Dynamics of PDE, Vol.7, No.2, 161-173, 2010

Long Time Behavior for Radially Symmetric Solutions of the Kuramoto-Sivashinsky Equation

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Communicated by Y. Charles Li, received November 3, 2009.

Abstract. In this paper, we consider the radially symmetric solutions of the Kuramoto-Sivashinsky equation in a shell domain $r_0 \leq r \leq R_0$ in any dimension n. Using Lyapunov function approach, we study the long time behavior of the solutions and prove that there exists a time independent bound for the L^2 norm of the solution. Thus there exists an absorbing ball when time tends to infinity. We also show that in the three dimensional case this bound is given by $C(R_0 - r_0)^{3/2}$ and we give an estimate of the rate with which the constant C blows up when $r_0 \to 0$. Similar results hold for any n-dimensional shell domain which does not contain the origin, which means that radially symmetric solution does not blow up at the origin if the dimension is sufficiently high.

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1. Introduction

The Kuramoto-Sivashinsky equation,

$$\varphi_t = -\Delta^2\varphi - \Delta\varphi - \frac{1}{2}|\nabla\varphi|^2 \tag{1}$$

has been introduced three decades ago as a model of nonlinear evolution of linearly unstable interfaces in various contexts such as phase turbulence and flame front

2000 *Mathematics Subject Classification.* Primary 35B35, 35B40; Secondary 37L30, 35G30.
Key words and phrases. Kuramoto-Sivashinsky equation, long time behavior, radially symmetric solutions.
Stanislavova supported in part by NSF-DMS 0807894.

propagation in combustion theory. The question of global regularity for the two and three-dimensional Kuramoto-Sivashinsky equation is still considered one of the major open problems in nonlinear analysis.

In one space dimension, the equation has been studied extensively. It is interesting mathematically because the linearization about the zero state has a large number of exponentially growing modes, whose growth corresponds to the development of nontrivial structures. The Kuramoto-Sivashinsky equation has become a canonical model for spatio-temporal chaos in $1+1$ dimensions. In [**20**], the instability of the travelling waves is a hint of the complexity of the dynamics of the equation if the domain is $\mathbf{R}$. When considered on a bounded domain with appropriate initial and boundary conditions, there are many important results, some of which we will explain here briefly. In this case it is convenient to work with the differentiated form of the equation, where $u = d\phi/dx$ and the equation becomes

$$u_t = -u_{xxxx} - u_{xx} - uu_x.$$

Using Lyapunov function approach, the authors of [**14**] gave the first long-time behavior result showing that $\limsup_{t\to\infty} \|u\|_2 \le CL^{5/2}$ for odd initial data. In [**4**], the exponent was improved to $\frac{8}{5}$ for any mean-zero initial data. Most recently, the authors of [**2**] improved the exponent from $\frac{8}{5}$ to $\frac{3}{2}$ for any mean-zero initial data. While all of the above results used the Lyapunov function framework, there are recent results in [**9**], [**12**] that do not use this approach.

Our main goal is to treat the case of higher space dimensions, in particular the three dimensional case. This problem is difficult and even the global regularity on unbounded domain and in the periodic case is still open. Some of the available results have restrictions on the domain or work on a modified equation. In the two dimensional case, defining $U = (u_1, u_2) = \nabla\varphi$, the differentiated KS equation becomes,

$$\begin{aligned} \partial_t u_1 + \triangle^2 u_1 + \triangle u_1 + u_1\partial_x u_1 + u_2 \partial_x u_2 &= 0 \\ \partial_t u_2 + \triangle^2 u_2 + \triangle u_2 + u_1\partial_y u_1 + u_2 \partial_y u_2 &= 0 \\ \partial_y u_1 &= \partial_x u_2 \end{aligned} \tag{2}$$

The authors of [**17**] showed the existence of a bounded local absorbing set and an attractor in thin two-dimensional domain, but with restricted initial data. Later in [**13**] this result was made sharper and more transparent. Molinet showed that there exist positive constants $C_0, K \geqslant 1$ such that for any $L_x \geqslant 2\pi$, if $0 < L_y < 2\pi$ satisfies

$$\left(1 - \left(\frac{L_y}{2\pi}\right)^2\right)^{-4/9} L_y \leqslant (K^2 C_0^3)^{-4/7} L_x^{-67/35} \tag{3}$$

then the solution satisfies

$$\limsup_{t\to\infty} \|u_1\|_2 \le K L_x^{8/5} L_y^{1/2}, \qquad \lim_{t\to\infty} \|u_2\|_2 = 0 \tag{4}$$

provided

$$\|u_{1_0}\|_2 \leqslant C_0^{-1} \left(1 - \left(\frac{L_y}{2\pi}\right)^2\right) L_x^{-1/4} L_y^{-7/4}, \qquad \|u_{1_0}\|_2 \leqslant C_0^{-1} L_x^{-1/4} L_y^{1/4} \tag{5}$$

Using the results in [**13**] and [**2**] and assuming $L_y \leq CL_x^{13/7}$, one gets a better bound

$$\limsup_{t\to\infty} \|\vec{u}\|_2 \leq CL_x^{3/2}L_y^{1/2}$$

If one is willing to modify the equation, as in [**15**], where the equation

$$u_t = -\Delta^2 u - u_{xx} - uu_x$$

with periodic boundary conditions is studied, then the existence of an attractor can be proved. Similarly, in [**5**], the following Kuramoto-Sivashinsky type equation was studied in dimension two,

$$u_t = -\triangle^2 u - \triangle u - uu_x - uu_y + g(x,y) \tag{6}$$

where $g(x,y)$ is an external force. In this paper, the existence of a global attractor in $L^2([-L,L]\times[-L,L])$ was proved as well as the bound

$$\limsup_{t\to\infty} \|u\|_2 \leqslant CL^2 \tag{7}$$

The authors of [**1**] worked on the radially symmetric solutions of

$$\varphi_t + \Delta^2\varphi = |\nabla\varphi|^2 \tag{8}$$

in an annulus $\Omega = \{x \in \mathbb{R}^2 \text{ such that } 0 < r_0 < \|x\| < R_0\}$ with Neumann boundary conditions:

$$\frac{\partial\varphi}{\partial r} = \frac{\partial\triangle\varphi}{\partial r} = 0 \quad \text{on } \Gamma_\infty \tag{9}$$

Assuming that the initial condition ϕ_0 is radially symmetric, they proved the existence of radially symmetric solution $\varphi(r,t)$ such that

$$\varphi \in L^\infty_{loc}([0,\infty); W^{1,2}(\Omega)) \cap L^2_{loc}([0,\infty); W^{3,2}(\Omega)) \tag{10}$$

Furthermore φ satisfies an exponentially growing with time bound on the norm of the solution as follows

$$\int_{r_0}^{R_0} \varphi^2(r,t)dr \leqslant e^t \frac{R_0}{r_0}\int_{r_0}^{R_0} \varphi^2(r,0)dr + (te^t+1)\frac{16c^2R_0^2}{r_0^2}e^{4ct}\left(\int_{r_0}^{R_0} \varphi_r^2(r,0)dr\right)^3$$

This global existence result is remarked there to be also true in space dimension 3 in a shell domain between two concentric spheres. Inspired by the paper [**1**], our goal is to show that $\limsup_{t\to\infty} \|u\|_2 \leq C_{r_0}(R_0 - r_0)^{3/2}$ where u is the radially symmetric solution of the Kuramoto-Sivashinsky equation in a shell domain $\Omega = \{x \in \mathbb{R}^n \text{ such that } 0 < r_0 < \|x\| < R_0\}$.

We work with the differentiated Kuramoto-Sivashinsky equation in Ω with boundary conditions similar to [**1**]

$$u = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2}{r}u\right) = 0 \text{ at } r = r_0, \text{ and } r = R_0. \tag{11}$$

We assume that the initial condition u_0 is a radial function $u_0(x) = u_0(r)$, differentiate (1) and introduce a new variable $u = \dfrac{d\varphi}{dr}$. Thus we get the reduced radial system, which will be the subject of this paper

$$u_t + u_{rrrr} + \frac{4}{r}u_{rrr} + \left(1 - \frac{4}{r^2}\right)u_{rr} + \frac{2}{r}u_r - \frac{2}{r^2}u + uu_r = 0 \tag{12}$$

$$u = u_{rr} + \frac{2}{r}u_r = 0 \qquad \text{for} \quad r = r_0,\, r = R_0 \tag{13}$$

$$u(x,0) = u_0(x) = u_0(|x|) \qquad \text{in } \Omega \tag{14}$$

We will use the following notations. If Ω is a smooth, bounded domain in $\mathbb{R}^n$, then $\mathcal{Q}_t = \Omega \times (0,t)$, $\Gamma_t = \partial\Omega \times (0,t)$, $\Omega_t = \Omega \times \{t\}$, $|\nabla\varphi| = (\nabla\varphi, \nabla\varphi)^{1/2}$ and $(\cdot,\cdot)$ is the usual Euclidian dot product in $\mathbb{R}^n$. Changing the coordinates from rectangular to polar and assuming that φ is radially symmetric, we get the usual formulas

$$|\nabla\varphi|^2 = \left(\frac{\partial\varphi}{\partial r}\right)^2, \qquad \Delta\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)\varphi, \tag{15}$$

$$\Delta^2\varphi = \left(\frac{\partial^4}{\partial r^4} + \frac{2(n-1)}{r}\frac{\partial^3}{\partial r^3} + \frac{(n-1)(n-3)}{r^2}\frac{\partial^2}{\partial r^2} - \frac{(n-1)(n-3)}{r^3}\frac{\partial}{\partial r}\right)\varphi \tag{16}$$

Note that throughout this paper, we use $\|\cdot\|_2$ to denote $\|\cdot\|_{L^2[r_0,R_0]}$.We also use $\bar{H}^2[r_0,R_0]$ to denote the Sobolev space obtained by taking the completion with respect to the norm $\|\cdot\|_{\bar{H}^2}$ of smooth functions satisfying the boundary condition $\phi(r_0) = 0$. As introduced in [**14**] we take a dot above any space to denote the subspaces of functions of zero mean.

Using Lyapunov function approach, we prove the following theorem for the radial system (12)-(14).

Theorem 1.1. *Consider the Kuramoto-Sivashinsky equation (12) with* $0 < r_0 < R_0 < \infty$*, subject to the boundary and initial conditions given by (13), (14). Assume also* $(R_0 - r_0) \geq \alpha(1 + \frac{1}{r_0^2})^{-1/2}$ *for some* $\alpha > 0$*. Then, there is constant* $C = C_\alpha$*, so that*

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leqslant C_\alpha (R_0 - r_0)^{3/2}\left(1 + \frac{1}{r_0^2}\right)^3. \tag{17}$$

For the related problem (8) *with* $(R_0 - r_0) \geq \alpha(1 + \frac{1}{r_0^2})^{-1/2}$ *subject to the radial initial conditions and the boundary conditions* [**1**]*, we also have*

$$\limsup_{t\to\infty} \|\partial_r\varphi(t)\|_{L^2[r_0,R_0]} \leqslant C_\alpha (R_0 - r_0)^{3/2}\left(1 + \frac{1}{r_0^2}\right)^3. \tag{18}$$

If $(R_0 - r_0) \leq (1 + \frac{1}{r_0^2})^{-1/2}$*, then*

$$\limsup_{t\to\infty} \|u(t)\|_{L^2[r_0,R_0]} \leq C\frac{(1+\frac{1}{r_0^2})^2}{\sqrt{R_0 - r_0}}.$$

and similar estimate holds for the derivative of the solution φ_r *of* (8)*.*

2. Proof of Theorem 1.1

The Lyapunov function approach is based on the following main lemma, which was used traditionally in similar situations, see for example the recent paper [**2**]. Here $\phi(r)$ is a potential function, which will be constructed later on.

LEMMA 2.1. *Given $u = u(t;r) \in L^2([r_0, R_0])$ and $\phi \in L^2([r_0, R_0])$ satisfying the following inequality:*

$$\frac{d}{dt}\|u-\phi\|_2^2 \leqslant -\lambda_0\|u\|_2^2 + P^2 \tag{19}$$

for some constants $\lambda_0 > 0$ and P, then $B(0, R^{**})$, the ball of radius R^{**} centered about the origin, is an attracting region, where the radius R^{**} is given by

$$R^{**} = \sqrt{2\|\phi\|_2^2 + \frac{2P^2}{\lambda_0}} + \|\phi\|_2 \tag{20}$$

It is clear that $B(\phi, R^*)$, the ball of radius R^* centered about ϕ, is exponentially attracting, with ${R^*}^2 = 2\|\phi\|_2^2 + (\frac{2P^2}{\lambda_0})$. The triangle inequality implies $B(\phi, R^*) \subset B(0, R^{**})$.This guarantees the existence of an absorbing set.

2.1. An energy estimate. Next lemma will be our main energy estimate, which we will use in conjunction with Lemma 2.1.

LEMMA 2.2. *For any $\phi \in \dot{\bar{H}}^2[r_0, R_0]$ and $u(t;r)$ solving (12) we have the inequality*

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_{r_0}^{R_0}(u-\phi)^2 dr &\leqslant \textstyle\int_{r_0}^{R_0} -u_{rr}^2 + \left(4+\frac{16}{r_0^2}\right)u_r^2 + (1-\phi_r)u^2 dr \\ &\quad + \textstyle\int_{r_0}^{R_0} 4\phi_{rr}^2 + \left(\frac{1}{2}+\frac{18}{r_0^2}\right)\phi_r^2 dr \end{aligned} \tag{21}$$

Note that (2.1) and (2.2) show that if one can construct $\phi \in \bar{H}^2[r_0, R_0]$ such that the coercivity estimate

$$\langle u, Ku\rangle = \int_{r_0}^{R_0}\left(u_{rr}^2 - B_{r_0}u_r^2 + (\phi_r - 1)u^2\right) \geq \lambda_0\|u\|_2^2 > 0 \tag{22}$$

holds for some λ_0 independent of r_0 and R_0, where $B_{r_0} = 4 + \dfrac{16}{r_0^2}$, then one gets an estimate of the form

$$\limsup_{t\to\infty}\|u\|_2 \leqslant R^{**} = \sqrt{c_1\|\phi\|_2^2 + c_2\|\phi_r\|_2^2 + c_3\|\phi_{rr}\|_2^2} + \|\phi\|_2 \leqslant C\|\phi\|_{\bar{H}^2} < \infty.$$

Next, we prove the lemma.

PROOF. A straightforward calculation gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 &= \int_{r_0}^{R_0} u_t(u-\phi)dr \\ &= \int_{r_0}^{R_0}[-u_{rrrr} - \frac{4}{r}u_{rrr} - \left(1-\frac{4}{r^2}\right)u_{rr} - \frac{2}{r}u_r + \frac{2}{r^2}u - uu_r](u-\phi)dr \end{aligned}$$

After integration by parts, applying periodic boundary conditions and simplifying, one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0) \\ &+ \int_{r_0}^{R_0}\left(-u_{rr}^2 + \frac{4}{r}u_{rr}u_r + u_r^2 + \frac{2}{r}u_r u\right)dr - u_{rr}(R_0)\phi_r(R_0) + u_{rr}(r_0)\phi_r(r_0) \\ &+ \int_{r_0}^{R_0}\left(u_{rr}\phi_{rr} - \frac{4}{r}u_{rr}\phi_r - u_r\phi_r - \frac{2}{r}u\phi_r - \frac{1}{2}u^2\phi_r\right)dr \end{aligned}$$

Using the boundary conditions, one can find estimate for $u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0)$ as follows:

$$u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0) = -\frac{2}{R_0}u_r^2(R_0) + \frac{2}{r_0}u_r^2(r_0) = -2\int_{r_0}^{R_0}\left(\frac{u_r^2}{r}\right)' dr$$
$$= 2\int_{r_0}^{R_0}\frac{u_r^2}{r^2}dr - 4\int_{r_0}^{R_0}\frac{u_r u_{rr}}{r}dr$$

Similarly

$$-u_{rr}(R_0)\phi_r(R_0) + u_{rr}(r_0)\phi_r(r_0) = -2\int_{r_0}^{R_0}\frac{u_r\phi_r}{r^2}dr + 2\int_{r_0}^{R_0}\frac{u_{rr}\phi_r + u_r\phi_{rr}}{r}dr$$

Next combine these terms and rewrite again to get

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int_{r_0}^{R_0}\left(-u_{rr}^2 + u_r^2 + \frac{2}{r}uu_r + \frac{2}{r^2}u_r^2 + u_{rr}\phi_{rr} - \frac{2}{r}u_{rr}\phi_r\right)dr$$
$$+ \int_{r_0}^{R_0}\left(\frac{2}{r}u_r\phi_{rr} - u_r\phi_r - \frac{2}{r}u\phi_r - \frac{2}{r^2}u_r\phi_r - \frac{1}{2}u^2\phi_r\right)dr$$

Applying the Cauchy-Schwartz inequality $\langle f,g\rangle \le p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 \leqslant \int_{r_0}^{R_0}\left(-1+\frac{m}{2}+p\right)u_{rr}^2 + \left(1+\frac{q}{r_0^2}+\frac{2}{r_0^2}+\frac{d}{2}+\frac{k}{r_0^2}+\frac{c}{r_0^2}\right)u_r^2 dr$$
$$+ \int_{r_0}^{R_0}\left(\frac{1}{q}+s-\frac{1}{2}\phi_r\right)u^2 + \left(\frac{1}{2m}+\frac{1}{c}\right)\phi_{rr}^2 + \left(\frac{1}{pr_0^2}+\frac{1}{2d}+\frac{1}{sr_0^2}+\frac{1}{kr_0^2}\right)\phi_r^2 dr$$

The choice $m = 1/2$, $p = 1/4$, $q = 4$, $s = 1/4$, $k = 1$, $d = 2$ and $c = 1$ gives (2.2). □

This shows that for the proof of Theorem 1.1, it remains to establish the coercivity estimate (22).

2.2. Constructing the function ϕ. In order to prove the coercivity estimate (22) for $L \ge 1$, we will use the following result, which is in essence what was proved in [**2**].

LEMMA 2.3. *(see Theorem 1, [**2**]) Let $z \in C^3[0,L], L \ge 1$ with $z(0) = 0$. Then there exists a function $\psi \in C^\infty[0,L]$, so that one has the estimate*

$$\int_0^L (z_{xx}^2 + \psi' z^2)dx \ge 10\int_0^L z^2 dx.$$

In addition, ψ is in the form $\psi'(x) = L^{4/3}\chi(L^{1/3}x) - \int_0^L \chi(y)dy$, where $\chi \in C^\infty$, supported on a set with diameter $O(1)$ and so that $\sup_x |\chi^{(\alpha)}(x)| \le C_\alpha, \alpha = 0,1,\ldots$.

Our next result will address the question for the coercivity estimates when $L < 1$. We shall need this result to finish the proof of the theorem in one of the two cases considered. Although it's proof reduces in a simple fashion to Lemma 2.3, we include it for completeness.

LEMMA 2.4. *Let $z \in C^3[0,\varepsilon], \varepsilon \le 1$ with $z(0) = 0$. Then there exists a function $\psi \in C_0^\infty[0,\varepsilon]$, so that*

$$\int_0^\varepsilon (z_{xx}^2 + \psi' z^2)dx \ge 10\int_0^\varepsilon z^2 dx.$$

In addition, ψ is in the form $\psi'(x) = \chi(x/\varepsilon) - \int_0^1 \chi(y)dy$, where $\chi \in C^\infty$, supported on $(0,1)$ and so that $\sup_{0\leq x<1} |\chi^{(\alpha)}(x)| \leq C_\alpha, \alpha = 0, 1, \ldots$.

Proof. Introduce v, so that $z(x) = v(x/\varepsilon)$. Clearly $v \in C^3[0,1] : v(0) = 0$ and we need to show

$$\int_0^1 (v_{yy}^2(y) + \varepsilon^4 \psi'(\varepsilon y) v^2(y))dy \geq 10\varepsilon^4 \int_0^1 v^2(y)dy.$$

Clearly, that puts us in the situation of Lemma 2.3 with $L = 1$ and thus, it will suffice to take $\psi : \psi'(\varepsilon y) = \chi(y) - \int_0^1 \chi(x)dx$. Indeed,

$$\int_0^1 (v_{yy}^2(y) + \varepsilon^4 \psi'(\varepsilon y) v^2(y))dy \geq \varepsilon^4 \int_0^1 (v_{yy}^2(y) + \psi'(\varepsilon y) v^2(y))dy \geq 10\varepsilon^4 \int_0^1 v^2(y)dy,$$

where we have used the construction of Lemma 2.3 in the last inequality. Thus,

$$\psi'(x) = \chi(x/\varepsilon) - \int_0^1 \chi(y)dy.$$

and the proof of Lemma 2.4 is complete. □

2.3. Completion of the proof of Theorem 1.1. We will do a rescaling argument, which will show how to obtain (22) from Lemma 2.3 or Lemma 2.4. To prove the theorem, we need to construct ϕ_r such that

$$\int_{r_0}^{R_0} (u_{rr}^2 - B_{r_0} u_r^2 + (\phi_r - 1 - \lambda_0)u^2)dr \geqslant 0 \tag{23}$$

After applying the Cauchy-Schwartz inequality to estimate

$$-B_{r_0} \int_{r_0}^{R_0} u_r^2 dr \geqslant -\frac{1}{2} \int_{r_0}^{R_0} u_{rr}^2 dr - \frac{B_{r_0}^2}{2} \int_{r_0}^{R_0} u^2 dr,$$

we see that it will be enough to show that for all $u \in C^3[r_0, R_0] : u(r_0) = 0$,

$$\int_{r_0}^{R_0} (u_{rr}^2 + \phi_r u^2)dr \geq K \int_{r_0}^{R_0} u^2 dr \tag{24}$$

where $K = 10 + B_{r_0}^2$. Let $L = R_0 - r_0$. Introduce $v \in C^3[0, L] : v(r) = u(r + r_0)$. Clearly $v(0) = 0$ and we need to show (for appropriate ϕ)

$$\int_0^L (v_{rr}^2 + \phi'(r + r_0)v^2)dr \geq K \int_0^L v^2 dr$$

Next, introduce $w : v(r) = w(K^{1/4} r)$. Again $w(0) = 0$ and we need

$$\int_0^{LK^{1/4}} (w_{rr}^2 + \frac{1}{K}\phi'(K^{-1/4} r + r_0)w^2)dr \geq \int_0^{LK^{1/4}} w^2 dr, \tag{25}$$

At this point, we will have to consider two separate cases, depending on the relative size of $LK^{1/4}$. These will be handled either by Lemma 2.3 or by Lemma 2.4. We will be mainly interested in the first case which holds always when r_0 is small and we are tracking the dependence of the constant on $\frac{1}{r_0}$ in this case.

2.4. Case I: $LK^{1/4} \geq 1$. By Lemma 2.3, the following choice of ϕ (recall $LK^{1/4} \geq 1$)

$$\frac{1}{K}\phi'(K^{-1/4}r + r_0) = \psi'(r) = (LK^{1/4})^{4/3}\chi((LK^{1/4})^{1/3}x) - c_0,$$

will guarantee (22). Note that $c_0 = \int_0^1 \chi(y)dy = O(1)$, according to Lemma 2.3. We get

$$\phi_r(r) = (LK)^{4/3}\chi(L^{1/3}K^{1/3}(r - r_0)) - c_0K : [r_0, R_0] \to \mathbf{R}^1.$$

Clearly now $\|\phi_{rr}\|_{L^2} \leq C(LK)^{3/2}$, $\|\phi_r\|_{L^2} \leq C(LK)^{7/6}$, while since $\|\phi\|_{L^\infty} \leq C(LK)$, we get $\|\phi\|_{L^2} \leq CL^{3/2}K$.

From Lemma 2.2 it follows that

$$\limsup_{t\to\infty} \|u\|_2 \leqslant R^{**} \leq \sqrt{2\|\phi\|_2^2 + c_1(1 + \frac{1}{r_0^2})\|\phi_r\|_2^2 + c_2\|\phi_{rr}\|_2^2} + \|\phi\|_2,$$

where the constants are independent of r_0. Thus, we get the estimate

$$\limsup_{t\to\infty} \|u\|_2 \leq C(R_0 - r_0)^{3/2}\left(1 + \frac{1}{r_0^2}\right)^3,$$

whenever $R_0 - r_0 = L \geq K^{-1/4}$.

2.5. Case II: $LK^{1/4} < 1$. Going back to the proof of of (25), we now have $LK^{1/4} < 1$ and hence, we use Lemma 2.4 with $\varepsilon = LK^{1/4} < 1$. Thus,

$$\frac{1}{K}\phi'(r_0 + K^{-1/4}r) = \psi'(r) = \chi(r/\varepsilon) - c_0.$$

or written otherwise

$$\phi_r(r) = K\chi(\frac{(r - r_0)}{L}) - Kc_0 : [r_0, R_0] \to \mathbf{R}^1.$$

Clearly, $\|\phi_{rr}\|_{L^2[r_0,R_0]} \leq CKL^{-1/2}$, $\|\phi_r\|_{L^2[r_0,R_0]} \leq CKL^{1/2}$ and $\|\phi\|_{L^\infty} < C(KL)$, which implies $\|\phi\|_{L^2} \leq CL^{3/2}K$. Hence

$$\limsup_{t\to\infty} \|u\|_2 \leq C\frac{(1 + \frac{1}{r_0^2})^2}{\sqrt{R_0 - r_0}}.$$

whenever $R_0 - r_0 = L \leq K^{-1/4}$.

3. n-dimensional case

In this section we will describe similar results for the general n dimensional case. The statement of the theorem remains the same as in the three-dimensional case, even though after the tedious computations some additional terms appear. In what follows we will show that the same lemmas can be applied and the coefficients remain similar and produce same result for the dependence of the limit on $\frac{1}{r_0}$. As before, we differentiate (1), define $u = \dfrac{d\varphi}{dr}$ and use the same boundary conditions as in (11). Thus we get the following reduced radial system, where n is the dimension.

$$u_t + u_{rrrr} + \tfrac{2(n-1)}{r}u_{rrr} + \left(\tfrac{n^2-6n+5}{r^2} + 1\right)u_{rr} + \left(\tfrac{n-1}{r} - \tfrac{3(n^2-4n+3)}{r^3}\right)u_r + \left(\tfrac{3(n^2-4n+3)}{r^4} - \tfrac{n-1}{r^2}\right)u + uu_r = 0 \tag{26}$$

$$u = u_{rr} + \left(\frac{n-1}{r}\right) u_r = 0 \qquad \text{for} \quad r = r_0,\, r = R_0 \tag{27}$$

$$u(x,0) = u_0(x) = u_0(|x|) \qquad \text{in } \Omega \tag{28}$$

3.1. An energy estimate. Similar to what we did in Lemma 2.2, we will find the energy estimate for the equation (26), which we will use in conjunction with the coercivity to prove that Theorem 1.1 holds in this case as well.

LEMMA 3.1. *For any $\phi \in \dot{\tilde{H}}^2[r_0, R_0]$ and $u(t;r)$ solving (26) we have the inequality*

$$\begin{aligned}\tfrac{d}{dt}\|u-\phi\|_2^2 \leqslant \textstyle\int_{r_0}^{R_0} -u_{rr}^2 + \left(\tfrac{12(n-1)}{r_0^2}+3\right)u_r^2 \\ + \left((n-1)\tfrac{(n-3)^2(4n-1)+9|n-3|+1}{r_0^4}\right)u^2 dr \\ + \textstyle\int_{r_0}^{R_0}\left(\tfrac{2(n-1)}{r_0^2} + (n-1-\phi_r)\right)u^2 + (n+3)\phi_{rr}^2 + \tfrac{(n-1)(4n-3)+1}{r_0^2}\phi_r^2 dr \\ + \textstyle\int_{r_0}^{R_0}\left(\tfrac{|n^2-4n+3|((n-3)(4n-1)+3)}{r_0^4} + 2(n-1)\right)\phi^2 dr\end{aligned}$$

PROOF. A straightforward calculation gives

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = \int_{r_0}^{R_0} u_t(u-\phi)dr$$

$$= \int_{r_0}^{R_0}[-u_{rrrr} - \frac{2(n-1)}{r}u_{rrr} - \left(\frac{n^2-6n+5}{r^2}+1\right)u_{rr}$$

$$-\left(\frac{n-1}{r} - \frac{3(n^2-4n+3)}{r^3}\right)u_r - \left(\frac{3(n^2-4n+3)}{r^4} - \frac{n-1}{r^2}\right)u - uu_r](u-\phi)dr$$

After integration by parts, applying periodic boundary conditions and simplifying, one gets

$$\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 = u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0) +$$

$$+\int_{r_0}^{R_0} -u_{rr}^2 + 2(n-1)\left(\frac{u_{rr}u_r}{r} - \frac{u_{rr}u}{r^2}\right) - \frac{(n^2-6n+5)}{r^2}u_{rr}u + u_r^2\, dr$$

$$+\int_{r_0}^{R_0}\left(\frac{3(n^2-4n+3)}{r^3} - \frac{n-1}{r}\right)u_r u + \left(\frac{n-1}{r^2} - \frac{3(n^2-4n+3)}{r^4}\right)u^2 dr$$

$$- u_{rr}(R_0)\phi_r(R_0) + u_{rr}(r_0)\phi_r(r_0)$$

$$+\int_{r_0}^{R_0} u_{rr}\phi_{rr} - 2(n-1)\left(\frac{u_{rr}\phi_r}{r} - \frac{u_{rr}\phi}{r^2}\right) + (n^2-6n+5)\frac{u_{rr}\phi}{r^2} - u_r\phi_r dr$$

$$+\int_{r_0}^{R_0}(n-1)\frac{u_r\phi}{r} - 3(n^2-4n+3)\frac{u_r\phi}{r^3}$$

$$+ 3(n^2-4n+3)\frac{u\phi}{r^4} - (n-1)\frac{u\phi}{r^2} - \frac{u^2\phi_r}{2}dr$$

Using the boundary conditions, one can find estimate for $u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0)$ as follows:

$$u_{rr}(R_0)u_r(R_0) - u_{rr}(r_0)u_r(r_0) = -\frac{(n-1)}{R_0}u_r^2(R_0) + \frac{(n-1)}{r_0}u_r^2(r_0) =$$

$$= -(n-1)\int_{r_0}^{R_0}\left(\frac{u_r^2}{r}\right)' dr = (n-1)\int_{r_0}^{R_0}\frac{u_r^2}{r^2}dr - 2(n-1)\int_{r_0}^{R_0}\frac{u_r u_{rr}}{r}dr$$

Similarly

$$\begin{aligned}&-u_{rr}(R_0)\phi_r(R_0) + u_{rr}(r_0)\phi_r(r_0)\\&= -(n-1)\int_{r_0}^{R_0}\frac{u_r\phi_r}{r^2}dr + (n-1)\int_{r_0}^{R_0}\frac{u_{rr}\phi_r + u_r\phi_{rr}}{r}dr.\end{aligned}$$

Next combine these terms and rewrite again to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 &= \int_{r_0}^{R_0}(n-1)\frac{u_r^2}{r^2} - u_{rr}^2 - \frac{(n^2-4n+3)}{r^2}u_{rr}u + u_r^2 dr\\&+\int_{r_0}^{R_0}\left(\frac{3(n^2-4n+3)}{r^3} - \frac{n-1}{r}\right)u_r u + \left(\frac{n-1}{r^2} - \frac{3(n^2-4n+3)}{r^4}\right)u^2 dr\\&+\int_{r_0}^{R_0} -(n-1)\frac{u_{rr}\phi_r}{r} + (n-1)\frac{\phi_{rr}u_r}{r} - (n-1)\frac{u_r\phi_r}{r^2} + u_{rr}\phi_{rr}dr\\&+\int_{r_0}^{R_0}(n^2-4n+3)\frac{u_{rr}\phi}{r^2} - u_r\phi_r + (n-1)\frac{u_r\phi}{r}dr\\&+\int_{r_0}^{R_0} -3(n^2-4n+3)\frac{u_r\phi}{r^3} + 3(n^2-4n+3)\frac{u\phi}{r^4} - (n-1)\frac{u\phi}{r^2} - \frac{u^2\phi_r}{2}dr\end{aligned}$$

Applying the Cauchy-Schwartz inequality $\langle f,g\rangle \le p/2\langle f,f\rangle + 1/2p\langle g,g\rangle$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u-\phi\|_2^2 \leqslant\\&\int_{r_0}^{R_0}\left(-1 + |n^2-4n+3|\frac{p}{2} + (n-1)\frac{m}{2} + \frac{f}{2} + |n^2-4n+3|\frac{h}{2}\right)u_{rr}^2 dr +\\&\int_{r_0}^{R_0}\left(\frac{(n-1)}{r_0^2} + 1 + \frac{3|n^2-4n+3|}{r_0^2}\frac{q}{2}\right.\\&\left.+\frac{(n-1)}{r_0^2}\frac{z}{2} + \frac{(n-1)}{r_0^2}\frac{c}{2} + \frac{(n-1)}{r_0^2}\frac{d}{2}\right)u_r^2 dr\\&+\int_{r_0}^{R_0}\left(\frac{y}{2} + \frac{(n-1)}{r_0^2}\frac{w}{2} + \frac{3|n^2-4n+3|}{r_0^2}\frac{j}{2}\right)u_r^2 dr\\&+\int_{r_0}^{R_0}\left(\frac{|n^2-4n+3|}{r_0^4}\frac{1}{2p} + \frac{3|n^2-4n+3|}{r_0^4}\frac{1}{2q} + \frac{3|n^2-4n+3|}{r_0^4}\frac{t}{2}\right)u^2 dr\\&+\int_{r_0}^{R_0}\left(\frac{(n-1)}{2z} + \frac{(n-1)}{r_0^2} + \frac{3|n^2-4n+3|}{r_0^4} + \frac{(n-1)}{r_0^4}\frac{\tilde{w}}{2} - \frac{\phi_r}{2}\right)u^2 dr\\&+\int_{r_0}^{R_0}\left(\frac{(n-1)}{2c} + \frac{1}{2f}\right)\phi_{rr}^2 + \left(\frac{(n-1)}{r_0^2}\frac{1}{2d} + \frac{1}{2y} + \frac{(n-1)}{r_0^2}\frac{1}{2m}\right)\phi_r^2 dr +\\&\int_{r_0}^{R_0}\left(\frac{|n^2-4n+3|}{r_0^4}\frac{1}{2h} + \frac{3|n^2-4n+3|}{r_0^4}\frac{1}{2j}\right.\\&\left.+\frac{3|n^2-4n+3|}{r_0^4}\frac{1}{2t} + \frac{n-1}{2w} + \frac{n-1}{2\tilde{w}}\right)\phi^2 dr\end{aligned}$$

Choosing $p = \frac{1}{4|n^2-4n+3|}$, $m = \frac{1}{4(n-1)}$, $f = \frac{1}{4}$, $h = \frac{1}{4|n^2-4n+3|}$, $q = \frac{1}{|n-3|}$, $c = 1$, $d = 1$, $j = \frac{1}{|n-3|}$, $z = 1$, $y = 1$, $w = 1$, $t = 1$, $z = 1$, $\tilde{w} = 1$, we get

$$\frac{d}{dt}\|u-\phi\|_2^2 \leqslant$$
$$\int_{r_0}^{R_0} -u_{rr}^2 + \left(\frac{12(n-1)}{r_0^2} + 3\right) u_r^2 + \left((n-1)\frac{(n-3)^2(4n-1) + 9|n-3| + 1}{r_0^4}\right) u^2 dr$$
$$+ \int_{r_0}^{R_0} \left(\frac{2(n-1)}{r_0^2} + (n-1-\phi_r)\right) u^2 + (n+3)\phi_{rr}^2 + \left(\frac{(n-1)(4n-3)}{r_0^2} + 1\right)\phi_r^2 dr$$
$$+ \int_{r_0}^{R_0} \left(\frac{|n^2-4n+3|((n-3)(4n-1)+3)}{r_0^4} + 2(n-1)\right)\phi^2 dr$$

□

We will prove the coercivity in the n-dimensional case using the lemmas from the previous section. For the modified coefficients $B_{r_0} = \frac{12(n-1)}{r_0^2} + 3$, and $C_{r_0} = (n-1)\frac{(n-3)^2(4n-1)+9|n-3|+1}{r_0^4} + \frac{2(n-1)}{r_0^2} + n - 1$ we have to show that there exists potential function ϕ_r, such that we have the following coercivity estimate.

$$\int_{r_0}^{R_0} (u_{rr}^2 - B_{r_0}u_r^2 + (\phi_r - C_{r_0})u^2)dr \geqslant \lambda_0 \int_{r_0}^{R_0} u^2 dr \tag{29}$$

Applying Cauchy-Schwartz inequality once again,

$$-B_{r_0}\int_{r_0}^{R_0} u_r^2 dr \geqslant -\frac{1}{2}\int_{r_0}^{R_0} u_{rr}^2 dr - \frac{B_{r_0}^2}{2}\int_{r_0}^{R_0} u^2 dr,$$

(29) will be equivalent to

$$\int_{r_0}^{R_0} (\frac{1}{2}u_{rr}^2 + (\phi_r - D_{r_0})u^2)dr \geqslant \lambda_0 \int_{r_0}^{R_0} u^2 dr \tag{30}$$

where $D_{r_0} = (n-1)\frac{(n-3)^2(4n-1)+72(n-1)+9|n-3|+1}{r_0^4} + \frac{38(n-1)}{r_0^2} + n + \frac{7}{2}$. Once again we have to prove that

$$\int_{r_0}^{R_0} (u_{rr}^2 + \phi_r u^2)dr \geq K\int_{r_0}^{R_0} u^2 dr \tag{31}$$

where $K = 10 + D_{r_0} \sim (1 + \frac{1}{r_0^2})^2$, which follows from Lemma 2.3 or Lemma 2.4. To finish the proof of the theorem, notice that now by Lemma 3.1 one has

$$\limsup_{t\to\infty}\|u\|_2 \leqslant R^{**} \leq \sqrt{c_1(1+\frac{1}{r_0^2})^2\|\phi\|_2^2 + c_2(1+\frac{1}{r_0^2})\|\phi_r\|_2^2 + c_3\|\phi_{rr}\|_2^2} + \|\phi\|_2,$$

where the constants are independent of r_0. Using the estimates for $\|\phi\|_2^2, \|\phi_r\|_2^2, \|\phi_{rr}\|_2^2$ in this inequality gives the same results as in the three-dimensional case and proves the theorem.

4. Summary, remarks and open questions

Kuramoto-Sivashinsky equation arises when studying the propagation of instabilities in combustion theory and hydrodynamics and is well studied in dimension one. A major characteristic of the periodic case in dimension one is the existence of globally invariant, exponentially attracting inertial manifold, which is finite-dimensional. Thus the long-term dynamics is well-known in this case. For the higher-dimensional Kuramoto-Sivashinsky equation the question of long-term dynamics is still open for any general solution, but some results are available when the equation is considered on a thin domain or restricted to a periodic solution on a shell domain that excludes zero. In this paper, we worked with the radially symmetric solutions of Kuramoto-Sivashinsky equation in a shell domain $\Omega = \{x \in \mathbb{R}^n \text{ such that } 0 < r_0 < \|x\| < R_0\}$ and established a time-independent bound for the L^2 norm of the radially symmetric solutions. In particular, we proved that $\limsup_{t\to\infty} \left(\int_{r_0}^{R_0} |u(t,r)|^2 dr \right)^{1/2} \leq C_{r_0}(R_0 - r_0)^{3/2}$ and we explicitly calculate the dependence of the constant C_{r_0} on $\frac{1}{r_0}$. This is important when r_0 tends to 0 since it might shed some light on the potential formation of singularity at the origin and is subject of future research. Thus we were not able to prove similar bounds for the whole disk/ball, but our results can be interpreted as showing that if the dimension is high enough there is no singularity at the origin. In particular if one considers the standard L^2-norm in polar coordinates on R^n as $\int_{r_0}^{R_0} |u(t,r)|^2 r^{n-1} dr$ instead of the norm that we have used one gets no singularity at zero in dimension seven and above immediately. This result does not seem optimal and we are currently working on the regularity and long time behavior for axisymmetric solutions of the same equation of the form $r^s u(r)$ for an appropriate power s in the standard norm. We have written the paper using the same norm and Neumann boundary conditions as in [**1**].

Although these boundary conditions are quite standard when dealing with radial and axisymmetric solutions, it might be of interest to consider similar problem with different boundary conditions. Our initial calculations show that one can get analogous results in many different situations and the question becomes which boundary conditions are most interesting for the applications.

Finally, it might be feasible to reconsider the general solutions of the Kuramoto-Sivashinsky equation in higher dimensions. We have proved the result using Lyapunov function methods that work fairly well in the case of one variable only. If one considers general solution in dimension two and higher the resulting equations contain mixed nonlinear terms that are very hard to treat using coercivity. It might be possible to drop the radial symmetry assumptions, but still use polar coordinates to respect the geometry of the circle to prove similar results by extending the methods used in this paper. This is going to require additional estimates beyond the scope of this work. These questions will be the subject of a future investigation.

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