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Existence and multiplicity of solutions to elliptic equations of fourth order on compact manifolds.

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Abstract. This paper deals with a fourth order elliptic equation on compact Riemannian manifolds, the function f involved in the nonlinearity is of changing sign which makes the analysis more difficult than the case where f is of constant sign.We prove the multiplicity of solutions in the subcritical case which is the subject of the first theorem. In the second one we establish the existence of solutions to the equation with critical Sobolev growth.

Contents

1. Introduction

Let (M, g) be a Riemannian compact smooth n- manifold $n \geq 5$ with the metric g, we let $H_2^2(M)$ be the standard Sobolev space which is the completion of the space

$$C_2^2(M) = \left\{ u \in C^\infty(M) \colon \ \|u\|_{2,2} < +\infty \right\}$$

with respect to the norm $\|u\|_{2,2} = \sum_{l=0}^{2} \left\|\nabla^l u\right\|_2$.

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Let H_2 be the space H_2^2 endowed with the equivalent norm

$$\|u\|_{H_2} = \left(\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2 \right)^{\frac{1}{2}}.$$

where, $\Delta(u) = -div(\nabla u)$, denotes the Riemannian Laplacian.

First we establish the existence of at least two solutions of the subcritical equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)\,|u|^{q-2}\,u \tag{1.1}$$

where $2 < q < N$. Next we investigate solutions of the critical equation

$$\Delta^2 u + \nabla^i(a(x)\nabla_i u) + h(x)u = f(x)\,|u|^{N-2}\,u \tag{1.2}$$

where a, h and f are smooth functions on M and $N = \frac{2n}{n-4}$ is the critical exponent.

The function f involved in the nonlinearity is of changing sign which makes the analysis more difficult than the case where f is of constant sign.

The equation (1.1) has a geometric roots, in fact while the conformal Laplacian

$$L_g(u) = \Delta u + \frac{n-2}{4(n-1)} R u$$

where R is the scalar curvature of the metric g, is associated to the scalar curvature; the Paneitz operator as discovered by Paneitz ([**10**]) on 4-dimension manifolds and extended by Branson ([**3**]) to higher dimensions ($n \geq 5$) reads as

$$PB_g(u) = \Delta^2 u + div(-\frac{(n-2)^2+4}{2(n-1)(n-2)} R.g + \frac{4}{n-2} Ric) du + \frac{n-4}{2} Q^n u$$

where Ric is the Ricci curvature of g and where

$$Q^n = \frac{1}{2(n-1)} \Delta R + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2 - \frac{2}{(n-2)^2} |Ric|^2$$

is associated to the notion of Q -curvature, good references on the subject are Chang ([**5**]) and Chang-Yang ([**6**]). When the manifold (M, g) is Einstein, the Paneitz-Branson operator has constant coefficients. It expresses as

$$PB_g = \Delta^2 u + \alpha \Delta u + a u$$

with

$$\alpha = \frac{n^2 - 2n - 4}{2n(n-1)} R \quad \text{and} \quad a = \frac{(n-4)(n^2-4)}{16n(n-1)^2} R^2$$

and this operator is a special case of what it is usually referred as a Paneitz- Branson type operator with constant coefficients.

Since 1990 many results have been established for precise functions a, h and f. D.E. Edmunds, D. Fortunato, E. Jannelli ([**8**]) proved for $n \geq 8$ that if $\lambda \in (0, \lambda_1)$, with λ_1 is the first eigenvalue of Δ^2 on the euclidean open ball B, the problem

$$\begin{cases} \Delta^2 u - \lambda u = u\,|u|^{\frac{8}{n-4}} \text{ in } B \\ u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B \end{cases}$$

has a non trivial solution.

In 1995, R. Van der Vorst ([**12**]) obtained the same results as D.E. Edmunds, D. Fortunato, E. Jannelli. when applied to the problem

$$\begin{cases} \Delta^2 u - \lambda u = u\,|u|^{\frac{8}{n-4}} \text{ in } \Omega \\ u = \Delta u = 0 \text{ on } \partial\Omega \end{cases}$$

where Ω is an open bounded set of R^n and moreover he showed that the solution is positive

In ([**7**]) D.Caraffa studied the equation (1.1) on compact manifolds in the case $f(x)$ =constant; and in the particular case where the functions $a(x)$ and $h(x)$ are precise constants she obtained the existence of positive regular solutions.

In the case of second order equation related to the prescribed scalar curvature, that is

$$\Delta u + \frac{n-2}{4(n-1)} R u = f u^{2^*-1} \tag{1.3}$$

where $2^* = \frac{2n}{n-2}$, A. Rauzy [**11**] stated, in the case where the scalar curvature R of the manifold (M,g) is a negative constant and f is a changing sign function, the following results.

Let f be a C^∞ function on M, $f^- = -\inf(f,0)$, $f^+ = \sup(f,0)$ and

$$\lambda_f = \inf_{u \in A} \frac{\int_M |\nabla u|^2 \, dv_g}{\int_M u^2 dvg}$$

where $A = \left\{ u \in H_1^2(M), u \geq 0, \;\; u \not\equiv 0 \text{ s.t. } \int_M f^- u dv_g = 0 \right\}$, and $\lambda_f = +\infty$ if $A = \phi$.

THEOREM 1. *Let (M,g) be a smooth manifold with constant scalar curvature $R < 0$ and let f be a smooth changing sign function on M. Suppose that there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int_M f^- dv_g}$ such that if f fulfills the following conditions*

(1) $|R| < \frac{4(n-1)}{n-2} \lambda_f$
(2) $\frac{\sup f^+}{\int f^- dv_g} < C.$

Then, the equation (1.3) admits a positive solution.

THEOREM 2. *Let (M,g) be a smooth manifold with constant scalar curvature $R < 0$ and let f be a smooth changing sign function on M. Suppose that there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int_M f^- dv_g}$ such that if f fulfills the following conditions*

(1) $|R| < \frac{4(n-1)}{n-2} \lambda_f$
(2) $\frac{\sup f^+}{\int f^- dv_g} < C$
(3) $\sup_M f > 0.$

Then the subcritical equation $\Delta_g u + Ru = f u^{q-1}$, $q \in \,]2, 2^*[$ admits two nontrivial distinct solutions.

More recently [**2**] the authors have extended the work of Rauzy to the case of the so called generalized prescribed scalar curvature type equation

$$\Delta_p u + a u^{p-1} = f u^{p^*-1} \tag{1.4}$$

where $p^* = \frac{np}{n-p}$, $\Delta_p u = -div(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator on a compact manifold M of dimension $n \geq 3$, with negative scalar curvature, $p \in (1, n)$, $u \in H_1^p(M)$ is a positive function, f is a changing sign function and a is a negative constant. Let

$$\lambda_f = \inf_{u \in A} \frac{\int_M |\nabla u|^p \, dv_g}{\int_M u^p dvg}$$

where $A = \left\{u \in H_1^p(M), u \geq 0, \, u \not\equiv 0 \text{ s.t. } \int_M f^- u dv_g = 0\right\}$, and $\lambda_f = +\infty$ if $A = \phi$

Theorem 3. *(Critical case) There is a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if $f \in C^\infty$ on M fulfills the following conditions*

(1) $|a| < \lambda_f$

(2) $\left(supf^+/\int f^- dv_g\right) < C$.

Then the equation (1.4) has a positive solution of class $C^{1,\alpha}(M)$.

Theorem 4. *(Subcritical case) For every C^∞- function on M there is a constant $C > 0$ which depends only on $f^-/(\int f^- dv_g)$ such that if f fulfills the following conditions*

(1) $|a| < \lambda_f$

(2) $\left(supf^+/\int f^- dv_g\right) < C$

(3) $\sup f > 0$.

Then the subcritical equation

$$\Delta_p u + a u^{p-1} = f u^{q-1} \quad q \in \left]p, p^*\right[$$

has at least two non trivial positive solutions of class $C^{1,\alpha}(M)$.

For a, f , C^∞ -functions M, we let

$$\lambda_{a,f} = \inf_{u \in A} \frac{\int_M (\Delta u)^2 dv_g - \int_M a \, |\nabla u|^2 \, dv_g}{\int_M u^2 dvg}$$

where $A = \left\{u \in H_2, \, u \geq 0, \, u \not\equiv 0 \text{ s. t. } \int_M f^- u dv_g = 0\right\}$, and

$$\lambda_{a,f} = +\infty \quad \text{if} \quad A = \phi.$$

Let h be a smooth negative function on M, we consider the functional F_q defined on H_2 by

$$F_q(u) = \|\Delta u\|_2^2 - \int_M a \, |\nabla u|^2 \, dv_g + \int_M h u^2 dv_g - \int_M f \, |u|^q \, dv_g, \qquad q \in (2, N] \, .$$

In the case of fourth order elliptic equations on manifolds with changing sign right hand side, no work is done at least I know off. While we borrow ideas from the paper of Rauzy ([**12**]), our method is not an adaptation of that of Rauzy, since the behavior of fourth order operators differs from that of second order ones. It is essentially due to the structures of the spaces $H_1^2(M)$ and $H_2^2(M)$: indeed if $u \in H_1^2(M)$ so does $|u|$ and the gradient of $|u|$ satisfies $|\nabla |u|| = |\nabla u|$ and also the analysis on $H_2^2(M)$ is more complicated than on $H_1^2(M)$. In this paper we state the following results

Theorem 5. *Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- dv_g > 0$, there exists a constant $C > 0$ which depends* *only on* $\frac{f^-}{\int f^- dv_g}$ *such that if* f *satisfies the following conditions*

(1) $|h(x)| < \lambda_{a,f}$ *for any* $x \in M$
(2) $\frac{\sup f^+}{\int f^- dv_g} < C$
(3) $\sup_M f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i (a \nabla_i u) + hu = f\, |u|^{q-2}\, u, \qquad q \in]2, N[$$

has at least two distinct solutions u *and* v *satisfying* $F_q(u) < 0 < F_q(v)$ *and of class* $C^{4,\alpha}$, *for some* $\alpha \in (0,1)$.

Theorem 6. *Let* a, h *be* C^∞ *functions on* M *with* h *negative. For every* C^∞ *function* f *on* M *with* $\int_M f^- dv_g > 0$ *there exists a constant* $C > 0$ *which depends only on* $\frac{f^-}{\int f^- dv_g}$ *such that if* f *satisfies the following conditions*

(1) $|h(x)| < \lambda_{a,f}$ *for any* $x \in M$
(2) $\frac{\sup f^+}{\int f^- dv_g} < C$

the critical equation

$$\Delta^2 u + \nabla^i (a \nabla_i u) + hu = f\, |u|^{N-2}\, u$$

has a solution of class $C^{4,\alpha}$, *for some* $\alpha \in (0,1)$, *with negative energy.*

To have applications to conformal geometry, we must obtain positive solutions but this is a difficult problem because of the lack of a maximum principle, This will be treated in a separated work.

If the set $A = \phi$, the condition (1) of Theorem 5 and 6 is fulfilled.

Suppose that $A \neq \phi$ and let $\mu = \inf_{u \in A} \frac{\int_M |\nabla u| dv_g}{\int_M u^2 dv_g}$.

Remark 1. *We get smooth functions for which we have solutions by observing that* $\int_{\{x \in M \,:\, f(x) \geqslant 0\}} dv_g < \left(K_2^2 + \epsilon\right) \|h\|_\infty + A_2(\epsilon) + \mu \|a\|_\infty$ *implies* $\lambda_{a,f} > \|h\|_\infty$ *(See Lemma 2) where* ϵ *is any positive real number and* K_2, $A_2(\epsilon)$ *are the constants of the Sobolev inequality given by Lemma 1.*

Let $B_{k,q} = \left\{ u \in H_2 : \|u\|_q^q = k \right\}$, where $\|\|_q$ denotes the L^q-norm, and put $\mu_{k,q} = \inf_{u \in B_{k,q}} F_q(u)$. The method used in this paper consists in the case of Theorem 5, to show that the curve $k \to \mu_{k,q}$ is continuous as a function of the argument k, starts at 0 goes by a relative negative minimum, which is attained, and takes positive values for k in some interval l_q and finally goes to $-\infty$, to do so many a priori estimates are given, then we deduce the existence of two solutions of the subcritical equation, one of negative energy and the other of positive energy. For the proof of Thorem 6, we show that the sequence of solutions of the subcritical equations, with negative energies,obtained in Theorem 5 is bounded in H_2 as q tends to $N = \frac{2n}{n-4}$, the critical Sobolev exponent. By classical arguments, we show that up to a subsequence u_q converges weakly to a solution u of the critical equation. After, we show that u is of negative energy i.e. $u \not\equiv 0$.

2. Preliminaries

Let a, h be C^{∞} functions on M with h negative. We suppose without lost of generality that the Riemannian manifold (M,g) is of volume equals to 1. Since it is equivalent to solve the equation (1.1) with f or αf (α a real number $\neq 0$), we consider the functional F_q defined on H_2 by

$$F_q(u) = \|\Delta u\|_2^2 - \int_M a\,|\nabla u|^2\,dv_g + \int_M hu^2 dv_g - \int_M f\,|u|^q\,dv_g, \qquad q\in(2,N)$$

and set

$$B_{k,q} = \left\{u \in H_2(M),\ \ \|u\|_q^q = k\right\}$$

where k is some constant. Let

$$\mu_{k,q} = \inf_{u\in B_{k,q}} F_q(u),$$

we state

Proposition 1. *The infimum $\mu_{k,q}$ is achieved. Futhermore any minimizer of the functional F_q is of class $C^{4,\alpha}$, $\alpha\in(0,1)$.*

Proof. We have

$$F_q(u) \geq \|\Delta u\|_2^2 - \|a_+\|_\infty \|\nabla u\|_2^2 + k^{\frac{2}{q}} \min_{x\in M} h(x) - k\max_{x\in M} f(x). \tag{2.1}$$

where $a_+(x) = \max[a(x),0]$ and $\|.\|_\infty$ is the supremum norm.

The following formula is well known on compact manifolds

$$\left\|\nabla^2 u\right\|_2^2 \leq \|\Delta u\|_2^2 - \int_M Ric_{ij}\nabla u_i \nabla u_j dv_g$$

$$\leq \|\Delta u\|_2^2 + \beta\,\|\nabla u\|_2^2\,. \tag{2.2}$$

where β is some constant. As it is shown in ([**1**] p.93), for any $\eta>0$, there exists a constant $C(\eta)$ depending on η such that

$$\|\nabla u\|_2^2 \leq \eta\left\|\nabla^2 u\right\|_2^2 + C(\eta)\,\|u\|_2^2 \tag{2.3}$$

Plugging (2.2) in (2.3), we get

$$\|\nabla u\|_2^2 \leq \eta\,\|\Delta u\|_2^2 + \eta\beta\,\|\nabla u\|_2^2 + C(\eta)\,\|u\|_2^2 \tag{2.4}$$

and choosing η such that $\eta\beta\leq\frac{1}{2}$, we obtain

$$\|\nabla u\|_2^2 \leq 2\eta\,\|\Delta u\|_2^2 + 2C(\eta)\,\|u\|_2^2\,. \tag{2.5}$$

The inequality (2.1) reads then

$$F_q(u) \geq \|\Delta u\|_2^2\,(1-2\eta\,\|a_+\|_\infty)$$
$$+k^{\frac{2}{q}}\left(\min_{x\in M} h(x) - 2C(\eta)\,\|a_+\|_\infty\right) - k\max_{x\in M} f(x)$$

and then, with η small enough, we have

$$1-2\eta\,\|a_+\|_\infty = \alpha > 0$$

so

$$F_q(u) \geq \alpha\,\|\Delta u\|_2^2 + C_1 \tag{2.6}$$

where α is some positive constant and C_1 is a constant independent of u. Let (u_j) be a minimizing sequence of the functional F_q in $B_{k,q}$; so for j sufficiently large $F_q(u_j) \leq \mu_{k,q} + 1$ and by (2.6), we get

$$\|\Delta u_j\|_2^2 \leq \frac{1}{\alpha}\left(\mu_{k,q} + 1 - C_1\right).$$

By formula (2.5) and the fact

$$\|u_j\|_2^2 \leq k^{\frac{2}{q}},$$

we obtain that $\|\nabla u_j\|_2^2$ is bounded. It follows that the sequence (u_j) is bounded in H_2. Consequently u_j converges weakly in H_2, the compact embedding of H_2 in L^q and the unicity of the weak limit allow us to claim that there is a subsequence of (u_j) still denoted (u_j) such that

$$u_j \to u \text{ strongly in } L^s \quad \text{for any } s < N$$

$$\nabla u_j \to \nabla u \text{ strongly in } L^2$$

and

$$\|u\|_{H_2} \leq \liminf_j \|u_j\|_{H_2} \ .$$

Consequently

$$F_q(u) = \|\Delta u\|_2^2 - \int_M a\left|\nabla u\right|^2 dv_g + \int_M hu^2 dv_g - \int_M f\left|u\right|^q dv_g$$

$$\leq \liminf_j \|\Delta u_j\|_2^2 - \lim_j \int_M a\left|\nabla u_j\right|^2 dv_g + \lim_j \int_M hu_j^2 dv_g - \lim_j \int_M f\left|u_J\right|^q dv_g$$

$$= \lim_J F_q(u_j) = \mu_{k,q}$$

and since clearly

$$\|u\|_q^q = k$$

we obtain that

$$F_q(u) = \mu_{k,q} \ .$$

So u fulfills

$$\int_M \Delta u.\Delta v dv_g - \int_M a(x)\nabla^i u.\nabla_i v dv_g + \int_M h(x) uv dv_g$$

$$-\frac{q}{2}\int_M f(x)\left|u\right|^{q-2} uv dv_g = \lambda_{k,q}\int_M \left|u\right|^{q-2} uv dv_g$$

for any $v \in H_2$; where $\lambda_{k,q}$ is the Lagrange multiplier and u is a weak solution of the equation

$$\Delta^2 u + \nabla^i(a\nabla_i u) + hu = \left(\lambda_{k,q} + \frac{q}{2}f\right)\left|u\right|^{q-2} u. \tag{2.7}$$

Using the bootstrap method, we show that $u \in L^s(M)$ for any $s < N$, so $P(u) = \Delta^2 u + \nabla^i(a\nabla_i u) + hu \in L^s(M)$ for any $s < N$ and since P is a fourth order elliptic operator, it follows by a well known regularity theorem that $P(u) \in C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$. Then $u \in C^{4,\alpha}(M)$. □

PROPOSITION 2. *$\mu_{k,q}$ is continuous as a function of the argument k .*

Proof. For any k , $l \in R^{+}$, let u and v be two functions of norm 1 in L^{q} such that $F_q(k^{\frac{1}{q}}u) = \mu_{k,q}$ and $F_q(l^{\frac{1}{q}}v) = \mu_{l,q}$.

Then

$$\begin{aligned}\mu_{l,q} - \mu_{k,q} &= F_q(l^{\frac{1}{q}}v) - F_q(k^{\frac{1}{q}}v) + F_q(k^{\frac{1}{q}}v) - \mu_{k,q} \\ &= F_q(k^{\frac{1}{q}}v) - \mu_{k,q} \\ &+(l^{\frac{2}{q}} - k^{\frac{2}{q}})\left(\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g + \int_M hv^2 dv_g\right) \\ &-(l-k)\int_M f\,|v|^q\,dv_g.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mu_{l,q} = F_q(l^{\frac{1}{q}}v) &= l^{\frac{2}{q}}\left(\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g + \int_M hv^2 dv_g\right) - l\int_M f\,|v|^q\,dv_g \\ &\leq F_q(l^{\frac{1}{q}}) = l^{\frac{2}{q}}\int_M h dv_g - l\int_M f dv_g\end{aligned}$$

i.e.

$$\begin{aligned}\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g + \int_M hv^2 dv_g \leq \\ \int_M h dv_g - l^{1-\frac{2}{q}}\int_M f dv_g + l^{1-\frac{2}{q}}\int_M f\,|v|^q\,dv_g.\end{aligned}$$

Since $\|v\|_q^q = 1$, it follows that the term $\int_M f\,|v|^q\,dv_g$ is bounded for any l in a neighborhood of k and so the term $\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g + \int_M hv^2 dv_g$ is upper bounded. Also since $\mu_{l,q}$ is lower bounded, it follows that $\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g + \int_M hv^2 dv_g$ is bounded in a neighborhood of k.

Consequently

$$\lim_{l\to k}\inf(\mu_{l,q} - \mu_{k,q}) \geq \lim_{l\to k}\inf\left(F_q(k^{\frac{1}{q}}v) - \mu_{k,q}\right)$$

and by the definition of $\mu_{k,q}$, we get

$$\lim_{l\to k}\inf(\mu_{l,q} - \mu_{k,q}) \geq 0\ . \tag{2.8}$$

By writing

$$\begin{aligned}\mu_{l,q} - \mu_{k,q} &= \mu_{l,q} - F_q(l^{\frac{1}{q}}u) + F_q(l^{\frac{1}{q}}u) - F_q(k^{\frac{1}{q}}u) \\ &= \mu_{l,q} - F_q(l^{\frac{1}{q}}u) \\ &+(l^{\frac{2}{q}} - k^{\frac{2}{q}})\left(\|\Delta u\|_2^2 - \int_M a\,|\nabla u|^2\,dv_g + \int_M hu^2 dv_g\right) \\ &-(l-k)\int_M f\,|u|^q\,dv_g\end{aligned}$$

we get

$$\lim_{l\to k}\sup(\mu_{l,q} - \mu_{k,q}) \leq 0$$

and taking into account of (2.8), we obtain

$$\lim_{l\to k}\mu_{l,q} = \mu_{k,q}\ .$$

□

3. A priori estimates

First, we quote the following Lemma due to Djadli-Hebey-Ledoux and improved by Hebey [**9**].

LEMMA 1. *Let M be a Riemannian compact manifold with dimension $n \geq 5$. For any $\epsilon > 0$ there is a constant $A_2(\epsilon)$ such that for any $u \in H_2$ $\|u\|_N^2 \leq K_2^2(1+\epsilon)\|\Delta u\|_2^2 + A_2(\epsilon)\|u\|_2^2$ with $K_2^{-2} = \pi^2 n(n-4)(n^2-4)\Gamma\left(\frac{n}{2}\right)^{\frac{4}{n}}\Gamma(n)^{-\frac{4}{n}}$.*

Suppose that the set $A = \left\{u \in H_2,\ u \not\equiv 0 \text{ s. t. } \int_M f^- |u|\, dv_g = 0\right\}$ is non empty.

LEMMA 2. *If $\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g$ as a function of the variable f tends to 0, $\lambda_{a,f}$ goes to $+\infty$. In particular the condition $\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g < K_2^2(1+\epsilon)\|h\|_\infty + A_2(\epsilon) + \mu\|a\|_\infty$ implies that $\lambda_{a,f} > \|h\|_\infty$.*

PROOF. For any $u \in A$, we obtain by applying successively the Hölder inequality and the Sobolev one given by Lemma 1,

$$\int_{\{x \in M\ :\ f(x) \geqslant 0\}} u^2 dv_g \leq \left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} |u|^N dv_g\right)^{\frac{2}{N}} \left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g\right)^{1-\frac{2}{N}}$$

$$= \left(\int_M |u|^N dv_g\right)^{\frac{2}{N}} \left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g\right)^{\frac{4}{n}}$$

$$\leq \left(K_2^2(1+\epsilon)\|\Delta u\|_2^2 + A_2(\epsilon)\|u\|_2^2\right) \left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g\right)^{\frac{4}{n}}.$$

So

$$\left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g\right)^{-\frac{4}{n}} \leq K_2^2(1+\epsilon)\lambda_{a,f} + A_2(\epsilon) + \inf_{x \in A} \frac{\int_M a(x)|\nabla u|^2 dv_g}{\|u\|_2^2}$$

and letting $\mu = \inf_{x \in A} \frac{\int_M |\nabla u|^2 dv_g}{\|u\|_2^2}$, we get that

$$\lambda_{a,f} \geqslant \frac{1}{K_2^2(1+\epsilon)} \left(\left(\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g\right)^{-\frac{4}{n}} - A_2(\epsilon) - \mu\|a\|_\infty\right)$$

where $\|a\|_\infty = \sup_{x \in M}|a(x)|$.

Hence if $\int_{\{x \in M\ :\ f(x) \geqslant 0\}} dv_g$ tends to 0 as a function of the variable f , $\lambda_{a,f}$ goes to $+\infty$. □

Denote also by $\|h\|_\infty = \sup_{x \in M}|h(x)|$ the supremum norm.

As in [**11**], we define the quantities,

$$\lambda_{a,f,\eta,q} = \inf_{u \in A(\eta,q)} \frac{\|\Delta u\|_2^2 - \int_M a|\nabla u|^2 dvg}{\|u\|_2^2}$$

with

$$A(\eta,q) = \left\{u \in H_2 :\ \ \|u\|_q = 1,\ \int_M f^-|u|^q dv_g = \eta \int_M f^- dv_g\right\}$$

for a real $\eta > 0$,

and

$$\lambda'_{a,f,\eta,q} = \inf_{u \in A'(\eta,q)} \frac{\|\Delta u\|_2^2 - \int_M a\,|\nabla u|^2\,dvg}{\|u\|_2^2}$$

where

$$A'(\eta,q) = \left\{ u \in H_2 : \ \|u\|_q^q = 1, \ \int_M f^- \,|u|^q \,dv_g \le \eta \int_M f^- dv_g \right\}.$$

Now, we will study $\lambda_{a,f,\eta,q}$, to do so, we distinguish (as it is done in [**11**]) the case where the set $\{x \in M : f(x) \ge 0\}$ is of positive measure with respect to Riemannian measure and the case where the set is negligible and $\sup_{x\in M} f = 0$.

Case $f^+ > 0$.

Claim 1. *For any real $\eta > 0$, the set $A(\eta,q)$ is not empty .*

Indeed, the set $A'(\eta,q)$ is not empty since it includes the set of functions $u \in H_2$ such that $\|u\|_q = 1$ and with supports in the set $\left\{x \in M : f^-(x) < \eta \int_M f^- dv_g\right\}$. The same arguments as in [**11**] show that $\lambda'_{a,f,\eta,q}$ is achieved by a function $v \in A'(\eta,q)$ and moreover v satisfies $\int_M f^- \,|u|^q \,dv_g = \eta \int_M f^- dv_g$.

The following facts which are proved in [**11**], for the Laplacian operator remain valid in the case of the bi-Laplacian operator: *$\lambda'_{a,f,\eta,q}$ is a decreasing function with respect to η, bounded by $\lambda_{a,f}$ and $\lambda_{a,f,\eta,q} = \lambda'_{a,f,\eta,q}$*, so $\lambda_{a,f,\eta,q}$ is also a decreasing function with respect to η, and bounded by $\lambda_{a,f}$.

Lemma 3. *For any $q \in \,]2,N[\,$, $\lambda_{a,f,\eta,q}$ goes to $\lambda_{a,f}$ whenever η goes to zero.*

Proof. $\lambda_{a,f,\eta,q}$ is attained by a family of functions labelled $v_{\eta,q}$. The functions $v_{\eta,q}$ indexed by η are bounded in H_2^2: since

$$\|v_{\eta,q}\|_2^2 \le \|v_{\eta,q}\|_q^2 \,Vol(M)^{1-\frac{2}{q}} = 1$$

and

$$\|\Delta v_{\eta,q}\|_2^2 - \|a_+\|_\infty \|\nabla v_{\eta,q}\|_2^2 \le \lambda_{a,f,\eta,q} \|v_{\eta,q}\|_2^2$$
$$\le \lambda_{a,f} \|v_{\eta,q}\|_2^2 \le \lambda_{a,f}.$$

By formula (2.5), for a well chosen $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that

$$\|\nabla v_{\eta,q}\|_2^2 \le 2\varepsilon \|\Delta v_{\eta,q}\|_2^2 + 2C(\varepsilon) \|v_{\eta,q}\|_2^2$$

so

$$\|\Delta v_{q,\eta}\|_2^2 \le \lambda_{a,f} + \|a_+\|_\infty \|\nabla v_{n,q}\|_2^2$$
$$\le \lambda_{a,f} + 2\|a_+\|_\infty \left(\varepsilon \|\Delta v_{\eta,q}\|_2^2 + C(\varepsilon)\|v_{\eta,q}\|_2^2\right)$$

and

$$\|\Delta v_{q,\eta}\|_2^2 (1 - 2\varepsilon \|a_+\|_\infty) \le \lambda_{a,f} + 2\|a_+\|_\infty C(\varepsilon).$$

By choosing $\varepsilon > 0$ small enough such that

$$1 - 2\varepsilon \|a_+\|_\infty > 0$$

we get that

$$\|\Delta v_{q,\eta}\|_2^2 \le C'(\lambda_{a,f}, \|a_+\|_\infty, \varepsilon)$$

where $C'(\lambda_{a,f}, \|a_+\|_\infty, \varepsilon)$ is a constant depending of $\lambda_{a,f}, \|a_+\|_\infty, \varepsilon$.

$$\|\nabla v_{q,\eta}\|_2^2 \le 2\varepsilon C(\lambda_{a,f}, \|a_+\|_\infty, \varepsilon) + 2C(\varepsilon) \le C'(\lambda_{a,f}, \|a_+\|_\infty, \varepsilon).$$

Consequently the sequence $(v_{q,\eta})_\eta$ is bounded in H_2 and we have

$$v_{q\eta} \longrightarrow v_q \text{ weakly in } H_2.$$
$$v_{q\eta} \longrightarrow v_q \text{ strongly in } H_r^2, \qquad r = 0, 1$$
$$v_{q\eta} \longrightarrow v_q \text{ strongly in } L^q$$

and

$$\|\Delta v_q\|_2^2 \leq \lim_{\eta \longrightarrow 0} \inf \|\Delta v_{q\eta}\|_2^2$$

Also

$$\|v_q\|_q = 1.$$

On the other hand

$$\int_M f^- |v_{q\eta}|^q \, dv_g = \eta \int_M f^- dv_g$$

so

$$\int_M f^- |v_q|^q \, dv_g = 0.$$

Hence

$$v_q \in A$$

and

$$\|v_q\|_2^2 \lambda_{a,f} \leq \|\Delta v_q\|_2^2 - \int_M a |\nabla v_q|^2 \, dv_g$$
$$\leq \lim_{\eta \longrightarrow 0} \inf \left(\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 \, dv_g \right) = \lim_{\eta \longrightarrow 0} \inf \|v_{q\eta}\|_2^2 (\lambda_{a,f,q,\eta})$$

and since by construction

$$\lambda_{a,f} \geq \lambda_{a,f,q,\eta}$$

we get that

$$\lim_{\eta \longrightarrow 0} \lambda_{a,f,q,\eta} = \lambda_{a,f}.$$

□

LEMMA 4. *Let $\varepsilon > 0$, there exists η_o such that for any $\eta < \eta_o$, there is q_η such that $\lambda_{a,f,q,\eta} \geq \lambda_{a,f} - \varepsilon$ for any $q > q_\eta$.*

PROOF. We proceed by contradiction. Suppose that there is a $\varepsilon_o > 0$, such that for any η there exists an $\eta_o < \eta$ and for any $q_{\eta o}$ there is $q > q_{\eta o}$ with $\lambda_{a,f,q,\eta} < \lambda_f - \varepsilon$. If $v_{q\eta}$ is the function in H_2 which achieves $\lambda_{a,f,q,\eta}$, then

$$\lambda_{a,f,q,\eta} = \frac{\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 \, dv_g}{\|v_{q\eta}\|_2^2}$$

with $\|v_{q\eta}\|_q^q = 1$. For a convenient η, we choose a sequence q converging to N such that

$$\|\Delta v_{q\eta}\|_2^2 - \int_M a |\nabla v_{q\eta}|^2 \, dv_g < \lambda_{a,f} - \varepsilon_o.$$

By the same argument as in the proof of Lemma 3, we get that the sequence $v_{q\eta}$ indexed by q is bounded in H_2 so up to a subsequence $v_{q\eta}$ converges weakly to v_η in H_2 and strongly in H_r^2 , $r = 0, 1$. Also we have

$$\|\Delta v_\eta\|_2^2 \leq \lim_{q \longrightarrow N} \inf \|\Delta v_{q\eta}\|_2^2$$

and by the strong convergence in H_r^2, $r = 0, 1$, we get

$$\|\Delta v_\eta\|_2^2 - \int_M a\,|\nabla v_\eta|^2\,dv_g < (\lambda_{a,f} - \varepsilon_o)\,\|v_\eta\|_2^2 .$$

By the Sobolev inequality given in the Lemma 1 we have for any $\varepsilon_1 > 0$ there is a constant $A(\varepsilon_1) > 0$ such that

$$1 = \|v_{q\eta}\|_q^2 \leq \|v_{q\eta}\|_N^2 \quad (\text{ since the manifold } M \text{ is of volume } 1\ )$$

$$\leq K_2^2\,(1+\varepsilon_1)\,\|\Delta v_{q\eta}\|_2^2 + A(\varepsilon_1)\,\|v_{q\eta}\|_2^2$$

$$\leq \left[K_2^2\,(1+\varepsilon_1)\,\lambda_{a,f} + A(\varepsilon_1)\right]\|v_{q\eta}\|_2^2 + +(K_2^2+\varepsilon_1)\,\|a_+\|_\infty\,\|\nabla v_{q\eta}\|_2^2$$

$$\leq \left[K_2^2\,(1+\varepsilon_1)\,(1+\|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)\right]\|v_{q\eta}\|_{H_1^2}^2 .$$

Consequently

$$\|v_\eta\|_2^2 \geq \frac{1}{[K_2^2\,(1+\varepsilon_1)\,(1+\|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)])}.$$

As in [**11**] we can show that

$$\int_M |v_\eta|^N\,dv_g \leq 1 \quad \text{and} \quad \int_M f^-\,|v_\eta|^N\,dv_g \leq \eta \int_M f^- dv_g.$$

Consider the sequence of η such that for any q_η, there is a $q > q_\eta$ with

$$\lambda_{a,f,q,\eta} \leq \lambda_{a,f} - \varepsilon.$$

Now tending η to 0, if v_η is the sequence corresponding to η previously considered, v_η is bounded in H_2 and

$$\|v_\eta\|_2^2 \geq \frac{1}{[K_2^2\,(1+\varepsilon_1)\,(1+\|a_+\|_\infty)\lambda_{a,f} + A(\varepsilon_1)])}.$$

so v_η converges weakly to $v \neq 0$ in H_2 and strongly to v in H_r^2, $r = 0, 1$ and v satisfies

$$\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g \leq (\lambda_{a,f} - \varepsilon_o)\,\|v\|_2^2 . \tag{3.1}$$

On the other hand

$$0 \leq \int_M f^-\,|v|^N\,dv_g \leq \liminf_{\eta\longrightarrow 0} \int_M f^-\,|v_\eta|^N\,dv_g \leq \lim_{\eta\to 0} \eta \int_M f^- dv_g = 0$$

then $\int_M f^-\,|v|\,dv_g = 0$ and v belongs to the domain A of definition of $\lambda_{a,f}$. Hence

$$\lambda_{a,f} \leq \frac{\|\Delta v\|_2^2 - \int_M a\,|\nabla v|^2\,dv_g}{\int_M |v|^2\,dv_g}.$$

A contradiction with the inequality (3.1) and Lemma 4 is proved. □

Case $f^+ = 0$.

In this case $\lambda_{a,f}$ is not defined so $\lambda_{a,f} = +\infty$. First, we give the lemma equivalent to Lemma 3

Lemma 5. *Let $q \in\,]2, N[$. For any positive constant R, there exists η_o such that for any $\eta < \eta_o$, $\lambda_{a,f,\eta,q} \geq R$.*

PROOF. We argue by contradiction. It is easy to show that $\lambda_{a,f,q,\eta}$ is achieved by a function $v_{q\eta}$ in H_2 with $\|v_{q,\eta}\|_q = 1$. Suppose that there is $\lambda_{a,f,\eta,q}$ bounded when η goes to 0. Then

$$\|\Delta v_{q,\eta}\|_2^2 - \|a_+\|_\infty \|\nabla v_{q,,\eta}\|_2^2 \leq \frac{\|\Delta v_{q,\eta}\|_2^2 - \|a_+\|_\infty \|\nabla v_{q,,\eta}\|_2^2}{\|v_{q,,\eta}\|_2^2}$$
$$\leq \lambda_{a,f,q,\eta} < +\infty.$$

and proceeding as in the proof of Lemma 3 we get that the sequence $v_{q\eta}$ indexed by η is bounded in H_2. Consequently the sequence $v_{q\eta}$ converges weakly to v_q in H_2 and converges strongly to v_q in H_r^2 , $r = 0, 1$, and strongly to v_q in L^q as η goes to 0. $\int_M f^- |v_q|^q \, dv_g = 0$ which implies that $v_q = 0$ almost everywhere and $\|v_q\|_q = 1$ which are in contradiction with each other. □

Now we give an analogue to Lemma 4.

LEMMA 6. *There exists an η_o such that for any $\eta < \eta_o$ there is q_η such that for any $q > q_\eta$ we have $\lambda_{a,f,q,\eta} > \|h\|_\infty$.*

The proof of this lemma is similar to the previous ones so we omit it.

Let $\sigma > 0$, any sufficient small real number, with the previous notations we obtain by using the lemmas quoted above the following

LEMMA 7. *(1) Suppose that $\sup_M f > 0$ and $\|h\|_\infty < \lambda_{a,f}$. There exists η such that $\lambda_{a,f,q,\eta} - \|h\|_\infty = \varepsilon_o > 0$.*

Let $b = \frac{(1-2\sigma\|a_+\|_\infty)\varepsilon_o}{[(\varepsilon_o+\|h\|_\infty+2\|a_+\|_\infty C(\sigma))K_2^2(1+\varepsilon)+(1-2\sigma\|a_+\|_\infty)A(\varepsilon)]}$

$\mu = \inf(b, \|h\|_\infty + 2\|a_+\|_\infty C(\sigma))$ and suppose that

$\frac{\sup_M f}{\int_M f^- dv_g} < \frac{\mu\eta}{8(\|h\|_\infty+2\|a_+\|_\infty C(\sigma))}$, where and K_2^2, $A(\epsilon)$ are the constants appearing in the Sobolev inequality given by Lemma 1. For any $q \in]2, N[$ there exists a non empty interval $I_q \subset R^+$ such that for every $u \in H_2$ with L^q-norm $k^{\frac{1}{q}}$ and $k \in I_q = [k_{1,q}, k_{2,q}]$ we have $F_q(u) \geq \frac{1}{2}\mu k^{\frac{2}{q}}$.

(2) Suppose that $\sup_M f = 0$ and $\|h\|_\infty < \lambda_{a,f}$, there exists an interval $I_q = [k_{1,q}, +\infty[$ such that for any $k \in I_q$ and any $u \in H_2$ with $\|u\|_q^q = k$, we have $F_q(u) \geq \frac{1}{2}\mu k^{\frac{2}{q}}$.

PROOF. *Case:* $f^+ > 0$.

Let $u \in H_2$ such that $\|u\|_q^q = k$.

Putting

$$G_q(u) = \|\Delta u\|_2^2 - \int_M a |\nabla u|^2 \, dv_g + \int_M hu^2 dv_g + \int_M f^- |u|^q \, dv_g,$$

we get

$$G_q(u) \geq \|\Delta u\|_2^2 - \|a_+\|_\infty \|\nabla u\|_2^2 - \|h\|_\infty \|u\|_2^2 + \int_M f^- |u|^q \, dv_g$$

and taking account of (2.5), we obtain that for any suitable real $\sigma > 0$, there is a constant $C(\sigma) > 0$ such that

$$G_q(u) \geq (1 - 2\sigma \|a_+\|_\infty) \|\Delta u\|_2^2$$

$$-\left(\|h\|_\infty + 2C(\sigma)\|a_+\|_\infty\right)\|u\|_2^2 + \int_M f^- |u|^q\, dv_g.$$

So if

$$\int_M f^- |u|^q\, dvg \geq \eta k \int_M f^- dv_g$$

then

$$G_q(u) \geq \left(1 - 2\sigma\|a_+\|_\infty\right)\|\Delta u\|_2^2$$

$$-\left(\|h\|_\infty + 2C(\sigma)\|a_+\|_\infty\right)\|u\|_2^2 + \eta k\int_M f^- dv_g \tag{3.2}$$

with $\sigma > 0$ sufficiently small so that

$$1 - 2\sigma\|a_+\|_\infty > 0.$$

Now since

$$\|u\|_2^2 \leq \|u\|_q^{\frac{2}{q}} Vol(M)^{1-\frac{2}{q}} = k^{\frac{2}{q}}$$

we get

$$G_q(u) \geq k^{\frac{2}{q}}\left[-\left(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)\right) + \eta k^{1-\frac{2}{q}}\int_M f^- dv_g\right]$$

$$\geq k^{\frac{2}{q}}\left(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)\right)\left(\frac{\eta k^{1-\frac{2}{q}}}{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}\int_M f^- dv_g - 1\right)$$

and choosing k such that

$$\frac{\eta k^{1-\frac{2}{q}}}{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}\int_M f^- dv_g - 1 \geq 1$$

that is

$$k \geq \left[2\frac{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}{\eta\int_M f^- dv_g}\right]^{\frac{q}{q-2}}$$

we obtain

$$G_q(u) \geq k^{\frac{2}{q}}\left(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)\right).$$

Let

$$k_{1,q} = \left[2\frac{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}{\eta\int_M f^- dv_g}\right]^{\frac{q}{q-2}}.$$

In the case $\int_M f^- |u|^q\, dv_g < \eta k\int_M f^- dv_g$, we have

$$\|\Delta u\|_2^2 - \int_M a|\nabla u|^2\, dv_g \geq \lambda_{a,f,q,\eta}\|u\|_2^2$$

so

$$G_q(u) \geq \lambda_{a,f,\eta,q}\|u\|_2^2 + \int_M hu^2 dv_g + \int_M f^- |u|^q\, dv_g$$

$$\geq \left(\lambda_{a,f,\eta,q} - \|h\|_\infty\right)\|u\|_2^2 + \int_M f^- |u|^q\, dv_g$$

by Lemma 4 and 6 there exists η such that

$$\lambda_{a,f,\eta,q} - \|h\|_\infty = \varepsilon_o > 0.$$

Now, putting $\delta_1 + \delta_2 = \varepsilon_o$, where δ_1 and δ_2 are positive real numbers, and solving $\|u\|_2^2$ in (3.2), we get

$$\|u\|_2^2 \geq \frac{1}{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}\left[\left(1 - 2\sigma\|a_+\|_\infty\right)\|\Delta u\|_2^2 - G_q(u) + \int_M f^- |u|^q\, dv_g\right].$$

Consequently

$$\left(1+\frac{\delta_2}{\|h\|_\infty+2\|a_+\|_\infty C(\sigma)}\right)G_q(u)\geq\delta_1\|u\|_2^2$$
$$+\frac{\delta_2}{\|h\|_\infty+2\|a_+\|_\infty C(\sigma)}\left(1-2\sigma\|a_+\|_\infty\right)\|\Delta u\|_2^2$$

so

$$G_q(u)\geq\frac{\delta_1\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)}{\|h\|_\infty+2\|a_+\|_\infty C(\sigma)+\delta_2}\|u\|_2^2+\frac{\delta_2\left(1-2\sigma\|a_+\|_\infty\right)}{\|h\|_\infty+2\|a_+\|_\infty C(\sigma)+\delta_2}\|\Delta u\|_2^2$$

and where σ is sufficiently small and such that $1-2\|a_+\|_\infty\sigma>0$.

Or

$$G_q(u)\geq\frac{\delta_2\left(1-2\sigma\|a_+\|_\infty\right)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)+\delta_2\right)\left(K_2^2+\varepsilon\right)}$$
$$\times\left[K_2^2(1+\varepsilon)\|\Delta u\|_2^2+\frac{\delta_1\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)\left(K_2^2+\varepsilon\right)}{\delta_2\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}A(\varepsilon)\|u\|_2^2\right]$$

where for any fixed $\varepsilon>0$, K_2^2 denotes the best Sobolev constant in the embedding of $H_2^2(R^n)$ in $L^q(R^n)$.

Taking δ_1 and δ_2 such that

$$\frac{\delta_1\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)\left(K_2^2+\varepsilon\right)}{\delta_2\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}=1$$

we get

$$\delta_1=\frac{\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}\varepsilon_o$$

and

$$\delta_2=\frac{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)\left(K_2^2+\varepsilon\right)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}\varepsilon_o.$$

Consequently

$$G_q(u)\geq\frac{\delta_2\left(1-2\sigma\|a_+\|_\infty\right)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)+\delta_2\right)\left(K_2^2+\varepsilon\right)}\|u\|_q^2$$

and since

$$\|h\|_\infty+2\|a_+\|_\infty C(\sigma)+\delta_2=\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)$$
$$\times\left[1+\frac{K_2^2(1+\varepsilon)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}\varepsilon_o\right]$$
$$=\frac{\left(\varepsilon_o+\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}{\left(\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)}$$

we get that

$$G_q(u)\geq\frac{\left(1-2\sigma\|a_+\|_\infty\right)\varepsilon_o}{\left[\left(\varepsilon_o+\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)\right]}k^{\frac{2}{q}}.$$

Letting

$$b=\frac{\left(1-2\sigma\|a_+\|_\infty\right)\varepsilon_o}{\left[\left(\varepsilon_o+\|h\|_\infty+2\|a_+\|_\infty C(\sigma)\right)K_2^2(1+\varepsilon)+\left(1-2\sigma\|a_+\|_\infty\right)A(\varepsilon)\right]}$$

we get

$$F_q(u) = G_q(u) - \int_M f^+ |u|^q \, dv_g$$

$$\geq bk^{\frac{2}{q}} - \int_M f^+ |u|^q \, dv_g \geq bk^{\frac{2}{q}} - k \sup f^+ = k^{\frac{2}{q}} (b - k^{1-\frac{2}{q}} \sup f^+).$$

So if $\sup_M f > 0$, let $\mu = \inf(b, \|h\|_\infty + 2\|a_+\|_\infty C(\sigma))$. For any $k \geq k_{1,q}$, we have

$$F_q(u) \geq k^{\frac{2}{q}} (\mu - k^{1-\frac{2}{q}} \sup f)$$

Now if we put $C_q = \frac{\eta}{8\left(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)\right)} \mu$ and suppose that $\sup_M f \leq C_q \int_M f^-$, we obtain that the inequality is fulfilled provided that

$$k \leq \left[\frac{4\left(\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)\right)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}} = 2^{\frac{q}{q-2}} k_{1,q}.$$

and

$$F_q(u) \geq \frac{1}{2} \mu k^{\frac{2}{q}}$$

provided that

$$k \leq \left[\frac{\mu}{2 \sup f} \right]^{\frac{q}{q-2}}.$$

We put

$$k_{2,q} = 2^{\frac{q}{q-2}} k_{1,q}.$$

Case $f^+ = 0$.

In this case, for any $k \geq k_{1,q}$,

$$F_q(u) \geq \frac{1}{2} \mu k^{\frac{2}{q}}.$$

□

4. Subcritical case

First, we show the existence of a solution to the subcritical equation with negative energy.

LEMMA 8. *For each* $t > 0$, *small enough,* $\inf_{\|u\|_{H_2} \leq t} F_q(u) < 0, \quad q \in]2, N]$.

In fact $F_q(t) \leq t^2 \left(h - t^{q-2} \int_M f dv_g\right)$, where $h = \max_M h(x)$, and since $h < 0$, there is $t_o > 0$ small enough such that $\inf_{\|u\|_{H_2} \leq t} F_q(u) < 0$ for each $t \in]0, t_o[$.

PROPOSITION 3. *Let* a, h *be* C^∞ *functions on* M , *with* h *negative. For every* C^∞ *function,* f *on* M *with* $\int_M f^- dv_g > 0$, *there exists a constant* $C > 0$ *which depends only on* $\frac{f^-}{\int f^- dv_g}$ *such that if* f *satisfies the following conditions*

(1) $|h(x)| < \lambda_{a,f}$ *for any* $x \in M$
(2) $\frac{\sup f^+}{\int f^- dv_g} < C$

then the subcritical equation

$$\Delta^2 u_q + \nabla^i (a \nabla_i u_q) + h u_q = f |u_q|^{q-2} u_q \quad \text{with } q \in]2, N[\tag{4.1}$$

admits a $C^{4,\alpha}$, *for some* $\alpha \in (0,1)$, *solution* u_q *with negative energy.*

Proof. For any $q \in]2, N[$ and $k > 0$, let $\mu_{k,q} = \inf_{\|w\|_q^q = k} F_q(w)$. First we remark that if k is close to 0, $k > 0$, $\mu_{k,q} < 0$: indeed

$$\mu_{k,q} \leq F_q(k^{\frac{1}{q}}) = k^{\frac{2}{q}} \left(\int_M h dv_g - k^{1-\frac{2}{q}} \int_M f dv_g \right) < 0.$$

By Proposition 2 the real valued function $k \to \mu_{k,q}$ is continuous and $\mu_{k,q}$ goes to 0 , when $k \to 0$. So by Lemma 7 and 8 the function $k \to \mu_{k,q}$ starts at 0, takes a negative minimum, say at k_q, then takes positive values. Let $l_q = k_{1,q} = \left[2 \frac{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g} \right]^{\frac{q}{q-2}}$ the lower bound of the interval I_q given in the proof of Lemma 7, then

$$\mu_{k_q,q} = \inf_{\|u\|_q^q \leq l_q} F_q(u).$$

By Proposition 1 the infimum $\mu_{kq,q}$ is attained by a function $v_q \in H_2$ with $\|v_q\|_q^q = k_q$, so

$$F_q(v_q) = \inf_{\|u\|_q^q \leq l_q} F_q(u) \ .$$

Now since for any $k_q \in I_q$, and any $u \in H_2$ with $\|u\|_q^q = k_q$, $F_q(u) \geq 0$, it follows that $k_q < l_q$. So v_q is a critical point of F_q , that is for any $\varphi \in H_2$

$$\int_M \Delta v_q \Delta \varphi dv_g - \int_M a \nabla v_q \nabla \varphi dv_g +$$

$$\int_M h v_q \varphi dv_g - \frac{q}{2} \int_M f \left| v_q \right|^{q-2} v_q \varphi dv_g = 0$$

then $u_q = (\frac{q}{2})^{\frac{1}{q-2}} v_q$ is a weak solution of the subcritical equation with negative energy such that

$$\|u_q\|_q^q \leq (\frac{q}{2})^{\frac{q}{q-2}} l_q.$$

Moreover, arguing as in the proof of the Proposition 1, $u_q \in C^{4,\alpha}(M)$ with $\alpha \in (0, 1)$. □

Now we are going to seek a second solution to the subcritical equation with positive energy.

We start by showing that F_q with $q \in]2, N[$ satisfies the Palais-Smale condition.

Lemma 9. *Let c be a real number, then each Palais-Smale sequence at level c for the functional F_q satisfies the Palais -Smale condition.*

Proof. First, we show that each Palais-Smale sequence is bounded: we argue by contradiction. Suppose that there exists a sequence (u_j) such that $F_q(u_j)$ tends to a finite limit c, $F'_q(u_j)$ goes strongly to zero and u_j to infinite in the H_2-norm. More explicitly we have

$$\int_M \left((\Delta u_j)^2 - a \left| \nabla u_j \right|^2 + h u_j^2 \right) dv_g - \int_M f \left| u \right|_j^q dv_g \to c$$

and

$$\int_M \left((\Delta u_j)^2 - a \left| \nabla u_j \right|^2 + h u_j^2 \right) dv_g - \frac{q}{2} \int_M f \left| u \right|_j^{q-1} v dv_g \to 0$$

so for any $\varepsilon > 0$ there exists a positive integer A such that for every $j \geq A$ we have

$$\left| \int_M \left((\Delta u_j)^2 - a \left| \nabla u_j \right|^2 + h u_j^2 \right) dv_g - \int_M f \left| u \right|_j^q dv_g - c \right| \leq \varepsilon$$

and

$$\left|\int_M \left((\Delta u_j)^2 - a\,|\nabla u_j|^2 + hu_j^2\right) dv_g dv_g - \frac{q}{2}\int_M f\,|u|_j^{q-1}\, v dv_g\right| \le \varepsilon.$$

Hence, we get

$$\left|(q-2)\int_M (\Delta u_j)^2 - a\,|\nabla u_j|^2 + hu_j^2 dv_g - qc\right| \le (q+2)\epsilon \tag{4.2}$$

and

$$\left|(q-2)\int_M f\,|u_j|^q - 2c\right| \le 4\varepsilon. \tag{4.3}$$

By Lemma 7, we can choose k to be an L^q- norm such that

$$\inf_{\|u\|_q^q = k} F_q(u) > 0.$$

Letting $v_j = k^{\frac{1}{q}} \frac{u_j}{\|u_j\|_q}$, we obtain from (4.2) and (4.3) that

$$\left|(q-2)\int_M f\,|v_j|^q\, dv_g - \frac{2ck^{\frac{2}{q}}}{\|u_j\|_q^2}\right| \le 4\varepsilon \frac{k^{\frac{2}{q}}}{\|u_j\|_q^2} \tag{4.4}$$

and

$$\left|(q-2)\int_M (\Delta v_j)^2 - a\,|\nabla v_j|^2 + hv_j^2 dv_g - qc\frac{k^{\frac{2}{q}}}{\|u_j\|_q^2}\right|$$

$$\le (q+2)\epsilon \frac{k^{\frac{2}{q}}}{\|u_j\|_q^2}. \tag{4.5}$$

Now since $(\|v_j\|_q)_j$ is a bounded sequence, it follows by (4.5) that (v_j) is bounded in H_2. If $\|u_j\|_q$ goes to infinity, it follows from (4.4) and (4.5) that $F_q(v_j)$ goes to zero. And since $\|v_j\|_q^q = k$, we have

$$\inf_{\|u\|_q^q = k} F_q(u) \le F_q(v_j)$$

so

$$\inf_{\|u\|_q^q = k} F_q(u) \le 0.$$

Hence a contradiction. Then the sequence (u_j) is bounded in H_2. Since $q < N$, the Sobolev injections are compact. Consequently the Palais-Smale condition is satisfied. □

Lemma 10. *Let $u \in H_2$. If the L_q-norm $\|u\|_q^q = k$ goes to $+\infty$, then $\mu_{k,q} = \inf_{\|u\|_q^q = k} F_q(u) \to -\infty$.*

Proof. In fact since $\sup_{x\in M} f(x) > 0$ let u be a function of class C^2 with support contained in the open subset $\{x \in M : f(x) > 0\}$ of the manifold M such that $\|u\|_q^q = 1$, then $\int_M f\,|u|^q\, dv_g > 0$ and

$$F_q(ku) = k^{\frac{2}{q}}\left(\int_M \left((\Delta u)^2 - a\,|\nabla u|^2 + hu^2\right) dv_g - k^{\frac{q-2}{q}}\int_M f\,|u|^q\, dv_g\right).$$

So $\lim_{k\to+\infty} F_q(ku) = -\infty$. □

Proposition 4. *Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions*

(1) $|h(x)| < \lambda_{a,f}$ *for any $x \in M$*
(2) $\frac{\sup f^+}{\int f^-} < C$
(3) $\sup f > 0$,

then the subcritical equation

$$\Delta^2 u + \nabla^i(a\nabla_i u) + hu = f\,|u|^{q-2}\,u, \qquad q \in \,]2, N[$$

admits a nontrivial solution of class $C^{4,\alpha}$, for some $\alpha \in (0,1)$, with positive energy.

Proof. By Lemma 7, 8 and 10 the curve $k \to \mu_{k,q}$ starts at 0, takes a negative minimum, then takes positive maximum and goes to minus infinite. Mimicking which is done in ([**11**]), let l_o be an L^q-norm such that $\mu_{l_o,q}$ is a maximum and l_1, l_2 two L^q-norms such that $\mu_{l_1,q} = \mu_{l_2,q} = 0$ with $l_1 < l_o$ and $l_2 > l_o$.

Set

$$\Gamma = \{\gamma \in C\left([0,1], H_2\right) :\ \gamma(0) = u_{l_1,q}, \gamma(1) = u_{l_2,q}\},$$

where $u_{l_i,q} \in B_{l_i,q}$, $i = 1, 2$, are such that $\mu_{l_i,q} = F_q\left(u_{l_i,q}\right) = \inf_{w \in B_{l_i,q}} F_q\left(w\right)$ and

$$\nu_q = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} F_q\left(\gamma(t)\right) \ .$$

Arguing as in [**11**], we show that ν_q is a critical level of the functional F_q and $\nu_q \geq \mu_{l,q} > 0$. Consequently the subcritical equation (1.2) admits a weak solution of positive energy. This solution is in fact of class $C^{4,\alpha}$ with $\alpha \in (0,1)$. □

Theorem 5 follows from Proposition 3 and 4.

5. Critical case

Now, we are going to investigate solutions of the critical equation.

Theorem 7. *Let a, h be C^∞ functions on M with h negative. For every C^∞ function, f on M with $\int_M f^- > 0$, there exists a constant $C > 0$ which depends only on $\frac{f^-}{\int f^-}$ such that if f satisfies the following conditions*

(1) $|h(x)| < \lambda_{a,f}$ *for any $x \in M$*
(2) $\frac{\sup f^+}{\int f^-} < C$

then the critical equation

$$\Delta^2 u + \nabla^i(a\nabla_i u) + hu = f\,|u|^{N-2}\,u \tag{5.1}$$

admits a $C^{4,\alpha}$, for some $\alpha \in (0,1)$, solution u with negative energy.

Proof. For each $q \in (2, N)$, let u_q be the solution to the subcritical equation (4.1) given by Proposition 3, u_q is of negative energy. We have already shown in the proof of Proposition 3 that

$$\|u_q\|_q^q = k_q \leq l_q = \left[2\frac{\|h\|_\infty + 2\,\|a_+\|_\infty\,C(\sigma)}{\eta\int_M f^- dv_g}\right]^{\frac{q}{q-2}}$$

and since l_q goes to $l_N = \left[2\frac{\|h\|_\infty + 2\|a_+\|_\infty C(\sigma)}{\eta \int_M f^- dv_g}\right]^{\frac{4}{n}}$ as q goes to N, (u_q) is bounded in L^q, so it is in L^2 and since u_q are of negative energy then

$$\|\Delta u_q\|_2^2 \leq \int_M a\,|\nabla u|^2\,dv_g - \int_M h u_q^2 dv_g + \int_M f\,|u_q|^q\,dv_g$$

$$\leq \|a_+\|_\infty \|\nabla u_q\|_2^2 + \|h\|_\infty \|u_q\|_q^2 + \|f\|_\infty \|u_q\|_q^q .$$

Now since for any sufficiently $\sigma > 0$, there exists a constant $C(\sigma)$ such that

$$\|\nabla u_q\|_2^2 \leq 2\sigma \|\Delta u_q\|_2^2 + 2C(\sigma)\|u_q\|_2^2$$

we get

$$(1 - 2\sigma\|a_+\|_\infty)\|\Delta u_q\|_2^2 \leq (2\|a_+\|_\infty C(\sigma) + \|h\|_\infty)\|u_q\|_q^2 + \|f\|_\infty \|u_q\|_q^q$$

$$\leq (2\|a_+\|_\infty C(\sigma) + \|h\|_\infty)\, l_q^{\frac{2}{q}} + \|f\|_\infty l_q.$$

So (u_q) is a bounded sequence in H_2. Consequently $u_q \to v$ weakly in H_2, up to a subsequence, we have

$$u_q \to v \text{ strongly in } L^s(M) \quad \text{for } s < N$$

$$\nabla u_q \to \nabla v \text{ strongly in } L^2$$

$$u_q(x) \to v(x) \quad \text{for a.e. } x \in M.$$

On the other hand for any $q \in]2, N[$, u_q satisfies, for any $\varphi \in H_2$

$$\int_M \Delta u_q \Delta\varphi dv_g - \int_M a\nabla^i u_q \nabla_i \varphi dv_g + \int_M h u_q \varphi dv_g$$

$$= \frac{q}{2}\int_M f\,|u_q|^{q-2}\,u_q\ \varphi dv_g \tag{5.2}$$

and since the convergence of (u_q) is weak in H_2, it follows that for any $\varphi \in H_2$

$$\int_M \Delta u_q \Delta\varphi dv_g - \int_M a\nabla^i u_q \nabla_i \varphi dv_g + \int_M h u_q \varphi dv_g$$

$$\to \int_M \Delta v \Delta\varphi dv_g - \int_M a\nabla^i v \nabla_i \varphi) dv_g + \int_M h v \varphi dv_g\ . \tag{5.3}$$

Moreover since $u_q(x) \to v(x)$ for a.e. $x \in M$ and (u_q) is bounded in H_2 we have

$$u_q(x)\,|u_q(x)|^{q-2} \to v(x)\,|v(x)|^{N-2} \quad \text{for a.e. } x \in M$$

and

$$\left\| u_q\,|u_q|^{q-2} \right\|_{\frac{N}{N-1}} = \|u_q\|_{(q-1)\frac{N}{N-1}}^{q-1} \leq C_1 \|u_q\|_N^{N-1} \leq C\,\|u_q\|_{H_2}^{N-1}.$$

consequently (u_q) is bounded in $L^{\frac{N}{N-1}}$ and by a well known theorem [**1**] u_q converges weakly to v in $L^{\frac{N}{N-1}}$. Now for any $\varphi \in H_2 \subset L^N$, and any smooth function f, $f\varphi \in L^N$ (the dual space of $L^{\frac{N}{N-1}}$), then

$$\int_M f\,|u_q|^{q-2}\,u_q \varphi dv_g \to \int_M f\,|v|^{N-2}\,v\varphi dv_g. \tag{5.4}$$

So by (5.3) and (5.4) $u = \left(\frac{N}{2}\right)^{\frac{1}{N-2}} v$ is a weak solution of the critical equation.

It remains to check that $u \neq 0$. We let

$$\mu_{k_q,q} = \inf_{w \in \bar{B}_{k,q}} F_q(w)$$

where

$$\bar{B}_{k,q} = \left\{ w \in H_2(M) : \|w\|_q^q \leq l_q \right\}.$$

By Proposition 1, $\mu_{k_q,q}$ is attained by by a function $u_q \in H_2(M)$ with $\|u_q\| = k_q \leq l_q$ that is $\mu_{k_q,q} = F_q(u_q)$.

Claim 2. *$\mu_{k_q,q}$ are uniformly lower bounded, as q goes to N.*

Indeed, in one hand we have $\mu_{k_q,q} < 0$ and on the other hand if $\min_{x\in M} a(x) \leq 0$ we obtain

$$\mu_{k_q,q} = F_q(u_q)$$
$$= \|\Delta u_q\|_2^2 - \int_M a |\nabla u_q|^2 dv_g + \int_M h u_q^2 dv_g - \int_M f |u_q|^q dv_g$$
$$\geq \min_{x\in M} h(x) k_q^{\frac{2}{q}} - \max_{x\in M} f^+(x) k_q.$$

Letting

$$C_q = \max(l_q, 1)$$

we get

$$\mu_{k_q,q} \geq \left(\min_{x\in M} h(x) - \max_{x\in M} f^+(x) \right) C_q$$

so

$$\lim_{q\to N} \inf \mu_{k_q,q} \geq \left(\min_{x\in M} h(x) - \max_{x\in M} f^+(x) \right) C_N.$$

In the case $\min_{x\in M} a(x) > 0$, thanks to formula (2.5), we obtain for any sufficiently small $\sigma > 0$

$$\mu_{k_q,q} \geq (1 - \sigma \min_{x\in M} a(x)) \|\Delta u_q\|_2^2 + \left(\min_{x\in M} h(x) + \min_{x\in M} a(x) C(\sigma) - \max_{x\in M} f^+(x) \right) C_q$$

and taking σ small so that $(1 - \sigma \min_{x\in M} a(x)) \geq 0$, we obtain

$$\mu_{k_q,q} \geq \left(\min_{x\in M} h(x) + \min_{x\in M} a(x) C(\sigma) - \max_{x\in M} f^+(x) \right) C_q$$

and $\mu_{k_q,q}$ are lower bounded as $q \to N$.

Claim 3. *Up to a subsequence we have*

$$\lim_{q\to N} \mu_{k_q,q} = \mu_{k_N,N} < 0 \ .$$

For q close to N, we let

$$0 < k < \min\left(l_q, \left[\frac{\left|\int_M h dv_g\right|}{2\int_M f^- dv_g} \right]^{\frac{q}{q-2}} \right).$$

Since

$$\mu_{k_q,q} = \inf_{u \in \bar{B}_{k,q}} F_q(u)$$

with

$$\bar{B}_{k_q,q} = \left\{ u \in H_2 : \|u\|_q^q \leq l_q \right\}$$

we get

$$\mu_{k_q,q} \leq F_q(k^{\frac{1}{q}}) = k^{\frac{2}{q}} \left(\int_M h dv_g + k^{1-\frac{2}{q}} \int_M f^- dv_g \right)$$
$$\leq \frac{1}{2} k^{\frac{2}{q}} \int_M h dv_g$$

hence up to a subsequence

$$\mu_{k_N,N} = \lim_{q \to N} \mu_{k_q,q} \leq \frac{1}{2} k^{\frac{2}{N}} \int_M h dv_g < 0. \tag{5.5}$$

Now, we are in position to show that $u = \left(\frac{N}{2}\right)^{\frac{1}{N-2}} v \neq 0$.

Claim 4. *The weak solution of the critical equation (5.1) is non trivial.*

In fact since u is a solution of the equation (5.1) and the sequence (u_q), of solutions to the subcritical equations, converges weakly to v in H_2, we have

$$\frac{N}{2} \int_M f |v|^N = \left(\|\Delta v\|_2^2 - \int_M a |\nabla v|^2 dv_g + \int_M h v^2 dv_g \right) \tag{5.6}$$
$$\leq \lim \inf_{q \to N} \left(\|\Delta u_q\|_2^2 - \int_M a |\nabla u_q|^2 dv_g + \int_M h u_q^2 dv_g \right)$$
$$= \lim \inf_{q \to N} \left(\frac{2}{q} \int_M f |u_q|^q dv_g \right).$$

The function u_q solution of the subcritical equation achieves the minimum $\mu_{k_q,q} = \inf_{u \in \bar{B}_{k,q}} F_q(u)$, where $\bar{B}_{k_q,q} = \left\{ u \in H_2 : \|u\|_q^q \leq l_q \right\}$.

So

$$\mu_{k_q,q} = F_q(u_q) = \left(\frac{q}{2} - 1 \right) \int_M f |u_q|^q dv_g$$

and taking account of (5.5) and (5.6), we get

$$\int_M f |v|^N dv_g < 0$$

hence

$$u = \left(\frac{N}{2} \right)^{\frac{1}{N-2}} v \neq 0.$$

By the bootstrap method and a method imagined by Vaugon see [**12**], we get that u is of class $C^{4,\alpha}$ for some $\alpha \in (0,1)$. □

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