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Stabilization of the viscoelastic Euler-Bernoulli type equation with a local nonlinear dissipation

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Abstract. In this paper, we consider the viscoelastic Euler-Bernoulli type equation

$$u_{tt} + \Delta^2 u - M(||\nabla u||^2)\Delta u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + \rho(x, u_t) = 0.$$

This work is devoted to prove the existence of global solutions and decay for the energy of solutions of the Euler-Bernoulli type equation with nonlinear localized dissipation term.

Contents

1. Introduction

In this paper, we are concerned with global existence and decay for the energy of solutions of viscoelastic Euler-Bernoulli type equation with a localized damping term:

(1.1)
$$\begin{cases} u_{tt} + \Delta^2 u - M(||\nabla u||^2)\Delta u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + \rho(x, u_t) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega, \end{cases}$$

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where $\Omega \subset \mathbb{R}^n$ is an bounded domain, $n \geq 1$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, where Γ_0 and Γ_1 are closed and disjoint and $M \in C^1(\mathbb{R}_+)$. $g(s)$ is a bounded C^2 function and $\rho(x,s)$ is almost everywhere differentiable and nondecreasing function in s. We shall denote by ν the unit outward normal vector to Γ. Δ and ∇ stand for the Laplacian and gradian with respect to the spatial variables respectively, $'$ denotes the derivative with respect to time t, and $\mathbb{R}_+ = [0, \infty)$.

The problem of proving existence of solutions has been studied from old times. There are many methods to solve existence of solutions, but recently many authors use the Galerkin's method. This paper is used Galerkin's method solving existence of solutions, too.

The problem of stabilization of partial differential equation has recently attracted a lot of attention and various results are available (see [**1**], [**2**], [**4**], [**8**], [**9**], [**10**], [**18**], [**19**], [**20**], [**21**]). When $\rho \equiv 0$, the problem has been treated many authors (cf. [**8**], [**9**], [**10**], [**20**] and a list of references therein). However, this paper put great emphasis on $\rho(x,u_t)$ term.

For the case of wave equation, Zuazua [**22**] had treat the linear case $\rho(x,v) = a(x)v$ with $a(x)$ vanishing somewhere on $\bar{\Omega}$. Zuazua proved that any energy finite solution $u(t)$ satisfies the exponential decay

$$E(t) \leq CE(0)e^{-\lambda t}$$

for some $\lambda > 0$. For the nonlinear case of $\rho(x,v)$ like $\rho(x,v) = a(x)|v|^r v$, Nakao has many treated (cf. [**14**], [**15**], [**16**]). In this case, the energy of solutions goes to zero, as $t \to \infty$, with a polynomial rate of decay.

For the case of Euler-Bernoulli type equation, Tucsnak [**21**] studied the linear case $\rho(x,v) = a(x)v$. By using appropriate Lyapounov functional, Tucsnak [**21**] found the result like Zuazua [**22**]. For the nonlinear case, Cavalcanti et al. [**3**] considered the following problem

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + a(t)u_t = 0 \quad \text{in} \quad \Omega \times (0,\infty),$$

where $a(t)$ is a nonlocal nonlinearity type function. In this case, $M \equiv 0$ in (1.1). Using the perturbed energy method by constructing a suitable Liapunov functional, [**3**] proved the exponential energy of the Euler-Bernoulli equation with a nonlocal dissipation in general domains. And Charão et al. [**4**] considered

$$\begin{cases} u_{tt} + \Delta^2 u - \alpha\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + \rho(x,u_t) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \eta} = 0, \quad \text{on} \quad \Gamma \times (0,\infty), \end{cases}$$

where α is a positive constant and $\rho(x,u_t)$ is a localized damping term. In this case, M is a constant and $g \equiv 0$ in (1.1). By using the Nakao's lemma, it was proved that polynomial decay rate of solution.

This paper leads to special difference inequalities for the energy of solutions and allows to apply the method developed in [**4**] and [**14**]. However, method of [**14**] produces some lower order terms that we manage with compactness. In order to obtain some identities, [**4**] and [**14**] were used the multiplier technique but the multiplier method is not suitable when dealing with the memory term $\int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau$. To overcome this point we use well-known inequalities and Sobolev imbedding theorem properly. The problem is then reduced to showing that the unique solution of (1.1) such that $u \equiv 0$ in $\omega \times \mathbb{R}_+$ is the trivial one, which requires the application of a unique continuation result in [**7**]. At this point, we observe that the unique continuation result in [**7**] applies only when ω is neighborhood of the whole boundary, which leads us to require such assumption in our present proofs. In other words, the decay of solutions of (1.1) is obtained localizing the damping function in a neighborhood of the whole boundary.

To prove the decay rats of the energy

$$E(t) = \frac{1}{2}\int_\Omega |u'|^2 dx + \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\hat{M}(||\nabla u||^2), \tag{1.2}$$

where

$$\hat{M}(t) = \int_0^t M(s)ds$$

we need to define a modified energy function. Indeed, a formal computation gives

$$E'(t) = -(\rho(x,u'),u') + \int_0^t g(t-\tau)(\Delta u(\tau), \Delta u'(t))d\tau,$$

which shows that we do not have any information about the sign of $E'(t)$. To solve this problem we use an argument from Dafermos [**5**] to define a new energy function $e(t)$ such that $e'(t) \le 0$ and $E(t) \le Ce(t)$ for some positive constant C. This will be discussed in section 4.

This paper is organized as follows : In section 2, we recall the notation and hypotheses and introduce our main results and lemma to prove our main results. In section 3, using the Galerkin's method we prove the existence and uniqueness of regular and weak solutions to problem (1.1). In section 4, we estimates some identities and inequalities and then using lemmas, we prove the energy decay.

2. Notations and main results

We begin this section introducing some notations and our main results. Throughout this paper we define $V = \{v \in H^2(\Omega); v = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma\}$ equipped with the norm $||v||_V = ||\Delta v||$, where $||\cdot||$ is a L^2-norm, $W = \{v \in V; \Delta^2 v \in L^2(\Omega)\}$ equipped with the norm$||w||_W = ||w||_V + ||\Delta^2 w||$ and $(u,v) = \int_\Omega u(x)v(x)dx$. From the Poincare's inequality, it follows that $||\cdot||_V$ and $||\cdot||_W$are equivalent to the standard norms of $H^2(\Omega)$ and $H^4(\Omega)$, respectively. Now we give the hypotheses for the main results.

($\mathbf{H_1}$) Hypotheses on $\boldsymbol{\Omega}$.

Let $\Omega \subset \mathbb{R}^n$ be an bounded domain, $n \ge 1$, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Here Γ_0 and Γ_1 are closed and disjoint, $\Gamma_0 \ne \emptyset$, satisfying the following condition:

$$\begin{aligned} &m\cdot\nu \ge \delta > 0 \text{ on } \Gamma_1, \quad m\cdot\nu \le 0 \text{ on } \Gamma_0, \\ &m(x) = x - x^0 (x^0 \in \mathbb{R}^n) \quad \text{and} \quad R = \max_{x\in\overline{\Omega}} |m(x)|, \end{aligned} \tag{2.1}$$

where ν represents the unit outward normal vector to Γ.

($\mathbf{H_2}$) Hypotheses on $\mathbf{M}$.

We consider M is a real-valued nondecreasing function satisfying the conditions

$$M \in C^1(\mathbb{R}_+) \quad \text{and} \quad M(s) \ge s_0 > 0 \quad \text{for all} \quad s \ge 0. \tag{2.2}$$

($\mathbf{H_3}$) Hypotheses on $\mathbf{g}$.

We assume the $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded C^2 function satisfying

$$1 - \int_0^\infty g(s)ds = \ell > 0 \tag{2.3}$$

and such that

$$-c_1 g(t) \le g'(t) \le -c_2 g(t), \tag{2.4}$$

$$0 \le g''(t) \le c_3 g(t), \tag{2.5}$$

where c_1, c_2 and c_3 are positive constants.

($\mathbf{H_4}$) Hypotheses on ρ.

Let $\rho(x,s)$ is almost everywhere differentiable and nondecreasing function in s and satisfies

$$\begin{aligned} &c_4 a(x)|s|^{r+1} \le |\rho(x,s)| \le c_5 a(x)(|s|^{r+1} + |s|), \quad \text{if} \quad |s| \le 1 \\ &\text{and} \\ &c_6 a(x)|s|^{p+1} \le |\rho(x,s)| \le c_7 a(x)(|s|^{p+1} + |s|), \quad \text{if} \quad |s| \ge 1, \end{aligned} \tag{2.6}$$

wherec_4, c_5, c_6 and c_7 are positive constants, $-1 < r < \infty$, $-1 < p \le \frac{2}{n-2}$ if $n \ge 3$ ($-1 < p < \infty$ if $n = 1, 2$). Also $a(x) \in L^\infty(\Omega)$ satisfies

$$a(x) \ge a_0 > 0 \quad \text{on} \quad \omega,$$

where ω is a neighborhood of Γ.

In addition, we assume that

$$\begin{aligned} &\rho(x,s)s \ge 0 \quad \text{and} \quad \frac{\partial \rho(x,s)}{\partial s} \ge 0, \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R}; \\ &\rho(\cdot,s) \quad \text{and} \quad \frac{\partial \rho(\cdot,s)}{\partial s} \in C(\bar{\Omega}). \end{aligned} \tag{2.7}$$

A typical example of $\rho(x,s)$ is

$$\rho(x,s) = \begin{cases} a(x)L^{-r-1}|s|^r s & \text{if} \quad |s| \le L \\ a(x)L^{-p-1}|s|^p s & \text{if} \quad |s| \ge L \end{cases}$$

with $L > 0$ (cf. [**14**]).

Now, we are in a position to state our main results.

Theorem 2.1. *Let the initial data $\{u_0(x), u_1(x)\}$ belong to $V \times L^2(\Omega)$ and assume that $(H_1) - (H_4)$ hold. Then problem* (1.1) *admits an unique weak solution u having the regularity*

$$u \in C(\mathbb{R}_+; V) \cap C^1(\mathbb{R}_+; L^2(\Omega)).$$

If we show Theorem 2.1, then we can assume the following hypothesis.

($\mathbf{H_5}$)

Let u is a solution of (1.1) and for any $\Phi \in W^{1,\infty}(\mathbb{R}_+)$, $\Psi \in L^\infty(\mathbb{R}_+)$, $\alpha \in \mathbb{R}_+$, the only function $v \in L^2(\Omega \times \mathbb{R}_+)$ satisfies the conditions

$$\begin{cases} v_{tt} + \Delta^2 v - \Phi(t)\Delta v - \alpha\Delta^2 u - \int_0^t \Psi(t-\tau)\Delta^2 u(\tau)d\tau = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+, \\ v = 0 \quad \text{in} \quad \omega \times \mathbb{R}_+, \end{cases}$$

then $u = v \equiv 0$ in $\Omega \times \mathbb{R}_+$.

Remark 2.1. *For $\Phi \in W^{1,\infty}(\mathbb{R}_+)$ and $\Psi \in L^\infty(\mathbb{R}_+)$, (H_5) holds true at least if $\Gamma_1 = \Gamma$ (which can be true for star-shaped domains), according to* [**7**], [**21**]. *Moreover, if Ω is an interval of the real line (H_5) holds for any open subset $\omega \subset \Omega$ (cf.* [**6**]*).*

In order to state another main result, we define the associated energy of problem (1.1) by

$$E(t) = \frac{1}{2}\int_{\Omega} |u'|^2 dx + \frac{1}{2}\int_{\Omega} |\Delta u|^2 dx + \frac{1}{2}\hat{M}(||\nabla u||^2),$$

where

$$\hat{M}(t) = \int_0^t M(s)ds.$$

Theorem 2.2. *Let* $(u_0(x), u_1(x)) \in V \times L^2(\Omega)$ *and* $R > 0$ *such that*

$$||(u_0(x), u_1(x))||_{V \times L^2(\Omega)} \leq R$$

. *Then the energy* $E(t)$ *associated with the solutions of* (1.1) *has the decay property*

$$E(t) \leq C(1+t)^{-\eta_1}, \qquad (i = 1, 2, 3, 4),$$

where $C = C(R, E(0))$ *is a positive constant and the decay rate* η *is given as follows corresponding to the cases;*

case 1
If $r \geq 0$ *and* $0 \leq p \leq \frac{2}{n-2}$, *then* $\eta_1 = \frac{2}{r}$.

case 2
If $r \geq 0$ *and* $-1 < p < 0$, *then*

$$\eta_2 = \min\left\{\frac{2}{r+2}, \frac{1}{p+2}\right\}.$$

case 3
If $-1 < r < 0$ *and* $0 \leq p \leq \frac{2}{n-2}$, *then*

$$\eta_3 = \min\left\{\frac{1}{r+1}, \frac{-r}{2(r+1)}\right\}.$$

case 4
If $-1 < r < 0$ *and* $-1 < p < 0$, *then*

$$\eta_4 = \min\left\{\frac{1}{r+1}, \frac{-r}{2(r+1)}, \frac{1}{p+1}\right\}.$$

In order to prove of above theorem, we need the following lemmas.

Lemma 2.1. *(Gagliardo-Nirenberg). Let* $1 \leq r < p < \infty$, $1 \leq q \leq p$ *and* $0 \leq m$. *Then,*

$$||v||_{W^{k,p}} \leq C||v||^{\theta}_{W^{m,q}}||v||^{1-\theta}_{L^r}$$

for $v \in W^{m,p}(\Omega) \cap L^r(\Omega)$, $\Omega \subset \mathbb{R}^N$, *where* C *is a positive constant and*

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \leq 1$.

Lemma 2.2. *(Nakao* [**13**]*) Let* $\phi(t)$ *be a nonnegative function on* $\mathbb{R}_+$ *satisfying*

$$\sup_{t \leq s \leq t+T} \phi(s)^{1+\gamma} \leq \psi(t)\Big\{\phi(t) - \phi(t+T)\Big\}$$

with $T > 0$, $\gamma > 0$ *and* $\psi(t)$ *a nondecreasing continuous function. Then* $\phi(t)$ *has the decay property*

$$\phi(t) \leq \left\{\phi(0)^{-\gamma} + \int_T^t \psi(s)^{-1} ds\right\}^{\frac{-1}{\gamma}} \qquad for \quad t \geq T.$$

If $\gamma = 0$ *in the above we have*

$$\phi(t) \le C\phi(0)e^{-\lambda t}$$

for some $\lambda > 0$.

3. Existence of solutions

In this section we prove the existence and uniqueness of regular and weak solutions to problem (1.1). Firstly we consider regular solutions and then, using density arguments we extend the same results for weak solutions.

Let us solve the variational problem associated with (1.1), which is given by: find $u(t) \in W$ such that

$$(u_{tt}(t), w) + (\Delta u(t), \Delta w) + M(||\nabla u||^2)(\nabla u(t), \nabla w) \\ - \int_0^t g(t-\tau)(\Delta u(\tau), \Delta w)d\tau + (\rho(u_t(t)), w) = 0$$

for all $w \in V$. Let $\{w^j\}$ be a complete orthogonal system of W. For each $m \in \mathbb{N}$, let V_m be the subspace generated by $\{w^1, w^2, \cdots, w^m\}$. We search for a function

$$u^m(t) = \sum_{j=1}^{m} \delta_m^j(t) w^j$$

satisfying the approximate equation

$$(u_{tt}^m(t), w) + (\Delta u^m(t), \Delta w) + M(||\nabla u^m||^2)(\nabla u^m(t), \nabla w) \\ - \int_0^t g(t-\tau)(\Delta u^m(\tau), \Delta w)d\tau + (\rho(u_t^m(t)), w) = 0 \tag{3.1}$$

with initial data

$$u^m(0) = u_0^m \to u_0 \quad \text{in} \quad W \quad \text{and} \quad u_t^m(0) = u_1^m \to u_1 \quad \text{in} \quad V. \tag{3.2}$$

By standard methods in differential equation, we prove the existence of solutions to the approximate equation (3.1) on some interval $[0, t_m)$. Then, this solution can be extended to the whole interval $[0, T]$, where $T = \infty$, by using the following first estimate.

3.1. The first estimate. Replacing w by $u_t^m(t)$ in equation (3.1) we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(||u_t^m(t)||^2 + ||\Delta u^m(t)||^2 + \hat{M}(||\nabla u^m(t)||^2)\Big) \\ &= \frac{d}{dt}\left(\int_0^t g(t-\tau)(\Delta u^m(\tau), \Delta u^m(t))d\tau\right) \\ &\quad - \int_0^t g'(t-\tau)(\Delta u^m(\tau), \Delta u^m(t))d\tau \\ &- g(0)||\Delta u^m(t)||^2 - (\rho(u_t^m(t)), u_t^m(t)). \end{aligned} \tag{3.3}$$

Considering the Cauchy-Schwartz inequality and taking hypotheses of g into account, we deduce

$$\begin{aligned}&\left|\int_0^t g'(t-\tau)(\Delta u^m(\tau), \Delta u^m(t))d\tau\right|\\ &\le ||\Delta u^m(t)|| \int_0^t |g'(t-\tau)|||\Delta u^m(\tau)||d\tau \qquad (3.4)\\ &\le \frac{c_1^2}{2}||\Delta u^m(t)||^2 + \frac{1}{2}||g||_{L^1(0,\infty)} \int_0^t g(t-\tau)||\Delta u^m(\tau)||^2 d\tau.\end{aligned}$$

From (2.2), (2.7), (3.3) and (3.4), we deduce by integration over $(0,t)$

$$\begin{aligned}&\frac{1}{2}(||u_t^m(t)||^2 + ||\Delta u^m(t)||^2 + s_0||\nabla u^m(t)||^2)\\ &\le \frac{1}{2}(||u_1^m||^2 + ||\Delta u_0^m||^2 + \hat{M}(||\nabla u_0^m||^2) + \int_0^t g(t-\tau)(\Delta u^m(\tau), \Delta u^m(t))d\tau \qquad (3.5)\\ &\quad + \frac{c_1^2}{2}\int_0^t ||\Delta u^m(s)||^2 ds + \frac{1}{2}||g||^2_{L^1(0,\infty)} \int_0^t ||\Delta u^m(s)||^2 ds.\end{aligned}$$

On the other hand, using the inequality $ab \le \frac{1}{4\epsilon}a^2 + \epsilon b^2$, we have

$$\begin{aligned}(3.6)\qquad &\int_0^t g(t-\tau)(\Delta u^m(\tau), \Delta u^m(t))d\tau\\ &\le \epsilon||\Delta u^m(t)||^2 + \frac{1}{4\epsilon}||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)} \int_0^t ||\Delta u^m(\tau)||^2 d\tau.\end{aligned}$$

Replacing (3.6) in (3.5) with $\epsilon > 0$ sufficiently small and employing Gronwall's lemma we obtain the first estimate

$$(3.7)\qquad ||u_t^m(t)||^2 + ||\Delta u^m(t)||^2 + ||\nabla u^m(t)||^2 \le C_1,$$

where C_1 is a positive constant. Therefore, the approximate solution $u^m(t)$ can be extended to the whole interval $[0,T]$, where $T = \infty$.

3.2. The second estimate. Preliminary to the second estimate, we introduce the useful lemma. The following lemma (with $t = 0$) will be used to estimate $||u_{tt}^m(t)||$.

Lemma 3.1. *(cf.* **[17]***)* $||\rho(\cdot, u_t^m(t))|| \le C$ *with* $C = C(u_0, u_1)$ *a positive constant (independent of* t, m*).*

In order to estimate $||u_{tt}^m(t)||^2$, we need to estimate $||u_{tt}^m(0)||$.

First of all, we are estimating $u_{tt}^m(0)$ in the L^2-norm. Considering $t = 0$ and $w = u_{tt}^m(0)$ in (3.1), we obtain

$$||u_{tt}^m(0)||^2 \le \Big(||\Delta^2 u_0^m|| + M(||\nabla u_0^m||^2)||\Delta u_0^m|| + ||\rho(u_1^m)||\Big)||u_{tt}^m(0)||.$$

From the previous lemma and hypotheses on the initial data, it follows that

$$(3.8)\qquad ||u_{tt}^m(0)|| \le C_2 \quad \text{for all} \quad m \in \mathbb{N},$$

where C_2 is a positive constant.

Now we are going to obtain an estimate for u_{tt}^m and Δu_t^m in L^2-norm. Finally, differentiating (3.1) with respect to t and substituting $w = u_{tt}^m(t)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(||u_{tt}^m(t)||^2 + ||\Delta u_t^m(t)||^2\Big)\\
&= +2M'(||\nabla u^m(t)||^2)(\nabla u^m(t), \nabla u_t^m(t))(\Delta u^m(t), u_{tt}^m(t))\\
&+ M(||\nabla u^m(t)||^2)(\Delta u^m(t), u_{tt}^m(t))\\
&\quad + g(0)\frac{d}{dt}(\Delta u^m(t), \Delta u_t^m(t)) - g(0)||\Delta u_t^m(t)||^2\\
&+ \frac{d}{dt}\Big(\int_0^t g'(t-\tau)(\Delta u^m(\tau), \Delta u_t^m(t))d\tau\Big)\\
&\quad - \int_0^t g''(t-\tau)(\Delta u^m(\tau), \Delta u_t^m(t))d\tau - g'(0)(\Delta u^m(t), \Delta u_t^m(t))\\
&- \Big(\frac{\partial\rho(x,s)}{\partial s}u_{tt}^m(t), u_{tt}^m(t)\Big).
\end{aligned} \tag{3.9}$$

Since $M \in C^1(\mathbb{R}_+)$ and (3.7), using the Young's inequality and Sobolev imbedding theorem we get

$$\Big|2M'(||\nabla u^m(t)||^2)(\nabla u^m(t), \nabla u_t^m(t))(\Delta u^m(t), u_{tt}^m(t))\Big| \le d_1(||\Delta u_t^m(t)||^2 + ||u_{tt}^m(t)||^2), \tag{3.10}$$

where d_1 is a positive constant. Similarly, we can easily check that

$$\Big|M(||\nabla u^m(t)||^2)(\Delta u^m(t), u_{tt}^m(t))\Big| \le d_2(\epsilon||\Delta u^m(t)||^2 + \frac{1}{4\epsilon}||u_{tt}^m(t)||^2), \tag{3.11}$$

where d_2 is a positive constant.
On the other hand, from (2.5), we easily obtain as similar calculation of (3.4)

$$\begin{aligned}
&\Big|\int_0^t g''(t-\tau)(\Delta u^m(\tau), \Delta u_t^m(t))d\tau\Big|\\
&\qquad \le \frac{c_3^2}{4\epsilon}||\Delta u_t^m(t)||^2 + \epsilon||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)||\Delta u^m(\tau)||^2 d\tau.
\end{aligned} \tag{3.12}$$

Replacing (3.10) - (3.12) in (3.9) and using the positivity of $\frac{\partial\rho}{\partial s}$ (cf. (2.7)), and then integrating (3.9) over $(0,t)$ we have

$$\begin{aligned}
&\frac{1}{2}\Big(||u_{tt}^m(t)||^2 + ||\Delta u_t^m(t)||^2\Big)\\
&\le \frac{1}{2}||u_{tt}^m(0)||^2 + \frac{1}{2}||\Delta u_1^m||^2 + d_1\int_0^t (||u_{tt}^m(s)||^2 + ||\Delta u_t^m(s)||^2)ds\\
&\quad + \frac{d_2}{4\epsilon}\int_0^t ||u_{tt}^m(s)||^2 ds + d_2\epsilon\int_0^t ||\Delta u^m(s)||^2 ds + g(0)(\Delta u^m(t), \Delta_t^m(t))\\
&\quad + \frac{c_3^2}{4\epsilon}\int_0^t ||\Delta u_t^m(s)||^2 ds + \epsilon||g||_{L^1(0,\infty)}^2\int_0^t ||\Delta u^m(s)||^2 ds\\
&\quad + \int_0^t g'(t-\tau)(\Delta u^m(\tau), \Delta u_t^m(t))d\tau - \int_0^t g'(0)(\Delta u^m(s), \Delta u_t^m(s))ds.
\end{aligned} \tag{3.13}$$

We note that

$$\Big|g(0)(\Delta u^m(t), \Delta u_t^m(t))\Big| \le \frac{(g(0))^2}{4\epsilon}||\Delta u^m(t)||^2 + \epsilon||\Delta u_t^m(t)||^2, \tag{3.14}$$

$$\left|\int_0^t g'(t-\tau)(\Delta u^m(\tau), \Delta u_t^m(t))d\tau\right| \tag{3.15}$$
$$\leq \epsilon||\Delta u_t^m(t)||^2 + \frac{c_1^2}{4\epsilon}||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)}\int_0^t ||\Delta u^m(\tau)||^2 d\tau$$

and

$$\left|\int_0^t g'(0)(\Delta u^m(s), \Delta u_t^m(s))ds\right|$$
$$\leq (g'(0))^2\epsilon\int_0^t ||\Delta u^m(s)||^2 ds + \frac{1}{4\epsilon}\int_0^t ||\Delta u_t^m(s)||^2 ds. \tag{3.16}$$

Substituting (3.14) - (3.16) in (3.13) with $\epsilon > 0$ sufficiently small and taking into account (3.2), (3.7) and (3.8), from Gronwall's lemma we obtain the second estimate

$$||u_{tt}^m(t)||^2 + ||\Delta u_t^m(t)||^2 \leq C_2, \tag{3.17}$$

where C_2 is a positive constant.

By estimates (3.8) and (3.17), we obtain

$$(u^m) \quad \text{is bounded in} \quad L^\infty(0,T;V),$$
$$(u_t^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),$$
$$(u_t^m) \quad \text{is bounded in} \quad L^\infty(0,T;V),$$
$$(u_{tt}^m) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)).$$

Therefore, we get a subsequence of (u^m), which from now on will be represented by the same notation, such that

$$u^m \to u \quad \text{weak star in} \quad L^\infty(0,T;V), \tag{3.18}$$

$$u_t^m \to u_t \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.19}$$

$$u_t^m \to u_t \quad \text{weak star in} \quad L^\infty(0,T;V), \tag{3.20}$$

$$u_{tt}^m \to u_{tt} \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \tag{3.21}$$

From Aubin-Lions lemma, we deduce that

$$u^m \to u \quad \text{strongly in} \quad C([0,T];V), \tag{3.22}$$

$$u_t^m \to u_t \quad \text{strongly in} \quad C([0,T];L^2(\Omega)). \tag{3.23}$$

The above convergences (3.18) - (3.23) and the fact that $(\rho(x,u_t^m),v) \to (\rho(x,u_t),v)$ in $D'(0,T)$, for all $v \in V$(cf. Lemma 4.4. in [**17**]) are enough to pass to the limit in (3.1). Then it is a matter of routine to conclude the existence of global solutions in $[0,T]$.

3.3. Uniqueness. Let u^1 and u^I be two solutions to problem (1.1). Then, $z := u^1 - u^I$ verifies

$$(z_{tt}(t), w) + (\Delta z(t), \Delta w) + (\rho(u_t^1(t)), w) - (\rho(u_t^I(t)), w) = \int_0^t g(t-\tau)(\Delta z(\tau), \Delta w)d\tau + M\Big(||\nabla u^I||^2\Big)(\nabla u^I(t), \nabla w) - M\Big(||\nabla u^1||^2\Big)(\nabla u^1(t), \nabla w) \tag{3.24}$$

for all $w \in V$. Replacing $w = z_t(t)$ in (3.24) and adding the term

$$M\Big(||\nabla u^I||^2\Big)(\nabla u^1(t), \nabla z_t(t))$$

both sides of (3.24), it follows that

$$\frac{1}{2}\frac{d}{dt}\Big(||z_t(t)||^2 + ||\Delta z(t)||^2\Big) + M\Big(||\nabla u^I||^2\Big)(\nabla z(t), \nabla z_t(t)) + (\rho(u_t^1(t)) - \rho(u_t^I(t)), z_t(t)) = \int_0^t g(t-\tau)(\Delta z(\tau), \Delta z_t(t))d\tau + \Big(M\Big(||\nabla u^1||^2\Big) - M\Big(||\nabla u^I||^2\Big)\Big)(\Delta u^1(t), z_t(t)). \tag{3.25}$$

On the other hand, we note that

$$\frac{d}{dt}\Big[M\Big(||\nabla u^I||^2\Big)||\nabla z(t)||^2\Big] = 2M'\Big(||\nabla u^I||^2\Big)(\nabla u^I(t), \nabla u_t^I(t))||\nabla z(t)||^2 + 2M\Big(||\nabla u^I||^2\Big)(\nabla z(t), \nabla z_t(t)).$$

Replacing above equality in (3.25), we get

$$\frac{1}{2}\frac{d}{dt}\Big(||z_t(t)||^2 + ||\Delta z(t)||^2 + M\Big(||\nabla u^I||^2\Big)||\nabla z(t)||^2\Big) + (\rho(u_t^1(t)) - \rho(u_t^I(t)), z_t(t)) = \int_0^t g(t-\tau)(\Delta z(\tau), \Delta z_t(t))d\tau + \Big(M\Big(||\nabla u^1||^2\Big) - M\Big(||\nabla u^I||^2\Big)\Big)(\Delta u^1(t), z_t(t)) + M'\Big(||\nabla u^I||^2\Big)(\nabla u^I(t), \nabla u_t^I(t))||\nabla z(t)||^2. \tag{3.26}$$

We observe that

$$\int_0^t g(t-\tau)(\Delta z(\tau), \Delta z_t(t))d\tau = -g(0)||\Delta z(t)||^2 - \int_0^t g'(t-\tau)(\Delta z(\tau), \Delta z(t))d\tau + \frac{d}{dt}\Big(\int_0^t g(t-\tau)(\Delta z(\tau), \Delta z(t))d\tau\Big), \tag{3.27}$$

$$\begin{aligned}&\Big|M\Big(||\nabla u^1||^2\Big) - M\Big(||\nabla u^I||^2\Big)\Big| \\ &= \Big|\int_{||\nabla u^I||^2}^{||\nabla u^1||^2} M'(s)ds\Big| \leq d_3\big|||\nabla u^1||^2 - ||\nabla u^I||^2\big| \leq d_4||\nabla z(t)||,\end{aligned} \tag{3.28}$$

and

$$M'\Big(||\nabla u^I||^2\Big)(\nabla u^I(t), \nabla u^I_t(t))||\nabla z(t)||^2 \leq d_5||\nabla u^I_t(t)||||\nabla z(t)||^2, \tag{3.29}$$

where d_3, d_4 and d_5 are positive constants.
By continuity of $\rho(\cdot, s)$ and $\frac{\partial \rho}{\partial s}$ and the mean value theorem for vector-valued differential functions, we conclude that

$$(\rho(u^1_t(t)) - \rho(u^I_t(t)), z_t(t)) = \Big(\frac{\partial \rho(\bar{s})}{\partial s} z_t(t), z_t(t)\Big), \tag{3.30}$$

for some $\bar{s}$ in the line between u^1_t and u^I_t.

Replacing (3.27)-(3.30) in (3.26) and using the positivity of $\frac{\partial \rho}{\partial s}$ and the first and second estimate, we arrive that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(||z_t(t)||^2 + ||\Delta z(t)||^2 + M\Big(||\nabla u^I||^2\Big)||\nabla z(t)||^2\Big) \\ &\leq \frac{c_1^2}{2}||\Delta z(t)||^2 + \frac{1}{2}||g||_{L^1(0,\infty)}\int_0^t g(t-\tau)||\Delta z(\tau)||^2 d\tau \\ &\quad + \frac{d}{dt}\Big(\int_0^t g(t-\tau)(\Delta z(\tau), \Delta z(t))d\tau\Big) + d_6||z_t(t)||^2 + d_7||\nabla z(t)||^2,\end{aligned} \tag{3.31}$$

where d_6 and d_7 are positive constants.
Now, integrating (3.31) over $(0, t)$ and noting that

$$\int_0^t g(t-\tau)(\Delta z(\tau), \Delta z(t))d\tau \leq \epsilon||\Delta z(t)||^2 + \frac{1}{4\epsilon}||g||_{L^1(0,\infty)}||g||_{L^\infty(0,\infty)}\int_0^t ||\Delta z(\tau)||^2 d\tau.$$

Then, we conclude by choosing $\epsilon > 0$ sufficiently small and employing Gronwall's lemma $||z_t(t)|| = ||\nabla z(t)|| = ||\Delta z(t)|| = 0$.

3.4. Weak solutions. Let us $\{u_0, u_1\} \in V \times L^2(\Omega)$. Then, by density, there exists $\{u^m_0, u^m_1\} \subset W \times V$ such that

$$u^m_0 \to u_0 \quad \text{in} \quad V \quad \text{and} \quad u^m_1 \to u_1 \quad \text{in} \quad L^2(\Omega). \tag{3.32}$$

Therefore, for each $m \in \mathbb{N}$, there exists u^m, smooth solution of problem (1.1) verifying

$$\begin{cases} u^m_{tt} + \Delta^2 u^m - M(||\nabla u^m||^2)\Delta u^m - \int_0^t g(t-\tau)\Delta^2 u^m(\tau)d\tau + \rho(x, u^m_t) = 0 \\ u^m(0) = u^m_0, \quad u^m_t(0) = u^m_1. \end{cases} \tag{3.33}$$

Repeating the same argument used in the first estimate, we obtain

$$||u^m_t(t)||^2 + ||\Delta u^m(t)||^2 + ||\nabla u^m(t)||^2 \leq C_3, \tag{3.34}$$

where C_3 is a positive constant.

Let $z^{m,l} = u^m - u^l$ with $m, l \in \mathbb{N}$, where u^m and u^l are regula solutions of (3.33). Then following the same already used in the uniqueness of regular solutions and taking the (3.32) into account, we deduce that there exists u such that

$$u^m \to u \quad \text{strongly in} \quad C([0, T]; V), \tag{3.35}$$

$$u_t^m \to u_t \quad \text{strongly in} \quad C([0,T];L^2(\Omega)). \tag{3.36}$$

From (3.34) - (3.36), we can pass to the limit using standard arguments in order to obtain

$$u_{tt} + \Delta^2 u - M(||\nabla u||^2)\Delta u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + \rho(x,u_t) = 0 \tag{3.37}$$

in $L^2(0,\infty,V')$, where V' is a dual space of V. The uniqueness of weak solutions can be also obtained by same argument of subsection 3.3.

4. Energy decay

In this section we prove the energy decay rate to problem (1.1) using the lemma 2.2. It is enough to consider $u_0 \in W \cap V$, $u_1 \in V$ and then to use a density argument.

We define the energy $E(t)$ of the problem (1.1) by

$$E(t) = \frac{1}{2}\int_\Omega |u'|^2 dx + \frac{1}{2}\int_\Omega |\Delta u|^2 dx + \frac{1}{2}\hat{M}(||\nabla u||^2). \tag{4.1}$$

Then the derivative of the energy is given by

$$E'(t) = -(\rho(x,u'),u') + \int_0^t g(t-\tau)(\Delta u(\tau),\Delta u'(t))d\tau.$$

Defining

$$(g\Box\Delta u)(t) = \int_0^t g(t-\tau)||\Delta u(\tau) - \Delta u(t)||^2 d\tau.$$

A direct computation shows that

$$\int_0^t g(t-\tau)(\Delta u(\tau),\Delta u'(t))d\tau = \frac{1}{2}(g'\Box\Delta u)(t) - \frac{1}{2}(g\Box\Delta u)'(t)$$
$$+\frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(t-\tau)d\tau||\Delta u||^2\right\} - \frac{1}{2}g(t)||\Delta u||^2.$$

We define the modified energy by

$$\begin{aligned} e(t) \ &= \ \frac{1}{2}\int_\Omega |u'|^2 dx + \frac{1}{2}\Big(1 - \int_0^t g(t-\tau)d\tau\Big)\int_\Omega |\Delta u|^2 dx + \\ &\quad \frac{1}{2}\hat{M}(||\nabla u||^2) + \frac{1}{2}(g\Box\Delta u)(t). \end{aligned} \tag{4.2}$$

Then

$$e'(t) = -(\rho(x,u'),u') + \frac{1}{2}(g'\Box\Delta u)(t) - \frac{1}{2}g(t)||\Delta u||^2.$$

We observe that in view of assumption (2.3) we have $e(t) \geq 0$, and according to hypotheses on g we deduce that $e'(t) \leq 0$. Moreover,

$$E(t) \leq \ell^{-1}e(t) \quad \text{for all} \quad t \geq 0.$$

Therefore, it is enough to obtain the decay for the modified energy $e(t)$.

Firstly, in order to prove the decay of $e(t)$ we introduce useful properties.

4.1. Some identities and the basic inequalities. Let u be the solution of (1.1) and $T > 0$ fixed.

Firstly, multiplying the equation by u' and integrating over $[t, t+T] \times \Omega$, we have

$$\int_t^{t+T} \int_\Omega \rho(x, u')u' dx ds - \frac{1}{2} \int_t^{t+T} (g' \Box \Delta u)(s) ds \\ + \frac{1}{2} \int_t^{t+T} g(s) ||\Delta u||^2 ds = e(t) - e(t+T). \tag{4.3}$$

Second, multiplying the equation by u and integrating we have

$$\int_t^{t+T} \int_\Omega (-|u'|^2 + |\Delta u|^2) dx ds + \int_t^{t+T} M(||\nabla u||^2) ||\nabla u||^2 ds \\ - \int_t^{t+T} \int_0^s g(s-\tau)(\Delta u(\tau), \Delta u(s)) d\tau ds \\ = - \int_t^{t+T} \int_\Omega \rho(x, u') u dx ds + (u'(t), u(t)) - (u'(t+T), u(t+T)). \tag{4.4}$$

Third, multiplying the equation by $m(x) \cdot \nabla u$ we have

$$\frac{n}{2} \int_t^{t+T} \int_\Omega |u'|^2 dx ds + \left(2 - \frac{n}{2}\right) \int_t^{t+T} \int_\Omega |\Delta u|^2 dx ds \\ + \left(1 - \frac{n}{2}\right) \int_t^{t+T} M(||\nabla u||^2) ||\nabla u||^2 ds \\ - 2 \int_t^{t+T} \int_0^s g(s-\tau)(\Delta u(\tau), \Delta u(s)) d\tau ds \\ - \int_t^{t+T} \int_0^s g(s-\tau)(\Delta u(\tau), m \cdot \nabla(\Delta u(s))) d\tau ds \\ + \int_t^{t+T} \int_0^s g(s-\tau) \int_\Gamma (m \cdot \nu) \Delta u(\tau) \Delta u(s) d\Gamma d\tau ds \\ + \int_t^{t+T} \int_\Omega \rho(x, u')(m(x) \cdot \nabla u) dx ds \\ = (u'(t), m(x) \cdot \nabla u(t)) - (u'(t+T), m(x) \cdot \nabla u(t+T)) \\ + \frac{1}{2} \int_t^{t+T} \int_\Gamma (m \cdot \nu) |\Delta u|^2 d\Gamma ds. \tag{4.5}$$

Next, take a function $\zeta \in W^{2,\infty}(\Omega)$ such that $\frac{|\nabla \zeta|^2}{\zeta}$ and $\frac{|\Delta \zeta|^2}{\zeta}$ are bounded and

$$0 \le \zeta(x) \le 1 \quad \text{in} \quad \Omega, \qquad \zeta = 1 \quad \text{in} \quad \tilde{\omega} \quad \text{and} \quad \zeta = 0 \quad \text{in} \quad \bar{\Omega} \setminus \omega, \tag{4.6}$$

where $\tilde{\omega}$ is an open set in $\bar{\Omega}$ with $\Gamma_1 \subset \tilde{\omega} \subset \omega \subset \bar{\Omega}$ (cf. [**11**]). Then, multiplying the equation ζu and integrating we have

$$\begin{aligned}
&\int_t^{t+T}\int_\Omega \zeta\Big(|\Delta u|^2 - |u'|^2 + M(||\nabla u||^2)|\nabla u|^2\Big)dxds \\
&= (u'(t),\zeta u(t)) - (u'(t+T),\zeta u(t+T)) \\
&- \int_t^{t+T}\int_\Omega \Big(\Delta u\Delta\zeta u + 2\Delta u(\nabla\zeta\cdot\nabla u)\Big)dxds \\
&\quad - \int_t^{t+T} M(||\nabla u||^2)\int_\Omega u(\nabla u\cdot\nabla\zeta)dxds \\
&- \int_t^{t+T}\int_\Omega \rho(x,u')\zeta u dxds \\
&\quad + \int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\Delta\zeta u(s) \\
&+ 2\nabla\zeta\cdot\nabla u(s) + \zeta\Delta u(s))d\tau ds.
\end{aligned} \tag{4.7}$$

Finally, take a vector field $h = (h^1, h^2, \cdots, h^n) : \bar{\Omega} \to \mathbb{R}^n$ of C^2 class such that

$$h = \nu \quad \text{on} \quad \Gamma_1, \qquad h\cdot\nu \geq 0 \quad \text{on} \quad \Gamma \quad \text{and} \quad h = 0 \quad \text{in} \quad \Omega\setminus\hat{\omega}, \tag{4.8}$$

where $\hat{\omega}$ is and open set in $\mathbb{R}^n$ with $\Gamma_1 \subset \hat{\omega}\cap\bar{\Omega} \subset \omega$ (cf. [**11**]). Then, multiplying the equation by $h\cdot\nabla u$ and integrating we have

$$\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\int_\Gamma (h\cdot\nu)|\Delta u|^2 d\Gamma ds \\
&= \frac{1}{2}\int_t^{t+T}\int_\Omega divh\Big(|u'|^2 - |\Delta u|^2 - M(||\nabla u||^2)|\nabla u|^2\Big)dxds \\
&+ \int_t^{t+T}\int_\Omega (\Delta h\cdot\nabla u)\Delta u dxds \\
&\quad + 2\int_t^{t+T}\int_\Omega \Delta u\sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxds \\
&+ \int_t^{t+T} M(||\nabla u||^2)\int_\Omega \sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxds \\
&\quad - \int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\Delta h\cdot\nabla u(s))d\tau ds \\
&\quad - \int_t^{t+T}\int_0^s g(s-\tau)\int_\Omega \Delta u(\tau)\sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u(s)}{\partial x_j}\Big)dxd\tau ds
\end{aligned}$$

$$
\begin{aligned}
(4.9)\qquad & + \int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma (h\cdot\nu)\Delta u(\tau)\Delta u(s)d\Gamma d\tau ds \\
& - \int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), \nabla h\cdot \Delta u(s))d\tau ds \\
& - \int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), h\cdot\nabla(\Delta u(s)))d\tau ds \\
& + (u'(t+T), h\cdot\nabla u(t+T)) - (u'(t), h\cdot\nabla u(t)) \\
& + \int_t^{t+T}\int_\Omega \rho(x,u')(h\cdot\nabla u)dxds.
\end{aligned}
$$

Now, our basic inequalities read as follows. And in the section the symbol C indicates positive constants, which may be different.

Proposition 4.1. *There exists a fixed $T > 0$ such that the modified energy $e(t)$ satisfies*

$$
\begin{aligned}
(4.10)\quad e(t) \le C\Big\{ & e(t) - e(t+T) + \int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \\
& + \int_t^{t+T}\int_\omega (|u'|^2 + |u|^2 + |\nabla u|^2)dxds\Big\},
\end{aligned}
$$

for all $t > 0$.

Proof. Let β be a positive number such that $\frac{n\beta}{2} - 1 > 0$. If $n \ge 4$, we also take β such that $(n-2)\beta < 2$. Then multiplying (4.5) by β and adding (4.4) we have

$$
\begin{aligned}
(4.11)\qquad & \Big(\frac{n\beta}{2} - 1\Big)\int_t^{t+T}\int_\Omega |u'|^2 dxds + \Big(\beta\Big(2 - \frac{n}{2}\Big) + 1\Big)\int_t^{t+T}\int_\Omega |\Delta u|^2 dxds \\
& + \Big(\beta\Big(1 - \frac{n}{2}\Big) + 1\Big)\int_t^{t+T} M(||\nabla u||^2)||\nabla u||^2 ds \\
& \underbrace{-(2\beta+1)\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), \Delta u(s))d\tau ds}_{:=I_1} \\
& \underbrace{-\beta\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), m\cdot\nabla(\Delta u(s)))d\tau ds}_{:=I_2} \\
& + \beta\int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma (m\cdot\nu)\Delta u(\tau)\Delta u(s)d\Gamma d\tau ds \\
& = (u'(t), \beta(m(x)\cdot\nabla u(t)) + u(t)) \\
& - (u'(t+T), \beta(m(x)\cdot\nabla u(t+T)) + u(t+T)) \\
& - \int_t^{t+T}\int_\Omega [\beta(m(x)\cdot\nabla u) + u]\rho(x,u')dxds \\
& + \frac{\beta}{2}\int_t^{t+T}\int_\Gamma (m\cdot\nu)|\Delta u|^2 d\Gamma ds.
\end{aligned}
$$

Now we will estimate I_1 and I_2.

Estimates for $I_1 := -(2\beta+1)\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\Delta u(s))d\tau ds$; Similarly to (3.7) and using Young's inequality, we have

$$\begin{aligned}
&\int_0^s g(s-\tau)(\Delta u(\tau),\Delta u(s))d\tau \\
&= \int_0^s g(s-\tau)(\Delta u(\tau)-\Delta u(s),\Delta u(s))d\tau + \int_0^s g(s-\tau)d\tau\int_\Omega |\Delta u|^2dx \\
&\le \frac{1}{2}\int_\Omega |\Delta u|^2dx + \frac{1}{2}||g||_{L^1(0,\infty)}(g\Box\Delta u)(s) + \int_0^s g(s-\tau)d\tau\int_\Omega |\Delta u|^2dx.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&-(2\beta+1)\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\Delta u(s))d\tau ds \\
&\ge -(2\beta+1)\Big\{\frac{1}{2}\int_t^{t+T}\int_\Omega |\Delta u|^2dxds \\
&+ \frac{1}{2}||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds \\
&+ \int_t^{t+T}\int_0^s g(s-\tau)d\tau\int_\Omega |\Delta u|^2dxds\Big\}.
\end{aligned} \tag{4.12}$$

Estimates for $I_2 := -\beta\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), m\cdot\nabla(\Delta u(s)))d\tau ds$; Similarly to I_1, we have

$$\begin{aligned}
&\int_0^s g(s-\tau)(\Delta u(\tau), m\cdot\nabla(\Delta u(s)))d\tau \\
&= \int_0^s g(s-\tau)(\Delta u(\tau)-\Delta u(s), m\cdot\nabla(\Delta u(s)))d\tau \\
&+ \int_0^s g(s-\tau)(\Delta u(s), m\cdot\nabla(\Delta u(s)))d\tau \\
&\le \epsilon\int_\Omega |\nabla(\Delta u)|^2dx + C(\epsilon)||g||_{L^1(0,\infty)}(g\Box\Delta u)(s) \\
&\quad + \frac{1}{2}\int_0^s g(s-\tau)d\tau\int_\Gamma (m\cdot\nu)|\Delta u|^2d\Gamma - \frac{n}{2}\int_0^s g(s-\tau)d\tau\int_\Omega |\Delta u|^2dx.
\end{aligned}$$

Hence,taking $\epsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}
&-\beta\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), m\cdot\nabla(\Delta u(s)))d\tau ds \\
&\ge -\beta\Big\{\frac{1}{2}\int_t^{t+T}\int_0^s g(s-\tau)d\tau\int_\Gamma (m\cdot\nu)|\Delta u|^2d\Gamma ds \\
&- \frac{n}{2}\int_t^{t+T}\int_0^s g(s-\tau)d\tau\int_\Omega |\Delta u|^2dxds \\
&\qquad + C||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds\Big\}.
\end{aligned} \tag{4.13}$$

Replacing (4.12) and (4.13) in (4.11) and nondecreasing property of M, we get

$$\begin{aligned}
&\Big(\frac{n\beta}{2}-1\Big)\int_t^{t+T}\int_\Omega |u'|^2 dxds+\Big(\big(1-\frac{n}{2}\big)\beta+\frac{1}{2}\Big)\\
&\int_t^{t+T}\int_\Omega\Big(1-\int_0^s g(s-\tau)d\tau\Big)|\Delta u|^2 dxds\\
&\quad+\Big(\big(1-\frac{n}{2}\big)\beta+1\Big)\hat{M}(||\nabla u||^2)+\frac{1}{2}||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds\\
&\le (u'(t),\beta(m(x)\cdot\nabla u(t))+u(t))\\
&-(u'(t+T),\beta(m(x)\cdot\nabla u(t+T))+u(t+T))\\
&\quad-\int_t^{t+T}\int_\Omega[\beta(m(x)\cdot\nabla u)+u]\rho(x,u')dxds\\
&-\beta\int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma(m\cdot\nu)\Delta u(\tau)\Delta u(s)d\Gamma d\tau ds\\
&\quad+\Big(\frac{3\beta}{2}+C+\frac{1}{2}\Big)||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds\\
&+\beta\int_t^{t+T}\int_{\Gamma_1}(m\cdot\nu)|\Delta u|^2 d\Gamma ds.
\end{aligned}$$

Choosing $\gamma=\min\Big\{2\Big(\frac{n\beta}{2}-1\Big),2\Big(\big(1-\frac{n}{2}\big)\beta+\frac{1}{2}\Big),||g||_{L^1(0,\infty)}\Big\}$ and using Poincare's inequality, we arrive that

$$\begin{aligned}
&\gamma\int_t^{t+T}e(s)ds\\
&\le C(e(t)+e(t+T))+\int_t^{t+T}\int_\Omega(\beta R|\nabla u|+|u|)|\rho(x,u')|dxds\\
&\quad\underbrace{-\beta\int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma(m\cdot\nu)\Delta u(\tau)\Delta u(s)d\Gamma d\tau ds}_{:=I_3}\\
&\quad\underbrace{+\Big(\frac{3\beta}{2}+C+\frac{1}{2}\Big)||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds}_{:=I_4}\\
&+\beta\int_t^{t+T}\int_{\Gamma_1}(m\cdot\nu)|\Delta u|^2 d\Gamma ds.
\end{aligned}\tag{4.14}$$

Using Young's inequality and from the fact $m\cdot\nu>0$ on Γ_1, we obtain

$$\begin{aligned}
&\int_0^s g(s-\tau)\int_\Gamma(m\cdot\nu)\Delta u(\tau)\Delta u(s)d\Gamma d\tau\\
&\qquad\le R\epsilon\int_0^s g(s-\tau)\int_{\Gamma_1}|\Delta u(\tau)|^2 d\Gamma d\tau\\
&\qquad\qquad+C(\epsilon)\int_0^s g(s-\tau)d\tau\int_{\Gamma_1}(m\cdot\nu)|\Delta u|^2 d\Gamma.
\end{aligned}$$

Hence,

$$\begin{aligned}|I_3| \le \epsilon R\beta \int_t^{t+T}\int_0^s g(s-\tau)\int_{\Gamma_1}|\Delta u(\tau)|^2 d\Gamma d\tau ds\\ +C(\epsilon)\beta(1-\ell)\int_t^{t+T}\int_{\Gamma_1}(m\cdot\nu)|\Delta u|^2 d\Gamma ds.\end{aligned} \tag{4.15}$$

By definition of $e(t)$, we can easily check that

$$I_4 \le Ce(t), \tag{4.16}$$

where C is a positive constant that depend on β, $||g||_{L^1(0,\infty)}$ and T.

Replacing (4.15) and (4.16) in (4.14) with $\epsilon > 0$ sufficiently small, we get

$$\begin{aligned}\gamma\int_t^{t+T} e(s)ds \le C(e(t)+e(t+T)) + \int_t^{t+T}\int_\Omega (\beta R|\nabla u| + |u|)|\rho(x,u')|dxds\\ + C\int_t^{t+T}\int_{\Gamma_1}(m\cdot\nu)|\Delta u|^2 d\Gamma ds.\end{aligned} \tag{4.17}$$

Next, we shall estimate the last term in (4.17). Since (4.8) and (4.9), the following holds.

$$\begin{aligned}&\frac{1}{2}\int_t^{t+T}\int_{\Gamma_1}|\Delta u|^2 d\Gamma ds \le \frac{1}{2}\int_t^{t+T}\int_\Gamma (h\cdot\nu)|\Delta u|^2 d\Gamma ds\\ &= (u'(t+T), h\cdot\nabla u(t+T)) - (u'(t), h\cdot\nabla u(t))\\ &+\int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma (h\cdot\nu)\Delta u(\tau)\Delta u(s) d\Gamma d\tau ds\\ &\quad+\int_t^{t+T}\int_\Omega \rho(x,u')(h\cdot\nabla u)dxds\\ &+\frac{1}{2}\int_t^{t+T}\int_\Omega divh\Big(|u'|^2 - |\Delta u|^2 - M(||\nabla u||^2)|\nabla u|^2\Big)dxds\\ &\quad+\int_t^{t+T}\int_\Omega (\Delta h\cdot\nabla u)\Delta u dxds\\ &+2\int_t^{t+T}\int_\Omega \Delta u\sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxds\end{aligned} \tag{4.18}$$

$$
\begin{aligned}
&+\int_t^{t+T} M(||\nabla u||^2)\int_\Omega \sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxds\\
&-\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\Delta h\cdot\nabla u(s))d\tau ds\\
&-\int_t^{t+T}\int_0^s g(s-\tau)\int_\Omega \Delta u(\tau)\sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u(s)}{\partial x_j}\Big)dxd\tau ds\\
&-\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),\nabla h\cdot\Delta u(s))d\tau ds\\
&-\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau),h\cdot\nabla(\Delta u(s)))d\tau ds\\
&:=(u'(t+T),h\cdot\nabla u(t+T))-(u'(t),h\cdot\nabla u(t))\\
&\quad+I_5+I_6+I_7+I_8+I_9+I_{10}-I_{11}-I_{12}-I_{13}-I_{14}.
\end{aligned}
$$

Since $h\in C^2(\bar\Omega)$ and $h\equiv 0$ in $\Omega\setminus\hat\omega$, we have

$$
|I_5|\le\epsilon\int_t^{t+T}\int_0^s g(s-\tau)\int_\Gamma|\Delta u(\tau)|^2d\Gamma d\tau ds+C(\epsilon)\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2dxds, \tag{4.19}
$$

$$
|(u'(t+T),h\cdot\nabla u(t+T))-(u'(t),h\cdot\nabla u(t))|\le C(e(t)+e(t+T)), \tag{4.20}
$$

$$
|I_6|=\left|\int_t^{t+T}\int_\Omega \rho(x,u')(h\cdot\nabla u)dxds\right|\le C\int_t^{t+T}\int_\Omega|\rho(x,u')||\nabla u|dxds \tag{4.21}
$$

and

$$
\begin{aligned}
|I_7|&=\left|\frac{1}{2}\int_t^{t+T}\int_\Omega divh\Big(|u'|^2-|\Delta u|^2-M(||\nabla u||^2)|\nabla u|^2\Big)dxds\right|\\
&\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|u'|^2+|\Delta u|^2+M(||\nabla u||^2)|\nabla u|^2dxds.
\end{aligned} \tag{4.22}
$$

Also using Hölder's and Poincare's inequalities, we obtain

$$
|I_8|=\left|\int_t^{t+T}\int_\Omega(\Delta h\cdot\nabla u)\Delta udxds\right|\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2dxds, \tag{4.23}
$$

$$
|I_9|=\left|2\int_t^{t+T}\int_\Omega\Delta u\sum_{i,j=1}^n\frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j}\Big)dxds\right|\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2dxds \tag{4.24}
$$

and

$$
\begin{aligned}
|I_{10}|&=\left|\int_t^{t+T}M(||\nabla u||^2)\int_\Omega\sum_{i,j=1}^n\frac{\partial h^j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dxds\right|\\
&\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}M(||\nabla u||^2)|\nabla u|^2dxds.
\end{aligned} \tag{4.25}
$$

Similarly to I_1 and I_8 and by hypotheses of g, we have

$$\begin{aligned}
&\int_0^s g(s-\tau)(\Delta u(\tau), \Delta h\cdot\nabla u(s))d\tau \\
&\le \frac{1}{2}\int_\Omega |\Delta h\cdot\nabla u|^2 dx + \frac{1}{2}||g||_{L^1(0,\infty)}(g\Box\Delta u)(s) \\
&+ \int_0^s g(s-\tau)d\tau \int_\Omega \Delta u(s)\Delta h\cdot\nabla u(s)dx \\
&\le C\int_{\hat\omega\cap\bar\Omega} |\Delta u|^2 dx + \frac{1}{2}||g||_{L^1(0,\infty)}(g\Box\Delta u)(s).
\end{aligned}$$

Hence,

$$\begin{aligned}
|I_{11}| &= \left|\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), \Delta h\cdot\nabla u(s))d\tau ds\right| \\
&\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2 dxds + \frac{1}{2}||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds.
\end{aligned} \tag{4.26}$$

By same method of I_{11}, we easily check that

$$\begin{aligned}
|I_{12}| &= \left|\int_t^{t+T}\int_0^s g(s-\tau)\int_\Omega \Delta u(\tau)\sum_{i,j=1}^n \frac{\partial h^j}{\partial x_i}\Big(\frac{\partial}{\partial x_i}\frac{\partial u(s)}{\partial x_j}\Big)dxd\tau ds\right| \\
&\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2 dxds + \frac{1}{2}||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
|I_{13}| &= \left|\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), \nabla h\cdot\Delta u(s))d\tau ds\right| \\
&\le C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2 dxds + \frac{1}{2}||g||_{L^1(0,\infty)}\int_t^{t+T}(g\Box\Delta u)(s)ds
\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}
|I_{14}| &= \left|\int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), h\cdot\nabla(\Delta u(s)))d\tau ds\right| \\
&\le \epsilon\int_t^{t+T}\int_\Omega |\nabla(\Delta u)|^2 dxds + \frac{1}{2}||g||_{L^1(0,\infty)}C(\epsilon)\int_t^{t+T}(g\Box\Delta u)(s)ds \\
&+ C\int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2 dxds.
\end{aligned} \tag{4.29}$$

Replacing (4.19) - (4.29) in (4.18) with $\epsilon > 0$ sufficiently small and again calculating I_4, we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\int_{\Gamma_1}|\Delta u|^2 d\Gamma ds \\
&\le C\Big\{e(t) + e(t+T) + \int_t^{t+T}\int_\Omega |\rho(x,u')||\nabla u|dxds \\
&+ \int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|u'|^2 dxds + \int_t^{t+T}\int_{\hat\omega\cap\bar\Omega}|\Delta u|^2 + M(||\nabla u||^2)|\nabla u|^2 dxds\Big\}.
\end{aligned} \tag{4.30}$$

In the sequel we will find boundedness for the last term of the right-hand side of (4.30). First, we use (4.7) with (4.6), then we can write that

$$
\begin{aligned}
&\int_t^{t+T}\int_\Omega \zeta\Big(|\Delta u|^2 + M(||\nabla u||^2)|\nabla u|^2\Big)dxds\\
&\le \int_t^{t+T}\int_\omega |u'|^2 dxds + (u'(t),\zeta u(t)) - (u'(t+T),\zeta u(t+T))\\
&- \int_t^{t+T}\int_\Omega \rho(x,u')\zeta u dxds\\
&\quad - \int_t^{t+T} M(||\nabla u||^2)\int_\Omega u(\nabla u\cdot\nabla\zeta)dxds\\
&- \int_t^{t+T}\int_\omega \Big(\Delta u\Delta\zeta u + 2\Delta u(\nabla\zeta\cdot\nabla u)\Big)dxds\\
&\quad + \int_t^{t+T}\int_0^s g(s-\tau)(\Delta u(\tau), \Delta\zeta u(s) + 2\nabla\zeta\cdot\nabla u(s)\\
&+ \zeta\Delta u(s))d\tau ds\\
&:= \int_t^{t+T}\int_\omega |u'|^2 dxds + (u'(t),\zeta u(t))\\
&- (u'(t+T),\zeta u(t+T)) - I_{15} - I_{16} - I_{17} + I_{18}.
\end{aligned} \tag{4.31}
$$

Similarly to (4.20), we get

$$
|(u'(t),\zeta u(t)) - (u'(t+T),\zeta u(t+T))| \le C(e(t)+e(t+T)). \tag{4.32}
$$

Assumption on ζ (see (4.6)) and using Höler's and Poincare's inequalities, we obtain

$$
|I_{15}| = \left|\int_t^{t+T}\int_\Omega \rho(x,u')\zeta u dxds\right| \le \int_t^{t+T}\int_\Omega |\rho(x,u')||u|dxds, \tag{4.33}
$$

$$
\begin{aligned}
|I_{16}| &= \left|\int_t^{t+T} M(||\nabla u||^2)\int_\Omega u(\nabla u\cdot\nabla\zeta)dxds\right|\\
&\le \int_t^{t+T} M(||\nabla u||^2)\left[\int_\omega \zeta^*|u|^2dx + \int_\omega \frac{1}{4\zeta^*}|\nabla u|^2|\nabla\zeta|^2dx\right]ds\\
&\le C\int_t^{t+T}\int_\omega |u|^2dxds + \frac{1}{4}\int_t^{t+T}\int_\Omega M(||\nabla u||^2)\zeta|\nabla u|^2dxds
\end{aligned} \tag{4.34}
$$

and

$$
\begin{aligned}
|I_{17}| &= \Big| \int_t^{t+T} \int_\omega \Big(\Delta u \Delta \zeta u + 2\Delta u (\nabla \zeta \cdot \nabla u) \Big) dx ds \Big| \\
&\le \int_t^{t+T} \int_\omega \sqrt{\frac{2}{\zeta}} |\Delta \zeta| |u| \sqrt{\frac{\zeta}{2}} |\Delta u| \\
&+ 2 \frac{|\nabla \zeta|}{\sqrt{\zeta}} |\nabla u| \sqrt{\zeta} |\Delta u| dx ds \qquad (4.35) \\
&\le \zeta^* \int_t^{t+T} \int_\omega |u|^2 dx ds + 2\zeta^* \int_t^{t+T} \int_\omega |\nabla u|^2 dx ds \\
&+ \frac{3}{4} \int_t^{t+T} \int_\Omega \zeta |\Delta u|^2 dx ds,
\end{aligned}
$$

where $\zeta^* = \max\Big\{ \sup_{x\in\Omega} \frac{|\nabla \zeta(x)|^2}{\zeta(x)}, \sup_{x\in\Omega} \frac{|\Delta \zeta(x)|^2}{\zeta(x)} \Big\}$.
Similarly to I_1 and I_{17} and using $ab \le a^2 + \frac{1}{4}b^2$ we have

$$
\begin{aligned}
|I_{18}| &= \Big| \int_t^{t+T} \int_0^s g(s-\tau)(\Delta u(\tau), \Delta \zeta u(s) + 2\nabla \zeta \cdot \nabla u(s) + \zeta \Delta u(s)) d\tau ds \Big| \\
&\le (2\zeta^* + 1) \int_t^{t+T} (g \Box \Delta u)(s) ds + (\zeta^* + \frac{1}{4}) \int_t^{t+T} \int_\omega |u|^2 dx ds \qquad (4.36) \\
&+ (2\zeta^* + 1) \int_t^{t+T} \int_\omega |\nabla u|^2 dx ds + \int_t^{t+T} \int_\Omega \zeta |\Delta u|^2 dx ds.
\end{aligned}
$$

Replacing (4.32) - (4.36) in (4.31) and again calculating I_4, we obtain that

$$
\begin{aligned}
&\int_t^{t+T} \int_\Omega \zeta \Big(|\Delta u|^2 + M(||\nabla u||^2) |\nabla u|^2 \Big) dx ds \\
&\le C \Big\{ e(t) + e(t+T) + \int_t^{t+T} \int_\Omega |\rho(x, u')| |u| dx ds \\
&+ \int_t^{t+T} \int_\omega |u'|^2 + |u|^2 + |\nabla u|^2 dx ds \Big\}.
\end{aligned}
$$

Moreover, since $0 \le \zeta(x) \le 1$, it follows that

$$
\begin{aligned}
&\int_t^{t+T} \int_{\hat{\omega} \cap \bar{\Omega}} |\Delta u|^2 + M(||\nabla u||^2) |\nabla u|^2 dx ds \\
&\le C \Big\{ e(t) + e(t+T) + \int_t^{t+T} \int_\Omega |\rho(x, u')| |u| dx ds \qquad (4.37) \\
&+ \int_t^{t+T} \int_\omega |u'|^2 + |u|^2 + |\nabla u|^2 dx ds \Big\}.
\end{aligned}
$$

Finally, noting that $\hat{\omega} \cap \bar{\Omega} \subset \omega$, replacing (4.37) in (4.30) we get

$$\begin{aligned} &\frac{1}{2}\int_t^{t+T}\int_{\Gamma_1} |\Delta u|^2 d\Gamma ds \\ &\leq C\Big\{ e(t) + e(t+T) + \int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \\ &\qquad + \int_t^{t+T}\int_\omega |u'|^2 + |u|^2 + |\nabla u|^2 dxds \Big\}. \end{aligned} \tag{4.38}$$

Thus we replace (4.38) in (4.17) and take $T \geq \frac{2C}{\gamma} + 1$, then the proof of Proposition 4.1 is completed. □

Proposition 4.2. *Let u be the solution of (1.1) and Δe be given by*

$$\Delta e \equiv e(t) - e(t+T).$$

Then, for $T > 0$ given in Proposition 4.1, the followings hold :

case 1

If $r \geq 0$, $0 \leq p \leq \frac{2}{n-2}$ and $n \geq 3$, then

$$\int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \leq C(\Delta e)^{\frac{1}{r+2}}\sqrt{e(t)} + C(\Delta e)^{\frac{p+1}{p+2}}\sqrt{e(t)}.$$

When $n = 2$, this estimate holds for the case $r \geq 0$ and $p \geq 0$.

case 2

If $r \geq 0$, $-1 < p < 0$ and $n \geq 2$, then

$$\int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \leq C(\Delta e)^{\frac{1}{r+2}}\sqrt{e(t)} + C(\Delta e)^{\frac{1}{2(p+2)}}\sqrt{e(t)}.$$

case 3

If $-1 < r < 0$, $0 \leq p \leq \frac{2}{n-2}$ and $n \geq 3$, then

$$\int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \leq C(\Delta e)^{\frac{r+1}{r+2}}\sqrt{e(t)} + C(\Delta e)^{\frac{p+1}{p+2}}\sqrt{e(t)}.$$

When $n = 2$, this estimate holds for the case $-1 < r < 0$ and $p \geq 0$.

case 4

If $-1 < r < 0$, $-1 < p < 0$ and $n \geq 2$, then

$$\int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \leq C(\Delta e)^{\frac{r+1}{r+2}}\sqrt{e(t)} + C(\Delta e)^{\frac{1}{2(p+2)}}\sqrt{e(t)}.$$

For $n = 1$ the above estimates are the same as for the case $n = 2$.

Proof. By the hypotheses on ρ, we have

$$\begin{aligned} &\int_t^{t+T}\int_\Omega |\rho(x,u')|(|u| + |\nabla u|)dxds \\ &\leq \int_t^{t+T}\int_{\Omega_1} c_5 a(x)(|u'|^{r+1} + |u'|)(|u| + |\nabla u|)dxds \\ &+ \int_t^{t+T}\int_{\Omega_2} c_7 a(x)(|u'|^{p+1} + |u'|)(|u| + |\nabla u|)dxds \\ &:= I_{19} + I_{20}, \end{aligned} \tag{4.39}$$

where $\Omega_1 = \{(x,t) \in \Omega \times \mathbb{R}_+ : |u'| \leq 1\}$ and $\Omega_2 = \Omega \setminus \Omega_1$.

Now, we will estimate I_{19} and I_{20}.

(i) Estimating I_{19} for $r \geq 0$ and $n \geq 2$.

In this case we see, by Poincare's inequality, Sobolev imbedding theorem,(2.6) and (4.3),

$$
\begin{aligned}
I_{19} &\leq 2c_5 ||\sqrt{a(x)}||_{L^\infty(\Omega)} \int_t^{t+T} \int_{\Omega_1} \sqrt{a(x)}|u'|(|u| + |\nabla u|)dxds \\
&\leq C\Big(\int_t^{t+T} \int_{\Omega_1} a(x)|u'|^2 dxds\Big)^{\frac{1}{2}} \Big(\int_t^{t+T} e(s)ds\Big)^{\frac{1}{2}} \\
&\leq C\Big(\int_t^{t+T} \int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{1}{r+2}} \sqrt{e(t)} \\
&\leq C\Big(\int_t^{t+T} \int_{\Omega_1} \rho(x, u')u' dxds\Big)^{\frac{1}{r+2}} \sqrt{e(t)} \\
&\leq C(\Delta e)^{\frac{1}{r+2}} \sqrt{e(t)}.
\end{aligned} \tag{4.40}
$$

(ii) Estimating I_{19} for $-1 \leq r \leq 0$ and $n \geq 2$.

Similarly to (i) and using $L^2 \hookrightarrow L^{r+2}$, we have

$$
\begin{aligned}
I_{19} &\leq 2c_5 \int_t^{t+T} \int_{\Omega_1} a(x)|u'|^{r+1}(|u| + |\nabla u|)dxds \\
&\leq C\Big(\int_t^{t+T} \int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{r+1}{r+2}} \\
&\quad \Big(\int_t^{t+T} \int_{\Omega_1} (|u| + |\nabla u|)^{r+2} dxds\Big)^{\frac{1}{r+2}} \\
&\leq C\Big(\int_t^{t+T} \int_{\Omega_1} \rho(x, u')u' dxds\Big)^{\frac{1}{r+2}} \\
&\quad \Big(\int_t^{t+T} \int_{\Omega_1} (|u| + |\nabla u|)^2 dxds\Big)^{\frac{1}{2}} \\
&\leq C(\Delta e)^{\frac{r+1}{r+2}} \sqrt{e(t)}.
\end{aligned} \tag{4.41}
$$

(iii) Estimating I_{20} for $0 \leq p \leq \frac{2}{n-2}$ and $n \geq 3$.

By Hölder's inequality and Sobolev imbedding theorem, we get

$$
\begin{aligned}
I_{20} &\le 2c_7 \int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+1}(|u|+|\nabla u|)dxds \\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2}dxds\Big)^{\frac{p+1}{p+2}} \\
&\quad \Big(\int_t^{t+T}\int_{\Omega_2} (|u|+|\nabla u|)^{p+2}dxds\Big)^{\frac{1}{p+2}} \\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2}dxds\Big)^{\frac{p+1}{p+2}} \\
&\quad \Big(\int_t^{t+T}\int_{\Omega} |\nabla u|^{p+2}dxds\Big)^{\frac{1}{p+2}}.
\end{aligned} \tag{4.42}
$$

Now we use the Lemma 2.1, then

$$
\begin{aligned}
||\nabla u||_{L^{p+2}(\Omega)} &\le C||\nabla u||^{\theta}_{H^1(\Omega)}||\nabla u||^{1-\theta}_{L^2(\Omega)} \\
&\le C||\Delta u||^{\theta}_{L^2(\Omega)}||\nabla u||^{1-\theta}_{L^2(\Omega)} \le C\Big(\frac{2}{\ell}e(t)\Big)^{\frac{\theta}{2}}\Big(\frac{2}{s_0}e(t)\Big)^{\frac{1-\theta}{2}} \le C\sqrt{e(t)}.
\end{aligned}
$$

Replacing above inequality in (4.42), it follows that

$$
I_{20} \le C\Big(\int_t^{t+T}\int_{\Omega_2} \rho(x,u')u'dxds\Big)^{\frac{p+1}{p+2}}\sqrt{e(t)} \le C(\Delta e)^{\frac{p+1}{p+2}}\sqrt{e(t)}. \tag{4.43}
$$

If $n=2$, then we can obtain same result for $p \ge 0$.

(iv) Estimating I_{20} for $-1 < p < 0$ and $n \ge 2$.

By Hölder's and Poincare's inequalities, we have

$$
\begin{aligned}
I_{20} &\le 2c_7 \int_t^{t+T}\int_{\Omega_2} a(x)|u'|(|u|+|\nabla u|)dxds \\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^2dxds\Big)^{\frac{1}{2}}\Big(\int_t^{t+T}\int_{\Omega_2} |\nabla u|^2dxds\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} |u'|^{\frac{p+2}{p+1}}dxds\Big)^{\frac{p+1}{p+2}} \\
&\quad \Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2}dxds\Big)^{\frac{1}{2(p+2)}}\Big(\int_t^{t+T} e(s)ds\Big)^{\frac{1}{2}} \\
&\le C\Big(\int_t^{t+T}\int_{\Omega_2} \rho(x,u')u'dxds\Big)^{\frac{1}{2(p+2)}}\sqrt{e(t)} \\
&\le C(\Delta e)^{\frac{1}{2(p+2)}}\sqrt{e(t)}
\end{aligned} \tag{4.44}
$$

because $u' \in L^\infty(0,\infty;H^1(\Omega)) \hookrightarrow L^\infty(0,\infty;L^{\frac{p+2}{p+1}}(\Omega))$.

Therefore, replacing (4.40), (4.41), (4.43) and (4.44) in (4.39) we conclude the proof of Proposition 4.2. □

Using Young's inequality and Propositions 4.1, 4.2, we obtain the next result.

Proposition 4.3. *Let u be the solution of* (1.1). *Then for $T > 0$ given in Proposition 4.1, the modified energy associated with* (1.1) *satisfies*

$$e(t) \le C\Big\{A_i(t)^2 + \int_t^{t+T}\int_\omega \Big(|u'|^2 + |u|^2 + |\nabla u|^2\Big)dxds\Big\} \tag{4.45}$$

,for $i = 1, 2, 3, 4$, where

case 1

If $r \ge 0$ and $0 \le p \le \frac{2}{n-2}$ ($0 \le p < \infty$ if $n = 2$),

$$A_1(t)^2 = \Delta e + (\Delta e)^{\frac{2}{r+2}} + (\Delta e)^{\frac{2(p+1)}{p+2}}.$$

case 2

If $r \ge 0$and $-1 < p < 0$,

$$A_2(t)^2 = \Delta e + (\Delta e)^{\frac{2}{r+2}} + (\Delta e)^{\frac{1}{p+2}}.$$

case 3

If $-1 < r < 0$ and $0 \le p \le \frac{2}{n-2}$ ($0 \le p < \infty$ if $n = 2$),

$$A_3(t)^2 = \Delta e + (\Delta e)^{\frac{2(r+1)}{r+2}} + (\Delta e)^{\frac{2(p+1)}{p+2}}$$

case 4

If $-1 < r < 0$ and $-1 < p < 0$,

$$A_4(t)^2 = \Delta e + (\Delta e)^{\frac{2(r+1)}{r+2}} + (\Delta e)^{\frac{1}{p+2}}.$$

To arrive at the desired difference inequality on $e(t)$ we must estimate further the last two terms in (4.45). Concerning the last two terms of the right hand side in (4.45) we show :

Proposition 4.4. *According to each $A_i(t)^2$ given in Proposition 4.3 there exists a constant $C > 0$ such that*

$$\int_t^{t+T}\int_\Omega (|u|^2 + |\nabla u|^2)dxds \le C\Big\{A_i(t)^2 + \int_t^{t+T}\int_\omega |u'|^2 dxds\Big\}. \tag{4.46}$$

Before the proof of Proposition 4.4, we shall show the following result.

Lemma 4.1. *Consider $\varphi \in W^{1,\infty}(0,T)$, $\varphi \ge 0$. Then if the function*

$$v \in W^{1,\infty}(0,T; L^2(\Omega)) \cap L^\infty(0,T;V)$$

satisfies the conditions

$$\begin{cases} v_{tt} + \Delta^2 v - \varphi(t)\Delta v - \int_0^t g(t-\tau)\Delta^2 v(\tau)d\tau = 0 & \text{in} \quad \Omega\times(0,T), \\ v = \frac{\partial v}{\partial\nu} = 0 \quad \text{on} \quad \Gamma\times(0,T), \\ v_t = 0 \quad \text{in} \quad \omega\times(0,T), \end{cases} \tag{4.47}$$

we have that $v \equiv 0$ in $\Omega\times(0,T)$.

Proof. If $\varphi(t) = \varphi_0$(constant), for any $t \in [0,T]$, by taking the derivative of (4.47) with respect to t we obtain that $w = v_t$ satisfies (in the distributions sense) the equation

$$\begin{cases} w_{tt} + \Delta^2 w - \varphi_0\Delta w - g(0)\Delta^2 v - \int_0^t g'(t-\tau)\Delta^2 v(\tau)d\tau = 0 & \text{in} \quad \Omega\times(0,T), \\ w = \frac{\partial w}{\partial\nu} = 0 \quad \text{on} \quad \Gamma\times(0,T), \\ w = 0 \quad \text{in} \quad \omega\times(0,T). \end{cases}$$

By (H_5) we have that $v_t = w \equiv 0$ in $\Omega \times (0,T)$. From (4.47) it follows that

$$\begin{cases} \Delta^2 v - \varphi_0 \Delta v - \int_0^t g(t-\tau)\Delta^2 v(\tau)d\tau = 0 \quad \text{in} \quad \Omega \times (0,T), \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T). \end{cases}$$

By the fact $v_t = 0$, (2.3) and standard elliptic uniqueness result, it follows that the above equation imply the conclusion of the Lemma.

Now let us suppose that $\varphi_t(t) \neq 0$ for t varying in a subset of strictly positive measure of $[0,T]$. By (4.47) and the fact that $v(x,t) = v(x)$ if $x \in \omega$ we get

$$\begin{cases} \left(1 - \int_0^t g(t-\tau)d\tau\right)\Delta^2 v - \varphi(t)\Delta v = 0 \quad \text{in} \quad \omega \times (0,T), \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T). \end{cases}$$

Applying to (2.3), and then deriving above equation with respect to t, we have

$$\Delta v = 0 \quad \text{in} \quad \omega,$$

since $\varphi_t(t) \neq 0$. Hence by Holmgren's uniqueness theorem, we obtain that

$$v \equiv 0 \quad \text{in} \quad \omega.$$

We can use (H_5) again with $\alpha = 0$ to obtain that

$$v \equiv 0 \quad \text{in} \quad \Omega.$$

□

Proof of Proposition 4.4. We prove (4.46) by contradiction. If (4.46) was false, there exist a sequence $\{t_n\} \subset \mathbb{R}$ and let $\{u_n(0), u_n'(0)\}$ be a sequence of initial data where the corresponding solutions $\{u_n\}$ of (1.1) with $E_n(0)$ uniform bounded in n, verifies

$$\lim_{n\to\infty} \frac{\int_{t_n}^{t_n+T}\int_\Omega (|u_n|^2 + |\nabla u_n|^2)dxds}{A_i(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_n'|^2 dxds} = \infty. \tag{4.48}$$

Setting

$$\lambda_n^2 = \int_{t_n}^{t_n+T}\int_\Omega (|u_n|^2 + |\nabla u_n|^2)dxds$$

and

$$v_n(t) = \frac{u_n(t+t_n)}{\lambda_n}, 0 \leq t \leq T.$$

Then, we get

$$Q_n^2 := \frac{1}{\lambda_n^2}\left\{A_i(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_n'|^2 dxds\right\} \to 0 \quad \text{as} \quad n \to \infty \tag{4.49}$$

and

$$\int_0^T \int_\Omega (|v_n|^2 + |\nabla v_n|^2)dxds = 1. \tag{4.50}$$

Thus, we have from (4.49), (4.50) and Proposition 4.3,

$$\begin{aligned} e(v_n(t)) &= e\Big(\frac{u_n(t+t_n)}{\lambda_n^2}\Big) = \frac{1}{\lambda_n^2}e(u_n(t+t_n)) \le \frac{1}{\lambda_n^2}e(u_n(t_n)) \\ &\le \frac{C}{\lambda_n^2}\Big\{A_i(t_n)^2 + \int_{t_n}^{t_n+T}\int_\omega |u_n'|^2 dxds + \int_{t_n}^{t_n+T}\int_\Omega (|u_n|^2 + |\nabla u_n|^2)dxds\Big\} \\ &= C\Big\{Q_n^2 + \int_0^T\int_\Omega (|v_n|^2 + |\nabla v_n|^2)dxds\Big\} \\ &\le C. \end{aligned}$$

Therefore,

$$||v_n'||, ||\nabla v_n||, ||\Delta v_n|| \le C.$$

Furthermore, using Poincare's inequality we obtain

$$\begin{aligned} \int_\Omega |v_n(x,t)|^2 dx &= \int_\Omega \frac{1}{\lambda_n^2}|u_n(x,t+t_n)|^2 dx \\ &\le C\int_\Omega \frac{1}{\lambda_n^2}|\nabla u_n(x,t+t_n)|^2 dx = C\int_\Omega |\nabla v(x,t)|^2 dx \le C. \end{aligned}$$

Combining the above estimates, we deduce that

$$\{v_n\} \quad \text{is bounded in} \quad W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V). \tag{4.51}$$

In order to take the limit of $\{v_n\}$ we shall first check that

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\rho(x, u'(t+t_n)) = 0 \quad \text{in} \quad L^1([0,T]\times\Omega). \tag{4.52}$$

Indeed, we consider in four cases (This divided cases is the same cases as Proposition 4.3).

For the case $r \ge 0$ and $0 \le p \le \frac{2}{n-2}$ if $n > 2$ and $0 \le p < \infty$ if $n = 1, 2$, by Proposition 4.2 (see case 1) and the definition of $A_1(t)$, we easily check that

$$\int_t^{t+T}\int_\Omega |\rho(x,u')|dxds \le C\Big((\Delta e)^{\frac{1}{r+2}} + (\Delta e)^{\frac{p+1}{p+2}}\Big) \le C(A_1(t) + A_1(t)) = 2CA_1(t).$$

Hence,

$$\frac{1}{\lambda_n}\int_{t_n}^{t_n+T}\int_\Omega |\rho(x,u_n')|dxds \le C \le CQ_n \to 0 \quad \text{as} \quad n\to\infty.$$

The remaining cases are treated similarly. We have proved that

$$\frac{1}{\lambda_n}\rho(x,u'(t+t_n)) \to 0 \quad \text{in} \quad L^1([0,T]\times\Omega).$$

Therefore, (4.52) is proved.

Now, using (4.51) and the Aubin-Lion's Lemma, there exists a function v and a subsequence, still denoted by $\{v_n\}$, such that

$$v_n \to v \quad \text{in} \quad L^2(0,T;H^1(\Omega)),$$

and by (4.50), we have

$$||v||_{L^2(0,T;H^1(\Omega))} = 1. \tag{4.53}$$

Furthermore, from (4.49) we get

$$\int_0^T \int_\omega |v'|^2 dxds = 0.$$

Then according to the previous analysis, the limit function v satisfies
(4.54)

$$\begin{cases} v \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V), \\ v_{tt} + \Delta^2 v - M(||\nabla u||^2)\Delta v - \int_0^t g(t-\tau)\Delta^2 v(\tau)d\tau = 0 \quad \text{in} \quad \Omega \times (0,T), \\ v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T), \\ v_t = 0 \quad \text{in} \quad \omega \times (0,T). \end{cases}$$

So, by Lemma 4.1 we have $v \equiv 0$ in $\Omega \times (0,T)$. This is a contradiction to (4.53). We complete the proof of Proposition 4.4.

□

4.2. Proof of the Theorem 2.2. Combining Proposition 4.3 and 4.4, we have

$$e(t) \le C\Big\{A_i(t)^2 + \int_t^{t+T} \int_\omega |u'|^2 dxds\Big\}, \tag{4.55}$$

where $A_i(t)$, $i = 1,2,3,4$, are given in Proposition 4.3.

Finally, we shall estimate the last term in (4.55) and derive the decay estimates stated in Theorem.

case 1 : $r \ge 0$, $0 \le p \le \frac{2}{n-2}$ and $n > 2$ ($0 \le p < \infty$ if $n = 1,2$) .

By hypothesis on $a(x)$, it follows that (see (4.40))

$$\begin{aligned} &\int_t^{t+T} \int_\omega |u'|^2 dxds \\ &\le \frac{1}{a_0} \int_t^{t+T} \int_\Omega a(x)|u'|^2 dxds \\ &\le C\Big\{\int_t^{t+T} \int_{\Omega_1} a(x)|u'|^2 dxds + \int_t^{t+T} \int_{\Omega_2} a(x)|u'|^2 dxds\Big\} \\ &\le C\Big\{\Big(\int_t^{t+T} \int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{2}{r+2}} + \int_t^{t+T} \int_{\Omega_2} a(x)|u'|^{p+2} dxds\Big\} \\ &\le C\Big\{\Big(\int_t^{t+T} \int_{\Omega_1} \rho(x,u')u' dxds\Big)^{\frac{2}{r+2}} + \int_t^{t+T} \int_{\Omega_2} \rho(x,u')u' dxds\Big\} \\ &\le C\Big((\Delta e)^{\frac{2}{r+2}} + \Delta e\Big). \end{aligned}$$

Hence, we have from (4.55) and the definition of $A_1(t)^2$

$$e(t) \le C\Big\{\Delta e + (\Delta e)^{\frac{2}{r+2}} + (\Delta e)^{\frac{2(p+1)}{p+2}}\Big\} \le C(\Delta e)^{\kappa_1},$$

where $\kappa_1 = \min\Big\{\frac{2}{r+2}, \frac{2(p+1)}{p+2}\Big\} = \frac{2}{r+2}$. Therefore,

$$\sup_{t \le s \le t+T} e(s)^{\frac{1}{\kappa_1}} \le C(e(t) - e(t+T)). \tag{4.56}$$

Applying Lemma 2.2 to (4.56) we conclude

$$e(t) \leq C(1+t)^{-\eta_1} \tag{4.57}$$

with $\eta_1 = \frac{2}{r}$.

case 2 : $r \geq 0$ and $-1 < p < 0$.

$$\begin{aligned}
&\int_t^{t+T}\int_\omega |u'|^2 dxds \\
&\leq C\Big\{\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^2 dxds + \int_t^{t+T}\int_{\Omega_2} a(x)|u'|^2 dxds\Big\} \\
&\leq C\Big\{\Big(\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{2}{r+2}} + \Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2} dxds\Big)^{\frac{1}{p+2}}\Big\} \\
&\leq C\Big((\Delta e)^{\frac{2}{r+2}} + (\Delta e)^{\frac{1}{p+2}}\Big).
\end{aligned}$$

Hence, we have from (4.55) and the definition of $A_2(t)^2$

$$e(t) \leq C\Big\{\Delta e + (\Delta e)^{\frac{2}{r+2}} + (\Delta e)^{\frac{1}{p+2}}\Big\} \leq C(\Delta e)^{\kappa_2},$$

where $\kappa_2 = \min\Big\{\frac{2}{r+2}, \frac{1}{p+2}\Big\}$. Therefore,

$$\sup_{t\leq s\leq t+T} e(s)^{\frac{1}{\kappa_2}} \leq C(e(t) - e(t+T)). \tag{4.58}$$

Applying Lemma 2.2 to (4.58) we conclude

$$e(t) \leq C(1+t)^{-\eta_2} \tag{4.59}$$

with $\eta_2 = \min\Big\{\frac{2}{r}, \frac{1}{p+1}\Big\}$.

case 3 : $-1 < r < 0$, $0 \leq p \leq \frac{2}{n-2}$ and $n > 2$ ($0 \leq p < \infty$ if $n = 2$).

$$\begin{aligned}
&\int_t^{t+T}\int_\omega |u'|^2 dxds \\
&\leq C\Big\{\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^2 dxds + \int_t^{t+T}\int_{\Omega_2} a(x)|u'|^2 dxds\Big\} \\
&\leq C\Big\{\Big(\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{1}{r+2}} + \int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2} dxds\Big\} \\
&\leq C\Big((\Delta e)^{\frac{1}{r+2}} + (\Delta e)\Big).
\end{aligned}$$

Hence, we have from (4.55) and the definition of $A_3(t)^2$

$$e(t) \leq C\Big\{\Delta e + (\Delta e)^{\frac{1}{r+2}} + (\Delta e)^{\frac{2(r+1)}{r+2}} + (\Delta e)^{\frac{2(p+1)}{p+2}}\Big\} \leq C(\Delta e)^{\kappa_3},$$

where $\kappa_3 = \min\Big\{\frac{1}{r+2}, \frac{2(r+1)}{r+2}\Big\}$. Therefore,

$$\sup_{t\leq s\leq t+T} e(s)^{\frac{1}{\kappa_3}} \leq C(e(t) - e(t+T)). \tag{4.60}$$

Applying Lemma 2.2 to (4.60) we conclude

$$e(t) \le C(1+t)^{-\eta_3} \tag{4.61}$$

with $\eta_3 = \min\Big\{\frac{1}{r+1}, \frac{-r}{2(r+1)}\Big\}$.

case 4 : $-1 < r < 0$ and $-1 < p < 0$.

$$\begin{aligned}
&\int_t^{t+T}\int_\omega |u'|^2 dxds \\
&\le C\Big\{\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^2 dxds + \int_t^{t+T}\int_{\Omega_2} a(x)|u'|^2 dxds\Big\} \\
&\le C\Big\{\Big(\int_t^{t+T}\int_{\Omega_1} a(x)|u'|^{r+2} dxds\Big)^{\frac{1}{r+2}} + \Big(\int_t^{t+T}\int_{\Omega_2} a(x)|u'|^{p+2} dxds\Big)^{\frac{1}{p+2}}\Big\} \\
&\le C\Big((\Delta e)^{\frac{1}{r+2}} + (\Delta e)^{\frac{1}{p+2}}\Big).
\end{aligned}$$

Hence, we have from (4.55) and the definition of $A_4(t)^2$

$$e(t) \le C\Big\{\Delta e + (\Delta e)^{\frac{1}{r+2}} + (\Delta e)^{\frac{2(r+1)}{r+2}} + (\Delta e)^{\frac{1}{p+2}}\Big\} \le C(\Delta e)^{\kappa_4},$$

where $\kappa_4 = \min\Big\{\frac{1}{r+2}, \frac{2(r+1)}{r+2}, \frac{1}{p+2}\Big\}$. Therefore,

$$\sup_{t\le s\le t+T} e(s)^{\frac{1}{\kappa_4}} \le C(e(t) - e(t+T)). \tag{4.62}$$

Applying Lemma 2.2 to (4.62) we conclude

$$e(t) \le C(1+t)^{-\eta_4} \tag{4.63}$$

with $\eta_4 = \min\Big\{\frac{1}{r+1}, \frac{-r}{2(r+1)}, \frac{1}{p+1}\Big\}$.

Now, the proof of the Theorem 2.2 is complete.

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