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The Existence of Chaos in Infinite Dimensional Non-Resonant Systems

Michal Fečkan and Joseph Gruendler

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Abstract. This work is concerned with showing the existence of chaotic dynamics in the flow generated by an infinite system of strongly coupled ordinary differential equations with a finite dimensional hyperbolic part and an infinite dimensional center part. This theory can be applied to partial differential equations by using a Galerkin expansion which is illustrated by the problem of oscillations of a buckled elastic beam.

Contents

1. Introduction

To motivate the ideas of this work consider the partial differential equation

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos\omega_0 t \tag{1.1}$$

where P_0, μ_1, μ_2, ω_0 are constants and u is a real valued function of two variables $t \in \mathbb{R}$, $x \in [0,\pi]$, subject to the boundary conditions

$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0\,.$$

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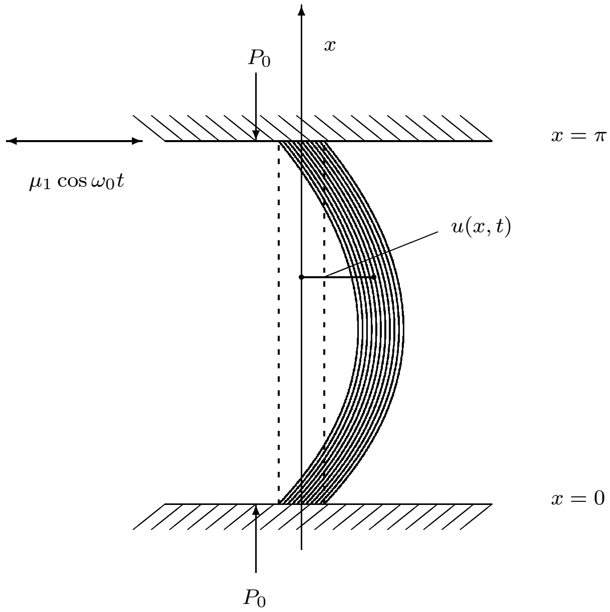

FIGURE 1. The forced buckled beam (1.1).

In (1.1), a superior dot denotes differentiation with respect to t and prime differentiation with respect to x. This is a model for oscillations of an elastic beam with a compressive axial load P_0 (see Figure 1). When P_0 is sufficiently large, (1.1) can exhibit chaotic behavior. The first work on this was done by Holmes and Marsden [**18**]. Some more recent work on the full equation is by Rodrigues and Silveira [**32**], by Berti and Carminati [**4**] and by Battelli, Fečkan and Franca [**2**]. An undamped buckled beam is investigated by Yagasaki [**41**] to show Arnold diffusion type motions. We will discuss some of this in more detail when we return to this problem in Section 5.

In (1.1) substitute $u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$, multiply by $\sin nx$ and integrate from 0 to π. This yields the infinite set of ordinary differential equations

$$\ddot{u}_n = n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2 \right] u_n - 2\mu_2 \dot{u}_n + 2\mu_1 \left[\frac{1-(-1)^n}{\pi n} \right] \cos \omega_0 t,$$
$$n = 1, 2, \dots .$$

We see that the linear parts of these equations are uncoupled and the equations divide into two types. The system of equations defined by $1 \le n^2 < P_0$ has a hyperbolic equilibrium at the origin whereas, for the system of equations satisfying $n^2 \ge P_0$, this equilibrium is a center.

For simplicity let us assume $1 < P_0 < 4$. Then only the equation with $n = 1$ is hyperbolic while the system of remaining equations has a center. To emphasize this let us define $p = u_1$ and $q_n = u_{n+1}$, $n = 1, 2, \dots$. The preceding equations now take the form

$$
\begin{aligned}
\ddot{p} &= a^2 p - \frac{\pi}{2}\left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] p - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos\omega_0 t, && \text{(1.2a)}\\
\ddot{q_n} &= -\omega_n^2 q_n - \frac{\pi}{2}(n+1)^2\left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] q_n \\
&\qquad -2\mu_2 \dot{q}_n + 2\mu_1\left[\frac{1-(-1)^{n+1}}{\pi(n+1)}\right]\cos\omega_0 t\,, && \text{(1.2b)}\\
&\quad n = 1, 2, \ldots
\end{aligned}
$$

where we have defined $a^2 = P_0 - 1$ and $\omega_n^2 = (n+1)^2\left[(n+1)^2 - P_0\right]$.

In (1.2) we project onto the hyperbolic subspace by setting $q = 0$ in (1.2a) to obtain what we shall call the reduced equation. In our example this is

$$
\ddot{p} = a^2 p - \frac{\pi}{2} p^3 - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos\omega_0 t. \tag{1.3}
$$

We see that this is the forced, damped Duffing equation with negative stiffness for which standard theory yields chaotic dynamics. The purpose of the present work is to show that the chaotic dynamics of (1.3) are, in some sense, shadowed in the dynamics of the full equation (1.2).

To put our example in first order form we define $x = (p, \dot{p})$ and

$$
y = (q_1, \dot{q}_1/\omega_1, q_2, \dot{q}_2/\omega_2, \ldots).
$$

The equations (1.2) now become

$$
\begin{aligned}
\dot{x}_1 &= x_2\,, && \text{(1.4a)}\\
\dot{x}_2 &= a^2 x_1 - \frac{\pi}{2}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right] x_1 \\
&\qquad -2\mu_2 x_2 + \frac{4}{\pi}\mu_1 \cos\omega_0 t\,, && \text{(1.4b)}\\
\dot{y}_{2n-1} &= \omega_n y_{2n}\,, && \text{(1.4c)}\\
\dot{y}_{2n} &= -\omega_n y_{2n-1} - \frac{\pi}{2}\frac{(n+1)^2}{\omega_n}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right] y_{2n-1} \\
&\qquad -2\mu_2 y_{2n} + 2\mu_1\left[\frac{1-(-1)^{n+1}}{\pi(n+1)\omega_n}\right]\cos\omega_0 t. && \text{(1.4d)}
\end{aligned}
$$

For these equations we define the Hilbert space

$$
\mathbb{Y} = \left\{ y = \{y_n\}_{n=1}^{\infty} \quad \middle| \quad y_n \in \mathbb{R}, \qquad \sum_{n=1}^{\infty} \omega_n^2 (y_{2n-1}^2 + y_{2n}^2) < \infty \right\}
$$

with inner product $\langle u, v\rangle = \sum_{n=1}^{\infty} \omega_n^2 (u_{2n-1} v_{2n-1} + u_{2n} v_{2n})$. By a weak solution to (1.4) we mean a pair of functions $x_0 : \mathbb{R} \to \mathbb{R}^2$, $y_0 : \mathbb{R} \to \mathbb{Y}$ such that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 \to \ell^2$ and which satisfy (1.4a), (1.4b) pointwise in $\mathbb{R}^2$; (1.4c), (1.4d) pointwise in ℓ^2. Note that in this case we have

$$\begin{aligned}(u_1,u_2,\ldots) &= (x,p_1,p_2,\ldots), \quad x^2+\sum_{n=1}^{\infty}\omega_n^2p_n^2<\infty,\\ (\dot u_1,\dot u_2,\ldots) &= (\dot x,\dot p_1,\dot p_2\ldots)\in\ell^2\end{aligned}$$

so that for the original differential equation (1.1), $u\in\mathbb{H}^2(0,\pi)\cap\mathbb{H}_0^1(0,\pi)$ and $\dot u\in\mathbb{L}^2(0,\pi)$. This is discussed in [**10**].

In the next section we will formulate an abstract problem for which the hypotheses will consist of the essential features of (1.4). We have already mentioned one of these: when y is set equal to zero in (1.4a) the resulting equation is the transverse perturbation of an autonomous equation with a homoclinic solution.

To see another important property we linearize (1.4c), (1.4d) about the origin which yields the system of equations

$$\left.\begin{aligned}\dot v_{2n-1} &= \omega_n v_{2n},\\ \dot v_{2n} &= -\omega_n v_{2n-1}-2\mu_2 v_{2n}\end{aligned}\right\} n=1,2,\ldots. \tag{1.5}$$

Note that for each n we get a pair of equations uncoupled from the others and for $|\mu_2|<\omega_n$ we have a fundamental solution for (v_{2n-1},v_{2n}) given by

$$V_n(t)=\begin{bmatrix}\cos\tilde\omega_n t+\dfrac{\mu_2}{\tilde\omega_n}\sin\tilde\omega_n t & \frac{\omega_n}{\tilde\omega_n}\sin\tilde\omega_n t\\ -\dfrac{\omega_n}{\tilde\omega_n}\sin\tilde\omega_n t & \cos\tilde\omega_n t-\frac{\mu_2}{\tilde\omega_n}\sin\tilde\omega_n t\end{bmatrix}\mathrm{e}^{-\mu_2 t}$$

where $\tilde\omega_n=\sqrt{\omega_n^2-\mu_2^2}$. This solution has the properties $V_n(0)=\mathbb{I}$ and

$$|V_n(t)V_n(s)^{-1}|=|V_n(t)V_n(-s)|=|V_n(t-s)|\le K\,\mathrm{e}^{\mu_2(s-t)}$$

where $K>0$ is independent of n.

Using the sequence $\{V_n\}_{n=1}^{\infty}$ we can define a group $\{V_{\mu_2}(t)\}$ of bounded operators from $\mathbb{Y}$ to $\mathbb{Y}$ by

$$\begin{bmatrix}(V_{\mu_2}(t)y)_{2n-1}\\ (V_{\mu_2}(t)y)_{2n}\end{bmatrix}=V_n(t)\begin{bmatrix}y_{2n-1}\\ y_{2n}\end{bmatrix}.$$

Then $|V_{\mu_2}(t)V_{\mu_2}(s)^{-1}|\le K\,\mathrm{e}^{\mu_2(s-t)}$. For $y^0\in\mathbb{Y}$, $y(t)=V_{\mu_2}(t)y^0$ is the weak solution to (1.5) satisfying $y(0)=y^0$.

If we retain the forcing term from (1.4d) we obtain the system of nonhomogeneous variational equations

$$\begin{aligned}\dot v_{2n-1} &= \omega_n v_{2n}\,,\\ \dot v_{2n} &= -\omega_n v_{2n-1}-2\mu_2 v_{2n}+\mu_1\nu_n\cos\omega_0 t\,,\end{aligned}$$

where $\nu_n=\dfrac{2[1-(-1)^{n+1}]}{\pi(n+1)\omega_n}$.

Here we encounter the question of resonance. In the nonresonant case, i.e. $\omega_n\ne\omega_0$, the preceding has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix}v_{2n-1}(t)\\ v_{2n}(t)\end{bmatrix}=\frac{\mu_1\nu_n}{(\omega_n^2-\omega_0^2)^2+4\mu_2^2\omega_0^2}\begin{bmatrix}\omega_n(\omega_n^2-\omega_0^2)\cos\omega_0 t+2\mu_2\omega_0\omega_n\sin\omega_0 t\\ -\omega_0(\omega_n^2-\omega_0^2)\sin\omega_0 t+2\mu_2\omega_0^2\cos\omega_0 t\end{bmatrix}$$

We make the existence of such a solution a separate hypothesis.

Finally, we mention others works on chaos in partial differential equations. For the complex Ginzburg-Landau equation in the near nonlinear Schrödinger regime (i.e. perturbed nonlinear Schrödinger equation), existence of homoclinic orbits is proved by Li, McLaughlin, Shatah and Wiggins [**21, 27, 28**], and existence of chaos is shown by Li [**22, 23**] under generic conditions. For perturbed sine-Gordon equation, existence of chaos and chaos cascade around a homoclinic tube was proved by Li [**24, 25, 26**]. For the reaction-diffusion equation, entropy study on the complexity of attractor is conducted by Zelik [**36, 37, 38**]. Chaotic oscillations of a linear wave equation with nonlinear boundary conditions are shown by Chen, Hsu and Zhou [**6**]. The development of chaos and its controlling for PDEs is summarized by Zhao [**39**]. Chaos for elastic beams is shown by Battelli and Fečkan [**1**].

2. The Abstract Problem

Using the example in the preceding section as a model we now develop an abstract theory. Let $\mathbb{Y}$ and $\mathbb{H}$ be separable real Hilbert spaces with $\mathbb{Y} \subset \mathbb{H}$.

We now consider differential equations of the form

$$\begin{aligned} \dot{x} &= f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,t) = Ay + g_0(x,y) + \mu_1 \nu \cos\omega_0 t + \mu_2 g_2(x,y,\mu)\,, \end{aligned} \tag{2.1}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\nu \in \mathbb{Y}$. We make the following assumptions about (2.1):

(H1) $A : \mathbb{Y} \to \mathbb{H}$ is a continuous and linear transformation.

(H2) The functions f_i and g_i are in the spaces:

$$\begin{aligned} &f_0 \in \mathcal{C}^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{R}^n); \qquad f_1, f_2 \in \mathcal{C}^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^n); \\ &g_0 \in \mathcal{C}^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{Y}); \qquad g_2 \in \mathcal{C}^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2, \mathbb{Y})\,. \end{aligned}$$

(H3) f_1 and f_2 are periodic in t with period $T = 2\pi/\omega_0$.

(H4) $f_0(0,0) = 0$ and $D_2 f_0(x,0) = 0$.

(H5) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.

(H6) The equation $\dot{x} = f_0(x,0)$ has a nontrivial solution homoclinic to $x = 0$.

(H7) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{12} g_0(0,0) = 0$ and $D_{22} g_0(x,0) = 0$.

(H8) There are constants $K > 0$, $\delta > 0$ and $b > 0$ so that when $0 \le |\mu_2| \le \delta$ the variational equation

$$\dot{v} = \big(A + \mu_2 D_2 g_2(0,0,0)\big) v$$

has a group $\{V_{\mu_2}(t)\}$ of bounded evolution operators from $\mathbb{Y}$ to $\mathbb{Y}$ satisfying $|V_{\mu_2}(t) V_{\mu_2}(s)^{-1}| \le K\, \mathrm{e}^{b\mu_2(s-t)}$.

(H9) There is a constant $K > 0$ such that the nonhomogeneous variational equation

$$\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v + \mu_1 \nu \cos\omega_0 t$$

has a particular solution $\psi : \mathbb{R} \to \mathbb{Y}$ satisfying $|\psi(t)| \le K|\mu_1||\nu|$.

By a weak solution to (2.1) we mean a pair of continuous functions $x_0 : \mathbb{R} \to \mathbb{R}^n$, $y_0 : \mathbb{R} \to \mathbb{Y}$ such that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 : \mathbb{R} \to \mathbb{H}$ and which satisfy (2.1) pointwise in $\mathbb{H}$.

By (H8) we mean that $V_{\mu_2}(s)^{-1} = V_{\mu_2}(-s)$, $V_{\mu_2}(s) \circ V_{\mu_2}(t) = V_{\mu_2}(s+t)$, $V_{\mu_2}(0) = \mathbb{I}$ and that for $y_0 \in \mathbb{Y}$, $y(t) = V_{\mu_2}(t) y_0$ is the weak solution to $\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v$ satisfying $y(0) = y_0$.

3. Chaos on the Hyperbolic Subspace

The reduced system of equations for (2.1) is

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{3.1}$$

with $x \in \mathbb{R}^n$. In [**3, 13, 14, 15**] a general Melnikov theory is developed for first order systems in $\mathbb{R}^n$. We summarize those results here as applied to (3.1).

By (H6), (3.1) has a nontrivial homoclinic solution γ when $\mu = 0$. By the variational equation along γ we mean the linear equation

$$\dot{u} = D_1 f_0(\gamma,0)u \tag{3.2}$$

and by the adjoint the system

$$\dot{v} = -D_1 f_0(\gamma,0)^* v\,. \tag{3.3}$$

We let $\{u_1,\dots,u_d\}$ denote a basis for the vector space of bounded solutions to (3.2) with $u_d = \dot{\gamma}$ and we let $\{v_1,\dots,v_d\}$ denote a basis for the vector space of bounded solutions to (3.3). Now define the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$, constants b_{ijk} and function

$$M : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^d$$

by

$$\begin{aligned} a_{ij}(\alpha) &= \textstyle\int_{-\infty}^{\infty} \langle v_i(t), f_j(\gamma(t),0,0,t+\alpha)\rangle\, dt\,; \\ &\qquad\qquad i = 1,\dots,d; \quad j = 1,2\,; \\ b_{ijk} &= \textstyle\int_{-\infty}^{\infty} \langle v_i, D_{11} f_0(\gamma,0)u_j u_k\rangle\, dt\,; \\ &\qquad\qquad i = 1,\dots,d; \quad j,k = 1,\dots,d-1\,; \\ M_i(\mu,\alpha,\beta) &= \sum_{j=1}^{2} a_{ij}(\alpha)\mu_j + \tfrac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k\,; \quad 1 \le i \le d\,. \end{aligned} \tag{3.4}$$

The function M is our bifurcation function and is used in Theorem 1 below.

Suppose that (3.2) has a $(d-1)$-parameter family of homoclinic orbits given by $t \to \gamma_\beta(t)$ with $\beta \in U_0$ where U_0 is an open neighborhood of the origin in $\mathbb{R}^{d-1}$. Then in (3.4) all $b_{ijk} = 0$, the hypotheses of Theorem 1 below cannot be satisfied and an alternate bifurcation function is required.

For each fixed β we let $\{v_{\beta 1},\dots,v_{\beta d}\}$ denote a basis for the vector space of bounded solutions to the adjoint equation $\dot{v} = -D_1 f_0(\gamma_\beta,0)^* v$. Without loss of generality we can assume that each $v_{\beta i}$ depends differentially on β. Now define functions $a_{ij} : \mathbb{R} \times U_0 \to \mathbb{R}$ and $M : \mathbb{R}^2 \times \mathbb{R} \times U_0 \to \mathbb{R}^d$ by

$$\begin{aligned} a_{ij}(\alpha,\beta) &= \textstyle\int_{-\infty}^{\infty} \langle v_{\beta i}(t), f_j(\gamma_\beta(t),0,0,t+\alpha)\rangle\, dt\,; \\ &\qquad\qquad i = 1,\dots,d; \quad j = 1,2\,; \\ M_i(\mu,\alpha,\beta) &= \sum_{j=1}^{2} a_{ij}(\alpha,\beta)\mu_j\,; \quad 1 \le i \le d\,. \end{aligned} \tag{3.5}$$

This function, M, is the bifurcation function for this situation.

The concept of exponential dichotomy is important in our next consideration so we state the definition for easy reference (cf. [**7**]).

Definition 1. *We say the linear differential equation* $\dot{x} = A_1(t)x$ *has an exponential dichotomy on* $\mathbb{R}$ *if its fundamental solution* U *has a projection* P *along with constants* $(\tilde{A}, a)$ *such that:*

i): $|U(t)PU(s)^{-1}| \leq \tilde{A}\,\mathrm{e}^{a(s-t)}$ *for* $s \leq t$,

ii): $|U(t)(\mathbb{I}-P)U(s)^{-1}| \leq \tilde{A}\,\mathrm{e}^{a(t-s)}$ *for* $t \leq s$.

The following result can be proved as in [**3, 9, 15, 29, 30**].

Theorem 1. *Let* M *be as in* (3.4) *or* (3.5) *and suppose* μ_0, α_0, β_0 *are such that* $M(\mu_0, \alpha_0, \beta_0) = 0$ *and* $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ *is nonsingular. Then there exists an interval* $J = (0, \xi_0]$ *such that for each* $\xi \in J$ *the equation* $\dot{x} = f(x, 0, \xi\mu_0, t)$ *has a homoclinic solution* γ_ξ *to a small hyperbolic periodic solution.*

Furthermore, γ_ξ *depends continuously on* ξ,

$$\lim_{\xi \to 0} \gamma_\xi(t) = \gamma(t - \alpha_0) \ \ (\text{or} = \gamma_{\beta_0}(t - \alpha_0), \text{ respectively})$$

uniformly in t *and the variational equation along* γ_ξ *has an exponential dichotomy on* $\mathbb{R}$.

We can use the preceding result to obtain chaos for (3.1) as follows: Let Σ denote the space of doubly infinite sequences with entries from the set of integers $\{0, 1\}$. The space Σ, endowed with the metric

$$d\left(\{\sigma_n\}_{n\in\mathbb{Z}}, \{\sigma'_n\}_{n\in\mathbb{Z}}\right) := \sum_{n\in\mathbb{Z}} \frac{|\sigma_n - \sigma'_n|}{2^{|n|+1}},$$

is a compact metric space. Let $\varphi : \Sigma \to \Sigma$ be the Bernoulli shift map defined by

$$\varphi\big(\{\sigma_j\}_{j\in\mathbb{Z}}\big) = \{\tilde{\sigma}_j\}_{j\in\mathbb{Z}}, \quad \tilde{\sigma}_j = \sigma_{j+1}.$$

The dynamics of φ is extremely rich as it is indicated in the next theorem [**8, 19, 31, 35**].

Theorem 2. φ *is a homeomorphism having*

i) *a countable infinity of periodic orbits of all possible periods,*
ii) *an uncountable infinity of nonperiodic orbits, and*
iii) *a dense orbit.*

Suppose Theorem 1 holds. Then we can show chaos for the differential equation $\dot{x} = f(x, 0, \xi\mu_0, t)$ by establishing a topological conjugacy between φ and some multiple of the period map of the flow for the differential equation [**34, 35**]. For this, first, for any $m \in \mathbb{N}$, $\xi \in J$ and $\sigma \in \Sigma$ define the function $\gamma_{\xi,\sigma,m} \in L^\infty(\mathbb{R}, \mathbb{R}^n)$ by

$$\gamma_{\xi,\sigma,m}(t) = \begin{cases} \gamma_\xi(t - 2jmT) & \text{if} \quad (2j-1)mT < t \leq (2j+1)mT \quad \text{and} \quad \sigma_j = 1 \\ 0 & \text{if} \quad (2j-1)mT < t \leq (2j+1)mT \quad \text{and} \quad \sigma_j = 0. \end{cases}$$

We now use Theorem 1 to show chaos for (3.1) following [**1, 11, 29, 30**].

Theorem 3. **a):** *Let* μ_0, α_0, β_0, ξ_0 *be as in Theorem* 1. *Fix* $\xi \in (0, \xi_0]$ *and let* γ_ξ *be obtained from Theorem* 1. *Then there exist an* $\varepsilon_0 > 0$ *and a function* $\varepsilon \to M(\varepsilon) \in \mathbb{N}$ *such that given* ε *with* $0 < \varepsilon \leq \varepsilon_0$ *and a positive integer* $m \geq M(\varepsilon)$ *the equation* $\dot{x} = f(x, 0, \xi\mu_0, t)$ *has for each* $\sigma \in \Sigma$ *a unique solution* $t \to x_\sigma(t)$ *satisfying*

$$|x_\sigma(t) - \gamma_{\xi,\sigma,m}(t)| \leq \varepsilon \quad \forall t \in \mathbb{R}.$$

b): *x_σ depends continuously on σ and $x_\sigma(t+2mT) = x_{\varphi(\sigma)}(t)$ where φ is the Bernoulli shift on Σ.*

c): *The correspondence $\phi(\sigma) = x_\sigma(0)$ is a homeomorphism of Σ onto the compact subset Λ of $\mathbb{R}^n$ given by*

$$\Lambda := \{x_\sigma(0) \mid \sigma \in \Sigma\}$$

on which the $2m$th iterate F^{2m} of the period map F of (3.1) is invariant and satisfies $F^{2m} \circ \phi = \phi \circ \varphi$.

Theorem 3 asserts that the following diagram is commutative

$$\begin{array}{ccc} \Sigma & \xrightarrow{\quad\varphi\quad} & \Sigma \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \Lambda & \xrightarrow[\quad F^{2m}\quad]{} & \Lambda \end{array}$$

This means that $F^{2m} : \Lambda \mapsto \Lambda$ has the same dynamics on Λ as the Bernoulli shift φ on Σ. Moreover, it is possible to show a sensitive dependence on initial conditions of F^{2m} on Λ in the sense [**8, 31, 35**] that there is an $c_0 > 0$ such that for any $x \in \Lambda$ and any neighborhood U of x, there exists $u \in U \cap \Lambda$ and an integer $q \geq 1$ such that

$$|F^{2mq}(x) - F^{2mq}(u)| > c_0\,.$$

Consequently, F^{2m} is chaotic on Λ, so (3.1) is also chaotic.

This construction is sometimes referred to as embedding a Smale horseshoe in the flow of the differential equation [**8, 31, 34**].

4. Chaos in the Full Equation

Since the homoclinic orbit γ_ξ obtained in Section 3 is hyperbolic the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ has an exponential dichotomy on $\mathbb{R}$ with constant K_ξ. Now, we show in [**12**] that the K_ξ tends to infinity as $\xi \to 0$. For this reason we consider the following modification of (2.1)

$$\begin{aligned} \dot{x} &= f(x,y,\mu,\lambda,t) &&:= f(x,\lambda y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,\lambda,t) &&:= Ay + g_0(x,y) + \lambda\mu_1\nu\cos\omega_0 t + \mu_2 g_2(x,y,\mu) \end{aligned} \tag{4.1}$$

for a parameter $\lambda \in [0,1]$.

Now let $(\mu_0, \alpha_0, \beta_0)$ with $\mu_{0,2} \neq 0$ and γ_ξ be as in Theorem 1. Following the arguments of [**12**, pp. 82-85], we obtain a constant $\bar{\xi}_0$ and for each $\xi \in (0, \bar{\xi}_0]$ a homoclinic orbit

$$\Gamma(\lambda,\xi)(t) = \big(\Gamma_1(\lambda,\xi)(t), \Gamma_2(\lambda,\xi)(t)\big)$$

for (4.1) with $\mu = \xi\mu_0$ such that

$$\begin{aligned} &\Gamma_1(\lambda,\xi)(t) \to \gamma(t-\alpha_0) \quad (\text{or} \to \gamma_{\beta_0}(t-\alpha_0), \text{ respectively})\,, \\ \text{and}\quad &\Gamma_2(\lambda,\xi)(t) \to 0 \end{aligned}$$

as $\xi \to 0$ uniformly for $\lambda \in [0,1]$. Moreover, we have $\Gamma(0,\xi) = (\gamma_\xi, 0)$ and $\Gamma(1,\xi)$ is a homoclinic solution for (2.1). The linearization of (4.1) with $\mu = \xi\mu_0$ along $\Gamma(\lambda,\xi)(t)$ has an exponential dichotomy on $\mathbb{R}$ with dichotomy constants uniform with respect to $0 \leq \lambda \leq 1$ and fixed ξ.

Analogous to the construction in Section 3, for each $\sigma \in \Sigma$, $\xi \in (0, \bar{\xi}_0]$ and $m \in \mathbb{N}$ we construct from $\Gamma(\lambda, \xi)$ a corresponding

$$\Gamma_\sigma(\lambda, \xi, m) = (\Gamma_{1,\sigma}(\lambda, \xi, m), \Gamma_{2,\sigma}(\lambda, \xi, m)).$$

Similarly, from γ_ξ we obtain $\gamma_{\xi,\sigma,m}$. Then we have $\Gamma_{1,\sigma}(0, \xi, m) = \gamma_{\xi,\sigma,m}$ and also $\Gamma_{2,\sigma}(0, \xi, m) = 0$. Using the uniform exponential dichotomy, following [**1, 11**], we now obtain the following extension of Theorem 3.

THEOREM 4. **a):** *Let μ_0, α_0, β_0 be as in Theorem 1 with $\mu_{0,2} \neq 0$. Fix $\xi \in (0, \bar{\xi}_0]$ and let $\Gamma(\lambda, \xi, m)(t)$ be obtained above. Then there exist an $\bar{\varepsilon}_0 > 0$ and a function $\varepsilon \to \bar{M}(\varepsilon) \in \mathbb{N}$ such that given ε with $0 < \varepsilon \leq \bar{\varepsilon}_0$ and a positive integer $m \geq \bar{M}(\varepsilon)$ the equation (4.1) with $\mu = \xi\mu_0$ has for each $\sigma \in \Sigma$ a unique weak solution $t \to (x_{\sigma,\lambda}(t), y_{\sigma,\lambda}(t))$ satisfying*

$$|x_{\sigma,\lambda}(t) - \Gamma_{1,\sigma}(\lambda, \xi, m)(t)| + |y_{\sigma,\lambda}(t) - \Gamma_{2,\sigma}(\lambda, \xi, m)(t)| \leq \varepsilon \quad \forall t \in \mathbb{R}.$$

b): *The functions $(x_{\sigma,\lambda}(t), y_{\sigma,\lambda}(t))$ depend continuously on σ, λ and we also have $x_{\sigma,\lambda}(t + 2mT) = x_{\varphi(\sigma),\lambda}(t)$, $y_{\sigma,\lambda}(t + 2mT) = y_{\varphi(\sigma),\lambda}(t)$ where φ is the Bernoulli shift on Σ.*

c): *The correspondence $\phi_\lambda(\sigma) = (x_{\sigma,\lambda}(0), y_{\sigma,\lambda}(0))$ is a homeomorphism of Σ onto the compact subset Λ_λ of $\mathbb{R}^n \times \mathbb{Y}$ given by*

$$\Lambda_\lambda := \{(x_{\sigma,\lambda}(0), y_{\sigma,\lambda}(0)) \mid \sigma \in \Sigma\}$$

on which the $2m$th iterate F_λ^{2m} of the period map F_λ of (4.1) is invariant and satisfies $F_\lambda^{2m} \circ \phi_\lambda = \phi_\lambda \circ \varphi$.

d): *$(x_{\sigma,0}(t), y_{\sigma,0}(t)) = (x_\sigma(t), 0)$ and $\phi_0 = \phi$ where ϕ is as in Theorem 3.*

Summarizing, we obtain the following main result.

THEOREM 5. *Suppose* (H1)-(H10) *hold. Let M be as in* (3.4) *or* (3.5) *and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exists $\bar{\xi}_0 > 0$ such that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in* (2.1) *are given by $\mu = \xi\mu_0$, and $\mu_{0,2} \neq 0$ then there exists a homeomorphism, ϕ_1, of Σ onto a compact subset of $\mathbb{R}^n \times \mathbb{Y}$ on which the $2m$th iterate, F_1^{2m}, of the period map F_1 of* (2.1) *is invariant and satisfies $F_1^{2m} \circ \phi_1 = \phi_1 \circ \varphi$ where φ is the Bernoulli shift on Σ.*

We might paraphrase Theorem 5, loosely, as saying that the Smale horseshoe embedded in the flow of the reduced equation (3.1) is shadowed by a horseshoe in the full equation (2.1).

5. Applications: Vibrating Elastic Beams

We now return to the example in Section 1 and apply our theory to the problem of vibrating elastic beams. We shall consider a number of different cases and generalizations. In each case our procedure will be:

i) Use a Galerkin expansion to convert the partial differential equation to an infinite set of ordinary differential equations as (2.1).
ii) Truncate the equation to get the finite problem (3.1).
iii) Apply Theorem 3 to get a Smale horseshoe for the finite problem. For this we must verify (H1) through (H6).
iv) Use Theorem 5 to lift the horseshoe to the flow of the original partial differential differential equation. This requires (H7)-(H9).

5.1. Planer Motion with One Buckled Mode. The boundary value problem for planer deflections of an elastic beam with a compressive axial load P_0 and pinned ends is

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos\omega_0 t,$$
$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0$$

where $u(x,t)$ is the transverse deflection at a distance x from one end at time t. We consider the μ_i terms as perturbations.

Our first step is to consider the linearized, unperturbed problem. We compute the eigenvalues at the origin to be $\lambda_n = n^2(n^2 - P_0)$ with corresponding eigenfunctions $\varphi_n(x) = \sin nx$ for $n = 1, 2, \ldots$. For small P_0 the origin is a center. As P_0 is increased the first bifurcation occurs at $P_0 = 1$, the first Euler buckling load. The corresponding eigenfunction, $\varphi_1(x) = \sin x$, is referred to as the first buckled mode. The second bifurcation occurs at $P_0 = 4$. Thus, the simplest case, which we now consider, consists of $1 < P_0 < 4$.

In the first equation we define $a^2 = \lambda_1 = P_0 - 1$. The eigenvalues for the center modes, or unbuckled modes, provide the frequencies used in (2.1) as we define $\omega_{n-1}^2 = \lambda_n = n^2[n^2 - P_0]$, $\quad n = 2, 3, \ldots$. We now use the eigenfunctions for the Galerkin expansion $u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$ and obtain the system of equations

$$\begin{aligned} \ddot{u}_n &= n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n \\ &\quad - 2\mu_2 \dot{u}_n + 2\mu_1 \left[\frac{1 - (-1)^n}{\pi n}\right] \cos\omega_0 t, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.1}$$

To obtain a first order system as in (2.1) we define

$$x = (u_1, \dot{u}_1), \quad y = (u_2, \dot{u}_2/\omega_1, u_3, \dot{u}_3/\omega_2, \ldots).$$

The reduced equations are

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= a^2 x_1 - \frac{\pi}{2} x_1^3 - 2\mu_2 x_2 + \frac{4}{\pi}\mu_1 \cos\omega_0 t \end{aligned} \tag{5.2}$$

obtained by setting $y = 0$ in the hyperbolic part. When $\mu = 0$, (5.2) has a homoclinic solution given by $\gamma = (r, \dot{r})$ where $r(t) = (2a/\sqrt{\pi})\,\mathrm{sech}\, at$. Equation (3.3) becomes

$$\dot{v}_1 = -(a^2 - \tfrac{3\pi}{2} r^2) v_2, \qquad \dot{v}_2 = -v_1$$

with solution $(v_1, v_2) = (-\ddot{r}, \dot{r})$.

In (3.4) we have $d = 1$ so the variable β does not appear, M is a scalar function, and the function $M = M_1$ becomes

$$M(\alpha) = \left[\tfrac{8\omega_0}{\sqrt{\pi}} \sin\omega_0\alpha\, \mathrm{sech}\, \tfrac{\pi\omega_0}{2a}\right]\mu_1 - \left(\tfrac{16a^3}{3\pi}\right)\mu_2.$$

Thus, the conditions $M(\mu_0, \alpha_0) = 0$, $(\partial M/\partial\alpha)(\mu_0, \alpha_0) \neq 0$ are satisfied for all μ_0 such that

$$\left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3}\, \mathrm{sech}\, \tfrac{\pi\omega_0}{2a}.$$

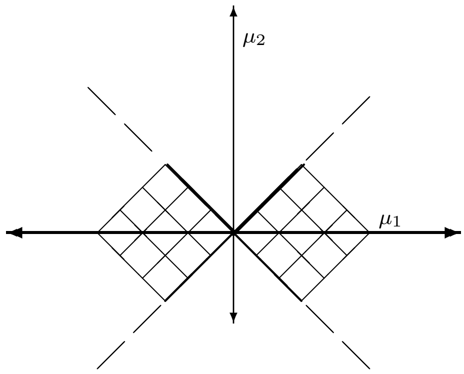

FIGURE 2. The chaotic open wedge-shaped region of (5.1) in $\mathbb{R}^2$.

Now we check condition (H9) which, for the present problem, requires us to consider the equation

$$\begin{aligned} \dot{v}_{2n-1} &= \omega_n v_{2n}\,, \\ \dot{v}_{2n} &= -\omega_n v_{2n-1} - 2\mu_2 v_{2n} + \mu_1 \nu_n \cos\omega_0 t \end{aligned}$$

where $\nu_n = \frac{2[1-(-1)^{n-1}]}{\pi(n+1)\omega_n}$.

This system has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix} v_{2n-1}(t) \\ v_{2n}(t) \end{bmatrix} = \frac{\mu_1\nu_n}{(\omega_n^2-\omega_0^2)^2+4\mu_2^2\omega_0^2} \begin{bmatrix} \omega_n(\omega_n^2-\omega_0^2)\cos\omega_0 t + 2\mu_2\omega_0\omega_n \sin\omega_0 t \\ -\omega_0(\omega_n^2-\omega_0^2)\sin\omega_0 t + 2\mu_2\omega_0^2\cos\omega_0 t \end{bmatrix}.$$

From this we see that (H9) is satisfied whenever $\omega_0 \neq \omega_n$ for all n.

We note that while the conditions $M(\alpha) = 0$, $M'(\alpha) \neq 0$ can be satisfied with $\mu_2 = 0$, $\alpha = 0$ we require $\mu_2 \neq 0$ in Section 4 where we use a weak exponential dichotomy to lift to the full equation. Thus, we obtain the following result using Theorem 5.

THEOREM 6. *If $\omega_0 \neq \omega_n$ for all n then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3}\operatorname{sech}\frac{\pi\omega_0}{2a}\,, \tag{5.3}$$

there exists a corresponding $\bar{\xi}_0 > 0$ such that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.1) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^2 \times \mathbb{Y}$ on which the $2m$th iterate, F^{2m}, of the period map F of* (5.1) *is invariant and conjugate to the Bernoulli shift on Σ.*

These results are stated in terms of the Galerkin equations (5.1) but they can be transferred back to the original partial differential equation. In this case we get a Bernoulli shift embedded in $\left[\mathbb{H}_0^1(0,\pi) \cap \mathbb{H}^2(0,\pi)\right] \times \mathbb{L}^2(0,\pi)$. This is discussed in [**10**]. In the μ_1-μ_2 plane we get from the condition (5.3) four small open wedge-shaped regions of parameter values for which the partial differential equation exhibits chaos (see Figure 2). These regions are bounded by the lines $\mu_1/\mu_2 = \pm\frac{3\sqrt{\pi}\omega_0}{2a^3}\operatorname{sech}\frac{\pi\omega_0}{2a}$ and $\mu_2 = 0$.

It is interesting to look at some history of this problem. The first work was by Holmes [**17**] in which he started with the partial differential equation and carried out the Galerkin expansion but restricted his analysis to the reduced equation (5.2). The significance of that work is that it introduced the idea of Melnikov analysis. In subsequent work [**18**] Holmes and Marsden extended the results to infinite dimension but abandoned the Galerkin approach in favor of nonlinear semigroup techniques directly in infinite dimensions. In our work we go back to the original, simpler analysis of the reduced equation and then show that the results apply to the original partial differential equation. Some advantages to this are that the Galerkin projection is a technique familiar to many engineers and physicists and, also, we are able to utilize our general Melnikov results in Section 3. This is illustrated further in the generalizations which follow. We note that equation (5.1) was treated also in [**4**].

5.2. Nonplaner Motion of a Symmetric Beam with One Buckled Mode. Let us consider a beam with symmetric cross section, pinned ends and compressive axial load P_0 and assume now that the beam is not constrained to defect in a plane. If $u(x,t)$ and $w(x,t)$ denote the transverse defections at position x and time t we obtain the following boundary value problem.

$$\begin{aligned}\ddot{u} &= -u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right)\, ds\right] u'' \\ &\qquad - 2\mu_2 \dot{u}\cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\ \ddot{w} &= -w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right)\, ds\right] w'' \\ &\qquad - 2\mu_2 \dot{w}\sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,,\end{aligned}$$

$$\begin{aligned}u(0,t) &= u(\pi,t) = u''(0,t) = u''(\pi,t) = w(0,t) \\ &= w(\pi,t) = w''(0,t) = w''(\pi,t) = 0\end{aligned}$$

where η, ζ are constants.

The parameters μ_1, μ_2 represent the coefficients of, respectively, total transverse forcing and total viscous damping. These effects are distributed between the two directions of motion. The quantity $\tan\zeta$ represents the ratio of forcing in the u-direction to forcing in the w-direction while $\tan\eta$ plays the same role for the damping. We suppose $\eta, \zeta \in (0, \pi/2)$ in order to avoid certain degeneracies.

In these equations we use the Galerkin expansions

$$u(x,t) = \sum_{k=1}^\infty u_k(t)\sin kx\,, \quad w(x,t) = \sum_{k=1}^\infty w_k(t)\sin kx$$

and proceed as before. This yields the system of equations

$$\begin{aligned}\ddot{u}_n &= n^2(P_0 - n^2)u_n - \tfrac{\pi}{2}n^2\left[\sum_{k=1}^\infty k^2(u_k^2 + w_k^2)\right]u_n \\ &\qquad -2\mu_2\dot{u}_n\cos\eta + 2\mu_1\cos\zeta\left[\tfrac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,, \\ \ddot{w}_n &= n^2(P_0 - n^2)w_n - \tfrac{\pi}{2}n^2\left[\sum_{k=1}^\infty k^2(u_k^2 + w_k^2)\right]w_n \\ &\qquad -2\mu_2\dot{w}_n\sin\eta + 2\mu_1\sin\zeta\left[\tfrac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,.\end{aligned} \tag{5.4}$$

As before, we assume $1 < P_0 < 4$ and define $a^2 = P_0 - 1$ and

$$\omega_{n-1}^2 = n(n^2 - P_0),\ n = 2, 3, \dots .$$

The equations (5.4) take the form of (2.1) when we define $x = (u_1, \dot{u}_1, w_1, \dot{w}_1)$ and $y = (u_2, \dot{u}_2/\omega_1, w_2, \dot{w}_2/\omega_1, u_3, \dot{u}_3/\omega_2, w_3, \dot{w}_3/\omega_2, \dots)$.

The reduced equations are

$$\begin{aligned}
\dot{x}_1 &= x_2\,,\\
\dot{x}_2 &= a^2 x_1 - \frac{\pi}{2}(x_1^2 + x_3^2)x_1 - 2\mu_2 x_2 \cos\eta + \frac{4}{\pi}\mu_1 \cos\zeta \cos\omega_0 t\,,\\
\dot{x}_3 &= x_4\,,\\
\dot{x}_4 &= a^2 x_3 - \frac{\pi}{2}(x_1^2 + x_3^2)x_3 - 2\mu_2 x_4 \sin\eta + \frac{4}{\pi}\mu_1 \sin\zeta \cos\omega_0 t\,.
\end{aligned}$$

When $\mu = 0$ we have a two-dimensional homoclinic manifold given by $\gamma_\beta = (r\cos\beta, \dot{r}\cos\beta, r\sin\beta, \dot{r}\sin\beta)$ where, as before, $r(t) = (2a/\sqrt{\pi})\,\mathrm{sech}\, at$ and β is a parameter. The adjoint equations (3.3) take the form

$$\begin{aligned}
\dot{v}_1 &= \left[-a^2 + \frac{\pi}{2}(3r^2\cos^2\beta + r^2\sin^2\beta)\right] v_2 + \left(\pi r^2 \sin\beta\cos\beta\right) v_4\,,\\
\dot{v}_2 &= -v_1,\\
\dot{v}_3 &= \left(\pi r^2 \sin\beta\cos\beta\right) v_2 + \left[-a^2 + \frac{\pi}{2}(r^2\cos^2\beta + 3r^2\sin^2\beta)\right] v_4\,,\\
\dot{v}_4 &= -v_3.
\end{aligned}$$

A one-parameter family of bounded solutions to these equations is given by

$$\begin{aligned}
v_{\beta 1} &= (-\dot{r}\sin\beta, r\sin\beta, \dot{r}\cos\beta, -r\cos\beta)\,,\\
v_{\beta 2} &= (-\ddot{r}\cos\beta, \dot{r}\cos\beta, -\ddot{r}\sin\beta, \dot{r}\sin\beta)
\end{aligned} \tag{5.5}$$

and the function, M, as in (3.5) becomes

$$M_1(\mu, \alpha, \beta) = \left[\frac{8}{\sqrt{\pi}} \sin(\beta - \zeta)\cos\omega_0\alpha\, \mathrm{sech}\, \tfrac{\pi\omega_0}{2a}\right]\mu_1\,,$$

$$M_2(\mu, \alpha, \beta) = \left[\frac{8\omega_0}{\sqrt{\pi}} \cos(\beta - \zeta)\sin\omega_0\alpha\, \mathrm{sech}\, \tfrac{\pi\omega_0}{2a}\right]\mu_1 - \left[\frac{16a^3(\cos\eta\cos^2\beta + \sin\eta\sin^2\beta)}{3\pi}\right]\mu_2\,.$$

Next, the conditions $M(\mu_0, \alpha_0, \beta_0) = 0$, $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ nonsingular are satisfied in two different cases. Of course, we suppose $\mu_{0,1} \neq 0$, $\mu_{0,2} \neq 0$ and then put $\lambda_0 = \frac{\mu_{0,2}}{\mu_{0,1}}$. We have the following two cases:

Case 1. We can choose either $\beta_0 = \zeta$ and then look for a simple root of the equation

$$\lambda_0 = m_1 \sin\omega_0\alpha\,, \tag{5.6}$$

or choose $\beta_0 = \zeta + \pi$ and look for a simple root of the equation

$$\lambda_0 = -m_1 \sin\omega_0\alpha \tag{5.7}$$

for

$$m_1 = \frac{3\sqrt{\pi}\omega_0}{2a^2(\cos\eta\cos^2\zeta + \sin\eta\sin^2\zeta)}\, \mathrm{sech}\, \frac{\pi\omega_0}{2a}\,.$$

Supposing the condition

$$0 < |\lambda_0| < m_1 \,, \tag{5.8}$$

there is a simple root α_0 of (5.6). Similarly, (5.7) has also a simple root $-\alpha_0$. According to the formulas (5.5) for v_{β_1} and v_{β_2}, these simple roots (ζ, α_0) and $(\zeta + \pi, -\alpha_0)$ give two different solutions of (5.4).

Case 2. In this case we begin by choosing $\omega_0\alpha_0 = (2k_0 + 1)\frac{\pi}{2}$ for $k_0 \in \{0, 1\}$ and then we look for a simple root $\beta_0 \neq \zeta + k\pi$, $\forall k \in \mathbb{Z}$ of

$$\lambda_0 = (-1)^{k_0}\Phi(\beta) \tag{5.9}$$

where

$$\Phi(\beta) = \frac{3\omega_0\sqrt{\pi}}{2a^3}\frac{\cos(\beta - \zeta)}{\cos\eta\cos^2\beta + \sin\eta\sin^2\beta}\operatorname{sech}\frac{\pi\omega_0}{2a}.$$

Let $m_2 = \max_{\beta\in\mathbb{R}}\Phi(\beta)$. We discuss the computation of the constant m_2 in Appendix A. Since $\Phi(\beta + \pi) = -\Phi(\beta)$, the range of Φ is the closed interval $[-m_2, m_2]$. We now split this case into two parts:

Part 2A). For $\eta = \pi/4$ we get $\Phi(\beta) = m_1\cos(\beta - \zeta)$, so $m_2 = m_1$. Equation (5.9) has now the form

$$(-1)^{k_0}m_1\cos(\beta - \zeta) = \lambda_0 \,,$$

so under condition (5.8), there is a simple root β_0 different from $\zeta + k\pi$, $\forall k \in \mathbb{Z}$. This holds for both cases $k_0 \in \{0, 1\}$ so we have two different solutions of (5.4). In addition, the results of Case 1 still apply here.

Thus, in this situation, we have in the μ_1-μ_2 plane four wedged-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which the partial differential equation exhibits chaos. In particular, (5.4) has four distinct homoclinic solutions, two from Case 1, two from Case 2A. These regions are labeled II in Figure 3. In this case there are no regions labeled I.

Part 2B). For $\eta \neq \pi/4$ we get $\Phi'(\zeta) \neq 0$, so $m_1 < m_2$. Certainly for the solvability of (5.9) we need $|\lambda_0| \leq m_2$. Now we claim:

Lemma 1. *If*

$$\lambda_0 \in (-m_2, m_2) \setminus \{\pm m_1, 0\} \,, \tag{5.10}$$

then equation (5.9) *has a simple root* $\beta_0 \in [0, 2\pi] \setminus \{\zeta, \zeta + \pi\}$.

Proof. Assume to the contrary that (5.9) has no simple roots for a $\lambda_0 \in (-m_2, m_2) \setminus \{\pm m_1, 0\}$. Then there are $0 \leq \beta_1 < \beta_2 \leq 2\pi$ such that

$$\Phi(\beta_{1,2}) = (-1)^{k_0}\lambda_0, \quad \Phi'(\beta_{1,2}) = 0, \quad \Phi''(\beta_{1,2}) = 0 \,. \tag{5.11}$$

Note then $\beta_{1,2} \neq \zeta + k\pi$ and $\beta_{1,2} \neq \zeta + \frac{2k+1}{2}\pi$, $\forall k \in \{0, 1\}$. After some calculation we derive from (5.11) that $\cos 2\beta_{1,2} \neq 0$, $\sin 2\beta_{1,2} \neq 0$ and that (5.11) is equivalent to

$$\begin{aligned} \frac{\cos(\beta_{1,2} - \zeta)}{\cos\eta\cos^2\beta_{1,2} + \sin\eta\sin^2\beta_{1,2}} &= \frac{\sin(\beta_{1,2} - \zeta)}{(\cos\eta - \sin\eta)\sin 2\beta_{1,2}} \\ &= \frac{\cos(\beta_{1,2} - \zeta)}{2(\cos\eta - \sin\eta)\cos 2\beta_{1,2}} = (-1)^{k_0}\frac{2a^3}{3\omega_0\sqrt{\pi}}\cosh\frac{\pi\omega_0}{2a}\lambda_0 \,. \end{aligned} \tag{5.12}$$

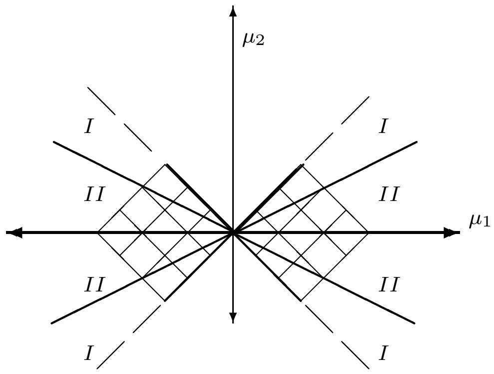

FIGURE 3. The chaotic wedge-shaped regions of (5.4) in $\mathbb{R}^2$.

From (5.12) we derive

$$\cos 2\beta_{1,2} = \frac{\cos\eta + \sin\eta}{3(\cos\eta - \sin\eta)}, \quad 2\tan(\beta_{1,2} - \zeta) = \tan 2\beta_{1,2}\,. \tag{5.13}$$

Hence

$$\beta_2 \in \{\pi - \beta_1, \pi + \beta_1, 2\pi - \beta_1\}\ .$$

If $\beta_2 = \pi - \beta_1$ then from $2\tan(\beta_2 - \zeta) = \tan 2\beta_2$ we get $2\tan(\beta_1 + \zeta) = \tan 2\beta_1$, but $2\tan(\beta_1 - \zeta) = \tan 2\beta_1$, so

$$\tan(\beta_1 + \zeta) = \tan(\beta_1 - \zeta)\,,$$

i.e. $\zeta = k\pi/2$, $k \in \{0, 1\}$. This contradicts $\zeta \in (0, \pi/2)$.

If $\beta_2 = \pi + \beta_1$ then

$$(-1)^{k_0}\lambda_0 = \Phi(\beta_2) = \Phi(\beta_1 + \pi) = -\Phi(\beta_1) = (-1)^{k_0+1}\lambda_0\,,$$

which implies $\lambda_0 = 0$, a contradiction.

If $\beta_2 = 2\pi - \beta_1$ then again we derive $\tan(\beta_1+\zeta) = \tan(\beta_1-\zeta)$, so that $\zeta = k\pi/2$, $k \in \{0, 1\}$, a contradiction to $\zeta \in (0, \pi/2)$. The proof is finished. □

Note $\beta_0 \in \{\zeta, \zeta + \pi\}$ for the Case 1, while $\beta_0 \in [0, 2\pi) \setminus \{\zeta, \zeta + \pi\}$ for the Case 2. Lemma 1 can be applied to both cases $\alpha_0 = \frac{\pi}{2\omega_0}(2k_0 + 1)$, $k_0 \in \{0, 1\}$ so Part 2B yields, in the μ_1-μ_2 plane, four wedge-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_2$, $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which (5.4) has two different homoclinic solutions. These regions are labeled I in Figure 3. Note we have four different solutions of (5.4) on regions labeled II, since there the Case 1 can be also applied (see (5.6) and (5.7)). This completes the analysis of the Melnikov function.

We now check for resonance. Because in the present problem all coupling terms are nonlinear, the linear equation in (H9) consists in two copies of the system of equations in the preceding example. This yields the following result obtained from Theorem 5.

THEOREM 7. *Suppose $\omega_0 \neq \omega_n$ for all n and let m_1, m_2 be as above.*

- **i)** *If* $m_0 \neq 0$ *satisfies one but not both of* $|m_0| < m_i$ *then if* $\mu_{0,2}/\mu_{0,1} = m_0$ *there exists a corresponding* $\bar{\xi}_0 > 0$ *such that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.4) *are given by* $\mu = \xi\mu_0$ *then there exist two homoclinic orbits which can be used to construct a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}*, of the period map* F *of* (5.4) *is invariant and conjugate to the Bernoulli shift on* Σ.
- **ii)** *If* $m_0 \neq 0$ *satisfies each of* $|m_0| < m_i$ *then there are four homoclinic orbits as in (i).*

Summarizing, we obtain eight open small wedge-shaped regions of parameter values in the μ_1-μ_2 plane bounded by the lines $\mu_2/\mu_1 = \pm m_1$, $\mu_2/\mu_1 = \pm m_2$ and $\mu_2 = 0$ with $m_1 \leq m_2$ for which the partial differential equation exhibits chaos (see Figure 3). In the regions labeled I there are two homoclinics while in regions II there exist four. It is interesting to note that in this case, by adjusting the parameters η and ζ, it is possible to make the size of the wedge arbitrarily close to filling the μ_1-μ_2 plane.

5.3. Nonplaner, Nonsymmetric Beam with One Buckled Mode in Each Plane. For the case of a nonsymmetric beam with nonplaner motion we have the boundary value problem

$$\begin{aligned} \ddot{u} &= -u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right)\, ds\right] u'' \\ &\qquad -2\mu_2 \dot{u}\cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\ \ddot{w} &= -R^2 w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right)\, ds\right] w'' \\ &\qquad -2\mu_2 \dot{w}\sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,, \\ & u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) \\ &= w(0,t) = w(\pi,t) = w''(0,t) = w''(\pi,t) = 0\,, \end{aligned}$$

where R^2 is constant representing the stiffness ratio for the two directions. We assume $R > 1$ which amounts to choosing w as the direction with stiffer cross-section. Note that $R = 1$ reduces to Section 5.2. As before we assume $\eta,\ \zeta \in (0, \pi/2)$.

The Galerkin expansion becomes

$$\begin{aligned} \ddot{u}_n &= n^2(P_0 - n^2)u_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right]u_n \\ &\quad - 2\mu_2 \dot{u}_n \cos\eta + 2\mu_1\cos\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,, \\ \ddot{w}_n &= n^2(P_0 - n^2R^2)w_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right]w_n \\ &\quad - 2\mu_2 \dot{w}_n \sin\eta + 2\mu_1\sin\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,. \end{aligned} \tag{5.14}$$

If P_0 is increased only enough to give one buckled mode, necessarily in the u direction, the problem reduces to Section 5.1. We shall assume here the next simplest case consisting of one buckled mode in each direction which occurs when $1 < P_0 < 4$

and $R^2 < P_0 < 4R^2$. Note that this requires $R < 2$ and we assume $R^2 < P_0 < 4$. If the stiffness ratio is too high there will be multiple buckled in the u (soft) direction before occurrence of the first buckled mode in the w (stiff) direction.

We define

$$a_1^2 = P_0 - 1, \qquad \omega_{n-1,1}^2 = n^2[(n^2 - P_0], \quad n = 2, 3, \dots ;$$
$$a_2^2 = P_0 - R^2, \qquad \omega_{n-1,2}^2 = n^2[n^2R^2 - P_0], \quad n = 2, 3, \dots .$$

We put (5.14) in the form of (2.1) by defining

$$x = (u_1, \dot{u}_1, w_1, \dot{w}_1)\,,$$
$$y = (u_2, \dot{u}_2/\omega_{1,1}, w_2, \dot{w}_2/\omega_{1,2}, u_3, \dot{u}_3/\omega_{2,1}, w_3, \dot{w}_3/\omega_{2,2}, \dots)\,.$$

The reduced equations are

$$\dot{x}_1 = x_2\,,$$
$$\dot{x}_2 = a_1^2 x_1 - \frac{\pi}{2}(x_1^2 + x_3^2)x_1 - 2\mu_2 x_2 \cos\eta + \frac{4}{\pi}\mu_1 \cos\zeta \cos\omega_0 t\,,$$
$$\dot{x}_3 = x_4\,,$$
$$\dot{x}_4 = a_2^2 x_3 - \frac{\pi}{2}(x_1^2 + x_3^2)x_3 - 2\mu_2 x_4 \sin\eta + \frac{4}{\pi}\mu_1 \sin\zeta \cos\omega_0 t\,.$$

For the unperturbed equations we have two homoclinic solutions given by

$$\gamma_1 = (r_1, \dot{r}_1, 0, 0), \qquad \gamma_2 = (0, 0, r_2, \dot{r}_2)$$

where $r_1(t) = (2a_1/\sqrt{\pi}\,)\operatorname{sech} a_1 t$ and $r_2(t) = (2a_2/\sqrt{\pi}\,)\operatorname{sech} a_2 t$.

Using γ_1 the adjoint equations (3.3) become

$$\begin{aligned}
\dot{v}_1 &= \left(-a_1^2 + \tfrac{3\pi}{2} r_1^2\right) v_2\,,\\
\dot{v}_2 &= -v_1\,,\\
\dot{v}_3 &= \left(-a_2^2 + \tfrac{\pi}{2} r_1^2\right) v_4\,,\\
\dot{v}_4 &= -v_3\,.
\end{aligned}$$

The essential question here is to determine the space of bounded solutions to these equations. We can write these in the form

$$\ddot{v}_2 = \left(a_1^2 - \tfrac{3\pi}{2} r_1^2\right) v_2, \qquad \ddot{v}_4 = \left(a_2^2 - \tfrac{\pi}{2} r_1^2\right) v_4.$$

The v_2 equation has a one-dimensional space of bounded solutions spanned by the solution $v_2 = \dot{r}_1$, obtained from $\dot{\gamma}_1$. For the v_4 equation we have the following result.

Lemma 2. *Let $\kappa > 0$. The equation*

$$\ddot{v} + (-\lambda + \kappa \operatorname{sech}^2 t)v = 0$$

has a bounded solution if and only if there exists an integer M such that

$$\begin{aligned}
&\lambda = \tfrac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 1\right)^2 \quad \textit{for} \quad 0 \le M < \tfrac{1}{4}\left(\sqrt{4\kappa+1} - 1\right)\\
\textit{or} \quad &\lambda = \tfrac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 3\right)^2 \quad \textit{for} \quad 0 \le M < \tfrac{1}{4}\left(\sqrt{4\kappa+1} - 3\right).
\end{aligned}$$

The idea for the proof of this lemma is to express the solution as the product of a power of $\operatorname{sech} t$ and a hypergeometric function with argument $-\sinh^2 t$. The condition for the existence of a bounded solution is that the hypergeometric series terminate and the resulting polynomial be of sufficiently small degree. The details for this have been worked out by Yagasaki in Appendix of [**40**]. See also Sections 23,25 of [**20**].

Applying Lemma 2 to the equation for v_4 we find that the condition for a bounded solution is $a_1 = a_2$ which is ruled out by the assumption $R > 1$. Hence, the system of equations for v has a one dimensional space of bounded solutions spanned by $v = (-\ddot{r}_1, \dot{r}, 0, 0)$ and the Melnikov function (3.4) is

$$M(\alpha) = \left[\frac{8\omega_0 \cos\zeta}{\sqrt{\pi}} \sin\omega_0\alpha \operatorname{sech}\tfrac{\pi\omega_0}{2a_1}\right]\mu_1 - \left(\frac{16a_1^3\cos\eta}{3\pi}\right)\mu_2.$$

The non-resonance hypothesis follows as in the previous examples which leads, in the present case, to the following result obtained from Theorem 5.

THEOREM 8. *If $\omega_0 \neq \omega_{n,i}$ for all n and for $i = 1, 2$, then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\cos\zeta}{2a_1^3\cos\eta}\operatorname{sech}\tfrac{\pi\omega_0}{2a_1}$$

there exists a corresponding $\bar{\xi}_0 > 0$ such that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.14) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^4 \times \mathbb{Y}$ on which the $2m$th iterate, F^{2m}, of the period map F of* (5.14) *is invariant and conjugate to the Bernoulli shift on Σ.*

Replacing γ_1 with γ_2 yields the following analogous result.

THEOREM 9. *If $\omega_0 \neq \omega_{n,i}$ for all n and for $i = 1, 2$, then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\sin\zeta}{2a_2^3\sin\eta}\operatorname{sech}\tfrac{\pi\omega_0}{2a_2}$$

there exists a corresponding $\bar{\xi}_0 > 0$ such that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.14) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^4 \times \mathbb{Y}$ on which the $2m$th iterate, F^{2m}, of the period map F of* (5.14) *is invariant and conjugate to the Bernoulli shift on Σ.*

In the μ_1-μ_2 plane in this case we get a diagram as in Figure 3. For parameter values in the regions labeled I there is one homoclinic orbit while for those in II there are two.

5.4. Multiple Buckled Modes. It remains to consider the situation where the axial load, P_0, is increased sufficiently to produce multiple buckled modes. We will look at the case of a beam constrained to planer motion. The calculations for the non-planer case are similar.

We return to the boundary value problem of Section 5.1 and use the same Galerkin equations

$$\begin{aligned}\ddot{u}_n =& n^2(P_0 - n^2)u_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2u_k^2\right]u_n \\ &- 2\mu_2\dot{u}_n + 2\mu_1\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t, \quad n = 1, 2, \dots .\end{aligned} \tag{5.15}$$

In the present case we assume there exists an integer N such that $N^2 < P_0 < (N+1)^2$. We then define

$$\begin{aligned} a_n^2 &= n^2(P_0 - n^2), \quad \text{for } n = 1, 2, \dots, N\,; \\ \omega_{n-N}^2 &= n^2(n^2 - P_0), \quad \text{for } n = N+1, N+2, \dots \end{aligned}$$

and put (5.15) in the form of (2.1) by defining

$$\begin{aligned} x &= (u_1, \dot{u}_1, u_2, \dot{u}_2, \ldots, u_N, \dot{u}_N)\,, \\ y &= (u_{N+1}, \dot{u}_{N+1}/\omega_1, u_{N+2}, \dot{u}_{N+2}/\omega_2, \ldots)\,. \end{aligned}$$

A truncated version of the resulting equations with $N = 2$ was studied in [**40**].

The reduced equations are

$$\left.\begin{aligned} \dot{x}_{2n-1} &= x_{2n} \\ \dot{x}_{2n} &= a_n^2 x_{2n-1} - \tfrac{\pi n^2}{2}\left(\sum_{k=1}^{N} k^2 x_{2k-1}^2\right) x_{2n-1} \\ &\qquad -2\mu_2 x_{2n} + 2\mu_1\left[\tfrac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t \end{aligned}\right\} n = 1, 2, \ldots, N\,.$$

When $\mu = 0$ we have N homoclinic solutions given by

$$\gamma_m = (0, \ldots, 0, \underbrace{r_m, \dot{r}_m}_{2m-1,2m}, 0, \ldots, 0), \quad m = 1, 2, \ldots, N$$

where $r_m(t) = (2a_m/m^2\sqrt{\pi})\operatorname{sech} a_m t$ and the adjoint equation (3.3) along γ_m is

$$\begin{aligned} \left.\begin{aligned} \dot{v}_{2n-1} &= \left(-a_n^2 + \tfrac{\pi m^2 n^2}{2} r_m^2\right) v_{2n}\,, \\ \dot{v}_{2n} &= -v_{2n-1}\,, \end{aligned}\right\} n \neq m \\ \dot{v}_{2m-1} &= \left(-a_m^2 + \tfrac{3\pi m^4}{2} r_m^2\right) v_{2m}\,, \\ \dot{v}_{2m} &= -v_{2m-1}\,. \end{aligned}$$

For the distinguished equation we have the bounded solution $v_{2m-1} = -\ddot{r}_m$, $v_{2m} = \dot{r}_m$ while for the equations with $n \neq m$ we must solve

$$\frac{d^2 v_{2n}}{dx^2} = \left(\frac{a_n^2}{a_m^2} - \frac{2n^2}{m^2}\operatorname{sech}^2 x\right) v_{2n}.$$

Using Lemma 2 we find that this last equation has a bounded solution if and only if there is an integer M such that one of the following conditions hold:

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \tfrac{1}{4}\left[\sqrt{\tfrac{8n^2}{m^2} + 1} - 4M - 1\right]^2 \qquad (5.16a)$$

$$\text{for } 0 \leq M < \tfrac{1}{4}\left(\sqrt{\tfrac{8n^2}{m^2} + 1} - 1\right),$$

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \tfrac{1}{4}\left[\sqrt{\tfrac{8n^2}{m^2} + 1} - 4M - 3\right]^2 \qquad (5.16b)$$

$$\text{for } 0 \leq M < \tfrac{1}{4}\left(\sqrt{\tfrac{8n^2}{m^2} + 1} - 3\right).$$

If, for some fixed m, none of the equations in (5.16) are satisfied for $n \neq m$ we can proceed much as in Section 5.1 since then the adjoint equation obtained from γ_m has a one-dimensional space of bounded solutions spanned by

$$v = (0, \ldots, 0, \underbrace{-\ddot{r}_m, \dot{r}_m}_{2m-1,2m}, 0, \ldots, 0).$$

One complication has been introduced by our assumption in the original partial differential equation that the transverse applied load is uniform in x. This assumption causes the μ_1 terms to drop out in (5.15) for n even which prohibits nonsingular solutions of $M(\alpha) = 0$ as can be seen by examining Section 5.1. For this reason, we must choose m odd. Theorem 5 now yields the following result.

THEOREM 10. *Let m be an odd integer, $1 \le m \le N$, and suppose P_0 is chosen so that none of the equations in* (5.16) *is satisfied. If $\omega_0 \neq \omega_n$ for all n, then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3} \operatorname{sech} \tfrac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ such that if $0 < \xi \le \bar{\xi}_0$, if the parameters in (5.15) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^{2N} \times \mathbb{Y}$ on which the $2k$th iterate, F^{2k}, of the period map F of* (5.15) *is invariant and conjugate to the Bernoulli shift on Σ.*

We can simplify the preceding results by finding cases where the equations in (5.16) can never have a solution. The following is a helpful result along these lines.

LEMMA 3. *The equations in* (5.16) *can never be satisfied for $n < m \le N$.*

Proof. For (5.16a) we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-1\right) < \frac{1}{2}$ so we have only one equation to consider with $M = 0$. But then we have, first, $\frac{n^2(P_0-n^2)}{m^2(P_0-m^2)} > \frac{n^2}{m^2}$, and also

$$\frac{1}{4}\left[\sqrt{\tfrac{8n^2}{m^2}+1}-1\right]^2 - \tfrac{n^2}{m^2} = \frac{2\frac{n^2}{m^2}\left(\frac{n^2}{m^2}-1\right)}{2\frac{n^2}{m^2}+1+\sqrt{\frac{8n^2}{m^2}+1}} < 0$$

so that the equation (5.16a) has no solution for any P_0.

Next we note that in when $n < m$, we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-3\right) < 0$ so that there are no equations for (5.16b). ∎

When $m = N$ the preceding result will eliminate any restriction, obtained from (5.16), on P_0. This fact was shown with a different technique by Berti and Carminati [**4**] where they used a more general transverse forcing term which allowed for the possibility of a μ_2 term for each n in (5.15) and, hence, also for each n in the reduced equation. They then take $m = N$. Since, for our specific form of loading, we must have m odd we have the following result.

THEOREM 11. *Let N and P_0 be as for* (5.15) *and suppose one of the following hold:*

(i) *N is odd and $m = N$.*
(ii) *N is even, $N \ge 4$, $m = N-1$ and*

$$P_0 \neq \frac{4N^2 - (N-1)^2\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}{4N^2 - \left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}.$$

(iii) *$N = 2$, $m = 1$ and*

$$P_0 \neq \frac{37+5\sqrt{33}}{16}, \quad P_0 \neq \frac{55+9\sqrt{33}}{16}.$$

Suppose in addition that $\omega_n \neq \omega_0$ for all n. Then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3} \operatorname{sech} \tfrac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *such that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.15) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^{2N} \times \mathbb{Y}$ *on which the* $2k$*th iterate,* F^{2k}*, of the period map* F *of* (5.15) *is invariant and conjugate to the Bernoulli shift on* Σ.

Proof. The result is obtained by using γ_m and proceeding as in Section 5.1. This is valid as long as the equations (5.16) have no solutions for $n \neq m$ so it remains to show this is true in each case. If (i) holds we can use Lemma 3.

If $m = N - 1$ then, using Lemma 3, we need check only $n = N$. Define

$$f_a(N) = \tfrac{1}{4}\left(\sqrt{\tfrac{8N^2}{(N-1)^2} + 1} - 1\right), \quad f_b(N) = \tfrac{1}{4}\left(\sqrt{\tfrac{8N^2}{(N-1)^2} + 1} - 3\right).$$

Then (5.16a) must be checked for integers $M \in [0, f_a(N))$ and (5.16b) for integers $M \in [0, f_b(N))$.

In case (ii) we have $N \geq 4$ which implies $1/2 < f_a(N) \leq (\sqrt{137} - 3)/12 < 1$ so we need consider only $M = 0$. In this case we solve

$$\frac{N^2(P_0 - N^2)}{(N-1)^2[P_0 - (N-1)^2]} = 4f_a(N)^2$$

for P_0 to get

$$P_0 = \frac{N^4 - 4f_a(N)^2(N-1)^4}{N^2 - 4f_a(N)^2(N-1)^2}.$$

But this value is negative and so can be discarded.

Similarly, we have for $N \geq 4$, $0 < f_b(N) \leq (\sqrt{137} - 9)/12 < 1$ so in (5.16b) we need also consider only $M = 0$. Here we get

$$\begin{aligned} P_0 &= \frac{N^4 - 4f_b(N)^2(N-1)^4}{N^2 - 4f_b(N)^2(N-1)^2} \\ &= \frac{4N^4 - (N-1)^2\left[\sqrt{9N^2 - 2N + 1} - 3(N-1)\right]^2}{4N^2 - \left[\sqrt{9N^2 - 2N + 1} - 3(N-1)\right]^2}. \end{aligned}$$

Next, we consider (iii) where $N = 2$, $m = 1$. Since $2 > f_a(2) = (\sqrt{33} - 1)/4 > 1$ we must consider $M = 0$ and $M = 1$ in (5.16a). When $M = 0$ we get the value $P_0 = -(7 + \sqrt{33})/2 < 0$ which can be discarded while for $M = 1$ we have $P_0 = (37 + 5\sqrt{33})/16$.

Finally, $0 < f_b(2) = (\sqrt{33} - 3)/4 < 1$ so only $M = 0$ must be considered in (5.16b) and this yields $P_0 = (55 + 9\sqrt{33})/16$. ∎

Appendix A. Appendix

We propose here a method for computing the constant m_2 of Section 5.2. For this reason, we first search for critical points of the function $\Phi(\beta)$ from equation (5.9). Note

$$\Phi(\beta) = \frac{3\omega_0\sqrt{\pi}}{2a^3}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\Psi(\beta)$$

for

$$\Psi(\beta) = \frac{\cos(\beta - \zeta)}{\cos\eta\cos^2\beta + \sin\eta\sin^2\beta}.$$

Since $\Psi(\beta+\pi) = -\Psi(\beta)$ and $\Psi(0) = \frac{\cos\zeta}{\cos\eta} > 0$, $\Psi'(0) = \frac{\sin\zeta}{\cos\eta} > 0$, we restrict our analysis to $\beta \in (0,\pi)$. Taking $x := \tan\beta/2$ and using the formulas

$$\sin\beta = \frac{2\tan(\beta/2)}{1+\tan^2(\beta/2)} = \frac{2x}{1+x^2}, \quad \cos\beta = \frac{1-\tan^2(\beta/2)}{1+\tan^2(\beta/2)} = \frac{1-x^2}{1+x^2},$$

we get $\Psi(\beta) = \Upsilon(x)$ for

$$\Upsilon(x) := \frac{(1-x^4)\cos\zeta + 2x(1+x^2)\sin\zeta}{(x^2-1)^2\cos\eta + 4x^2\sin\eta}. \tag{A.1}$$

We compute

$$\Upsilon'(x) = -\frac{2f(x)\sin\eta\sin\zeta}{\left((x^2-1)^2\cos\eta + 4x^2\sin\eta\right)^2}$$

for

$$f(x) := Gx^6 + x^5(4Z-2GZ) + x^4(5G-4)$$
$$+4GZx^3 + x^2(4-5G) + x(4Z-2GZ) - G,$$

where $Z = \cot\zeta > 0$ and $G = \cot\eta > 0$. Hence $\Upsilon'(x) = 0$, $x > 0$ if and only if

$$f(x) = 0. \tag{A.2}$$

Verifying

$$x^6 f(-1/x) = -f(x) \quad \forall x \neq 0, \tag{A.3}$$

$f(x)$ is an anti-reciprocal polynomial [**33**]. Due to $f(0) = -G < 0$, certainly $f(x) = 0$ has a positive root. Also if $f(x_0) = 0$ then $f(-1/x_0) = 0$. Using (A.3), we put

$$y := x - x^{-1}.$$

Note the mapping $x \mapsto x - x^{-1}$ is increasing on $(0,\infty)$ with the range $\mathbb{R}$. Its inverse is $x = \frac{y+\sqrt{y^2+4}}{2}$. Next we rewrite (A.2) as

$$Gx^3 - Gx^{-3} + (x^2+x^{-2})(4Z-2GZ) + (x-x^{-1})(5G-4) + 4GZ = 0. \tag{A.4}$$

From

$$y^2 = x^2 + x^{-2} - 2, \quad y^3 = x^3 - x^{-3} - 3(x - x^{-1}),$$

in the variable y, (A.4) has the form

$$p(y) := Gy^3 + 2Z(2-G)y^2 + 4(2G-1)y + 8Z = 0. \tag{A.5}$$

Rewriting (A.1) in the variable y we get

$$\Theta(y) = \frac{2\sin\zeta - y\cos\zeta}{y^2\cos\eta + 4\sin\eta}\sqrt{y^2+4}. \tag{A.6}$$

Recall $y = x - x^{-1}$, $y = -2\cot\beta$ and $\Psi(\beta) = \Upsilon(x) = \Theta(y)$ for $\beta \in (0,\pi)$, $x \in (0,\infty)$, $y \in \mathbb{R}$. We can easily check that

$$\Theta'(y) = -\frac{2p(y)\sin\zeta\sin\eta}{\sqrt{y^2+4}\,(y^2\cos\eta + 4\sin\eta)^2}.$$

Hence $\Theta'(y) = 0$ if and only if $p(y) = 0$. Note

$$f(x) = x^3 p\left(x - x^{-1}\right).$$

Summarizing, in order to find m_2, we need to solve (A.5) and then insert its solutions to (A.6).

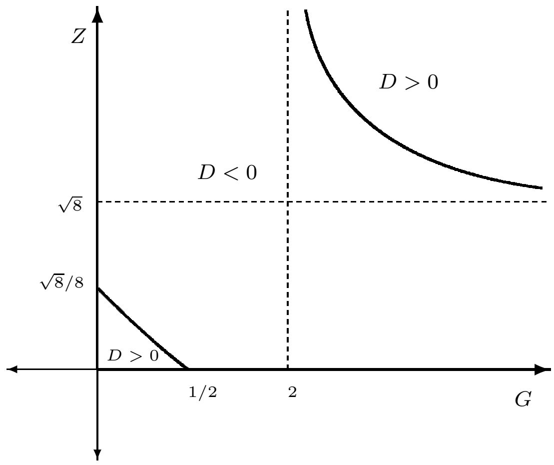

FIGURE 4. Sign regions of the discriminant D.

Next we compute the discriminant D of (A.5) (see [**5**]):

$$D = \frac{64}{27G^4}(Z^2 + G)((G-2)^3 Z^2 - (2G-1)^3)\,.$$

Now (see Figure 4) we have the following possibilities:

1) $D > 0 \Leftrightarrow (G-2)^3 Z^2 > (2G-1)^3$, and then (A.5) has 3 different real roots.
2) $D = 0 \Leftrightarrow (G-2)^3 Z^2 = (2G-1)^3$, and then (A.5) has 3 real roots, but one is double.
3) $D < 0 \Leftrightarrow (G-2)^3 Z^2 < (2G-1)^3$, and then (A.5) has 1 real root.

These roots are done by the Cardano formulas [**5**]. For general ζ and η these formulas are rather awkward, but for concrete values of ζ and η, we can easily check which one of the above cases (1)-(3) hold, and then we easily compute the roots by using these Cardano formulas. Inserting these roots into (A.6) we are able to find m_2 for concrete values of ζ and η. Moreover, in the cases (2), (3) there is a unique simple zero y_0 of $p(y)$ and then $m_2 = \frac{3\omega_0\sqrt{\pi}}{2a^3}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\Theta(y_0)$.

Summarizing, we suggest this simple algorithm/method for finding m_2: For general ζ and η, the formula for m_2 is very awkward, so we do not derive it, but for concrete values of ζ and η, we can directly calculate m_2 avoiding the use of some numerical/approximation methods.

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Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava - Slovakia, and Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava - Slovakia

E-mail address: Michal.Feckan@fmph.uniba.sk

37 Fountain Manor Drive, Suite B, Greensboro, North Carolina 27405-8077, USA