Dynamics of PDE, Vol.5, No.4, 329-347, 2008

On a Semi-Linear Wave Equation Associated with Memory Conditions at the Boundaries: Stability and Asymptotic Expansion

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Communicated by S.-T. Yau, received August 25, 2008.

Abstract. In this paper the stability and asymptotic expansion of the weak solution of an initial-boundary problem relating to a semi-linear wave equation and two integral equations at the boundaries are given.

Contents

1. Introduction

We study the solution $u(x,t)$ of following semi-linear equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial u}{\partial x}(x,t)\right) + G\left(u(x,t)\right) + H\left(\frac{\partial u}{\partial t}(x,t)\right) = F(x,t), \tag{1.1}$$

1991 *Mathematics Subject Classification.* 35L15, 35L20; 35L70.
Key words and phrases. Contracted procedure, semi-linear wave equation, stability, asymptotic expansion.
The author is currently supported by the Academy of Finland and Emil Aaltonen Foundation.

where $0 < x < 1$, $0 < t < T$, associated with initial-boundary values given by

$$\mu(0,t)\frac{\partial u}{\partial x}(0,t) = g_0(t) + \int_0^t k_0(t-s)u(0,s)ds, \tag{1.2}$$

$$-\mu(1,t)\frac{\partial u}{\partial x}(1,t) = g_1(t) + \int_0^t k_1(t-s)u(1,s)ds, \tag{1.3}$$

$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{1.4}$$

where μ, G, H, F, g_0, k_0, g_1, k_1, u_0 and u_1 are given real functions satisfying conditions specified later. The terms $G\left(u(x,t)\right)$, $H\left(\frac{\partial u}{\partial t}(x,t)\right)$ are, respectively, called the damping, the source term of Eq. (1.1), and we shortly call the sum $G\left(u(x,t)\right) + H\left(\frac{\partial u}{\partial t}(x,t)\right)$ the damping-source term.

The problems of wave equations associated with memory conditions or integral equations at the boundaries have interested many mathematicians (see [**5**], [**11**], [**14**], [**15**], [**19**]-[**25**], [**27**]).

When $\mu(x,t) \equiv 1$ and $F(x,t) = 0$, in [**19**], Nguyen and Alain considered problem (1.1), (1.4) in the case of the full nonlinear damping-source term of u and $\frac{\partial u}{\partial t}$ associated with the homogeneous boundary at $x = 0$ and the non-homogeneous boundary condition at $x = 1$ given by

$$-\frac{\partial u}{\partial x}(1,t) \equiv Q(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.5}$$

where h is a positive constant; Q, g and k are given functions. We note that (1.5) is deduced from a Cauchy problem for an ordinary differential equation at the boundary $x = 1$ as follows

$$\begin{cases} Q''(t) + \omega^2 Q(t) = h\frac{\partial^2 u}{\partial t^2}(1,t),\ t \in (0,T), \\ Q(0) = Q_0,\ Q'(0) = Q_1, \end{cases} \tag{1.6}$$

where $\omega > 0$, Q_0 and Q_1 are given constants. This problem is a mathematical model describing the shock of a rigid body and a nonlinear viscoelastic bar resting on a rigid base. In this article, the authors obtained the unique solvability of the weak solution.

In [**25**], Santos studied the asymptotic behavior of the solution of problem (1.1), (1.2), (1.4) in the case of $\mu(x,t) \equiv \mu(t)$, $G(u) = H\left(\frac{\partial u}{\partial t}\right) = 0$, $F(x,t) = 0$ associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)\frac{\partial u}{\partial x}(1,s)ds = 0,\ t > 0, \tag{1.7}$$

in which g and μ are given functions. It is noted that the boundary conditions (1.5) and (1.7) are similar since their formal differences can be crossed out after solving the Volterra equation with respect to the variable $u(1,t)$ given by (1.7).

In [**21, 22, 23**], Nguyen, Lê and T. Nguyen considered the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)-(1.4) when $\mu(x,t) \equiv \mu(t)$, $G(u) = Ku$, $H\left(\frac{\partial u}{\partial t}\right) = \lambda\frac{\partial u}{\partial t}$ and the boundary condition (1.2) is homogeneous and the boundary value at $x = 1$, (1.3), is

$$-\mu(t)\frac{\partial u}{\partial x}(1,t) = g(t) + K_1(t)u(1,t) + \lambda_1(t)\frac{\partial u}{\partial t}(1,t) + \int_0^t k(t-s)u(1,s)ds, \tag{1.8}$$

where g, K_1, λ_1 and k are given functions.

When $\mu = 1$, $G(u) = K|u|^\alpha u$ and $H\left(\frac{\partial u}{\partial t}\right) = \lambda\left|\frac{\partial u}{\partial t}\right|^\beta \frac{\partial u}{\partial t}$ for $K, \lambda, \alpha, \beta \geq 0$, Nguyen, Alain and Tran [**20**] studied the unique solvability, the regularity of the weak solution of problem (1.1), (1.4) associated with the boundary conditions as follows:

$$\begin{cases} \frac{\partial u}{\partial x}(0,t) = g(t) + hu(0,t) - \int_0^t k(t-s)u(0,s)ds, \\ \frac{\partial u}{\partial x}(1,t) + K_1 u(1,t) + \lambda_1 \frac{\partial u}{\partial t}(1,t) = 0, \end{cases} \tag{1.9}$$

where h, K_1, λ_1 are given constants and g, k are given functions. In the case of $\alpha = \beta = 0$, the authors obtained the asymptotic expansion of the weak solution with respect to non-negative constants K and λ.

In the above articles, the authors mainly applied Faedo-Galerkin approximation to study the unique solvability.

In the case of homogeneous boundaries, in [**26**], Sengul investigated the existence of the global attractor of Eq.(1.1) in the case of

$$\mu(x,t) = 1,\ H\left(\frac{\partial u}{\partial t}\right) = \alpha \frac{\partial u}{\partial t},\ \alpha > 0$$

associated homogeneous boundary conditions and the initial conditions similar to (1.4). In [**1**], Aassila and Benaissa obtained the global unique solvability, also by Faedo-Galerkin approximation, and the decay for the solution of the following problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Phi\left(\displaystyle\int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx\right) \displaystyle\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \\ \qquad + g\left(\dfrac{\partial^2 u}{\partial t^2}\right) + f(u) = 0 \text{ in } \Omega \times [0, +\infty[, \\ u(x,t) = 0 \quad \text{on } \Gamma \times [0, +\infty[, \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on } \Omega, \end{cases} \tag{1.10}$$

for Ω a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$, where Φ, g, f, u_0 and u_1 are given functions. In [**4**], Benaddi and Rao obtained the energy decay rate of the solution by a shooting method for problem (1.1)-(1.4) where $\mu(x,t) = 1$, G and H are linear, $F(x,t) = 0$, and $g_0 = g_1 = k_0 = k_1 \equiv 0$ (or (1.2)-(1.3) are homogeneous). In addition, Phung [**24**] studied the stabilization of the wave equation with a localized linear dissipation in a three-dimensional bounded domain on which exists a trapped ray given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \sum\limits_{i=1}^3 \frac{\partial^2 u}{\partial x_i^2} + \alpha(x)\frac{\partial u}{\partial t} = 0 \text{ in } \Omega \times \mathbb{R}^+, \\ u(x,t) = 0 \text{ on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0,\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ in } \Omega, \end{cases} \tag{1.11}$$

for Ω a bounded domain in $\mathbb{R}^3$ with a boundary $\partial\Omega$ at least Lipschitz; α, u_0 and u_1 are given functions.

Regarding Mikusiński calculus, D. Takači and A. Takači studied the existence of the solution of problem (1.1), (1.4) in the field of Mikusiński when $\mu(x,t) = 1$, G and H are linear, and the boundary conditions are non-homogeneous.

In this paper, we study the stability and asymptotic expansion of the weak solution of problem (1.1)-(1.4). What we obtain here is considered as both the generalization and the more effective approach of those in Aassila and Benaissa [**1**], Nguyen and Alain [**19**], in Nguyen, Alain and Tran [**20**], in Santos [**25**], in Sengul [**26**], D. Takači and A. Takači [**27**] and in mine [**11, 14, 21, 22, 23**].

Furthermore, to obtain the unique solvability of problem (1.1)-(1.4), we apply a contracted procedure (see [**12**] and [**15**]) which exceeds a routine application of usual methods, namely the Faedo-Galerkin method with the compactness argument and the monotone operator method, for semi-linear damped wave equations as popularized by Jacques-Louis Lions several years ago (see [**16**], [**17**]) and also by Songmu Zheng [**30**]. Moreover, by this contracted procedure, we can cover unsolvable cases related to problem (1.1)-(1.4) regarding the solvability, and if the solvability does not hold then we obviously fail to discuss neither the stability nor the asymptotic expansion involved in this paper.

2. Preliminary results and notations

First we introduce some preliminary results and notations used in this paper. We omit the definitions of usual function spaces: C^m, L^p, $W^{m,p}$, H^m for $p \in [1, +\infty]$ and $m \in \mathbb{N}$.

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(0,1)$ or pair of dual scalar product of a continuous linear functional with an element of a function space. We denote by $\| \cdot \|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, $T > 0$, the Banach space of the real measurable functions $v : (0,T) \to X$, such that

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|_X^p dt \right)^{1/p} < \infty \text{ for } 1 \leq p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = \operatorname*{esssup}_{0<t<T} \|v(t)\|_X \text{ for } p = \infty.$$

In $H^1(0,1)$, we use the norm

$$\|v\|_{H^1(0,1)} = \sqrt{\|v\|_{L^2(0,1)}^2 + \|v'\|_{L^2(0,1)}^2}, \ v \in H^1(0,1).$$

Then we have the following lemma whose proof is omitted:

Lemma 1. *The embedding $H^1(0,1) \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \sqrt{2} \|v\|_{H^1(0,1)}, \tag{2.1}$$

for all $v \in H^1(0,1)$.

Now let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{Z}_+^2$, we denote

$$\begin{cases} y \leq x \Leftrightarrow \begin{cases} y_1 \leq x_1, \\ y_2 \leq x_2, \end{cases} \\ x! = x_1! x_2!, \\ C_x^y = \frac{x!}{y!(x-y)!}. \end{cases}$$

Then there is a lemma as follows:

Lemma 2. *Let $m \in \mathbb{Z}_+$ and $n, i, j \in \mathbb{N}$. Then, the equality*

$$\left(\sum_{1 \le i+j \le n} a_{ij} \varepsilon^i \delta^j \right)^m = \sum_{m \le i+j \le mn} [a_{ij}]_m \, \varepsilon^i \delta^j \tag{2.2}$$

holds for ε, δ, $a_{ij} \in \mathbb{R}$ and

$$[a_{ij}]_m = \begin{cases} a_{ij}, i+j \le n, \ m = 1, \\ \sum\limits_{(k,h) \in [\mathbb{Z}_{ij}]_m} a_{(i-k)(j-h)} \, [a_{kh}]_{m-1} \,, \ m \le i+j \le mn, \ m \ge 2, \end{cases} \tag{2.3}$$

in which the family $[\mathbb{Z}_{ij}]_m$ is given by

$$\big\{ (k,h) \in \mathbb{Z}_+^2 : (k,h) \le (i,j), \, 1 \le i-k+j-h \le n, \\ m-1 \le k+h \le (m-1)n \big\}.$$

Proof. In the case of $m = 1$, it is clear that (2.2) holds with respect to $[a_{ij}]_1 = a_{ij}$, $1 \le i+j \le n$. When $m \ge 2$, by putting

$$f(\varepsilon, \delta) = \left(\sum_{1 \le i+j \le n} a_{ij} \varepsilon^i \delta^j \right)^m, \tag{2.4}$$

we have the Maclaurin formula of f up to order mn as follows

$$\begin{aligned} f(\varepsilon, \delta) &= \sum_{0 \le i+j \le mn} \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial \varepsilon^i \partial \delta^j}(0,0) \varepsilon^i \delta^j \\ &= \sum_{m \le i+j \le mn} \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial \varepsilon^i \partial \delta^j}(0,0) \varepsilon^i \delta^j. \end{aligned} \tag{2.5}$$

Therefore, we deduce from (2.2) and (2.5), that

$$[a_{ij}]_m = \frac{1}{i!j!} \frac{\partial^{i+j} f}{\partial \varepsilon^i \partial \delta^j}(0,0), \ m \le i+j \le mn. \tag{2.6}$$

Moreover, from (2.2) and (2.4), f can be rewritten as follows

$$\begin{cases} f(\varepsilon, \delta) = f_1(\varepsilon, \delta) \, f_2(\varepsilon, \delta), \\ f_1(\varepsilon, \delta) = \left(\sum\limits_{1 \le i+j \le n} a_{ij} \varepsilon^i \delta^j \right), \\ f_2(\varepsilon, \delta) = \left(\sum\limits_{m-1 \le i+j \le (m-1)n} [a_{ij}]_{m-1} \, \varepsilon^i \delta^j \right). \end{cases} \tag{2.7}$$

Since f is differentiable at $(0,0)$, it follows from (2.7), that

$$\frac{\partial^{i+j} f}{\partial \varepsilon^i \partial \delta^j}(0,0) = \sum_{(k,h) \le (i,j)} C_{(i,j)}^{(k,h)} \frac{\partial^{i-k+j-h} f_1}{\partial \varepsilon^{i-k} \partial \delta^{j-h}}(0,0) \frac{\partial^{k+h} f_2}{\partial \varepsilon^k \partial \delta^h}(0,0). \tag{2.8}$$

Since (2.2), $(2.7)_2$ and $[a_{ij}]_1 = a_{ij}$, we obtain

$$a_{ij} = \frac{1}{i!j!} \frac{\partial^{i+j} f_1}{\partial \varepsilon^i \partial \delta^j}(0,0), \ 1 \le i+j \le n. \tag{2.9}$$

Hence, we have

$$\frac{\partial^{i-k+j-h} f_1}{\partial \varepsilon^{i-k} \partial \delta^{j-h}}(0,0) = (i-k)!(j-h)! a_{(i-k)(j-h)}, \; 1 \le i-k+j-h \le n. \tag{2.10}$$

Moreover, we also obtain from (2.2), (2.6) and $(2.7)_3$, that

$$\frac{\partial^{k+h} f_2}{\partial \varepsilon^{k} \partial \delta^{h}}(0,0) = k!h! \left[a_{kh}\right]_{m-1}, \; m-1 \le k+h \le (m-1)n. \tag{2.11}$$

On account of (2.6), (2.8), (2.10) and (2.11), we deduce that (2.3) hlods. Thus, the proof of this lemma is complete. □

Remark 1. In [**18**], we obtained an original result which is more general than Lemma 2. However, since [**18**] was unpublished, we must give the detailed proof of Lemma 2 in this paper.

3. Unique solvability and regularity

In this section, we shortly list some results which were obtained in [**15**] and can be independently studied by the contracted procedure in [**12**].

We make some following essential assumptions:

(A_μ) $\frac{\partial \mu}{\partial t} \in L^1\left(0,T; L^\infty(0,1)\right)$, $\mu(x,t) \ge \mu_0 > 0$;

$(A_{GH}^{(1)})$ G, $H \in L^2(\mathbb{R})$;

$(A_{GH}^{(2)})$ $\exists K_G, K_H > 0$:

$$|G(u) - G(v)| \le K_G |u-v|,$$
$$|H(u) - H(v)| \le K_H |u-v|,$$

for u, $v \in \mathbb{R}$;

(A_F) $F \in L^2\left((0,1)\times(0,T)\right)$;

(A_g) $g_0, g_1 \in H^1(0,T)$;

(A_k) $k_0, k_1 \in W^{1,1}(0,T)$;

(A_u) $u_0 \in H^1(0,1)$, $u_1 \in L^2(0,1)$.

In this paper, we say that a function

$$u \in H^1\left(0,T; L^2(0,1)\right) \cap L^\infty\left(0,T; H^1(0,1)\right)$$

is a weak solution of problem (1.1)-(1.4) iff

$$\begin{cases} \frac{d}{dt}\left\langle \frac{\partial u}{\partial t}(t), v\right\rangle + \left\langle \mu(\cdot,t)\frac{\partial u}{\partial x}(t), v'\right\rangle + Q_0(t)v(0) + Q_1(t)v(1) \\ \qquad\qquad + \left\langle G\left(u(t)\right) + H\left(\frac{\partial u}{\partial t}(t)\right), v\right\rangle = \left\langle F(\cdot,t), v\right\rangle, \\ u(x,0) = u_0(x), \; \frac{\partial u}{\partial t}(x,0) = u_1(x), \\ Q_0(t) = g_0(t) + \int_0^t k_0(t-s)u(0,s)ds, \\ Q_1(t) = g_1(t) + \int_0^t k_1(t-s)u(1,s)ds, \end{cases}$$

for each $v \in H^1(0,1)$ and *a.e.* time $0 \le t \le T$. In this case we can also say that problem (1.1)-(1.4) is weakly solvable in $H^1\left(0,T; L^2(0,1)\right) \cap L^\infty\left(0,T; H^1(0,1)\right)$.

Then we have the following theorem:

Theorem 1. *Let* (A_μ), $(A_{GH}^{(1)})$, $(A_{GH}^{(2)})$, (A_F), (A_g), (A_k) *and* (A_u) *hold. Then, for* $T > 0$, *problem* (1.1)-(1.4) *has a unique weak solution* $u(x,t)$ *satisfying*

$$u \in H^1\left(0,T; L^2(0,1)\right) \cap L^\infty\left(0,T; H^1(0,1)\right). \tag{3.1}$$

REMARK 2. When $H\left(\frac{\partial u}{\partial t}\right) = \lambda \left|\frac{\partial u}{\partial t}\right|^p \frac{\partial u}{\partial t}$, $p > 0$, $\lambda < 0$, it fails to apply Faedo-Galerkin method for the unique solvability of problem (1.1)-(1.4). This method is only applicable for linear or some special nonlinear problems (as in [**5**], [**11**], [**14**], [**19**]-[**25**]).

REMARK 3. For some special nonlinear forms of $G(u) + H\left(\frac{\partial u}{\partial t}\right)$ such as

$$K\,|u|^p\,u + \lambda \left|\frac{\partial u}{\partial t}\right|^q \frac{\partial u}{\partial t},\ (p,q) \in \mathbb{R}_+^2 \setminus (0,0),\ (K,\lambda) \in \mathbb{R}_+^2,$$

$$G(u) + K\frac{\partial u}{\partial t},\ K \in \mathbb{R},$$

it is possible to apply Faedo-Galerkin approximation for the unique solvability of problem (1.1)-(1.4); however, some more assumptions for μ, G and H must be modified and unfortunately the proofs in such cases are usually not only very long but also truly messy such as a priori estimates or passing to the limit by monotone techniques as in [**16**] and [**17**].

To study the weak solution's smoothness with respect to the smoothness of given data, assumptions (A_μ), $(A_{GH}^{(1)})$, $(A_{GH}^{(2)})$, (A_F), (A_g), (A_k) and (A_u) are strengthened as follows:

$(\widetilde{A}_\mu)$ $\mu \in C^1([0,1]\times[0,T])$, $\frac{\partial^2\mu}{\partial t^2} \in L^2(0,T;L^\infty(0,1))$, $\mu(x,t) \geq \mu_0 > 0$;
$(\widetilde{A}_{GH})$ $G, H \in C^1(\mathbb{R})$, $|H'(\eta)| \leq C_H|\eta|^\alpha$, $\forall\eta \in \mathbb{R}$, and $\alpha > 0$, $C_H \geq 0$;
$(\widetilde{A}_F)$ F, $\frac{\partial F}{\partial t} \in L^2((0,1)\times(0,T))$;
$(\widetilde{A}_g)$ $g_0, g_1 \in H^2(0,T)$;
$(\widetilde{A}_k)$ $k_0, k_1 \in W^{2,1}(0,T)$;
$(\widetilde{A}_u)$ $u_0 \in H^2(0,1)$, $u_1 \in H^1(0,1)$.

It is clear that problem (1.1)-(1.4) has a unique weak solution

$$u \in H^1\left(0,T;L^2(0,1)\right) \cap L^\infty\left(0,T;H^1(0,1)\right)$$

for which $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$, $(\widetilde{A}_k)$ and $(\widetilde{A}_u)$ hold. The regularity of this weak solution is stated in the following theorem:

THEOREM 2. *Let $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$, $(\widetilde{A}_k)$ and $(\widetilde{A}_u)$ hold. Then, for $T > 0$, problem* (1.1)-(1.4) *has a unique weak solution $u(x,t)$ satisfying*

$$u \in H^2\left(0,T;L^2(0,1)\right) \cap H^1\left(0,T;H^1(0,1)\right) \cap L^\infty\left(0,T;H^2(0,1)\right). \tag{3.2}$$

4. The stability of the weak solution

In this section, we study the stability of the weak solution of problem (1.1)-(1.4) in the sense that this weak solution continuously depends on some given data.

By assuming the functions G, H, u_0 and u_1 satisfy $(\widetilde{A}_{GH})$ and $(\widetilde{A}_u)$, we have from Theorem 2 that problem (1.1)-(1.4) has a unique weak solution u depending on μ, F, g_0, g_1, k_0 and k_1.

Consider

$$u = u\left(\mu, F, g_0, g_1, k_0, k_1\right),$$

where μ, F, g_0, g_1, k_0 and k_1, respectively, satisfy $(\widetilde{A}_\mu)$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$ and $(\widetilde{A}_k)$. Let G, H, u_0 and u_1 be fixed functions such that $(\widetilde{A}_{GH})$ and $(\widetilde{A}_u)$ hold. For $\mu_0 > 0$

given, we put

$$\Xi(\mu_0) = \Big\{ (\mu, F, g_0, g_1, k_0, k_1) : \mu,\ F,\ g_0,\ g_1,\ k_0 \text{ and } k_1 \\ \text{satisfy } (\widetilde{A}_\mu),\ (\widetilde{A}_F),\ (\widetilde{A}_g) \text{ and } (\widetilde{A}_k),\ \text{respectively} \Big\}.$$

Right then, we have the following theorem:

THEOREM 3. *Let $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$, $(\widetilde{A}_k)$ and $(\widetilde{A}_u)$ hold. For every $T > 0$, the weak solution $u(x,t)$ of problem (1.1)-(1.4) is stable with respect to μ, F, g_0, g_1, k_0 and k_1 in the sense that if $(\mu, F, g_0, g_1, k_0, k_1)$, $(\mu_i, F_i, g_{0i}, g_{1i}, k_{0i}, k_{1i}) \in \Xi(\mu_0)$ and*

$$\begin{cases} \mu_i \to \mu & \text{in} \quad C^1\left([0,1]\times[0,T]\right), \\ F_i \to F,\ \frac{\partial F_i}{\partial t} \to \frac{\partial F}{\partial t} & \text{in} \quad L^2\left(0,T; L^2(0,1)\right), \\ g_{0i} \to g_0,\ g_{1i} \to g_1 & \text{in} \quad H^2(0,T), \\ k_{0i} \to k_0,\ k_{1i} \to k_1 & \text{in} \quad W^{2,1}(0,T), \end{cases} \tag{4.1}$$

when $i \to +\infty$, then

$$\begin{cases} u_i \to u & \text{in} \quad L^\infty\left(0,T; H^1(0,1)\right), \\ \frac{\partial u_i}{\partial t} \to \frac{\partial u}{\partial t} & \text{in} \quad L^\infty\left(0,T; L^2(0,1)\right), \end{cases} \tag{4.2}$$

as $i \to +\infty$, where $u_i = u_i\left(\mu_i, F_i, g_{0i}, g_{1i}, k_{0i}, k_{1i}\right)$.

PROOF. From Theorem 2, we can deduce that

$$\begin{cases} \left\|\frac{\partial u}{\partial t}(t)\right\|^2_{L^2(0,1)} + \mu_0 \left\|\frac{\partial u}{\partial x}(t)\right\|^2_{L^2(0,1)} \le M_T, \\ \left\|\frac{\partial^2 u}{\partial t^2}(t)\right\|^2_{L^2(0,1)} + \mu_0 \left\|\frac{\partial^2 u}{\partial x \partial t}(t)\right\|^2_{L^2(0,1)} \le M_T \end{cases} \tag{4.3}$$

for all $t \in [0,T]$ if the given data μ, F, g_0, g_1, k_0 and k_1 satisfy

$$\begin{cases} \|\mu\|_{C^1([0,1]\times[0,T])} \le \overline{\mu}, \\ \|F\|_{L^2((0,1)\times(0,T))} + \left\|\frac{\partial F}{\partial t}\right\|_{L^2((0,1)\times(0,T))} \le \overline{F}, \\ \|g_0\|_{H^2(0,T)} + \|g_1\|_{H^2(0,T)} \le \overline{g}, \\ \|k_0\|_{W^{2,1}(0,T)} + \|k_1\|_{W^{2,1}(0,T)} \le \overline{k}, \end{cases} \tag{4.4}$$

where $\overline{\mu}$, $\overline{F}$, $\overline{g}$ and $\overline{k}$ are fixed positive constants, and M_T is a positive constant depending only on T, μ_0, u_0, u_1, $\overline{\mu}$, $\overline{F}$, $\overline{g}$, $\overline{k}$ but is independent of μ, F, g_0, g_1, k_0, k_1.

From (4.1), it is clear that there exist positive constants $\overline{\mu}$, $\overline{F}$, $\overline{g}$ and $\overline{k}$ such that μ_i, F_i, g_{0i}, g_{1i}, k_{0i} and k_{1i} satisfy (4.4) for $(\mu, F, g_0, g_1, k_0, k_1) = (\mu_i, F_i, g_{0i}, g_{1i}, k_{0i}, k_{1i})$. Therefore, we conclude that the weak solution $u_i(x,t)$ of problem (1.1)-(1.4) with $(\mu, F, g_0, g_1, k_0, k_1) = (\mu_i, F_i, g_{0i}, g_{1i}, k_{0i}, k_{1i})$ satisfies (4.3), namely we obtain

$$\begin{cases} \left\|\frac{\partial u_i}{\partial t}(t)\right\|^2_{L^2(0,1)} + \mu_0 \left\|\frac{\partial u_i}{\partial x}(t)\right\|^2_{L^2(0,1)} \le M_T, \\ \left\|\frac{\partial^2 u_i}{\partial t^2}(t)\right\|^2_{L^2(0,1)} + \mu_0 \left\|\frac{\partial^2 u_i}{\partial x \partial t}(t)\right\|^2_{L^2(0,1)} \le M_T \end{cases} \tag{4.5}$$

for all $t \in [0,T]$.

Now, by letting $\widehat{\mu}_i$, $\widehat{F}_i$, $\widehat{g}_{0i}$, $\widehat{g}_{1i}$, $\widehat{k}_{0i}$ and $\widehat{k}_{1i}$ be functions given by

$$\begin{cases} \widehat{\mu}_i = \mu_i - \mu, \\ \widehat{F}_i = F_i - F, \\ \widehat{g}_{0i} = g_{0i} - g_0,\ \widehat{g}_{1i} = g_{1i} - g_1, \\ \widehat{k}_{0i} = k_{0i} - k_0,\ \widehat{k}_{1i} = k_{1i} - k_1, \end{cases} \tag{4.6}$$

we deduce that $w_i(x,t) = u_i(x,t) - u(x,t)$ satisfies the following variational problem:

$$\begin{cases} \left\langle \dfrac{\partial^2 w_i}{\partial t^2}(t), v \right\rangle + \left\langle \mu(\cdot,t)\dfrac{\partial w_i}{\partial x}(t), v' \right\rangle + \widehat{Q}_{0i}(t)v(0) + \widehat{Q}_{1i}(t)v(1) \\ \qquad = -\left\langle G\left(u_i(t)\right) - G\left(u(t)\right) + H\left(\dfrac{\partial u_i}{\partial t}(t)\right) - H\left(\dfrac{\partial u}{\partial t}(t)\right), v \right\rangle \\ \qquad\quad - \left\langle \widehat{\mu}_i(\cdot,t)\dfrac{\partial u_i}{\partial x}(t), v' \right\rangle + \left\langle \widehat{F}_i(\cdot,t), v \right\rangle, \\ w_i(x,0) = \frac{\partial w_i}{\partial t}(x,0) = 0, \\ \widehat{Q}_{0i}(t) = \check{g}_{0i}(t) + \int_0^t k_0(t-s)w_i(0,s)ds, \\ \widehat{Q}_{1i}(t) = \check{g}_{1i}(t) + \int_0^t k_1(t-s)w_i(1,s)ds, \\ \check{g}_{0i}(t) = \widehat{g}_{0i}(t) + \int_0^t \widehat{k}_0(t-s)u_i(0,s)ds, \\ \check{g}_{1i}(t) = \widehat{g}_{1i}(t) + \int_0^t \widehat{k}_1(t-s)u_i(1,s)ds \end{cases} \tag{4.7}$$

for each $v \in H^1(0,1)$ and a.e. time $0 \le t \le T$.

In (4.7), replacing v by $\frac{\partial w_i}{\partial t}$, then integrating from 0 to t, we obtain that

$$\begin{aligned} Ⓒ_i(t) = &\int_0^t \left\langle \frac{\partial \mu}{\partial s}(\cdot,s), \left|\frac{\partial w_i}{\partial x}(s)\right|^2 \right\rangle ds - 2\int_0^t \left\langle \widehat{\mu}_i(\cdot,s)\frac{\partial u_i}{\partial x}(s), \frac{\partial^2 w_i}{\partial x \partial s}(s) \right\rangle ds \\ &- 2\int_0^t \left\langle G\left(u_i(s)\right) - G\left(u(s)\right), \frac{\partial w_i}{\partial s}(s) \right\rangle ds \\ &- 2\int_0^t \left\langle H\left(\frac{\partial u_i}{\partial s}(s)\right) - H\left(\frac{\partial u}{\partial s}(s)\right), \frac{\partial w_i}{\partial s}(s) \right\rangle ds \\ &- 2\int_0^t \widehat{Q}_{0i}(s)\frac{\partial w_i}{\partial s}(0,s)ds - 2\int_0^t \widehat{Q}_{1i}(s)\frac{\partial w_i}{\partial s}(1,s)ds \\ &+ 2\int_0^t \left\langle \widehat{F}_i(\cdot,s), \frac{\partial w_i}{\partial s}(s) \right\rangle ds, \end{aligned} \tag{4.8}$$

in which

$$Ⓒ_i(t) = \left\|\frac{\partial w_i}{\partial t}(t)\right\|^2_{L^2(0,1)} + \left\|\sqrt{\mu(\cdot,t)}\frac{\partial w_i}{\partial x}(t)\right\|^2_{L^2(0,1)}. \tag{4.9}$$

From (4.5), (4.8), (4.9) and assumptions $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widetilde{A}_g)$, $(\widetilde{A}_k)$, we have some estimates as follows:

$$\int_0^t \left\langle \frac{\partial \mu}{\partial s}(\cdot,s), \left|\frac{\partial w_i}{\partial x}(s)\right|^2 \right\rangle ds \le \frac{1}{\mu_0}\int_0^t \left\|\frac{\partial \mu}{\partial s}(\cdot,s)\right\|_{L^\infty(0,1)} ©_i(s)ds, \tag{4.10}$$

$$-2\int_0^t \left\langle \widehat{\mu}_i(\cdot,s)\frac{\partial u_i}{\partial x}(s), \frac{\partial^2 w_i}{\partial x \partial s}(s)\right\rangle ds \le \widetilde{M}_i + \beta ©_i(t) + \int_0^t ©_i(s)ds, \tag{4.11}$$

$$-2\int_0^t \left\langle G(u_i(s)) - G(u(s)), \frac{\partial w_i}{\partial s}(s)\right\rangle ds \le \left(1+TG_0^2\right)\int_0^t ©_i(s)ds, \tag{4.12}$$

$$\begin{aligned} &-2\int_0^t \left\langle H\left(\frac{\partial u_i}{\partial s}(s)\right) - H\left(\frac{\partial u}{\partial s}(s)\right), \frac{\partial w_i}{\partial s}(s)\right\rangle ds \\ &\qquad\qquad \le \left(1+H_0^2\right)\int_0^t ©_i(s)ds, \end{aligned} \tag{4.13}$$

$$-2\int_0^t \widehat{Q}_{ji}(s)\frac{\partial w_i}{\partial s}(j,s)ds \le \widehat{q}_{j1i} + \frac{4\beta}{\mu_0}©_i(t) + \widehat{q}_{j2i}\int_0^t ©_i(s)ds,\ j=0,1, \tag{4.14}$$

$$2\int_0^t \left\langle \widehat{F}_i(\cdot,s), \frac{\partial w_i}{\partial s}(s)\right\rangle ds \le \int_0^t \left\|\widehat{F}_i(\cdot,s)\right\|^2_{L^2(0,1)} ds + \int_0^t ©_i(s)ds, \tag{4.15}$$

for some $\beta > 0$ and $\widetilde{M}_i$, G_0, H_0, $\widehat{q}_{j1i}$, $\widehat{q}_{j1i}$ are non-negative constants given by

$$\begin{cases} \widetilde{M}_i = \frac{M_T}{\mu_0^2}\left(\frac{1}{\beta}\|\widehat{\mu}_i\|^2_{L^\infty((0,1)\times(0,T))} + 4T\|\widehat{\mu}_i\|^2_{C^1((0,1)\times(0,T))}\right), \\ G_0 = \max|G'(y)|,\ |y| \le \sqrt{2\left(\|u_0\|^2_{L^2(0,1)} + \left(T^2+\frac{1}{\mu_0}\right)M_T\right)}, \\ H_0 = \max|H'(z)|,\ |z| \le \sqrt{2\left(1+\frac{1}{\mu_0}M_T\right)}, \\ \widehat{q}_{j1i} = \frac{1}{\beta}\|\breve{g}_{ji}\|^2_{C^0([0,T])} + \left\|\breve{g}'_{ji}\right\|^2_{L^2(0,T)},\ j=0,1, \\ \widehat{q}_{j2i} = \left(4\beta T + \left[T^2 + \dfrac{1}{\mu_0}\right]\left[4 + 4|k_j(0)| + \dfrac{2}{\beta}\|k_j\|^2_{L^2(0,T)} \right.\right. \\ \qquad\qquad \left.\left. + 2T\|k'_j\|^2_{L^2(0,T)}\right]\right),\ j=0,1. \end{cases} \tag{4.16}$$

As a result, we deduce from (4.8), (4.10)-(4.16), that

$$©_i(t) \le \widehat{M}_i + \beta\left(1+\frac{4}{\mu_0}\right)©_i(t) + \int_0^t p_i(s)©_i(s)ds, \tag{4.17}$$

where $\widehat{M_T}$ is a non-negative constant given by

$$\widehat{M}_i = \widetilde{M}_i + \widehat{q}_{01i} + \widehat{q}_{11i} + \int_0^t \left\|\widehat{F}_i(\cdot,s)\right\|^2_{L^2(0,1)} ds \tag{4.18}$$

and

$$p_i(t) = 4 + TG_0^2 + H_0^2 + q_{02} + q_{12} + \frac{1}{\mu_0}\left\|\frac{\partial \mu}{\partial s}(\cdot,s)\right\|_{L^\infty(0,1)}. \tag{4.19}$$

Hence, by choosing $\beta > 0$ such that $\beta \le \frac{\mu_0}{2(\mu_0+4)}$, we conclude from (4.17), that

$$©_i(t) \le 2\widehat{M}_i \exp\left(\int_0^t p_i(s)ds\right) \text{ for all } t \in [0,T]. \tag{4.20}$$

From $(\widetilde{A}_\mu)$ and (4.19), we have

$$p_i \in L^1(0,T),\ \forall i = 1,2,...,\ \text{and}\ \exp\left(\int_0^t p_i(s)ds\right) < +\infty. \tag{4.21}$$

In addition, on account of (4.1), $(4.7)_{5,6}$, $(4.16)_{1,4}$ and (4.18), we deduce that

$$\widehat{M}_i \to 0 \text{ as } i \to +\infty. \tag{4.22}$$

Finally, it is clear that (4.2) is obviously concluded from (4.1), (4.9) and (4.20)-(4.22). Hence, this proof is complete. □

5. Low-frequency asymptotic expansion of the weak solution

In this section, let $G(u(x,t)) \equiv \varepsilon G(u(x,t))$ and $H\left(\frac{\partial u}{\partial t}(x,t)\right) \equiv \delta H\left(\frac{\partial u}{\partial t}(x,t)\right)$ such that

$(A_{\varepsilon,\delta})$ $\quad \varepsilon, \delta \in \mathbb{R}$.

In addition, we modify smooth assumptions for G and H as follows:

$(\widehat{A}_{GH})$ $\quad G,\ H \in C^{n+1}(\mathbb{R})$ for $n \in \mathbb{N}$ given.

Now we consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial}{\partial x}\left(\mu(x,t)\dfrac{\partial u}{\partial x}(x,t)\right) + \varepsilon G(u(x,t)) \\ \qquad\qquad + \delta H\left(\dfrac{\partial u}{\partial t}(x,t)\right) = F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ \mu(0,t)\frac{\partial u}{\partial x}(0,t) = g_0(t) + \int_0^t k_0(t-s)u(0,s)ds, \\ -\mu(1,t)\frac{\partial u}{\partial x}(1,t) = g_1(t) + \int_0^t k_1(t-s)u(1,s)ds, \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{cases} \tag{5.1}$$

On account of $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(A_{\varepsilon,\delta})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$ and $(\widetilde{A}_k)$, by Theorem 2, problem (5.1) has a unique weak solution $u(x,t)$ depending on (ε,δ) such that

$$u = u(\varepsilon,\delta)$$

satisfying (3.1), namely,

$$u \in H^2\left(0,T;L^2(0,1)\right) \cap H^1\left(0,T;H^1(0,1)\right) \cap L^\infty\left(0,T;H^2(0,1)\right).$$

Our purpose here is to investigate the *low-frequency asymptotic expansion* (see [**9**]) of the weak solution $u(x,t)$ of problem (5.1) with respect to two parameters ε, δ.

By putting

$$\begin{cases} Uu \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial u}{\partial x}\right), \\ \partial U_0 u \equiv \mu(0,t)\frac{\partial u}{\partial x}(0,t), \\ \partial U_1 u \equiv -\mu(1,t)\frac{\partial u}{\partial x}(1,t), \end{cases} \tag{5.2}$$

we concern the solvable problem $\left(U_{\varepsilon\delta}\right)$ given by

$$\begin{cases} Uu = -\varepsilon G(u) - \delta H\left(\dfrac{\partial u}{\partial t}\right) + F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ \partial U_0 u = g_0(t) + \int_0^t k_0(t-s)u(0,s)ds, \\ \partial U_1 u = g_1(t) + \int_0^t k_1(t-s)u(1,s)ds, \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x), \\ u \equiv u(\varepsilon,\delta) \in H^2\left(0,T;L^2(0,1)\right) \cap H^1\left((0,T)\times(0,1)\right) \cap L^\infty\left(0,T;H^2(0,1)\right), \end{cases}$$

which is perturbed with respect to both small parameters ε, δ such that

$$|\varepsilon| \leq 1, \ \ |\delta| \leq 1. \tag{5.3}$$

Firstly we construct a sequence of weak solutions $\{u_{ij}\}$ from a family of solvable problems $\{(U_{ij})\}$, respectively, such that $i, j \in \mathbb{N}$ and $i + j \leq n$.

We begin with the case $i = j = 0$. Let u_{00} be the weak solution of problem (U_{00}) in regard to $\varepsilon = \delta = 0$, namely we have

(5.4)
$$\left(U_{00}\right) \begin{cases} Uu_{00} \equiv A_{00} \equiv F(x,t), \ 0 < x < 1, \ 0 < t < T, \\ \partial U_0 u_{00} = g_0(t) + \int_0^t k_0(t-s) u_{00}(0,s) ds, \\ \partial U_1 u_{00} = g_1(t) + \int_0^t k_1(t-s) u_{00}(1,s) ds, \\ u_{00}(x,0) = u_0(x), \ \frac{\partial u_{00}}{\partial t}(x,0) = u_1(x), \\ u_{00} \in H^2\left(0,T; L^2(0,1)\right) \cap H^1\left((0,T)\times(0,1)\right) \cap L^\infty\left(0,T; H^2(0,1)\right). \end{cases}$$

When $1 \leq i + j \leq n$, consider $\{u_{ij}\}$ as the sequence of the weak solutions of the family of solvable problems $\{(U_{ij})\}$, which is defined as follows

(5.5)
$$\left(U_{ij}\right) \begin{cases} Uu_{ij} \equiv A_{ij}, \ 0 < x < 1, \ 0 < t < T, \\ \partial U_0 u_{ij} = \int_0^t k_0(t-s) u_{ij}(0,s) ds, \\ \partial U_1 u_{ij} = \int_0^t k_1(t-s) u_{ij}(1,s) ds, \\ u_{ij}(x,0) = 0, \ \frac{\partial u_{ij}}{\partial t}(x,0) = 0, \\ u_{ij} \in H^2\left(0,T; L^2(0,1)\right) \cap H^1\left((0,T)\times(0,1)\right) \cap L^\infty\left(0,T; H^2(0,1)\right), \end{cases}$$

where A_{ij}, $1 \leq i + j \leq n$, are defined by the following recursive scheme:

$$\begin{cases} A_{10} = -G(u_{00}), \\ A_{01} = -H\left(\frac{\partial u_{00}}{\partial t}\right), \\ A_{i0} = -\sum\limits_{k=1}^{i+j-1} \frac{1}{k!} G^{(k)}(u_{00}) \left[u_{(i-1)0}\right]_k, \ 2 \leq i \leq n, \\ A_{0j} = -\sum\limits_{k=1}^{i+j-1} \frac{1}{k!} H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right) \left[\frac{\partial u_{0(j-1)}}{\partial t}\right]_k, \ 2 \leq j \leq n, \\ A_{ij} = -\sum\limits_{k=1}^{i+j-1} \dfrac{1}{k!} \left(G^{(k)}(u_{00}) \left[u_{(i-1)j}\right]_k - H^{(k)}\left(\dfrac{\partial u_{00}}{\partial t}\right) \left[\dfrac{\partial u_{i(j-1)}}{\partial t}\right]_k\right), \\ \qquad\qquad\qquad\qquad 2 \leq i + j \leq n, i \geq 1, \ j \geq 1, \end{cases} \tag{5.6}$$

in which $[a_{ij}]_k$ is defined as what in Lemma 2.

Then we obtain *low-frequency asymptotic expansion* of the weak solution $u(x,t)$ of problem (5.1) with respect to two parameters ε, δ in the following theorem:

THEOREM 4. *For ε, $\delta \in \mathbb{R}$, if $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widehat{A}_{GH})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$, $(\widetilde{A}_k)$ and (5.3) hold, then problem $(U_{\varepsilon\delta})$ has a unique weak solution $u = u(\varepsilon, \delta)$ satisfying the asymptotic estimates with respect to two parameters ε and δ up to order $n+1$ as* *follows:*

(5.7)
$$\left\| \frac{\partial u}{\partial t} - \sum_{0\le i+j\le n} \frac{\partial u_{ij}}{\partial t}\varepsilon^i\delta^j \right\|_{L^\infty(0,T;L^2(0,1))} + \left\| u - \sum_{0\le i+j\le n} u_{ij}\varepsilon^i\delta^j \right\|_{L^\infty(0,T;H^1(0,1))} \le C\left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}},$$

in which u_{ij} is the unique weak solution of problem (U_{ij}), respectively, for $0 \le i+j \le n$ and C is a positive constant independent of ε, δ.

Proof. By putting

$$\begin{cases} v \equiv \sum\limits_{i+j\le n} u_{ij}\varepsilon^i\delta^j, \\ w \equiv u - v, \end{cases} \tag{5.8}$$

we have that w satisfies the following problem

$$\begin{cases} Uw = -\,\varepsilon\left(G\left(w+v\right) - G\left(v\right)\right) - \delta\left(H\left(\dfrac{\partial}{\partial t}(w+v)\right) - H\left(\dfrac{\partial v}{\partial t}\right)\right) \\ \qquad + W_n(\varepsilon,\delta),\ 0<x<1,\ 0<t<T, \\ \partial U_0 w = \int_0^t k_0(t-s)w(0,s)ds, \\ \partial U_1 w = \int_0^t k_1(t-s)w(1,s)ds, \\ w(x,0) = 0,\ \frac{\partial w}{\partial t}(x,0) = 0, \\ w \in H^2\left(0,T;L^2(0,1)\right) \cap H^1\left((0,T)\times(0,1)\right) \cap L^\infty\left(0,T;H^2(0,1)\right) \end{cases} \tag{5.9}$$

where

$$W_n(\varepsilon,\delta) = F(x,t) - \varepsilon G(v) - \delta H\left(\frac{\partial v}{\partial t}\right) - \sum_{i+j\le n} A_{ij}\varepsilon^i\delta^j. \tag{5.10}$$

The estimate of $W_n(\varepsilon,\delta)$ is given in the following lemma:

Lemma 3. *Under assumptions $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widehat{A}_{GH)})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$ and $(\widetilde{A}_k)$, the following estimate*

$$\|W_n(\varepsilon,\delta)\|_{L^\infty(0,T;L^2(0,1))} \le W_T\left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}} \tag{5.11}$$

holds for $\varepsilon,\delta\in\mathbb{R}$ satisfying (5.3), where W_T is a non-negative constant depending only on G, H, u_{ij} and $\frac{\partial u_{ij}}{\partial t}$ for $0\le i+j\le n$.

The detailed proof of this lemma will be specified later.

Now, by multiplying the two sides of $(5.9)_1$ by $\frac{\partial w}{\partial t}$ and taking into account (5.2), we deduce after integrating with respect to time variable that

$$
\begin{aligned}
\Theta(t) = &\int_0^t \left\langle \frac{\partial \mu}{\partial s}(\cdot,s), \left(\frac{\partial w}{\partial x}(s)\right)^2 \right\rangle ds \\
&- 2\int_0^t \frac{\partial w}{\partial s}(0,s)\left(\int_0^s k_0(s-\tau)w(0,\tau)d\tau\right) ds \\
&- 2\int_0^t \frac{\partial w}{\partial s}(1,s)\left(\int_0^s k_1(s-\tau)w(1,\tau)d\tau\right) ds \\
&- 2\varepsilon \int_0^t \left\langle G(w+v) - G(v), \frac{\partial w}{\partial s}(s) \right\rangle ds \\
&- 2\delta \int_0^t \left\langle H\left(\frac{\partial}{\partial t}(w+v)\right) - H\left(\frac{\partial v}{\partial t}\right), \frac{\partial w}{\partial s}(s) \right\rangle ds \\
&+ 2\int_0^t \left\langle W_n(\varepsilon,\delta), \frac{\partial w}{\partial s}(s) \right\rangle ds
\end{aligned} \tag{5.12}
$$

where

$$
\Theta(t) = \left\| \frac{\partial w}{\partial t}(t) \right\|^2_{L^2(0,1)} + \left\| \sqrt{\mu(t)}\frac{\partial w}{\partial x}(t) \right\|^2_{L^2(0,1)}. \tag{5.13}
$$

From (2.1), $(\widetilde{A}_\mu)$, $(\widetilde{A}_g)$ and $(\widetilde{A}_k)$, also taking into account (5.12) and (5.13), we deduce that

$$
\begin{aligned}
&\int_0^t \left\langle \frac{\partial \mu}{\partial s}(\cdot,s), \left(\frac{\partial w}{\partial x}(s)\right)^2 \right\rangle ds \\
&\qquad\qquad \le \frac{1}{\mu_0}\int_0^t \left\| \frac{\partial \mu}{\partial s}(\cdot,s)\right\|_{L^\infty(0,1)} \Theta(s)ds,
\end{aligned} \tag{5.14}
$$

$$
-2\int_0^t \frac{\partial w}{\partial s}(\nu,s)\left(\int_0^s k_\nu(s-\tau)w(\nu,\tau)d\tau\right) ds \le \frac{2\varrho}{\mu_0}\Theta(t) + \overline{k}_\nu \int_0^t \Theta(s)ds, \tag{5.15}
$$

in which $\varrho > 0$ and $\overline{k}_\nu$, $\nu = 0, 1$, are non-negative constants defined as follows:

$$
\overline{k}_\nu = \left(T^2 + \frac{1}{\mu_0}\right)\left(2 + 2\varrho T + 4|k_\nu(0)| + \frac{2}{\varrho}\|k_\nu\|^2_{L^2(0,T)} + 2T\|k'_\nu\|^2_{L^2(0,T)}\right). \tag{5.16}
$$

In addition, also regarding (5.12) and (5.13), we obtain from (5.3), (5.8) and $(\widetilde{A}_{GH})$, that

$$
\begin{cases}
-2\varepsilon \int_0^t \left\langle G(w+v) - G(v), \frac{\partial w}{\partial s}(s)\right\rangle ds \le \overline{G}\int_0^t \Theta(s)ds, \\
-2\delta \int_0^t \left\langle H\left(\frac{\partial}{\partial t}(w+v)\right) - H\left(\frac{\partial v}{\partial t}\right), \frac{\partial w}{\partial s}(s)\right\rangle ds \le \overline{H}\int_0^t \Theta(s)ds,
\end{cases} \tag{5.17}
$$

for non-negative constants $\overline{G}$ and $\overline{H}$ given by

$$\text{(5.18)}\qquad \begin{cases} \overline{G} = 1 + T^2 \sup |G'(\eta)|^2, \\ \overline{H} = 1 + \sup |H'(\sigma)|^2, \\ |\eta| \le \sqrt{2}\left(\|u\|_{L^\infty(0,T;H^1(0,1))} + \sum\limits_{0\le i+j\le n} \|u_{ij}\|_{L^\infty(0,T;H^1(0,1))} 2^{i+j} \right), \\ |\sigma| \le \sqrt{2}\left(\|u\|_{L^\infty(0,T;H^2(0,1))} + \sum\limits_{0\le i+j\le n} \|u_{ij}\|_{L^\infty(0,T;H^2(0,1))} 2^{i+j} \right). \end{cases}$$

From Lemma 3 and (5.13), it obviously follows that

$$\text{(5.19)}\qquad 2\int_0^t \left\langle W_n(\varepsilon,\delta), \frac{\partial w}{\partial s}(s) \right\rangle ds \le T W_T \left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}} + \int_0^t \Theta(s) ds.$$

Combining (5.12)-(5.19) by choosing $\varrho \in \left(0, \frac{\mu_0}{4}\right)$, we conclude that

$$\text{(5.20)}\qquad \frac{\Theta(t)}{\min\{1,\mu_0\}} \le C\left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}}, \text{ for all } t\in[0,T]$$

for

$$\text{(5.21)}\quad C \equiv \frac{2TW_T}{\min\{1,\mu_0\}} \left[\exp\left(1 + \overline{G} + \overline{H} + \overline{k}_\nu + \frac{1}{\mu_0} \int_0^t \left\| \frac{\partial \mu}{\partial s}(\cdot, s) \right\|_{L^\infty(0,1)} ds \right) \right]^2.$$

Finally, it is clear that (5.7) is deduced from (5.13), (5.20) and (5.21). Hence, the proof is complete.

□

6. Appendix: The detailed proof of Lemma 3

Recall Lemma 3 in the previous section as follows:

Lemma 3. *Under assumptions* $(\widetilde{A}_\mu)$, $(\widetilde{A}_{GH})$, $(\widehat{A}_{GH})$, $(\widetilde{A}_F)$, $(\widetilde{A}_g)$ *and* $(\widetilde{A}_k)$, *the following estimate*

$$\text{(6.1)}\qquad \|W_n(\varepsilon,\delta)\|_{L^\infty(0,T;L^2(0,1))} \le W_T\left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}}$$

holds for $\varepsilon, \delta \in \mathbb{R}$ *satisfying* (5.3), *where* W_T *is a non-negative constant depending only on* G, H, u_{ij} *and* $\frac{\partial u_{ij}}{\partial t}$ *for* $0 \le i+j \le n$.

Proof. In the cases of $n = 0$ and $n = 1$, the proof of Lemma 3 is easy, hence we omit the details, here we mainly prove this lemma for $n \ge 2$.

By putting

$$\text{(6.2)}\qquad \widetilde{v} \equiv \sum_{1\le i+j\le n} u_{ij}\varepsilon^i\delta^j,$$

we have that Taylor's formulas of the functions $G(v) = G(u_{00}+\widetilde{v})$, $H\left(\frac{\partial v}{\partial t}\right) = H\left(\frac{\partial}{\partial t}(u_{00}+\widetilde{v})\right)$ about the points u_{00}, $\frac{\partial u_{00}}{\partial t}$, respectively, up to order n are given by

(6.3)

$$\begin{cases} G(v) = G(u_{00}) + \sum\limits_{k=1}^{n-1} \frac{1}{k!} G^{(k)}(u_{00})\widetilde{v}^k + \frac{1}{n!} G^{(n)}(u_{00}+\xi\widetilde{v})\widetilde{v}^n, \\ H\left(\frac{\partial v}{\partial t}\right) = H\left(\frac{\partial u_{00}}{\partial t}\right) + \sum\limits_{k=1}^{n-1} \frac{1}{k!} H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right)\left(\frac{\partial \widetilde{v}}{\partial t}\right)^k + \frac{1}{n!} H^{(n)}\left(\frac{\partial}{\partial t}(u_{00}+\zeta\widetilde{v})\right)\left(\frac{\partial \widetilde{v}}{\partial t}\right)^n, \end{cases}$$

where ξ, and $\zeta \in (0,1)$. Applying Lemma 2, we deduce from (6.3), that

$$
(6.4)\qquad \begin{cases} G(v) = G(u_{00}) + \displaystyle\sum_{k=1}^{n-1} \frac{1}{k!} G^{(k)}(u_{00}) \sum_{k \le i+j \le kn} [u_{ij}]_k \, \varepsilon^i \delta^j \\ \qquad\qquad + \displaystyle\frac{1}{n!} G^{(n)}(u_{00} + \xi \widetilde{v}) \sum_{n \le i+j \le n^2} [u_{ij}]_n \, \varepsilon^i \delta^j, \\ H\left(\displaystyle\frac{\partial v}{\partial t}\right) = H\left(\frac{\partial u_{00}}{\partial t}\right) + \sum_{k=1}^{n-1} \frac{1}{k!} H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right) \sum_{k \le i+j \le kn} \left[\frac{\partial u_{ij}}{\partial t}\right]_k \varepsilon^i \delta^j \\ \qquad\qquad + \displaystyle\frac{1}{n!} H^{(n)}\left(\frac{\partial}{\partial t}(u_{00} + \zeta \widetilde{v})\right) \sum_{n \le i+j \le n^2} \left[\frac{\partial u_{ij}}{\partial t}\right]_n \varepsilon^i \delta^j, \end{cases}
$$

in which

$$
\begin{cases} [u_{ij}]_k = \begin{cases} u_{ij}, \ 1 \le i+j \le n, \ k=1, \\ \sum_{(p,q) \in [\mathbb{Z}_{ij}]_k} u_{(i-p)(j-q)} \, [u_{pq}]_{k-1}, \ k \le i+j \le kn, \ k \ge 2, \end{cases} \\ \left[\frac{\partial u_{ij}}{\partial t}\right]_k = \begin{cases} \frac{\partial u_{ij}}{\partial t}, \ 1 \le i+j \le n, \ k=1, \\ \sum_{(p,q) \in [\mathbb{Z}_{ij}]_k} \frac{\partial u_{(i-p)(j-q)}}{\partial t} \left[\frac{\partial u_{pq}}{\partial t}\right]_{k-1}, \ k \le i+j \le kn, \ k \ge 2, \end{cases} \\ [\mathbb{Z}_{ij}]_k = \{(p,q) \in \mathbb{Z}_+^2 : (p,q) \le (i,j), \, 1 \le i-p+j-q \le n, \\ \qquad\qquad k-1 \le p+q \le (k-1)n\}, \\ [u_{ij}]_n = \begin{cases} u_{10}, \ i=n=1, \ j=0, \\ u_{01}, \ i=0, \ j=n=1, \\ \sum_{(p,q) \in [\mathbb{Z}_{ij}]_n} u_{(i-p)(j-q)} \, [u_{pq}]_{n-1}, \ n \le i+j \le n^2, \ n \ge 2, \end{cases} \\ \left[\frac{\partial u_{ij}}{\partial t}\right]_n = \begin{cases} \frac{\partial u_{10}}{\partial t}, \ i=n=1, \ j=0, \\ \frac{\partial u_{01}}{\partial t}, \ i=0, \ j=n=1, \\ \sum_{(p,q) \in [\mathbb{Z}_{ij}]_n} \frac{\partial u_{(i-p)(j-q)}}{\partial t} \left[\frac{\partial u_{pq}}{\partial t}\right]_{n-1}, \ n \le i+j \le n^2, \ n \ge 2, \end{cases} \\ [\mathbb{Z}_{ij}]_n = \{(p,q) \in \mathbb{Z}_+^2 : (p,q) \le (i,j), \, 1 \le i-p+j-q \le n, \\ \qquad\qquad n-1 \le p+q \le n^2-n\}. \end{cases}
$$

In addition, on account of these identities

$$
\varepsilon \sum_{k=1}^{n-1} \frac{1}{k!} G^{(k)}(u_{00}) \sum_{k \le i+j \le n-1} [u_{ij}]_k \, \varepsilon^i \delta^j = \sum_{1 \le i+j \le n-1} \sum_{k=1}^{i+j} \frac{1}{k!} G^{(k)}(u_{00}) \, [u_{ij}]_k \, \varepsilon^{i+1} \delta^j,
$$

$$
\begin{aligned} \delta \sum_{k=1}^{n-1} \frac{1}{k!} H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right) & \sum_{k \le i+j \le n-1} \left[\frac{\partial u_{ij}}{\partial t}\right]_k \varepsilon^i \delta^j \\ &= \sum_{1 \le i+j \le n-1} \sum_{k=1}^{i+j} \frac{1}{k!} H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right) \left[\frac{\partial u_{ij}}{\partial t}\right]_k \varepsilon^i \delta^{j+1}, \end{aligned}
$$

we obtain from (6.4), that

(6.5)

$$\begin{cases} \varepsilon G(v) = \varepsilon G(u_{00}) + \sum\limits_{2\le i+j\le n, i\ge 1} \sum\limits_{k=1}^{i+j-1} \frac{1}{k!} G^{(k)}(u_{00}) \left[u_{(i-1)j}\right]_k \varepsilon^i \delta^j + \widetilde{G}(G, \widetilde{v}, \varepsilon, \delta), \\ \delta H\left(\dfrac{\partial v}{\partial t}\right) = \delta H\left(\dfrac{\partial u_{00}}{\partial t}\right) + \sum\limits_{2\le i+j\le n, j\ge 1} \sum\limits_{k=1}^{i+j-1} \dfrac{1}{k!} H^{(k)}\left(\dfrac{\partial u_{00}}{\partial t}\right)\left[\dfrac{\partial u_{i(j-1)}}{\partial t}\right]_k \varepsilon^i \delta^j \\ \qquad + \widetilde{H}\left(H, \dfrac{\partial \widetilde{v}}{\partial t}, \varepsilon, \delta\right), \end{cases}$$

in which $\widetilde{G}(G, \widetilde{v}, \varepsilon, \delta)$ and $\widetilde{H}\left(H, \frac{\partial \widetilde{v}}{\partial t}, \varepsilon, \delta\right)$ are given as follows

$$(6.6) \quad \begin{cases} \widetilde{G}(G, \widetilde{v}, \varepsilon, \delta) = \varepsilon \sum\limits_{k=1}^{n-1} \dfrac{1}{k!} G^{(k)}(u_{00}) \sum\limits_{n\le i+j\le kn} [u_{ij}]_k \varepsilon^i \delta^j \\ \qquad + \dfrac{\varepsilon}{n!} G^{(n)}(u_{00} + \xi \widetilde{v}) \sum\limits_{n\le i+j\le n^2} [u_{ij}]_n \varepsilon^i \delta^j, \\ \widetilde{H}\left(H, \dfrac{\partial \widetilde{v}}{\partial t}, \varepsilon, \delta\right) = \delta \sum\limits_{k=1}^{n-1} \dfrac{1}{k!} H^{(k)}\left(\dfrac{\partial u_{00}}{\partial t}\right) \sum\limits_{n\le i+j\le kn} \left[\dfrac{\partial u_{ij}}{\partial t}\right]_k \varepsilon^i \delta^j \\ \qquad + \dfrac{\delta}{n!} H^{(n)}\left(\dfrac{\partial}{\partial t}(u_{00} + \zeta \widetilde{v})\right) \sum\limits_{n\le i+j\le n^2} \left[\dfrac{\partial u_{ij}}{\partial t}\right]_n \varepsilon^i \delta^j. \end{cases}$$

From (5.10), (6.5) and (6.6), we deduce that

$$(6.7) \qquad W_n(\varepsilon, \delta) = -\widetilde{G}(G, \widetilde{v}, \varepsilon, \delta) - \widetilde{H}\left(H, \frac{\partial \widetilde{v}}{\partial t}, \varepsilon, \delta\right).$$

Recall that u_{ij}, $0 \le i + j \le n$, are the weak solutions of problems (U_{ij}), respectively. From (2.1), (5.3)-(5.5) and (6.2), we deduce that

$$(6.8) \qquad \begin{cases} |u_{00}(x,t)| \le \sqrt{2}\, \|u_{00}\|_{L^\infty(0,T;H^1(0,1))}, \\ |u_{00}(x,t) + \xi \widetilde{v}(x,t)| \le \sqrt{2}\, \|u_{00} + \xi \widetilde{v}\|_{L^\infty(0,T;H^1(0,1))}, \end{cases}$$

for $(x,t) \in (0,1) \times (0,T)$ and $\xi \in (0,1)$.

Since $(\widehat{A}_{GH})$ and (6.8), it follows that C_k, $1 \le k \le n$, such that

$$(6.9) \qquad \begin{cases} C_k \equiv \sup\limits_{u_{00}} \left|G^{(k)}(u_{00})\right| \text{ for every } 1 \le k \le n-1, \\ C_n \equiv \sup\limits_{u_{00}+\xi\widetilde{v}} \left|G^{(n)}\left(u_{00} + \xi \widetilde{v}\right)\right| \ \xi \in (0,1) \end{cases}$$

are non-negative constants. Now, combining $(6.6)_1$ and (6.9), we conclude that

$$(6.10) \qquad \left\|\widetilde{G}(G, \widetilde{v}, \varepsilon, \delta)\right\|_{L^\infty(0,T;L^2(0,1))} \le G_0 \left(\varepsilon^2 + \delta^2\right)^{\frac{n+1}{2}}$$

for G_0 a non-negative constant given by

$$(6.11) \qquad \begin{aligned} G_0 = &\sum_{k=1}^{n-1} \sum_{n\le i+j\le kn} 2^{\frac{i+j-n}{2}} \frac{C_k}{k!} \|[u_{ij}]_k\|_{L^\infty(0,T;L^2(0,1))} \\ &+ \frac{C_n}{n!} \sum_{n\le i+j\le n^2} 2^{\frac{i+j-n}{2}} \|[u_{ij}]_n\|_{L^\infty(0,T;L^2(0,1))}. \end{aligned}$$

Moreover, let C'_k, $1 \le k \le n$, be non-negative constants defined as follows:

$$\tag{6.12} \begin{cases} C'_k \equiv \sup\limits_{\frac{\partial u_{00}}{\partial t}} \left| H^{(k)}\left(\frac{\partial u_{00}}{\partial t}\right)\right| \text{ for every } 1 \le k \le n-1, \\ C'_n \equiv \sup\limits_{\frac{\partial u_{00}}{\partial t}+\zeta\frac{\partial \widetilde{v}}{\partial t}} \left| H^{(n)}\left(\frac{\partial u_{00}}{\partial t}+\zeta\frac{\partial \widetilde{v}}{\partial t}\right)\right|, \ \zeta \in (0,1). \end{cases}$$

By the same way for the estimate of $\widetilde{G}(G, \widetilde{v}, \varepsilon, \delta)$ in $L^\infty\left(0, T; L^2(0,1)\right)$, we also obtain

$$\tag{6.13} \left\|\widetilde{H}(H, \widetilde{v}, \varepsilon, \delta)\right\|_{L^\infty(0,T;L^2(0,1))} \le H_0\left(\varepsilon^2+\delta^2\right)^{\frac{n+1}{2}},$$

in which H_0 is a non-negative constant given by

(6.14)

$$H_0 = \sum_{k=1}^{n-1} \sum_{n \le i+j \le kn} 2^{\frac{i+j-n}{2}} \frac{C'_k}{k!} \left\|\left[\frac{\partial u_{ij}}{\partial t}\right]_k\right\|_{L^\infty(0,T;L^2(0,1))} + \frac{C'_n}{n!} \sum_{n \le i+j \le n^2} 2^{\frac{i+j-n}{2}} \left\|\left[\frac{\partial u_{ij}}{\partial t}\right]_n\right\|_{L^\infty(0,T;L^2(0,1))}.$$

Finally, it is clear that (6.10) and (6.13) imply (6.1). □

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