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Limiting Profile of the Blow-up Solutions for the Fourth-order Nonlinear Schrödinger Equation

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ABSTRACT. This paper is concerned with the blow-up solutions of the focusing fourth-order mass-critical nonlinear Schrödinger equation. Establishing the profile decomposition of the bounded sequences in H^2, we obtain the variational characteristics of the corresponding ground state and a compactness lemma. Moreover, we obtain the L^2-concentration of the blow-up solutions and the limiting profile of the minimal mass blow-up solutions in the general case.

CONTENTS

1. Introduction

In this paper, we study the Cauchy problem of the focusing fourth-order mass-critical nonlinear Schrödinger equation

$$iu_t - \triangle^2 u + |u|^{\frac{8}{N}} u = 0,\ t \in [0,T),\ x \in R^N, \tag{1.1}$$

$$u(0,x) = u_0, \tag{1.2}$$

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where $i = \sqrt{-1}$; $\triangle^2 = \triangle\triangle$ is the biharmonic operator defined in R^N and $\triangle = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N; $u = u(t,x)$: $[0,T) \times R^N \to \mathcal{C}$ is the complex valued function and $0 < T \leq +\infty$; N is the space dimension. Fourth-order Schrödinger equations are introduced by Karpman [**11**] and Karpman, Shagalov [**12**] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity, and such fourth-order Schrödinger equation are written as

$$i\phi_t + \varepsilon\triangle^2\phi + \mu\triangle\phi + |\phi|^{p-1}\phi = 0, \quad \phi = \phi(t,x) : R \times R^N \to \mathcal{C}. \tag{1.3}$$

Note that Equation (1.1) is a special case of Equation (1.3) by taking $\varepsilon = 1$, $\mu = 0$ and $p = 1 + \frac{8}{N}$. For the Cauchy problem (1.1)-(1.2), Kenig, Ponce and Vega [**13**], Ben-Artzi, Koch and Saut [**2**] established the local well-posedness in $H^2 = H^2(R^N)$. Equation (1.1) is called the mass-critical due to the mass $M(u) = \int_{R^N} |u(t,x)|^2 dx$ and the equation itself are invariant under the rescaling symmetry $u \mapsto \lambda^{\frac{N}{2}} u(\lambda^4 t, \lambda x)$.

We recall some known results for the classical focusing mass-critical nonlinear Schrödinger equation

$$iv_t + \triangle v + |v|^{\frac{4}{N}} v = 0, \quad v(0,x) = \varphi \in H^1. \tag{1.4}$$

Ginibre and Velo [**8**] showed the local well-posedness in $H^1 = H^1(R^N)$. In fact, in this space energy arguments apply, and a blow-up theory has been developed in the last two decades (see [**4**], [**29**] and the references therein). This theory is connected to the notion of ground state: the unique positive radial solution of the elliptic problem

$$\triangle R - R + |R|^{\frac{4}{N}} R = 0, \quad R \in H^1. \tag{1.5}$$

Weinstein [**33**] exhibited the following refined Gagliardo-Nirenberg inequality:

$$\|f\|_{L^{2+\frac{4}{N}}}^{2+\frac{4}{N}} \leq (1 + \frac{2}{N}) \left(\frac{\|f\|_{L^2}}{\|R\|_{L^2}} \right)^{\frac{4}{N}} \|\nabla f\|_{L^2}^2, \quad f \in H^1. \tag{1.6}$$

Combined with the conservation of energy, this implies that: if the initial data $\|\varphi\|_{L^2} < \|R\|_{L^2}$, then the solution $v(t,x)$ exists globally; if the initial data $\|\varphi\|_{L^2} \geq \|R\|_{L^2}$, then the solution $v(t,x)$ may blow up. The value $\|R\|_{L^2}$ is the sharp value of blow-up and global existence of the solutions in terms of Merle's [**17**] results. Using the variational characteristic of the ground state elliptic equation (1.5), Weinstein [**34**] showed the structure and formation of singularity of the minimal-mass blow-up solution (i.e. $\|\varphi\|_{L^2} = \|R\|_{L^2}$). It reads that the corresponding blow-up solution $v(t,x)$ remains close to $R(x)$ in H^1 up to scaling and phase parameters, and also translation in the nonradial case. In other words, the blow-up solution has the same shape as the ground state $R(x)$. Thus, basing on this structure and formation of singularity, Merle and Raphaël [**18, 19**] obtained a large body of breakthrough work on the qualitative properties of blow-up solutions with the help of the Spectral Properties [**18**], such as blow-up rates, profiles of blow-up solutions, etc. On the other hand, for $\varphi \in H^1$, Merle and Tsutsumi [**20**,**30**](for radial data), Nawa [**23**] and Weinstein [**35**](for general data) proved the following L^2-concentration property of the blow-up solutions by using the variational characterization of the ground state:

there exists $x(t) \in R^N$ such that $\forall\ r > 0$

$$\liminf_{t\to\tau} \int_{|x-x(t)|\le r} |v(t,x)|^2 dx \ge \int R^2 dx,$$

where τ is the blow-up time. These results are extended to $\varphi \in H^s(R^N)$ for some $s_0 < s < 1$(see [**3**,**5**,**10**,**14**,**15**,**31**,**32**]) by using the variational characteristic of the ground state elliptic Equation (1.5) and harmonic analysis techniques.

In Equation (1.1), if we replace the nonlinearity $|u|^{\frac{8}{N}}u$ with $|u|^{p-1}u$, it is a class of semilinear fourth-order Schrödinger equations similar to Equation (1.1), which has been widely investigated. For $1 < p < \frac{2N}{(N-4)^+}$ (where $\frac{2N}{(N-4)^+} = +\infty$, when $N \le 4$; $\frac{2N}{(N-4)^+} = \frac{2N}{N-4}$, when $N > 4$), Ben-Artzi, Koch and Saut [**2**] established the local well-posedness in H^2. Fibich, Ilan and Papanicolaou [**6**] obtain the general results of global well-posedness in H^2. Pausader [**25**] and Segata [**28**] studied the global well-posedness and scattering of the fourth-order nonlinear Schrödinger equation with cubic nonlinearity. For $p = \frac{2N}{N-4}$, Miao, Xu and Zhao [**21**], Pausader [**26**] studied the global existence and scattering of the focusing fourth-order nonlinear Schrödinger equation; Miao, Xu and Zhao [**22**], Pausader [**24**] studied the global existence and scattering of the defocusing fourth-order nonlinear Schrödinger equation. The above studies focused on global solutions. From the view-point of physics, physicists are very interested in the elaborate description of the blow-up solutions in H^2, such as blow-up rate, L^2-concentration, limiting profile of the blow-up solutions, etc.

In this paper, we study the limiting profile of the blow-up solutions to the Cauchy problem (1.1)-(1.2) in H^2. Motivated by the study of the classical mass-critical nonlinear Schrödinger equation (1.4), we consider the ground state solution of the Equation (1.1)

$$\triangle^2 Q + Q - |Q|^{\frac{8}{N}} Q = 0, \quad Q \in H^2, \tag{1.7}$$

which is a special periodic solution of Equation (1.1) in the form $u(t,x) = Q(x)e^{it}$.

Fibich, Ilan and Papanicolaou [**6**] showed some numerical observations of the solution to the Cauchy problem (1.1)-(1.2), which implies that if the initial data $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the solution $u(t,x)$ exists globally; if the initial data $\|u_0\|_{L^2} \ge \|Q\|_{L^2}$, then the solution $u(t,x)$ may blow up in finite time. Since the effect of fourth-order dispersion $\triangle^2 u$, whether the variance identity arguments can be extended to show the existence of blow-up solutions for the biharmonic nonlinear Schrödinger equation is still unknown (see [**1**, **6**]). On the other hand, the numerical observations in [**6**] showed the existence of blow-up solutions. Baruch, Fibich and Mandelbaum [**1**] obtained some dynamical properties of the radially symmetric blow-up solutions, such as blow-up rate, L^2-concentration. However, to our knowledge, the existence of ground state of the elliptic Equation (1.7), the variational structure of the ground state solution Q and the limiting profile of the nonradially symmetric blow-up solutions are not addressed.

In the present paper, establishing the profile decomposition of the bounded sequences in H^2, we prove the existence of the ground state of elliptic Equation (1.7), and we obtain the variational characteristics of the ground state solution $Q(x)$. Moreover, we obtain a compactness lemma adapted to the analysis of the blow-up phenomenon of the fourth-order nonlinear Schrödinger equations in H^2, as follows.

Theorem 1.1. *Let $\{v_n\}_{n=1}^{\infty}$ be a bounded family of H^2 functions such that*

$$\lim_{n\to\infty}\sup\|\triangle v_n\|_{L^2}\le M \quad \text{and} \quad \lim_{n\to\infty}\sup\|v_n\|_{L^{2+\frac{8}{N}}}\ge m. \tag{1.8}$$

Then, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of R^N such that up to a subsequence

$$v_n(x+x_n)\rightharpoonup V(x) \quad \text{weakly in} \quad H^2. \tag{1.9}$$

with $\|V\|_{L^2}^{\frac{8}{N}}\ge\frac{\|Q\|_{L^2}^{\frac{8}{N}}m^{2+\frac{8}{N}}}{(1+\frac{4}{N})M^2}$, and Q is the solution of ground state Equation (1.7).

Finally, we apply them to obtain some dynamical properties of the blow-up solutions in the general case: the L^2-concentration of the blow-up solutions, limiting profile of the minimal-mass blow-up solutions, as follows.

Theorem 1.2. *Let $u(t,x)\in C([0,T);H^2)$ be the corresponding blow-up solution of the Cauchy problem (1.1)-(1.2). Suppose that $a(t)>0$ is any function such that*

$$\lim_{t\to T}a(t)=0 \quad \text{and} \quad \lim_{t\to T}\frac{(T-t)^{\frac{1}{4}}}{a(t)}=0. \tag{1.10}$$

Then, there exists $y(t)\in R^N$ such that

$$\lim_{t\to T}\inf\int_{|x-y(t)|\le a(t)}|u(t,x)|^2dx\ge\int|Q|^2dx, \tag{1.11}$$

where Q is the ground state solution of Equation (1.7).

Theorem 1.3. *Let $u_0\in H^2$ and $\|u_0\|_{L^2}=\|Q\|_{L^2}$. Suppose that $u(t,x)\in C([0,T);H^2)$ is the blow-up solution of the Cauchy problem (1.1)-(1.2) in finite time $0<T<+\infty$. Then $\forall\varepsilon>0$, $\exists\delta>0$ s.t. when $|t-T|<\delta$, there are functions $y(t)\in R^N$, $\gamma(t)\in R$ such that*

$$\left\|\lambda^{\frac{N}{2}}(t)u(t,\lambda(t)(x+y(t)))e^{i\gamma(t)}-Q(x)\right\|_{H^2}<\varepsilon, \tag{1.12}$$

where $\lambda(t)=\left(\frac{\|\triangle Q\|_{L^2}}{\|\triangle u\|_{L^2}}\right)^{\frac{1}{2}}$ and $Q(x)$ is the unique solution of ground state Equation (1.7).

In this paper, the main tools of the proof of the compactness lemma in this paper are an argument of profile decomposition, introduced by Gérard [**7**] and Hmidi and Keraani [**9**] to study the defect of compactness for Sobolev embedding. We obtain the existence of ground state of the elliptic Equation (1.7) and variational characteristic of the ground state $Q(x)$ of Equation (1.7) by establishing the profile decomposition of the bounded sequence in H^2, which which are important in studying the blow-up dynamic of the blow-up solutions for fourth-order nonlinear Schrödinger equations. Moreover, we extend the results in [**1**] to the nonradially blow-up solutions of the Cauchy problem (1.1)-(1.2).

In this paper, We use the denotes $L^q:=L^q(R^N)$, $\|\cdot\|_q:=\|\cdot\|_{L^q(R^N)}$, $H^s:=H^s(R^N)$, $\dot{H}^s:=\dot{H}^s(R^N)$ and $\int\cdot dx:=\int_{R^N}\cdot dx$. The various positive constants will be simply denoted by C.

2. Preliminary

For the Cauchy problem (1.1)-(1.2), the energy space H^2 is defined by

$$H^2 := \{u \in S(R^N) \mid \int_{R^N} (1+|\xi|^2)^2 |\widehat{u}(\xi)|^2 d\xi < +\infty\},$$

with the norm

$$\|u\|_{H^2} = (\|u\|_2^2 + \|\nabla u\|_2^2 + \|\triangle u\|_2^2)^{\frac{1}{2}}.$$

H^2 is a Hilbert space. It is easy to check that, there exist two positive constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1(\|u\|_2^2 + \|\triangle u\|_2^2) \le \|u\|_{H^2}^2 \le C_2(\|u\|_2^2 + \|\triangle u\|_2^2),$$

which implies that $(\|u\|_2^2 + \|\triangle u\|_2^2)^{\frac{1}{2}}$ is an equivalent norm of H^2. In this paper, we shall use this equivalent norm of H^2: $(\|u\|_2^2 + \|\triangle u\|_2^2)^{\frac{1}{2}}$ to study the profile decomposition of the bounded sequences in H^2. Moreover, we define the energy functional $E(u)$ on H^2 by

$$E(u) := \frac{1}{2}\int |\triangle u|^2 dx - \frac{1}{2+\frac{8}{N}}\int |u|^{2+\frac{8}{N}} dx.$$

The functional $E(u)$ is well-defined according to the Sobolev embedding theorem(see [**4**]). Ben-Artzi, Koch and Saut [**2**] established the local well-posedness of the Cauchy problem (1.1)-(1.2) in H^2, as follows.

Proposition 2.1. *Let $u_0 \in H^2$. There exists an unique solution $u(t,x)$ of the Cauchy problem (1.1)-(1.2) on the maximal time interval $[0,T)$ such that $u(t,x) \in C([0,T);H^2)$ and either $T = +\infty$(global existence), or else $0 < T < +\infty$ and $\lim\limits_{t\to T} \|u(t,x)\|_{H^2} = +\infty$ (blow-up). Furthermore, for all $t \in [0,T)$, $u(t,x)$ satisfies the following conservation laws:*

(i) *Conservation of mass*

$$\int |u(t,x)|^2 dx = \int |u_0|^2 dx. \tag{2.1}$$

(ii) *Conservation of energy*

$$E(u(t,x)) = E(u_0). \tag{2.2}$$

Baruch, Fibich and Mandelbaum [**1**] obtained the lower-bound for the blow-up rate of the blow-up solutions to the Cauchy problem (1.1)-(1.2).

Lemma 2.2. *Let $u(t,x)$ be the blow-up solution of the Cauchy problem (1.1)-(1.2) at finite time $0 < T < +\infty$. Then, there exists a constant $K = K(\|u_0\|_2) > 0$ such that*

$$\|\triangle u(t)\|_2 \ge \frac{K}{(T-t)^{\frac{1}{2}}}, \quad 0 < t < T. \tag{2.3}$$

In order to study the variational characteristic of the ground state corresponding to Equation (1.1), we need the following profile decomposition of the bounded sequences in H^2, which is also our main tool. Similar results for bounded sequences in L^2 and H^1 have appeared in Gérard [**7**] and Hmidi and Keraani [**9**].

Proposition 2.3. *Let $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in H^2. Then there exist a subsequence of $\{v_n\}_{n=1}^{\infty}$(still denoted $\{v_n\}_{n=1}^{\infty}$), a family $\{x_n^j\}_{j=1}^{\infty}$ of sequences in R^N and a sequence $\{V^j\}_{j=1}^{\infty}$ of H^2 functions such that*

(i) *for every* $k \neq j$,

$$|x_n^k - x_n^j| \to \infty \text{ as } n \to \infty, \tag{2.4}$$

(ii) *for every* $l \geq 1$ *and every* $x \in R^N$

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x), \tag{2.5}$$

with

$$\lim_{n\to\infty} \sup \|v_n^l\|_p \to 0, \text{ as } l \to \infty, \tag{2.6}$$

for every $p \in (2, \frac{2N}{(N-4)^+})$.

Moreover, as $n \to \infty$, *we have*

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1) \tag{2.7}$$

and

$$\|\triangle v_n\|_2^2 = \sum_{j=1}^{l} \|\triangle V^j\|_2^2 + \|\triangle v_n^l\|_2^2 + o(1). \tag{2.8}$$

Proof. Since H^2 is a Hilbert space, we denote $\mu(v_n)$ is the set of functions obtained as weak limits of subsequences of the translated $v_n(x + x_n)$ with $\{x_n\}_{n=1}^{\infty}$ in H^2. We denote

$$\eta(v_n) = \sup\{\|V\|_2 + \|\triangle V\|_2, V \in \mu(v_n)\}. \tag{2.9}$$

It is obvious that

$$\eta(v_n) \leq \lim_{n\to\infty} \sup\{\|V\|_2 + \|\triangle V\|_2\}. \tag{2.10}$$

Next, we shall prove that there exist a subsequence $\{V^j\}_{j=1}^{\infty}$ of $\mu(v_n)$ and a family $\{x_n^j\}_{j=1}^{\infty}$ of sequences of R^N such that

$$\forall k \neq j \quad |x_n^k - x_n^j| \to \infty \text{ as } n \to \infty \tag{2.11}$$

and up to extracting a subsequence, the sequence $\{v_n\}_{n=1}^{\infty}$ can be written as

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l, \ \eta(v_n^l) \to 0 (l \to \infty) \tag{2.12}$$

and (2.7) and (2.8) are true.

Indeed, if $\eta(v_n) = 0$, we can take $V^j = 0$ for all j, otherwise, we choose $V^1 \in \mu(v_n)$ such that

$$\|\triangle V^1\|_2 + \|V^1\|_2 \geq \frac{1}{2}\eta(v_n) > 0.$$

By the definition of $\mu(v_n)$, there exists a subsequence x_n^1 of R^N such that up to extracting a subsequence, we have

$$v_n(x + x_n^1) \rightharpoonup V^1(x) \quad \text{weakly in} \quad H^2. \tag{2.13}$$

Setting $v_n^1(x) = v_n(x) - V^1(x - x_n^1)$, by (2.13), we have $v_n^1(x + x_n^1) \rightharpoonup 0$ weakly in H^2 and

$$\|v_n\|_2^2 = \|V^1\|_2^2 + \|v_n^1\|_2^2 + o(1), \tag{2.14}$$

$$\|\triangle v_n\|_2^2 = \|\triangle V^1\|_2^2 + \|\triangle v_n^1\|_2^2 + o(1). \tag{2.15}$$

Now, replacing v_n by v_n^1 and repeating the same process. There exists $V^2 \in \mu(v_n)$ such that $\|\triangle V^2\|_2 + \|V^2\|_2 \geq \frac{1}{2}\eta(v_n^1) > 0$ and

$$v_n^1(x + x_n^2) \rightharpoonup V^2(x) \quad \text{weakly in} \quad H^2. \tag{2.16}$$

Setting $v_n^2(x) = v_n^1(x) - V^2(x - x_n^2)$, by (2.13), we have $v_n^2(x + x_n^2) \rightharpoonup 0$ weakly in H^2 and

$$\|v_n^1\|_2^2 = \|V^2\|_2^2 + \|v_n^2\|_2^2 + o(1), \tag{2.17}$$

$$\|\triangle v_n^1\|_2^2 = \|\triangle V^2\|_2^2 + \|\triangle v_n^2\|_2^2 + o(1), \tag{2.18}$$

and

$$|x_n^1 - x_n^2| \to \infty \quad \text{as} \quad n \to \infty. \tag{2.19}$$

Indeed, if (2.19) is not true, then

$$v_n^2(x + x_n^2) = v_n^1(x + x_n^2 - x_n^1 + x_n^1) + V^2(x), \tag{2.20}$$

which implies that $V^2 = 0$ by $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ and $v_n^2(\cdot + x_n^2) \rightharpoonup 0$ in H^2, a contradiction.

An argument of iteration and orthogonal extraction allows us to construct the families $\{x_n^j\}_{j=1}^\infty$ and $\{V^j\}_{j=1}^\infty$ satisfying the claim above. Furthermore, since the convergence of the series $\sum_{j=1}^{l}(\|V^j\|_2^2 + \|\triangle V^j\|_2^2)$, we have

$$\|V^j\|_2^2 + \|\triangle V^j\|_2^2 \to 0 \quad \text{as} \quad j \to \infty, \tag{2.21}$$

which implies that

$$\eta(v_n^j) \leq 2(\|V^{j-1}\|_2 + \|\triangle V^{j-1}\|_2) \to 0 \quad \text{as} \quad j \to \infty. \tag{2.22}$$

Therefore, (2.7), (2.8) and (2.12) are true.

In the end, we shall prove that for $p \in (2, \frac{2N}{(N-4)^+})$

$$\|v_n^l\|_p \to 0 \quad \text{as} \quad l \to \infty. \tag{2.23}$$

Let $\chi_R \in S(R^N)$ such that *supp* $\hat{\chi_R}(\xi) = \{\frac{1}{2R} \leq |\xi| \leq 2R\}$, $\hat{\chi_R} = 1$ on $\{\frac{1}{R} \leq |\xi| \leq R\}$, and $0 \leq \hat{\chi_R} \leq 1$ on *supp* $\hat{\chi_R}(\xi)$, where $\hat{}$ denotes the Fourier transform. It is obvious that

$$v_n^l = \chi_R * v_n^l + (\delta - \chi_R) * v_n^l, \tag{2.24}$$

where $*$ is the convolution and δ is the Dirac function. Using the the definition of χ_R, we have

$$\|(\delta - \chi_R) * v_n^l\|_2 \qquad \leq C\left(\int_{|\xi|\leq\frac{1}{R}} |\hat{v}_n^l(\xi)|^2 d\xi + \int_{|\xi|\geq R} |\hat{v}_n^l(\xi)|^2 d\xi\right)^{\frac{1}{2}}. \tag{2.25}$$

Since v_n^l is bounded in L^2, for any $n \geq 1$, we have

$$\int_{|\xi|\leq\frac{1}{R}} |\hat{v}_n^l(\xi)|^2 d\xi + \int_{|\xi|\geq R} |\hat{v}_n^l(\xi)|^2 d\xi \to 0, \quad \text{as} \quad R \to \infty. \tag{2.26}$$

Using the Sobolev embedding $\dot{H}^2 \hookrightarrow L^{\frac{2N}{(N-4)^+}}$, we have

$$\begin{aligned}
\|(\delta-\chi_R)*v_n^l\|_{L^{\frac{2N}{(N-4)^+}}} &\le C\|(\delta-\chi_R)*v_n^l\|_{\dot{H}^2} \\
&\le C\left(\int |\xi|^4|(1-\hat{\chi}_R(\xi))\hat{v}_n^l(\xi)|^2 d\xi\right)^{\frac{1}{2}} \qquad (2.27)\\
&\le C\left(\int_{|\xi|\le\frac{1}{R}} |\xi|^4|\hat{v}_n^l(\xi)|^2 d\xi + \int_{|\xi|\ge R} |\xi|^4|\hat{v}_n^l(\xi)|^2 d\xi\right)^{\frac{1}{2}}.
\end{aligned}$$

Since v_n^l is bounded in H^2, for any $n \ge 1$, we have

$$\int_{|\xi|\le\frac{1}{R}} |\xi|^4|\hat{v}_n^l(\xi)|^2 d\xi + \int_{|\xi|\ge R} |\xi|^4|\hat{v}_n^l(\xi)|^2 d\xi \to 0, \quad \text{as} \quad R\to\infty. \tag{2.28}$$

Taking $p \in (2, \frac{2N}{(N-4)^+})$ and using the Hölder interpolation inequality and (2.26)-(2.28), we have

$$\begin{aligned}
\|(\delta-\chi_R)*v_n^l\|_p &\le \|(\delta-\chi_R)*v_n^l\|_2^{\theta}\|(\delta-\chi_R)*v_n^l\|_{\frac{2N}{(N-4)^+}}^{1-\theta} \qquad (2.29)\\
&\to 0, \quad \text{as} \quad R\to\infty,
\end{aligned}$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{\frac{2N}{(N-4)^+}}$ and $0<\theta<1$.

On the other hand, by the definition of χ_R one can estimate

$$\|\chi_R * v_n^l\|_p \le C\|\chi_R * v_n^l\|_\infty^{\frac{p-2}{p}}\|\chi_R * v_n^l\|_2^{\frac{2}{p}}. \tag{2.30}$$

In view of the definition of $\mu(v_n)$, we have

$$\lim_{n\to\infty}\sup\|\chi_R * v_n^l\|_\infty \le \sup\{|\int \chi_R(-x)V(x)dx|, V\in\mu(v_n)\}. \tag{2.31}$$

Using the Parseval identity and Hölder inequality, we have

$$\begin{aligned}
\left|\int \chi_R(-x)V(x)dx\right| &= \left|\int \mathcal{F}^{-1}[\chi_R(-x)]\mathcal{F}[V(x)]d\xi\right| \\
&= \left|\int \widehat{\chi_R}(\xi)\widehat{V}(\xi)d\xi\right| \qquad (2.32)\\
&\le C\int_{\frac{1}{2R}\le|\xi|\le 2R} \frac{1}{|\xi|^2}\ |\xi|^2\ |\widehat{V}(\xi)|d\xi \\
&\le CR^{N+2}\|\triangle V\|_2 \le CR^{N+2}\eta(v_n^l).
\end{aligned}$$

Taking $R = \left(\frac{1}{\eta(v_n^l)}\right)^{\frac{1}{N+2}-\varepsilon}$ with $\forall\ \varepsilon>0$ sufficient small, we have

$$\left|\int \chi_R(-x)V(x)dx\right| \to 0 \quad \text{as} \quad l\to\infty. \tag{2.33}$$

It follows from (2.30), (2.31) and (2.33) that for $p\in(2,\frac{2N}{(N-4)^+})$,

$$\|\chi_R * v_n^l\|_p \to 0 \quad \text{as} \quad l\to\infty. \tag{2.34}$$

Applying (2.24), (2.29) and (2.34), we have

$$\lim_{n\to\infty}\sup\|v_n^l\|_p \to 0, \quad \text{as} \quad l\to\infty. \tag{2.35}$$

This completes the proof.

3. Variational Structure

In order to study the variational structure of the ground state, we consider the following elliptic equation

$$\triangle^2 Q + Q - |Q|^{\frac{8}{N}} Q = 0, \quad Q \in H^2. \tag{3.1}$$

Define the variational problem

$$J := \min\{J(u) : u \in H^2\}, \quad \text{where} \quad J(u) := \frac{(\int |u|^2 dx)^{\frac{4}{N}} (\int |\triangle u|^2 dx)}{\int |u|^{2+\frac{8}{N}} dx}. \tag{3.2}$$

By some basic calculations, it is easy to check that if W is the minimizer of $J(u)$, we have the following lemma, one can also see [**6**, **16**]. But we provide the detail here for the reader's convenience.

Lemma 3.1. *If W is the minimizer of $J(u)$, then W satisfies*

$$\|W\|_2^{\frac{8}{N}} \triangle^2 W + \frac{4}{N} \|\triangle W\|_2^2 \|W\|_2^{\frac{8}{N}-2} W - J(1 + \frac{4}{N}) |W|^{\frac{8}{N}} W = 0. \tag{3.3}$$

Proof. It follows from the fact that W is a minimizing function of $J(u)$ in H^2, and we have $\forall\ v \in C_0^\infty(R^N)$

$$\frac{d}{d\varepsilon} J(W + \varepsilon v) \mid_{\varepsilon=0} = 0. \tag{3.4}$$

By some computations, we have

$$\frac{d}{d\varepsilon}\{\|(W+\varepsilon v)\|_2^{\frac{8}{N}}\} \mid_{\varepsilon=0} = \frac{4}{N}\|W\|_2^{\frac{8}{N}-2} \int 2\Re W \overline{v} dx, \tag{3.5}$$

$$\frac{d}{d\varepsilon}\{\|\triangle(W+\varepsilon v)\|_2^2\} \mid_{\varepsilon=0} = \int 2\Re \triangle^2 W \overline{v} dx \tag{3.6}$$

and

$$\frac{d}{d\varepsilon}\{\|W+\varepsilon v\|_{2+\frac{8}{N}}^{2+\frac{8}{N}}\} \mid_{\varepsilon=0} = (1+\frac{4}{N}) 2\Re \int |W|^{\frac{8}{N}} W \overline{v} dx. \tag{3.7}$$

By (3.4)-(3.7), we have

$$\begin{aligned} &\tfrac{4}{N}\|W\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \|\triangle W\|_2^2 \|W\|_2^{\frac{8}{N}-2} \textstyle\int 2\Re W\overline{v}dx + \|W\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \|W\|_2^{\frac{8}{N}} \textstyle\int 2\Re\triangle^2 W\overline{v}dx \\ = \ & (1+\tfrac{4}{N})\|\triangle W\|_2^2 \|W\|_2^{\frac{8}{N}} 2\Re \textstyle\int |W|^{\frac{8}{N}} W\overline{v}dx, \end{aligned} \tag{3.8}$$

which implies that (3.3) is true.

Now, we use the profile decomposition of the bounded sequence in H^2 to obtain the following proposition.

Proposition 3.2. *J is attained at a function $U(x) \in H^2$ with the following properties:*

$$U(x) = aQ(\lambda x + x_0) \quad \text{for some} \quad a \in \mathcal{C}^*,\ \lambda > 0 \text{ and } \ x_0 \in R^N \tag{3.9}$$

where Q is the solution of ground state elliptic Equation (3.1). Moreover,

$$J = \frac{1}{1+\frac{4}{N}} \|Q\|_2^{\frac{8}{N}}. \tag{3.10}$$

Proof. If we set $u^{\lambda,\mu} = \mu u(\lambda x)$, where $\mu = \frac{\|u\|_2^{\frac{N}{4}-1}}{\|\triangle u\|_2^{\frac{N}{4}}}$ and $\lambda = \frac{\|u\|_2^{\frac{1}{2}}}{\|\triangle u\|_2^{\frac{1}{2}}}$, we have

$$\|u^{\lambda,\mu}\|_2 = 1,\ \|\triangle u^{\lambda,\mu}\|_2 = 1 \text{ and } J(u^{\lambda,\mu}) = J(u). \tag{3.11}$$

Now, choosing a minimizing sequence $\{u_n\}_{n=1}^{\infty} \subset H^2$ such that $J(u_n) \to J$ as $n \to \infty$, after scaling, we may assume

$$\|u_n\|_2 = 1 \text{ and } \|\triangle u_n\|_2 = 1, \tag{3.12}$$

and we have

$$J(u_n) = \frac{1}{\int |u_n|^{2+\frac{8}{N}} dx} \to J, \text{ as } n \to \infty. \tag{3.13}$$

Note that $\{u_n\}_{n=1}^{\infty}$ is bounded in H^2. It follows form the profile decomposition (Proposition 2.3)that

$$u_n(x) = \sum_{j=1}^{l} U^j(x - x_n^j) + r_n^l(x)$$

and

$$\sum_{j=1}^{l} \|U_n^j\|_2^2 \le 1, \quad \sum_{j=1}^{l} \|\triangle U_n^j\|_2^2 \le 1, \tag{3.14}$$

where $U_n^j = U^j(x - x_n^j)$. Moreover, since $2 < 2 + \frac{8}{N} < \frac{2N}{(N-4)^+}$, for r_n^l, we have

$$\int |r_n^l|^{2+\frac{8}{N}} dx \to 0, \text{ as } l \to +\infty. \tag{3.15}$$

Using the orthogonal conditions and the following elementary inequality ($p > 1$)

$$\left| |\sum_{j=1}^{l} a_j|^{1+p} - \sum_{j=1}^{l} |a_j|^{1+p} \right| \le C \sum_{j \ne k} |a_j||a_k|^p, \tag{3.16}$$

we have

$$\int |\sum_{j=1}^{l} U^j(x - x_n^j)|^{2+\frac{8}{N}} dx \to \sum_{j=1}^{l} \int |U^j|^{2+\frac{8}{N}} dx, \text{ as } n \to \infty. \tag{3.17}$$

Therefore, by (3.13), (3.15) and (3.17), we have

$$\sum_{j=1}^{l} \int |U^j|^{2+\frac{8}{N}} dx \to \frac{1}{J}, \text{ as } n \to \infty. \tag{3.18}$$

For another thing, by the definition of J, we have

$$J \int |U^j|^{2+\frac{8}{N}} dx \le \|U^j\|_2^{\frac{8}{N}} \|\triangle U^j\|_2^2. \tag{3.19}$$

Since the series $\sum_j \|U^j\|_2^2$ is convergent, there exists a $j_0 \ge 1$ such that

$$\|U^{j_0}\|_2 = \sup\{\|U^j\|_2 \mid j \ge 1\}. \tag{3.20}$$

It follows from (3.18)-(3.20) that

$$\begin{aligned}1\le J\left(\sum_{j=1}^{l}\int|U^j|^{2+\frac{8}{N}}dx\right) &\le \sup\{\|U^j\|_2^{\frac{8}{N}}\ |\ \ j\ge 1\}\left(\sum_{j=1}^{l}\|\nabla U^j\|_2^2\right)\\ &\le \|U^{j_0}\|_2^{\frac{8}{N}}\left(\sum_{j=1}^{l}\|\nabla U^j\|_2^2\right)\le \|U^{j_0}\|_2^{\frac{8}{N}}.\end{aligned}\tag{3.21}$$

It follows from (3.14) that $\|U^{j_0}\|_2=1$, which implies that there exists only one term $U^{j_0}\neq 0$ such that

$$\|U^{j_0}\|_2=1,\ \|\triangle U^{j_0}\|_2=1,\ \ \text{and}\quad \int|U^{j_0}|^{2+\frac{8}{N}}dx=\frac{1}{J}.\tag{3.22}$$

Therefore, we show that U^{j_0} is the minimizer of $J(u)$. It follows from Lemma 3.1 that

$$\triangle^2U^{j_0}+\frac{4}{N}U^{j_0}-(1+\frac{4}{N})J|U^{j_0}|^{\frac{8}{N}}U^{j_0}=0.\tag{3.23}$$

We take $U^{j_0}=aQ(\lambda x+x_0)$ for the reason of symmetric invariance of Equation (3.1), where $a\in C^*$, $\lambda>0$, $x_0\in R^N$ and Q is the solution of (3.1).

On the other hand, if Q is the solution of Equation (3.1), we claim

$$\int|\triangle Q|^2dx+\int|Q|^2dx-\int Q^{2+\frac{8}{N}}dx=0\tag{3.24}$$

and

$$(2-\frac{N}{2})\int|\triangle Q|^2dx-\frac{N}{2}\int|Q|^2dx+\frac{N}{2+\frac{8}{N}}\int Q^{2+\frac{8}{N}}dx=0.\tag{3.25}$$

Indeed, Multiplying (3.1) by Q and integrating by parts, we have that (3.24) is true.

Multiplying (3.1) by $x\cdot\nabla Q$ and integrating by parts, we have

$$\int\triangle^2Qx\cdot\nabla Qdx+\int Qx\cdot\nabla Qdx-\int|Q|^{\frac{8}{N}}Qx\cdot\nabla Qdx=0.$$

For another thing, we have

$$\int\triangle^2Qx\cdot\nabla Qdx=2\int|\triangle Q|^2dx+\int x\cdot\nabla(\frac{|\triangle Q|^2}{2})dx=(2-\frac{N}{2})\int|\triangle Q|^2dx,$$

$$\int Qx\cdot\nabla Qdx=-\frac{N}{2}\int Q^2dx$$

and

$$\int|Q|^{\frac{8}{N}}Qx\cdot\nabla Qdx=-\frac{N}{2+\frac{8}{N}}\int Q^{2+\frac{8}{N}}dx.$$

Collecting the above identities, we have that (3.25) is true.

Now, we return to the proof of Proposition 3.2. By some computations, we have that $\|U^{j_0}\|_2^2=\frac{|a|^2}{\lambda^N}\|Q\|_2^2=1$, $\|\triangle U^{j_0}\|_2^2=\frac{|a|^2}{\lambda^{N-4}}\|\triangle Q\|_2^2=1$ and $\int|U^{j_0}|^{2+\frac{8}{N}}dx=\frac{|a|^{2+\frac{8}{N}}}{\lambda^N}\int|Q|^{2+\frac{8}{N}}dx=\frac{1}{J}$. Applying Claim (3.24) and (3.25), we have

$$\frac{1}{1+\frac{4}{N}}\int|Q|^{2+\frac{8}{N}}dx=\int|\triangle Q|^2dx,$$

which implies that

$$J = \frac{\lambda^N}{|a|^{2+\frac{8}{N}}} \frac{1}{\int |Q|^{2+\frac{8}{N}} dx} = \frac{1}{1+\frac{4}{N}} \|Q\|_2^{\frac{8}{N}}. \tag{3.26}$$

This completes the proof Proposition 3.2.

Remark 3.3. *In [**6**], Fibich etal also showed the following sharp Gargliardo-Nirenberg inequality*

$$\frac{1}{1+\frac{4}{N}} \int |u|^{2+\frac{8}{N}} dx \le \frac{1}{(\int |Q|^2 dx)^{\frac{4}{N}}} (\int |u|^2 dx)^{\frac{4}{N}} (\int |\triangle u|^2 dx),$$

where Q is the solution of ground state Equation (3.1), but the existence of the ground state Equation (3.1) is not addressed in their paper. In this paper, we prove the existence of the ground state Equation (3.1). Our results are more strong than Fibich etal's results and the methods are different.

At the end of this section, we prove Theorem 1.1 by applying the profile decomposition of the bounded sequence in H^2 and the sharp Gagliardo-Nirenberg inequality (Proposition 3.2).

Proof of Theorem 1.1. By extracting a subsequence, we may replace lim sup in the assumption in Theorem 1.1 by lim. According to the profile decomposition in Proposition 2.3, the sequence $\{v_n\}_{n=1}^{\infty}$ can be written up to a subsequence, as

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x) \tag{3.27}$$

with

$$\lim_{l\to\infty} \lim_{n\to\infty} \sup \|v_n^l\|_p = 0,$$

for $p \in (2, \frac{2N}{(N-4)^+})$, and we have the following estimations

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1), \tag{3.28}$$

$$\|\triangle v_n\|_2^2 = \sum_{j=1}^{l} \|\triangle V^j\|_2^2 + \|\triangle v_n^l\|_2^2 + o(1). \tag{3.29}$$

This implies that

$$\begin{aligned} m^{2+\frac{8}{N}} &\le \lim_{n\to\infty} \sup \|v_n\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\ &\le \lim_{n\to\infty} \sup \|\sum_{j=1}^{l} V^j(x-x_n^j) + v_n^l(x)\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\ &\le \lim_{n\to\infty} \sup \left(\|\sum_{j=1}^{l} V^j(x-x_n^j)\|_{2+\frac{8}{N}} + \|v_n^l(x)\|_{2+\frac{8}{N}} \right)^{2+\frac{8}{N}} \\ &\le \|\sum_{j=1}^{l} V^j(x-x_n^j)\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \qquad \text{as } l \to \infty. \end{aligned} \tag{3.30}$$

Using the elementary inequality

$$\left| |\sum_{j=1}^{l} a_j|^{2+\frac{8}{N}} - \sum_{j=1}^{l} |a_j|^{2+\frac{8}{N}} \right| \leq C \sum_{j\neq k} |a_j||a_k|^{1+\frac{8}{N}}, \tag{3.31}$$

and the pairwise orthogonality of the family $\{x_n^j\}_{j=1}^{\infty}$, we have that the mixed terms in (3.30) vanish. Hence, we have

$$m^{2+\frac{8}{N}} \leq \sum_{j=1}^{\infty} \|V^j\|_{2+\frac{8}{N}}^{2+\frac{8}{N}}. \tag{3.32}$$

On the other hand, using the Gagliardo-Nirenberg inequality, we have

$$\sum_{j=1}^{\infty} \|V^j\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \leq \frac{1+\frac{4}{N}}{\|Q\|_2^{\frac{8}{N}}} \sup\{\|V^j\|_2^{\frac{8}{N}},\ j \geq 1\} \left(\sum_{j=1}^{\infty} \|\triangle V^j\|_2^2 \right). \tag{3.33}$$

By(1.8) and (3.29), we have

$$\sum_{j=1}^{\infty} \|\triangle V^j\|_2^2 \leq \lim_{n\to\infty} \sup \|\triangle v_n\|_2^2 \leq M^2. \tag{3.34}$$

Therefore, we have

$$\sup_{j\geq 1} \|V^j\|_2^{\frac{8}{N}} \geq \frac{\|Q\|_2^{\frac{8}{N}} m^{2+\frac{8}{N}}}{(1+\frac{4}{N})M^2}. \tag{3.35}$$

Since the series $\sum\limits_{j=1}^{\infty} \|V^j\|_2^2$ is convergent, we have that the supremum of $\{\|V^j\|_2^{\frac{8}{N}}; j \geq 1\}$ is attained. In particular, there exists a $j_0 \geq 1$ such that

$$\|V^{j_0}\|_2^{\frac{8}{N}} \geq \frac{\|Q\|_2^{\frac{8}{N}} m^{2+\frac{8}{N}}}{(1+\frac{4}{N})M^2}. \tag{3.36}$$

By a change of variables, we have

$$v_n(x + x_n^{j_0}) = V^{j_0}(x) + \sum_{j\neq j_0} V^j(x - x_n^j + x_n^{j_0}) + \tilde{v}_n^l(x), \tag{3.37}$$

where $\tilde{v}_n^l(x) = v_n^l(x + x_n^{j_0})$. Applying the pairwise orthogonality of the family x_n^j to (3.37), we have

$$V^j(x - x_n^j + x_n^{j_0}) \rightharpoonup 0 \ \text{ weakly in } \ H^2 \tag{3.38}$$

for $j \neq j_0$. Hence, we have

$$v_n(x + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{v}^l. \tag{3.39}$$

where $\tilde{v}^l$ denote the weak limit of $\tilde{v}_n^l$. Using the Proposition 2.3, we have

$$\|\tilde{v}^l\|_{2+\frac{8}{N}} \leq \lim_{n\to\infty} \sup \|\tilde{v}_n^l\|_{2+\frac{8}{N}} = \lim_{n\to\infty} \sup \|v_n^l\|_{2+\frac{8}{N}} \to 0, \ \text{ as } l \to \infty, \tag{3.40}$$

which implies that

$$\tilde{v}^l = 0 \quad \text{for} \ \ l \geq j_0, \tag{3.41}$$

by the uniqueness of the weak limit.

Therefore, we have

$$v_n(x + x_n^{j_0}) \rightharpoonup V^{j_0} \ \text{ weakly in } \ H^2, \tag{3.42}$$

which implies the sequence $x_n^{j_0}$ and the function V^{j_0} now fulfill the condition of Theorem 1.1. This completes the proof.

4. L^2-Concentration

In this section, we shall use the compactness results in Theorem 3.4 to study the L^2-concentration properties of blow-up solutions to the Cauchy problem (1.1)-(1.2) in the general case. This result extends the results in [**1**] to the non-radially symmetric blow-up solutions. More precisely, we have the following theorem.

Theorem 4.1. *Let $u(t,x) \in C([0,T); H^2)$ be the corresponding blow-up solution of the Cauchy problem (1.1)-(1.2) such that*

$$\lim_{t\to T} \|\triangle u(t,x)\|_2 = +\infty. \tag{4.1}$$

Suppose that $a(t) > 0$ is any function such that $a(t)\|\triangle u(t)\|_2^{\frac{1}{2}} \to +\infty$ as $t \to T$. Then, there exists $y(t) \in R^N$ such that

$$\liminf_{t\to T} \int_{|x-y(t)|\le a(t)} |u(t,x)|^2 dx \ge \int |Q|^2 dx, \tag{4.2}$$

where Q is the solution of ground state Equation (3.1).

Proof. Since $u(t,x) \in C([0,T); H^2)$ is the corresponding blow-up solution of the Cauchy problem (1.1)-(1.2) such that $\lim_{t\to T} \|\triangle u(t,x)\|_2 = +\infty$. For any $t_k \to T$ as $k \to +\infty$, we take

$$\frac{1}{\lambda_k^2} = \|\triangle u(x,t_k)\|_2 \to +\infty, \quad \text{as } \ \text{k} \to +\infty. \tag{4.3}$$

Considering $U_k = \lambda_k^{\frac{N}{2}} u(\lambda_k x)$, by direct computations, we have

$$\begin{cases} \|U_k\|_2 & = \|u(t_k)\|_2 = \|u_0\|_2, \\ \|\triangle U_k\|_2 & = \lambda_k^2 \|\triangle u(t_k)\|_2 = 1. \end{cases} \tag{4.4}$$

Therefore, U_k is a uniformly bounded sequence in H^2 by (4.4). Note that

$$\begin{aligned} E(U_k) & = \tfrac{1}{2}\int |\triangle U_k|^2 dx - \tfrac{1}{2+\frac{8}{N}} \int |U_k|^{2+\frac{8}{N}} dx \\ & = \lambda_k^4 E(u_0) \\ & \to 0, \ \text{as } k \to +\infty, \end{aligned} \tag{4.5}$$

by the conservation of energy. Combining (4.4) with (4.5), we have

$$\lim_{k\to\infty} \int |U_k|^{2+\frac{8}{N}} dx \ge 1 + \frac{4}{N}. \tag{4.6}$$

Applying Theorem 1.1 to the sequence U_k (with $M = 1$, $m^{2+\frac{8}{N}} = 1 + \frac{4}{N}$), we have that there exists $\{y_k\} \subset R^N$

$$U_k(x+y_k) \rightharpoonup U(x) \ \text{ weakly } \text{ in } H^2, \ \text{with } \|U\|_2 \ge \|Q\|_2, \tag{4.7}$$

where Q is the ground state solution of Equation (3.1). That is,

$$\lambda_k^{\frac{N}{2}} u(t_k, \lambda_k(x+y_k)) \rightharpoonup U(x) \ \text{ weakly } \text{ in } H^2, \tag{4.8}$$

which implies that for every $A > 0$

$$\lim_{k\to\infty}\inf\int_{|x|\le A}\lambda_k^N|u(t_k,\lambda_k(x+y_k))|^2dx \ge \int_{|x|\le A}|U|^2dx. \tag{4.9}$$

Since the assumption $\frac{a(t_k)}{\lambda(t_k)} \to +\infty$ as $k \to +\infty$, we have

$$\lim_{k\to\infty}\inf\sup_{y\in R^N}\int_{|x-y|\le a(t_k)}|u(t_k,x)|^2dx \ge \int_{|x|\le A}|U|^2dx.$$

For every $A > 0$, we have

$$\lim_{k\to\infty}\inf\sup_{y\in R^N}\int_{|x-y|\le a(t_k)}|u(t_k,x)|^2dx \ge \int |U|^2dx. \tag{4.10}$$

Therefore, since the sequence $\{t_k\}$ is arbitrary and (4.7), we have

$$\lim_{t\to T}\inf\sup_{y\in R^N}\int_{|x-y|\le a(t)}|u(t,x)|^2dx \ge \int |Q|^2dx. \tag{4.11}$$

On the other hand, for every $t \in [0,T)$, the function $y \mapsto \int_{|x-y|\le a(t)}|u(t,x)|^2dx$ is continuous and goes to 0 at infinity, and we have

$$\sup_{y\in R^N}\int_{|x-y|\le a(t)}|u(t,x)|^2dx = \int_{|x-y(t)|\le a(t)}|u(t,x)|^2dx, \tag{4.12}$$

for some $y(t) \in R^N$. This completes the proof.

Applying the lower blow-up rate of the solutions to the Cauchy problem (1.1)-(1.2) obtained by Baruch, Fibich and Mandelbaum [**1**](see Lemma 2.2), we have the following rate of L^2-concentration of the blow-up solutions to the Cauchy problem (1.1)-(1.2) (see Theorem 1.2). At the end of this section, we give the proof of Theorem 1.2

Proof of Theorem 1.2. The result follows immediately from Lemma 2.2 and Theorem 4.1.

5. Limiting Profile of Minimal Mass Blow-up Solutions

In this section, we assume that the ground state solution of Equation (3.1) is unique up to translations in space, phase and dilations, which is also denoted by $Q(x)$, where it is assumed as the same as in [**33**] for the classical Schrödinger equation (1.4). Using the compactness lemma obtained in Section 3 and the characterization of the corresponding ground state, we obtain the limiting profile of the blow-up solutions in H^2 for the Cauchy problem (1.1)-(1.2). More precisely, we obtain that if the initial data $u_0 \in H^2$ satisfies $\|u_0\|_2 = \|Q(x)\|_2$, then the corresponding blow-up solution of the Cauchy problem (1.1)-(1.2) $u(t,x)$ remains close to $Q(x)$ in H^2 up to scaling and phase parameters, and also translation in the nonradial case. At first, we consider the variational characterization of the ground state of Equation(3.1), as follows.

Lemma 5.1. *If $u \in H^2$ is such that $\|u\|_2 = \|Q\|_2$ and $E(u) = 0$, then $u(x)$ is of the following form*

$$u(x) = e^{i\gamma}\lambda^{\frac{N}{2}}Q(\lambda x + x_0), \quad \text{for some} \quad \gamma \in R, \lambda > 0, x_0 \in R^N, \tag{5.1}$$

where Q is the unique solution of ground state Equation (3.1).

Proof. Since $E(u)=0$, we have

$$\int |\triangle u|^2 dx = \frac{1}{1+\frac{4}{N}} \int |u|^{2+\frac{8}{N}} dx. \tag{5.2}$$

Hence, we have

$$J(u) = \frac{(\int |u|^2 dx)^{\frac{4}{N}} (\int |\triangle u|^2 dx)}{\int |u|^{2+\frac{8}{N}} dx} = \frac{1}{1+\frac{4}{N}} \|u\|_2^{\frac{8}{N}} = \frac{1}{1+\frac{4}{N}} \|Q\|_2^{\frac{8}{N}} = J, \tag{5.3}$$

which implies that u is a minimizer of $J(u)$. By Proposition 3.2 and the uniqueness of Q, we have that u is of the form $u(x) = aQ(\lambda x + x_0)$. On the other hand, by $\|u\|_2 = \|Q\|_2$, we have $|a| = \lambda^{\frac{N}{2}}$. Therefore, since the value of $u(x)$ is in $\mathcal{C}$, there exists $\gamma \in R$ such that

$$u(x) = e^{i\gamma} \lambda^{\frac{N}{2}} Q(\lambda x + x_0),$$

where $\lambda > 0$, $x_0 \in R^N$ and Q is the unique ground state solution of Equation (3.1). This completes the proof.

Now, we are in proposition to prove Theorem 1.3 by applying the variational characteristic of the ground state of Equation (3.1).

Proof of Theorem 1.3. We show that for any sequence $t_k \to T$, there is a subsequence t_{k_j}, y_{k_j} and $\gamma(t_{k_j})$ such that

$$\lambda^{\frac{N}{2}}(t_{k_j}) u(t_{k_j}, \lambda(t_{k_j})(x + y(t_{k_j}))) e^{i\gamma(t_{k_j})} \to Q(x) \quad \text{strongly in } H^2 \text{ as } j \to \infty, \tag{5.4}$$

where $Q(x)$ is the unique ground state solution of Equation (3.1). If not, then (5.4) does not holds along some sequence t_{k_r}. But then we can find a subsequence of t_{k_r} along with (5.4) holds, this is a contradiction. Since t_k is an arbitrary sequence approaching T, and (1.12) follows.

Since $u(t,x) \in C([0,T); H^2)$ is the blow-up solution of the Cauchy problem (1.1)-(1.2), there is a $0 < T < +\infty$ such that $\lim\limits_{t \to T} \|\triangle u\|_2 = +\infty$. For any $t_k \to T$ as $k \to +\infty$, we take

$$\frac{1}{\lambda_k^2} = \|\triangle u(x, t_k)\|_2 \to +\infty, \quad \text{as } \ \mathrm{k} \to +\infty. \tag{5.5}$$

Consider $U_k = \lambda_k^{\frac{N}{2}} u(\lambda_k x)$, by direct computations, we have

$$\begin{cases} \|U_k\|_2 & = \|u(t_k)\|_2 = \|u_0\|_2 = \|Q(x)\|_2, \\ \|\triangle U_k\|_2 & = \lambda_k^2 \|\triangle u(t_k)\|_2 = 1. \end{cases} \tag{5.6}$$

Therefore, U_k is a uniformly bounded sequence in H^2 and U_k has a weakly convergent subsequence U_k (still denoted by U_k). Note that

$$\begin{aligned} E(U_k) &= \tfrac{1}{2} \|\triangle U_k\|_2^2 - \tfrac{1}{2+\frac{8}{N}} \|U_k\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\ &= \lambda_k^4 E(u_0) \to 0, \text{ as } k \to +\infty. \end{aligned} \tag{5.7}$$

Combining (5.6) with (5.7), one has

$$\lim_{k\to\infty} \|U_k\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} = 1 + \frac{4}{N}.$$

Applying Theorem 1.1 to the sequence U_k ($M=1$, $m^{2+8/N}=1+\frac{4}{N}$), one has that there exist $\{y_k\}\subset R^N$ and $U\in H^2$ such that

$$U_k(x+y_k)\rightharpoonup U(x) \quad \text{weakly} \quad \text{in } H^2$$

with $\|U\|_2\geq\|Q\|_2$. Since $\|U\|_2\leq\|U_k(x+y_k)\|_2=\|Q\|_2$ and the Brézis-Lieb Lemma, one has

$$U_k(x+y_k)\rightarrow U(x) \quad \text{strongly} \quad \text{in } L^2. \tag{5.8}$$

By the Gagliardo-Nirenberg's inequality (see Proposition 3.2), there exists $\gamma_k\in R$ such that

$$\|U_k(x+y_k)e^{i\gamma_k}-U\|_{2+\frac{8}{N}}^{2+\frac{8}{N}}\leq C\|U_k(x+y_k)e^{i\gamma_k}-U\|_2^{\frac{8}{N}}\|\triangle(U_k(x+y_k)e^{i\gamma_k}-U)\|_2^2.$$

It follows from (5.6) and $\|\triangle U_k(x+y_k)\|_2\leq C$ that

$$U_k(x+y_k)e^{i\gamma_k}\rightarrow U \text{ strongly in } L^{2+\frac{8}{N}}. \tag{5.9}$$

Next, we shall show that $U_k(x+y_k)e^{i\gamma_k}$ converges to U strongly in H^2. We need only now show that $\|\triangle U\|_2=1$ by the Brézis-Lieb Lemma. Note that

$$\begin{aligned} 0 &= \lim_{k\to\infty}E(U_ke^{i\gamma_k}) \\ &= \frac{1}{2}-\frac{1}{2+\frac{8}{N}}\lim_{k\to\infty}\|U_k\|_{2+\frac{8}{N}}^{2+\frac{8}{N}} \\ &= \frac{1}{2}-\frac{1}{2+\frac{8}{N}}\|U\|_{2+\frac{8}{N}}^{2+\frac{8}{N}}. \end{aligned} \tag{5.10}$$

Therefore, we have $\|\triangle U\|_2=1$ and the fact $U\not\equiv 0$, which implies that $U_k(x+y_k)e^{i\gamma_k}$ converges strongly to U in H^2.

Therefore, applying the variational characteristic of the ground state of Equation(3.1), we have, $\exists y\in R^N$, $\gamma\in R$ such that $U(x)=Q(x+y)e^{i\gamma}$, which implies that

$$\lambda_k^{\frac{N}{2}}u(t_k,\lambda_k(x+y_k))e^{i\gamma_k}\rightarrow Q(x+y)e^{i\gamma} \quad \text{strongly} \quad \text{in} \quad H^2 \text{ as } k\rightarrow\infty. \tag{5.11}$$

By redefining the sequences y_k and γ_k, we have (5.4) is true. This completes the proof.

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