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Blow up of the solutions of the nonlinear parabolic equation

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Abstract. In this paper the Cauchy problem for nonlinear parabolic equation is investigated. We prove that the Cauchy problem has one nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r) \in \mathcal{C}([0,1)L^2([r_0,\infty)))$ for which $\lim_{t\longrightarrow 1} ||u||_{L^2([r_0,\infty)))} = \infty$, where $r = |x|$, $r_0 \geq 1$ is arbitrary chosen and fixed. Also, we prove that the solution map is not uniformly continuous.

Contents

1. Introduction

In this paper we consider the Cauchy problem

$$(1.1) \quad u_t - \Delta u = f(t,|x|,u), \quad t \in [0,1], \quad |x| \geq r_0, \quad n \geq 2,$$
$$(1.2) \quad u(0,x) = u_0(x) \in L^2(\mathcal{R}^n \backslash \{|x| < r_0\}),$$

where $r_0 \geq 1$ is arbitrary chosen and fixed, $f(t,|x|,u) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([r_0,\infty)) \times \mathcal{C}^1(\mathcal{R}^1)$, $a|u| \leq f'_u(t,|x|,u) \leq b|u|$ for every $t \in [0,1]$, for every $|x| \geq r_0$, a and b are fixed positive constants, $f(t,|x|,0) = 0$ for every $t \in [0,1]$, $\forall |x| \geq r_0$.

We will prove that the Cauchy problem (1.1), (1.2) has a nontrivial solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r) \in L^2([r_0,\infty))$ for every $t \in [0,1]$, for which $\lim_{t\longrightarrow 1} ||u||_{L^2([r_0,\infty))} = \infty$. Also we will prove that the solution map is not uniformly continuous. When we say that the solution map $u_\circ \longrightarrow u(t,r)$ is uniformly

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continuous we mean: *for every positive constant ϵ there exists positive constant δ such that for any two solutions u, v of the Cauchy problem* (1.1), (1.2)*, so that*

$$E(0, u-v) \leq \delta,$$

the following inequality holds

$$E(t, u-v) \leq \epsilon \quad for \quad \forall t \in [0,1],$$

where

$$E(t,u) := ||u(t,\cdot)||^2_{L^2([r_0,\infty))} + \Big|\Big|\frac{\partial}{\partial r}u(\cdot,r)||^2_{L^2([r_0,\infty))}.$$

Here we use the approach which is used in [1], [2], [3], [4]. In the accessible literature there are too many methods for investigations of this problem which are different than the method which we propose in this paper.

Our main result is:

Theorem 1.1. Let $n \geq 2$ is fixed, $r_0 \geq 1$ is fixed, $f(t,|x|,u) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([r_0,\infty)) \times \mathcal{C}^1(\mathcal{R}^1)$, $a|u| \leq f'_u(t,|x|,u) \leq b|u|$ for every $t \in [0,1]$, for every $|x| \geq r_0$, a and b are fixed positive constants, $f(t,|x|,0) = 0$. Then the problem of Cauchy (1.1), (1.2) has one nontrivial solution $u = u(t,r) \in \mathcal{C}([0,1)L^2([r_0,\infty)))$ for which $\lim_{t\longrightarrow 1} ||u||_{L^2([r_o,\infty))} = \infty$. Also the solution map is not uniformly continuous.

The paper is organized as follows. In section 2 we will prove our main result. In the appendix we will prove a result which we will use for the proof of our main result.

2. Proof of Main Result

Here $r_0 \geq 1$ is fixed, $n \geq 2$ is fixed.

Since we will search a positive solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ we rewrite the problem (1.1), (1.2) as follows

$$\begin{array}{ll} (2.1) & u_t - u_{rr} - \frac{n-1}{r}u_r = f(t,r,u), \quad t \in [0,1], \quad r \geq r_0, \\ (2.2) & u(0,r) = u_0(r) \in L^2([r_0,\infty)). \end{array}$$

For fixed positive constants $n \geq 1$, $r_0 \geq 1$, a, b we suppose that the constants A_1, A_2, A, B, c_1, d_1 satisfy the following conditions

$$(i1) \quad \begin{cases} 1 \leq r_0 \leq c_1 \leq d_1, 0 < A_1 \leq A_2, 0 < \frac{1}{A} \leq \frac{1}{B}, \\ A_1 > 2, \\ \Big(A_1 - 1\Big)\frac{d_1^n}{(1+d_1)^n} \geq 1. \end{cases}$$

Example. Let $n = 14$, $r_0 \geq 2$. Let also

$$\begin{array}{l} a = 2r_0^{11n}, b = 4r_0^{11n}, A = r_0^n, B = \frac{1}{2}r_0^n, \\ A_1 = r_0^{10n}, A_2 = 2r_0^{10n}, c_1 = r_0 + 1, d_1 = r_0 + 2. \end{array}$$

We note that $\frac{a}{2A} = A_1$. •

For fixed $n \geq 1$, $r_0 \geq 1$, a, b bellow we suppose that the constants A_1, A_2, A, B, c_1, d_1 satisfy the conditions $(i1)$. Also we will suppose that the function $v(t)$ is fixed function which satisfies the following hypotheses

$$\begin{array}{ll} (H1) & v(t) \in \mathcal{C}^3([0,1]), \quad v(t) > 0 \quad \forall t \in [0,1], \quad \frac{v'(t)}{v(t)} > 0 \quad \forall t \in [0,1], \\ (H2) & A_1 \leq \frac{v'(t)}{v(t)} \leq A_2 \quad \forall t \in [0,1], \quad \lim_{t\longrightarrow 1}\Big(\frac{v'(t)}{v(t)} - \frac{a}{2A}\Big) = +0. \end{array}$$

There exists a function $v(t)$ which satisfies the conditions $(H1)$, $(H2)$. For instance $v(t) = e^{\frac{a}{2A}(t-1)}$, where a, b, A, B, A_1, A_2, c_1, d_1 are the constants from the above example.

Let N be the set

$$\begin{aligned}
&N = \Big\{u(t,r) : u(t,r) \in \mathcal{C}^1([0,1]) \quad \forall r \geq r_0,\\
&u(t,\infty) = u_r(t,\infty) = 0 \quad \forall t \in [0,1],\\
&r^\alpha|\partial_r^\beta u(t,r)| \leq 1 \quad \forall t \in [0,1], \forall r \geq r_0, \quad \forall \alpha \in \mathcal{N}\cup\{0\}, \beta = 0,1,\\
&u(t,r) \geq 0 \quad \forall t \in [0,1], \forall r \geq r_0, u(t,r) \leq \tfrac{1}{B} \quad \forall t \in [0,1], \forall r \geq r_0,\\
&u(t,r) \geq \tfrac{1}{A} \quad \forall t \in [0,1], \forall r \in [c_1,d_1],\\
&u(t,r) \in L^2([r_0,\infty)) \quad \forall t \in [0,1]\Big\}.
\end{aligned}$$

For $u \in N$, fixed $n \geq 1$ and for every fixed $t \in [0,1]$ we define the operator

$$P(u) = \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t,\tau,u)\Big)d\tau ds, \quad r \geq r_0. \tag{2.1***}$$

We put

$$u_0(r) = \int_r^\infty \tfrac{1}{s^n}\int_s^\infty \tau^n\Big(\tfrac{v'(0)}{v(0)}u_0 - f(0,\tau,u_0)\Big)d\tau ds, \quad r \geq r_0.$$

Bellow we will prove that $u_0(r) \in L^2([r_0,\infty))$ exists.

Theorem 2.1. Let $n \geq 2$ be fixed, $r_0 \geq 1$ be fixed, the positive constants a, b, $a \leq b$, are fixed, $f(t,|x|,u) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([r_0,\infty)) \times \mathcal{C}^1(\mathcal{R}^1)$, $a|u| \leq f'_u(t,|x|,u) \leq b|u|$ for every $t \in [0,1]$, for every $|x| \geq r_0$, $f(t,|x|,0) = 0$ for every $t \in [0,1]$ and for every $|x| \geq r_0$. Let also the positive constants c_1, d_1, A_1, A_2, A, B are fixed which satisfy the conditions $(i1)$, the function $v(t)$ is fixed which satisfies the hypotheses $(H1)$, $(H2)$. Then the Cauchy problem (2.1), (2.2) has one unique solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ for which $u(t,r) \in N$, $\lim_{t\longrightarrow 1}||u||_{L^2([r_0,\infty))} = \infty$.

Proof. Here and bellow we will suppose that $t \in [0,1]$ is fixed.

First we will prove that $P : N \longrightarrow N$. Let $u \in N$ is fixed.

1) Since $u(t,r) \in \mathcal{C}^1([0,1])$ for every $r \geq r_0$, $f(t,r,u) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([r_0,\infty)) \times \mathcal{C}^1(\mathcal{R}^1)$, $v(t) \in \mathcal{C}^3([0,1])$, we have $P(u) \in \mathcal{C}^1([0,1])$ for every $r \geq r_0$. Also

$$\begin{aligned}
&P(u)_{|r=\infty} = 0,\\
&\tfrac{\partial P(u)}{\partial r} = -\tfrac{1}{r^n}\int_r^\infty \tau^n\Big(\tfrac{v'(t)}{v(t)}u - f(t,\tau,u)\Big)d\tau,\\
&\tfrac{\partial P(u)}{\partial r}_{|r=\infty} = 0.
\end{aligned}$$

We note that from the conditions of the Theorem 2.1 we have $f'_u(t,\tau,u) \leq bu$. From here and from $f(t,\tau,0) = 0$ we get $f(t,\tau,u) \leq \frac{b}{2}u^2$. From the defintion of the set N we have $u \leq \frac{1}{B}$. Therefore $f(t,\tau,u) \leq \frac{b}{2B}u$ for every $t \in [0,1]$, $\tau \geq r_0$. Also we have $f(t,r,u) \geq \frac{a}{2}u^2$ for every $t \in [0,1]$ and for every $r \geq r_0$.

Let $\alpha \in \mathcal{N}\cup\{0\}$ is arbitrary chosen and fixed, $k \in \mathcal{N}$ is enough large such that

$$\begin{aligned}
&k > \alpha + 3,\\
&A_2 + \tfrac{b}{2B} \leq k - 1.
\end{aligned}$$

Then for $r \geq r_0$ we have

$$\begin{aligned}
&|r^\alpha P(u)| = \Big| r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u - f(t,\tau,u) \Big) d\tau ds \Big| \leq \\
&\leq r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u + f(t,\tau,u) \Big) d\tau ds \leq \\
&here \quad we \quad use \quad that \quad f(t,\tau,u) \leq \frac{b}{2B} u \\
&\leq r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u + \frac{b}{2B} u \Big) d\tau ds \leq \\
&\leq r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(A_2 u + \frac{b}{2B} u \Big) d\tau ds = \\
&= r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(A_2 + \frac{b}{2B} \Big) u d\tau ds = \\
&= \Big(A_2 + \frac{b}{2B} \Big) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n u d\tau ds = \\
&= \Big(A_2 + \frac{b}{2B} \Big) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{\tau^{k+n} u}{\tau^k} d\tau ds \leq
\end{aligned}$$

here we use that from the definition of the set N we have $\tau^{k+n} u \leq 1$

$$\begin{aligned}
&\leq \Big(A_2 + \frac{b}{2B} \Big) r^\alpha \int_r^\infty \frac{1}{s^n} \int_s^\infty \frac{1}{\tau^k} d\tau ds = \\
&= \frac{1}{(k-1)(k+n-2)} \Big(A_2 + \frac{b}{2B} \Big) \frac{1}{r^{k+n-\alpha-2}} \leq \\
&\leq \frac{1}{(k-1)(k+n-2)} \Big(A_2 + \frac{b}{2B} \Big) \frac{1}{r_0^{k+n-\alpha-2}} \leq 1.
\end{aligned}$$

In the last inequality we use our choice of the constant k.

Let k is the same as above. Then for $r \geq r_0$ we have

$$\begin{aligned}
&\Big| r^\alpha \frac{\partial P(u)}{\partial r} \Big| = \Big| r^\alpha \frac{1}{r^n} \int_r^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u - f(t,\tau,u) \Big) d\tau \Big| \leq \\
&\leq r^\alpha \frac{1}{r^n} \int_r^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u + f(t,\tau,u) \Big) d\tau \leq \\
&here \quad we \quad use \quad that f(t,\tau,u) \leq \frac{b}{2B} u \\
&\leq r^\alpha \frac{1}{r^n} \int_r^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u + \frac{b}{2B} u \Big) d\tau \leq \\
&\leq r^\alpha \frac{1}{r^n} \int_r^\infty \tau^n \Big(A_2 u + \frac{b}{2B} u \Big) d\tau = \\
&= r^\alpha \frac{1}{r^n} \int_r^\infty \tau^n \Big(A_2 + \frac{b}{2B} \Big) u d\tau = \\
&= \Big(A_2 + \frac{b}{2B} \Big) r^\alpha \frac{1}{r^n} \int_r^\infty \frac{\tau^{k+n} u}{\tau^k} d\tau \leq
\end{aligned}$$

here we use that from the definition of the set N we have $\tau^{k+n} u \leq 1$

$$\begin{aligned}
&\leq \Big(A_2 + \frac{b}{2B} \Big) r^\alpha \frac{1}{r^n} \int_r^\infty \frac{1}{\tau^k} d\tau = \\
&= \Big(A_2 + \frac{b}{2B} \Big) \frac{1}{k-1} \frac{1}{r^{n+k-\alpha-1}} \leq \\
&\leq \Big(A_2 + \frac{b}{2B} \Big) \frac{1}{k-1} \frac{1}{r_0^{n+k-\alpha-1}} \leq 1.
\end{aligned}$$

In the last inequality we use our choice of the constant k.

2) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \geq r_0$ we have $P(u) \geq 0$.

Really, for $k \in \mathcal{N}$ for which $\frac{b}{r_0^{nk}} < 1$ we have

$$P(u) = \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u - f(t,\tau,u) \Big) d\tau ds =$$

now we apply the middle point theorem

$$= \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(t)}{v(t)} u - f'_u(t,\tau,\xi) u \Big) d\tau ds \geq$$

here we use that $f'_u(t,\tau,\xi) \leq b\xi \leq bu$

$$\geq \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)} - bu\Big)ud\tau ds =$$
$$= \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)} - b\frac{\tau^{nk}u}{\tau^{nk}}\Big)ud\tau ds \geq$$
$$here \quad we \quad use \quad \tau^{nk}u \leq 1$$
$$\geq \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_1 - \frac{b}{r_0^{nk}}\Big)ud\tau ds,$$

i.e.

$$P(u) \geq \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_1 - \frac{b}{r_0^{nk}}\Big)ud\tau ds. \tag{2.2'}$$

Since $u(t,r) \geq 0$ for every fixed $t \in [0,1]$ and for every $r \geq r_0$ and from our choice of the constant k we have that $P(u) \geq 0$ for every $t \in [0,1]$ and for every $r \geq r_0$.

3) Now we will see that for every fixed $t \in [0,1]$ and for every $r \in [c_1,d_1]$ we have $P(u) \geq \frac{1}{A}$. We suppose that k is same as in **2**). Let

$$g(u) = \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_1 - \frac{b}{r_0^{nk}}\Big)ud\tau ds.$$

Then

$$g'(u) = \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_1 - \frac{b}{r_0^{nk}}\Big)d\tau ds \geq 0.$$

In the last inequality we use our choice of the constant k. Consequently $g(u)$ is increase function of u. Since for every fixed $t \in [0,1]$ and for every $r \in [c_1,d_1]$ we have $u \geq \frac{1}{A}$ we get

$$g(u) \geq g(\tfrac{1}{A}) =$$
$$= g(u) = \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_1 - \frac{b}{r_0^{nk}}\Big)\frac{1}{A}d\tau ds \geq$$
$$\geq \int_{d_1}^{d_1+1} \frac{1}{s^n}\int_{d_1}^{d_1+1} \tau^n\Big(A_1 - 1\Big)\frac{1}{A}d\tau ds \geq$$
$$\geq (A_1-1)\frac{d_1^n}{(d_1+1)^n}\frac{1}{A} \geq \frac{1}{A}.$$

In the last inequality we use the conditions $(i1)$. From here and from $(2.2')$ we get that $P(u) \geq \frac{1}{A}$ for every fixed $t \in [0,1]$ and for every $r \in [c_1,d_1]$.

4) Now we will prove that for every fixed $t \in [0,1]$ and for every $r \geq r_0$ we have $P(u) \leq \frac{1}{B}$. Let $k \in \mathcal{N}$ is chosen such that

$$\Big(A_2 + \frac{b}{2B}\Big)\frac{1}{(k-1)(n+k-2)r_0^{n+k-2}} \leq \frac{1}{B}, \quad k \geq 2.$$

Then

$$P(u) = \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t,\tau,u)\Big)d\tau ds \leq$$
$$\leq \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_2u + f(t,\tau,u)\Big)d\tau ds \leq$$
$$\leq \int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(A_2u + \frac{b}{2B}u\Big)d\tau ds =$$
$$= \int_r^\infty \frac{1}{s^n}\int_s^\infty \Big(A_2 + \frac{b}{2B}\Big)\frac{\tau^{n+k}u}{\tau^k}d\tau ds \leq$$
$$here \quad we \quad use \quad \tau^{n+k}u \leq 1$$
$$\int_r^\infty \frac{1}{s^n}\int_s^\infty \Big(A_2 + \frac{b}{2B}\Big)\frac{1}{\tau^k}d\tau ds \leq$$
$$\leq \Big(A_2 + \frac{b}{2B}\Big)\frac{1}{(k-1)(n+k-2)r_0^{n+k-2}} \leq \frac{1}{B}.$$

In the last inequality we use our choice of the constant k.

5) Now we will prove that $P(u) \in L^2([r_0,\infty))$ for every fixed $t \in [0,1]$. We choose $k \in \mathcal{N}$ such that $n+k-4>0$.

$||P(u)||^2_{L^2([r_0,\infty))} = \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(\frac{v'(t)}{v(t)}u - f(t,\tau,u)\Big)d\tau ds\Big)^2 dr \leq$
$\leq \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(A_2+\frac{b}{2B}\Big)u d\tau ds\Big)^2 dr =$
$= \Big(A_2+\frac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{\tau^{2k+2n}u}\sqrt{u}\frac{1}{\tau^k}d\tau ds\Big)^2 dr \leq$
$\Big(A_2+\frac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{u}\frac{1}{\tau^k}d\tau ds\Big)^2 dr \leq$
now we use the Hölder's inequality
$\leq \Big(A_2+\frac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\Big(\int_s^{\infty}\frac{1}{\tau^{\frac{4k}{3}}}d\tau\Big)^{\frac{3}{4}}\Big(\int_s^{\infty}u^2 d\tau\Big)^{\frac{1}{4}}ds\Big)^2 dr \leq$
$\leq \Big(A_2+\frac{b}{2B}\Big)^2\frac{1}{(\frac{4}{3}k-1)^{\frac{3}{2}}}\cdot\frac{1}{(n+k-\frac{7}{4})^2}\frac{1}{2n+2k-\frac{9}{2}}\frac{1}{r_0^{2n+2k-\frac{9}{2}}}||u||_{L^2([r_0,\infty))} < \infty.$

From 1), 2), 3), 4), 5) we conclude that $P: N \longrightarrow N$ for every fixed $t \in [0,1]$.

Now we will prove that the operator P has one unique nontrivial fixed point in the set N.

Let $t \in [0,1)$ is fixed. Let also $u_1 \in N$, $u_2 \in N$ are fixed and $\alpha = ||u_1 - u_2||_{L^2([r_0,\infty))} \neq 0$.

We choose $k \in \mathcal{N}$, $k>3$ enough large so that

$$\frac{1}{\alpha}Q_1 = \frac{2\Big(A_2+\frac{b}{B}\Big)^2}{\alpha(\frac{4}{3}k-1)^{\frac{3}{2}}(k+n-\frac{7}{4})^2\Big(2n+2k-\frac{9}{2}\Big)r_0^{2n+2k-\frac{9}{2}}} < 1, \quad Q_1 < 1.$$

Then

$||P(u_1)-P(u_2)||^2_{L^2([r_0,\infty))} =$
$= \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(\frac{v'(t)}{v(t)}(u_1-u_2)-(f(t,\tau,u_1)-f(t,\tau,u_2))\Big)d\tau ds\Big)^2 dr \leq$
$\leq \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(\frac{v'(t)}{v(t)}|u_1-u_2|+|f(t,\tau,u_1)-f(t,\tau,u_2)|\Big)d\tau ds\Big)^2 dr \leq$
$\leq \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(A_2|u_1-u_2|+|f(t,\tau,u_1)-f(t,\tau,u_2)|\Big)d\tau ds\Big)^2 dr \leq$

now we use the middle point theorem

$\leq \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(A_2|u_1-u_2|+|f'_u(t,\tau,\xi)||u_1-u_2|\Big)d\tau ds\Big)^2 dr \leq$
$\leq \int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\Big(A_2|u_1-u_2|+\frac{b}{B}|u_1-u_2|\Big)d\tau ds\Big)^2 dr =$
$= \Big(A_2+\frac{b}{B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n|u_1-u_2|d\tau ds\Big)^2 dr =$
$= \Big(A_2+\frac{b}{B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\sqrt{\tau^{2k+2n}|u_1-u_2|}\frac{1}{\tau^k}\sqrt{|u_1-u_2|}d\tau ds\Big)^2 dr \leq$
here we use that $\sqrt{\tau^{2k+2n}|u_1-u_2|} \leq \sqrt{2}$
$\leq 2\Big(A_2+\frac{b}{B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\frac{1}{\tau^k}\sqrt{|u_1-u_2|}d\tau ds\Big)^2 dr \leq$
here we use the Hölder's inequality
$\leq 2\Big(A_2+\frac{b}{B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\frac{1}{s^n}\Big(\int_s^{\infty}\frac{1}{\tau^{\frac{4k}{3}}}d\tau\Big)^{\frac{3}{4}}\Big(\int_s^{\infty}|u_1-u_2|^2 d\tau\Big)^{\frac{1}{4}}ds\Big)^2 dr \leq$
$\leq Q_1||u_1-u_2||_{L^2([0,\infty))},$

i.e.

$$||P(u_1)-P(u_2)||^2_{L^2([r_0,\infty))} \leq Q_1||u_1-u_2||_{L^2([r_0,\infty))}.$$

Since we choose the constant k so that $\frac{1}{\alpha}Q_1 < 1$ we have

$$||P(u_1) - P(u_2)||^2_{L^2([r_0,\infty))} \leq \frac{Q_1}{\alpha}\alpha||u_1 - u_2||^2_{L^2([r_0,\infty))} \leq \frac{Q_1}{\alpha}||u_1 - u_2||^2_{L^2([r_0,\infty))}.$$

From here and from the following theorem

Theorem[5, p. 294]*Let B be the completely metric space for which $AB \subset B$ and for the operator A is hold the following condition*

$$\rho(Ax, Ay) \leq L(\alpha, \beta)\rho(x, y), \quad x, y \in B, \alpha \leq \rho(x, y) \leq \beta,$$

where $L(\alpha, \beta) < 1$ for $0 < \alpha \leq \beta < \infty$. Then the operator A has exactly one fixed point in the space B.

we conclude that the operator P has one unique fixed point u in the set N. We note that the set N is a closed subset of the space $L^2([r_0, \infty))$ for every fixed $t \in [0, 1]$ (see lemma 3.1 in the appendix of this paper) As in the proof of the Proposition 2.1, 2.2 [4] we have that the fixed point u satisfies the equation (2.1) with initial data

$$u_0 = \int_r^\infty \frac{1}{s^n} \int_s^\infty \tau^n \Big(\frac{v'(0)}{v(0)}u_0 - f(0, \tau, u_0)\Big)d\tau ds, r \geq r_0.$$

We have that $u_0 \in L^2([r_0, \infty))$.

Now we will prove that

$$\lim_{t \longrightarrow 1} ||u||_{L^2([r_0,\infty))} = \infty.$$

For $k \in \mathcal{N}$ we put

$$\begin{aligned} Q_2 &= \Big(A_2 + \frac{b}{2B}\Big)^2 \frac{1}{(\frac{4}{3}k-1)^{\frac{3}{2}}(n+k-\frac{7}{4})^2(2n+2k-\frac{9}{2})r_0^{2n+2k-\frac{9}{2}}}, \\ Q_3 &= 2A^2\Big(\frac{v'(t)}{v(t)} - \frac{a}{2A}\Big)^2 d_1^{2n}(d_1 - c_1)^3\frac{1}{c_1^{2n}}, \\ Q_4 &= \Big(A_1 - \frac{b}{2B}\Big)\frac{1}{A}\frac{c_1^n}{d_1^n}(d_1 - c_1)^{\frac{5}{2}}. \end{aligned}$$

We choose the constant $k \in \mathcal{N}$ such that

$$1 - 10\frac{Q_2}{Q_4} > 0.$$

Then

$$\begin{aligned} ||u||^2_{L^2([r_0,\infty))} &= \int_{r_0}^\infty \Big(\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t, \tau, u)\Big)d\tau ds\Big)^2 dr = \\ &= \int_{r_0}^{c_1} \Big(\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t, \tau, u)\Big)d\tau ds\Big)^2 dr + \\ &+ \int_{c_1}^\infty \Big(\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t, \tau, u)\Big)d\tau ds\Big)^2 dr. \end{aligned}$$

Let

$$\begin{aligned} J_1 &:= \int_{r_0}^{c_1} \Big(\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t, \tau, u)\Big)d\tau ds\Big)^2 dr, \\ J_2 &:= \int_{c_1}^\infty \Big(\int_r^\infty \frac{1}{s^n}\int_s^\infty \tau^n\Big(\frac{v'(t)}{v(t)}u - f(t, \tau, u)\Big)d\tau ds\Big)^2 dr. \end{aligned}$$

Then

$$||u||^2_{L^2([r_0,\infty))} = J_1 + J_2. \tag{2.3}$$

For J_1 we have the following estimate

$$
\begin{aligned}
& J_1 \le \int_{r_0}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u+f(t,\tau,u)\Big)d\tau ds\Big)^2 dr \le \\
& \le \int_{r_0}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(A_2u+\tfrac{b}{2B}u\Big)d\tau ds\Big)^2 dr = \\
& = \Big(A_2+\tfrac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\sqrt{\tau^{2k+2n}u}\tfrac{1}{\tau^k}\sqrt{u}d\tau ds\Big)^2 dr \le \\
& here \quad we \quad use \quad that \quad \sqrt{\tau^{2k+2n}u}\le 1 \\
& \le \Big(A_2+\tfrac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tfrac{1}{\tau^k}\sqrt{u}d\tau ds\Big)^2 dr \le \\
& now \quad use \quad the \quad H\ddot{o}lder's \quad inequlity \\
& \le \Big(A_2+\tfrac{b}{2B}\Big)^2\int_{r_0}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\Big(\int_s^{\infty}\tfrac{1}{\tau^{\frac{4k}{3}}}d\tau\Big)^{\frac{3}{4}}\Big(\int_s^{\infty}u^2d\tau\Big)^{\frac{1}{4}}ds\Big)^2 dr \le \\
& \le Q_2||u||_{L^2([r_0,\infty))}.
\end{aligned}
$$

$$J_1 \le Q_2||\tilde{u}||_{L^2([r_0,\infty))}. \tag{2.4}$$

Now we consider J_2. For it we have

$$
\begin{aligned}
& J_2 = \int_{c_1}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v''(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr = \\
& \int_{c_1}^{d_1}\Big(\int_{c_1}^{d_1}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds+ \\
& \int_{d_1}^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr+ \\
& +\int_{d_1}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr = \\
& = \int_{c_1}^{d_1}\Big(\int_{c_1}^{d_1}\tfrac{1}{s^n}\int_{c_1}^{d_1}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds+ \\
& \int_{c_1}^{d_1}\tfrac{1}{s^n}\int_{d_1}^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds+ \\
& +\int_{d_1}^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr+ \\
& +\int_{d_1}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr \le \\
& here \quad we \quad use \quad the \quad inequality \quad (a+b)^2\le 2(a^2+b^2) \\
& \le 4\int_{c_1}^{d_1}\Big(\int_{c_1}^{d_1}\tfrac{1}{s^n}\int_{d_1}^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr+ \\
& +4\int_{c_1}^{d_1}\Big(\int_{d_1}^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr+ \\
& +2\int_{c_1}^{d_1}\Big(\int_{c_1}^{d_1}\tfrac{1}{s^n}\int_{c_1}^{d_1}\tau^n\Big(\tfrac{v''(t)}{v(t)}\tilde{u}-f(\tilde{u})\Big)d\tau ds\Big)^2 dr+ \\
& +\int_{d_1}^{\infty}\Big(\int_r^{\infty}\tfrac{1}{s^n}\int_s^{\infty}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr \le \\
& 2\int_{c_1}^{d_1}\Big(\int_{c_1}^{d_1}\tfrac{1}{s^n}\int_{c_1}^{d_1}\tau^n\Big(\tfrac{v'(t)}{v(t)}u-f(t,\tau,u)\Big)d\tau ds\Big)^2 dr+9Q_2||u||_{L^2([r_0,\infty))} \le \\
& here \quad we \quad use \quad that \quad u\ge \tfrac{1}{A} \\
& \le 9Q_2||\tilde{u}||_{L^2([r_0,\infty))}+2\tfrac{d_1^{2n}}{c_1^{2n}}\Big(\tfrac{v'(t)}{v(t)}-\tfrac{a}{2A}\Big)^2(d_1-c_1)^3A^2\Big(\int_{c_1}^{d_1}u^2d\tau\Big)^2 \le \\
& \le 9Q_2||\tilde{u}||_{L^2([r_0,\infty))}+2\tfrac{d_1^{2n}}{c_1^{2n}}\Big(\tfrac{v'(t)}{v(t)}-\tfrac{a}{2A}\Big)^2(d_1-c_1)^3A^2||u||^4_{L^2([r_0,\infty))}.
\end{aligned}
$$

Then

$$J_2 \le 9Q_2||u||_{L^2([r_0,\infty))}+Q_3||u||^4_{L^2([r_0,\infty))}.$$

From here and from (2.3), (2.4) we get

$$||u||^2_{L^2([r_0,\infty))} \le 10Q_2||u||_{L^2([r_0,\infty))}+Q_3||u||^4_{L^2([r_0,\infty))}. \tag{2.5}$$

Also we note

$$||u||_{L^2([r_0,\infty))} = \left(\int_{r_0}^{\infty}\left(\int_r^{\infty}\frac{1}{s^n}\int_s^{\infty}\tau^n\left(\frac{v'(t)}{v(t)}u - f(t,\tau,u)\right)d\tau ds\right)^2 dr\right)^{\frac{1}{2}} \geq$$
$$\geq \left(\int_{c_1}^{d_1}\left(\int_{c_1}^{d_1}\frac{1}{s^n}\int_{c_1}^{d_1}\tau^n\left(\frac{v'(t)}{v(t)}u - f(t,\tau,u)\right)d\tau ds\right)^2 dr\right)^{\frac{1}{2}} \geq$$
$$\geq \left(\int_{c_1}^{d_1}\left(\int_{c_1}^{d_1}\frac{1}{s^n}\int_{c_1}^{d_1}\tau^n\left(A_1 - \frac{b}{2B}\right)\frac{1}{A}d\tau ds\right)^2 dr\right)^{\frac{1}{2}} \geq$$
$$\geq \frac{c_1^n}{d_1^n}\left(A_1 - \frac{b}{2B}\right)\frac{1}{A}(d_1-c_1)^{\frac{5}{2}}.$$

From here and from (2.5) we get

$$Q_4||u||_{L^2([r_0,\infty))} \leq ||u||^2_{L^2([r_0,\infty))} \leq 10Q_2||u||_{L^2([r_0,\infty))} + Q_3||u||^4_{L^2([r_0,\infty))}.$$

Then

$$(Q_4 - 10Q_2)||u||_{L^2([r_0,\infty))} \leq Q_3||u||^4_{L^2([r_0,\infty))}.$$

From our choice of the constant k we have that $Q_4 - 10Q_2 > 0$. Therefore

$$\frac{(Q_4 - 10Q_2)}{Q_3} \leq ||u||^3_{L^2([r_0,\infty))},$$

from where

$$\lim_{t\longrightarrow 1}||u||_{L^2([r_0,\infty))} = \infty,$$

because $\lim_{t\longrightarrow 1} Q_3 = 0$ (see $(H2)$). •

Theorem 2.2. Let $n \geq 2$ be fixed, $r_0 \geq 1$ be fixed, the positive constants a, b, $a \leq b$, are fixed, $f(t,|x|,u) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([r_0,\infty)) \times \mathcal{C}^1(\mathcal{R}^1)$, $a|u| \leq f'_u(t,|x|,u) \leq b|u|$ for every $t \in [0,1]$, for every $|x| \geq r_0$, $f(t,|x|,0) = 0$. Let also the positive constants c_1, d_1, A_1, A_2, A, B are fixed which satisfy the conditions $(i1)$, the function $v(t)$ is the same function as in the Theorem 2.1. Then the Cauchy problem (2.1), (2.2) has one unique solution $u(t,r)$ in the form $u(t,r) = v(t)\omega(r)$ for which $u(t,r) \in N$, $u(t,r) \in \dot{H}^1([r_0,\infty))$ for $\forall t \in [0,1]$, and the solution map is not uniformly continuous.

Proof. In the Theorem 2.1 was proved that the equation (2.1) has one unique nontrivial solution $\tilde{u}(t,r) = v(t)\omega(r)$ for which $\tilde{u}(t,r) \in N$. Also, for every $k \in \mathcal{N}$ and for every fixed $t \in [0,1]$ we have

$$\left|\left|\frac{\partial\tilde{u}}{\partial r}\right|\right|^2_{L^2([r_0,\infty))} =$$
$$= \int_{r_0}^{\infty}\left(\frac{1}{r^n}\int_r^{\infty}s^n\left(\frac{v'(t)}{v(t)}\tilde{u} - f(t,\tau,\tilde{u})\right)ds\right)^2 dr \leq$$
$$\leq \int_{r_0}^{\infty}\left(\frac{1}{r^n}\int_r^{\infty}s^n\left(\frac{v'(t)}{v(t)}\tilde{u} + f(t,\tau,\tilde{u})\right)ds\right)^2 dr \leq$$
$$\leq \int_{r_0}^{\infty}\left(\frac{1}{r^n}\int_r^{\infty}s^n\left(A_2 + \frac{b}{2B}\right)\tilde{u}ds\right)^2 dr =$$
$$= \left(A_2 + \frac{b}{2B}\right)^2\int_{r_0}^{\infty}\left(\frac{1}{r^n}\int_r^{\infty}\sqrt{s^{2n+2k}\tilde{u}}\sqrt{\tilde{u}}\frac{1}{s^k}ds\right)^2 dr \leq$$
$$\leq \left(A_2 + \frac{b}{2B}\right)^2\int_{r_0}^{\infty}\left(\frac{1}{r^n}\int_r^{\infty}\sqrt{\tilde{u}}\frac{1}{s^k}ds\right)^2 dr \leq$$
$$\leq \left(A_2 + \frac{b}{2B}\right)^2\int_{r_0}^{\infty}\left(\frac{1}{r^n}\left(\int_r^{\infty}\tilde{u}^2 ds\right)^{\frac{1}{4}}\left(\int_r^{\infty}\frac{1}{s^{\frac{4k}{3}}}ds\right)^{\frac{3}{4}}\right)^2 dr \leq$$
$$\leq \left(A_2 + \frac{b}{2B}\right)^2\frac{1}{\left(\frac{4k}{3}-1\right)^{\frac{3}{2}}(2n+2k-\frac{5}{2})r_0^{2n+2k-\frac{5}{2}}}||\tilde{u}||_{L^2([r_0,\infty))} < \infty,$$

because $\tilde{u} \in L^2([r_0,\infty))$. Consequently $\tilde{u} \in \dot{H}^1([r_0,\infty))$ for every fixed $t \in [0,1]$.

Now we suppose that the solution map $(u_0, u_1) \longrightarrow u(t, r)$ is uniformly continuous.

Let

$$0 < \epsilon < \Big(A_1 - \frac{b}{2B}\Big)^2 \frac{1}{A^2}(d_1 - c_1)^3 \frac{c_1^{2n}}{d_1^{2n}}. \tag{2.6}$$

Let also

$$u_1 = \tilde{u}, \quad u_2 = 0.$$

Then there exists positive constant δ such that

$$E(0, u_1 - u_2) \le \delta$$
$$and$$
$$E(1, u_1 - u_2) \le \epsilon.$$

From here

$$\epsilon \ge E(1, u_1 - u_2) = E(1, \tilde{u}) \ge \int_{r_0}^{\infty}\Big(\frac{1}{r^n}\int_r^{\infty} s^n\Big(\frac{v'(1)}{v(1)}\tilde{u} - f(1, s, \tilde{u})\Big)ds\Big)^2 dr \ge$$
$$\ge \int_{c_1}^{d_1}\Big(\frac{1}{r^n}\int_{c_1}^{d_1} s^n\Big(\frac{v'(1)}{v(1)}\tilde{u} - f(1.s, \tilde{u})\Big)ds\Big)^2 dr \ge$$
$$\ge \int_{c_1}^{d_1}\Big(\frac{1}{r^n}\int_{c_1}^{d_1} s^n\Big(\frac{v'(1)}{v(1)}\tilde{u} - \frac{b}{2B}\tilde{u}\Big)ds\Big)^2 dr \ge$$
$$\ge \int_{c_1}^{d_1}\Big(\frac{1}{r^n}\int_{c_1}^{d_1} s^n\Big(\frac{v'(1)}{v(1)} - \frac{b}{2B}\Big)\frac{1}{A}ds\Big)^2 dr \ge$$
$$\ge \Big(A_1 - \frac{b}{2B}\Big)^2 \frac{1}{A^2}(d_1 - c_1)^3 \frac{c_1^{2n}}{d_1^{2n}}$$

which is a contradiction with (2.6). •

3. Appendix

Lemma 3.1. The set N is a closed subset of $\mathcal{C}([0, 1]L^2([r_0, \infty)))$.

Proof. Let $t \in [0, 1]$ is fixed.

Let also $\{u_n\}$ is a sequence of elements of the set N for which

$$\lim_{n\longrightarrow\infty} ||u_n - \tilde{u}||_{L^2([r_0,\infty))} = 0,$$

where $\tilde{u} \in L^2([r_0, \infty))$. Since $P(u)$ is a continuous- differentiable function of u, for $r \in [r_0, r_0 + 1]$ and $u \in N$ we have

$$P'(u) = \int_r^{\infty} \frac{1}{s^n}\int_s^{\infty} \tau^n\Big(\frac{v'(t)}{v(t)} - f'_u(t, \tau, u)\Big)d\tau ds \ge$$
$$\ge \int_r^{\infty} \frac{1}{s^n}\int_s^{\infty} \tau^n\Big(A_1 - \frac{b}{B}\Big)d\tau ds \ge$$
$$\ge \int_{r_0+1}^{r_0+2} \frac{1}{s^n}\int_{r_0+1}^{r_0+2} \tau^n\Big(A_1 - \frac{b}{B}\Big)d\tau ds \ge$$
$$\ge \Big(A_1 - \frac{b}{B}\Big)\frac{(r_0+1)^n}{(r_0+2)^n}.$$

From here follows that for every $u \in N$ there exists

$$L = \min_{r\in[r_0,r_0+1]} |P'(u)(r)| > 0.$$

Let

$$M_1 = \max_{r\in[r_0,r_0+1]}\Big|\frac{\partial}{\partial r}P'(u)(r)\Big|.$$

Now we will prove that for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $|x - y| < \delta$ we have

$$|u_m(x) - u_m(y)| < \epsilon \quad for \quad \forall m \in \mathcal{N}.$$

We suppose that there exists $\tilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural m and $x, y \in [r_0, \infty)$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$. We choose $\tilde{\tilde{\epsilon}}$ such that $0 < \tilde{\tilde{\epsilon}} < L\tilde{\epsilon}$. We note that $P(u_m)(x)$ is uniformly continuous for $x \in [r_0, \infty)$(for $u \in N$ $P(u)(r)$ is uniformly continuous function for $r \in [r_0, \infty)$ because $P(u)(r) \in \mathcal{C}([r_0, \infty))$ and as in the proof of the Theorem 2.1 we have that there exists positive constant C such that $\left|\frac{\partial}{\partial r} P(u)(r)\right| \leq C$). Then there exists $\delta_1 = \delta_1(\tilde{\tilde{\epsilon}}) > 0$ such that for every natural m we have

$$|P(u_m)(x) - P(u_m)(y)| < \tilde{\tilde{\epsilon}}, \quad \forall x, y \in [r_0, \infty) : |x - y| < \delta_1.$$

Consequently we can choose

$$0 < \delta < min\Big\{1, \delta_1, \frac{(L\tilde{\epsilon} - \tilde{\tilde{\epsilon}})B}{M_1}\Big\}$$

such that there exist natural m and $x_1, x_2 \in [r_0, \infty)$ for which

$$|x_1 - x_2| < \delta, \quad |u_m(x_1 - x_2 + r_0) - u_m(r_0)| \geq \tilde{\epsilon}.$$

In particular

$$|P(u_m)(x_1 - x_2 + r_0) - P(u_m)(r_0)| < \tilde{\tilde{\epsilon}}. \tag{3.1}$$

Let us suppose for convinience that $x_1 - x_2 > 0$. Then $x_1 - x_2 < 1$ and for every $u \in N$ we have $P'(u)(x_1 - x_2 + r_0) \geq L$. Then from the middle point theorem we have

$$\begin{aligned}
&P(0) = 0, P(u_m)(x_1 - x_2 + r_0) = P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0),\\
&P(u_m)(r_0) = P'(\xi)(r_0)u_m(r_0),\\
&|P(u_m)(x_1 - x_2 + r_0) - P(u_m)(r_0)| =\\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(r_0)u_m(r_0)| =\\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)+\\
&+P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| \geq\\
&\geq |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)|-\\
&-|P'(\xi)(x_1 - x_2 + r_0)u_m(r_0) - P'(\xi)(r_0)u_m(r_0)| =\\
&= |P'(\xi)(x_1 - x_2 + r_0)u_m(x_1 - x_2 + r_0) - P'(\xi)(x_1 - x_2 + r_0)u_m(r_0)|-\\
&-\left|\frac{\partial}{\partial r} P'(\xi)\right||x_1 - x_2||u_m(r_0)| \geq\\
&\geq L\tilde{\epsilon} - M_1\delta\frac{1}{B} \geq \tilde{\tilde{\epsilon}},
\end{aligned}$$

which is a contradiction with (3.1). Therefore, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that from $|x - y| < \delta$ follows

$$|u_m(x) - u_m(y)| < \epsilon \quad \forall m \in \mathcal{N}. \tag{3.2}$$

On the other hand from the definition of the set N we have that for every natural m

$$u_m(r) \leq \frac{1}{B} \quad \forall r \geq r_0. \tag{3.3}$$

From (3.2) and (3.3) follows that the set $\{u_m\}$ is a compact subset of the space $\mathcal{C}([r_0, \infty))$. Therefore there is a subsequence $\{u_{n_k}\}$ and function $u \in \mathcal{C}([r_0, \infty))$ for which

$$|u_{n_k}(x) - u(x)| < \epsilon \quad \forall x \in [r_0, \infty).$$

Now we suppose that there is no true that $u = \tilde{u}$ a.e. in $[r_0, \infty)$. Then there exist $\epsilon_1 > 0$ and subinterval $\Delta \subset [r_0, \infty)$ such that $\mu(\Delta) > 0$ and

$$|u - \tilde{u}| > \epsilon_1 \quad for \quad r \in \Delta.$$

Let $\epsilon > 0$ is chosen such that

$$\epsilon < \frac{\epsilon_1 (\mu(\Delta))^{\frac{1}{2}}}{\mu(\Delta)^{\frac{1}{2}} + 1}. \tag{3.4}$$

Then, for every enough large $n_k \in \mathcal{N}$, we have

$$\begin{aligned}
&||u_{n_k} - \tilde{u}||_{L^2([r_0,\infty))} < \epsilon,\\
&\epsilon\mu(\Delta) = \epsilon \int_\Delta dx > \int_\Delta |u_{n_k} - u| dx =\\
&= \int_\Delta |u_{n_k} - \tilde{u} + \tilde{u} - u| dx \geq\\
&\geq \int_\Delta |\tilde{u} - u| dx - \int_\Delta |u_{n_k} - \tilde{u}| dx \geq\\
&\geq \epsilon_1 \mu(\Delta) - \Big(\int_\Delta |u_{n_k} - \tilde{u}|^2 dx\Big)^{\frac{1}{2}} \Big(\mu(\Delta)\Big)^{\frac{1}{2}} \geq\\
&\geq \epsilon_1 \mu(\Delta) - ||u_{n_k} - \tilde{u}||_{L^2([r_0,\infty))} \Big(\mu(\Delta)\Big)^{\frac{1}{2}} >\\
&> \epsilon_1 \mu(\Delta) - \epsilon \Big(\mu(\Delta)\Big)^{\frac{1}{2}},
\end{aligned}$$

which is a contradiction with (3.4). From here $u = \tilde{u}$ a.e. in $[r_0, \infty)$, $|u_n - u|^2 = |\tilde{u} - u_n|^2$ a.e. in $[r_0, \infty)$, $||u_n - u||_{L^2([r_0,\infty))} = ||u_n - \tilde{u}||_{L^2([r_0,\infty))}$.

Consequently, for every sequence $\{u_n\}$ from elements of the set N, which is convergent in $L^2([r_0, \infty))$, there exists a function $u \in \mathcal{C}([r_0, \infty))$, $u \in L^2([r_0, \infty))$ for which

$$\lim_{n \longrightarrow \infty} ||u_n - u||_{L^2([r_0,\infty))} = 0.$$

Bellow we will suppose that $\{u_n\}$ is a sequence from elements of the set N, which is convergent in $L^2([r_0, \infty))$. Then there exists a function $u \in \mathcal{C}([r_0, \infty))$, $u \in L^2([r_0, \infty))$ for which

$$\lim_{n \longrightarrow \infty} ||u_n - u||_{L^2([r_0,\infty))} = 0.$$

Now we suppose that $u(t, \infty) \neq 0$. Then there exist enough large $Q > 0$, enough large natural m and $\epsilon_2 > 0$ for which

$$u_m(t, r) = 0, \quad u(t, r) > \epsilon_2, \quad \forall r \geq Q.$$

We choose

$$0 < \epsilon_3 < \epsilon_2. \tag{3.5}$$

Then, for every enough large $n \in \mathcal{N}$ we have $|u_n(t, r) - u(t, r)| < \epsilon_3$ and

$$\begin{aligned}
&\epsilon_3 > \int_Q^{Q+1} |u_n(t, r) - u(t, r)| dr \geq\\
&\geq \int_Q^{Q+1} (|u(t, r)| - |u_n(t, r)|) dr =\\
&= \int_Q^{Q+1} |u(t, r)| dr > \epsilon_2,
\end{aligned}$$

which is a contradiction with (3.5). Therefore $u(t, \infty) = 0$.

Now we will prove that $\frac{\partial}{\partial t} u(t, r)$ exists for every $t \in [0, 1]$. Let us suppose that $r \in [r_0, \infty)$ is fixed and there exists $t_1 \in [0, 1]$ such that $\frac{\partial}{\partial t} u(t_1, r)$ no exists. Then for every $h > 0$, which is enough small, exists $\epsilon_4 > 0$ such that

$$\Big|\frac{u(t_1 + h, r) - u(t_1, r)}{h}\Big| > \epsilon_4,$$

and

$$0 < \epsilon_5 < \frac{h}{2}\epsilon_4, \tag{3.6}$$

such that

$$|u_n(t_1+h,r) - u(t_1,r)| < \epsilon_5.$$

From here

$$\begin{aligned}
&\epsilon_5 > |u_n(t_1+h,r) - u(t_1+h,r)| = \\
&= |u_n(t_1+h,r) - u(t_1,r) + u(t_1,r) - u(t_1+h,r)| \geq \\
&\geq |u(t_1,r) - u(t_1+h,r)|\tfrac{1}{h}h - |u_n(t_1+h,r) - u(t_1,r)| \geq \epsilon_4 h - \epsilon_5,
\end{aligned}$$

which is a contradiction of our choice of ϵ_5. Therefore $\frac{\partial}{\partial t}u(t,r)$ exists for every $t \in [0,1]$. As in above we can see that $u(t,r) \in \mathcal{C}^1([0,1])$ for every $r \geq r_0$, $u(t,r) \in \mathcal{C}^2([r_0,\infty))$ for every $t \in [0,1]$, $u_r(t,\infty) = 0$ for every $t \in [0,1]$.

Now we suppose that there exists interval $\Delta_2 \subset [r_0,\infty)$ such that

$$u(t,r) \geq \frac{1}{B} + \epsilon_7 \quad for \quad r \in \Delta_2.$$

Let $n \in \mathcal{N}$ is enough large and $\epsilon_8 > 0$ are chosen such that

$$|u_n(t,r) - u(t,r)| < \epsilon_8 \quad for \quad r \in \Delta_2, 0 < \epsilon_8 < \epsilon_7. \tag{3.7}$$

From here, for $r \in \Delta_2$ we have

$$\epsilon_8 > |u_n(t,r) - u(t,r)| \geq |u(t,r)| - |u_n(t,r)| \geq \frac{1}{B} + \epsilon_7 - \frac{1}{B} = \epsilon_7,$$

which is one contradiction with (3.7). Therefore we have $u(t,r) \leq \frac{1}{B}$ for every $r \geq r_0$.

Now we suppose that there exists interval $\Delta_3 \subset [c_1,d_1]$ for which $u(t,r) < \frac{1}{A}$ for every $r \in \Delta_3$. From here there exists $\epsilon_9 > 0$ such that $u(t,r) \leq \frac{1}{A} - \epsilon_9$ for $r \in \Delta_3$. Also, let

$$0 < \epsilon_{10} < \epsilon_9 \tag{3.8}$$

and $n \in \mathcal{N}$ is enough large such that $\epsilon_{10} > |u_n(t,r) - u(t,r)|$ for $r \in \Delta_3$. Then for $r \in \Delta_3$ we have

$$\epsilon_{10} > |u_n(t,r) - u(t,r)| \geq |u_n(t,r)| - |u(t,r)| \geq \frac{1}{A} - \frac{1}{A} + \epsilon_9,$$

which is one contradiction with (3.8). Consequently, for every $r \in [c_1,d_1]$ we have $u(t,r) \geq \frac{1}{A}$.

Now we suppose that there exist $\alpha \in \mathcal{N} \cup \{0\}$, interval $\Delta_4 \subset [r_0,\infty)$ and $\epsilon_{11} > 0$ such that

$$|r^\alpha u(t,r)| > 1 + \epsilon_{11} \quad for \quad r \in \Delta_4.$$

Let $\epsilon_{12} > 0$ and $n \in \mathcal{N}$ are chosen such that

$$|r^\alpha(u_n(t,r) - u(t,r))| < \epsilon_{12} \quad for \quad r \in \Delta_4, \quad 0 < \epsilon_{12} < \epsilon_{11}. \tag{3.9}$$

From here

$$\epsilon_{12} > |r^\alpha(u_n(t,r) - u(t,r))| \geq |r^\alpha u(t,r)| - r^\alpha|u_n(t,r)| \geq \epsilon_{11},$$

which is a contradiction with (3.9). Therefore for every $\alpha \in \mathcal{N} \cup \{0\}$ and for every $r \in [r_0,\infty)$ we have $r^\alpha u(t,r) \leq 1$. After we use the same arguments we can see that for every $\alpha \in \mathcal{N} \cup \{0\}$ and for every $r \in [r_0,\infty)$ we have $r^\alpha |u_r(t,r)| \leq 1$.

Now we suppose that there exist interval $\Delta_5 \subset [r_0, \infty)$ and $\epsilon_{13} > 0$ such that for $r \in \Delta_5$ we have

$$u(t, r) < -\epsilon_{13}.$$

Let $n \in \mathcal{N}$ is enough large and $\epsilon_{14} > 0$ are fixed for which

$$|u_n(t, r) - u(t, r)| < \epsilon_{14} \quad for \quad r \in \Delta_5, \quad 0 < \epsilon_{14} < \epsilon_{13}. \tag{3.10}$$

Then for $r \in \Delta_5$ we have

$$\epsilon_{14} > u_n(t, r) - u(t, r) > \epsilon_{13}$$

which is one contradiction with (3.10). •

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