{2}, 1\right], \mathcal{C}^2([r_0, r_0 + A])\right)$ and etc., the function

$$u^2 = \begin{cases} u^{21} & t \in \left[\frac{1}{2}, 1\right], r \in [r_0, r_0 + A], \\ u^{22} & t \in \left[\frac{1}{2}, 1\right], r \in [r_0 + A, r_0 + 2A], \\ \dots & \dots \end{cases}$$

is a solution to the Cauchy problem

$$u_{tt} - \frac{n-1}{r}u_r - u_{rr} = u_t^p, \quad t \in \left[\frac{1}{2}, 1\right], r \geq r_0,$$

$$u\left(\frac{1}{2}, r\right) = u^1\left(\frac{1}{2}, r\right), u_t\left(\frac{1}{2}, r\right) = u_t^1\left(\frac{1}{2}, r\right), \quad r \geq r_0,$$

which belongs to the space $\mathcal{C}^2\left(\left[\frac{1}{2}, 1\right], \mathcal{C}^2([r_0, \infty))\right)$ and etc. We note that in above definition of the solution u^2 we agglutinate continuous - differentiable u^{21} and u^{22} in $r = r_0 + A$, u^{22} and u^{23} in $r = r_0 + 2A$ and etc. if it is necessary. The function

$$u = \begin{cases} u^1 & t \in [0, \frac{1}{2}], r \geq r_0, \\ u^2 & t \in [\frac{1}{2}, 1], r \geq r_0, \\ \dots & \dots \end{cases}$$

is a solution to the Cauchy problem (1.1), (1.2) which belongs to the space

$$\mathcal{C}^2([0, \infty), \mathcal{C}^2([r_0, \infty))).$$

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