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A Hopf Bifurcation in a Radially Symmetric Free Boundary Problem

Sang-Gu Lee* and YoonMee Ham**

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Abstract. We consider an interface problem derived by a reaction-diffusion equation in two- and three- dimensional system with radial symmetry. Existence of Hopf bifurcation as a parameter varies will be studied in two- and three- dimensional spaces.

Contents

1. Introduction

Dynamics of interfacial patterns are discussed in many systems from biology, chemistry, physics and other fields [**1, 3, 5, 13, 19**]. Internal layers (or free boundary), which separate two stable bulk states by a sharp transition near interfaces, are often observed in bistable reaction-diffusion equations when the reaction rate is faster than the diffusion effect.

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** Corresponding author. Email:ymham@kyonggi.ac.kr; Tel.:+82-31-249-9611; Fax : +82-31-253-1165.

The typical patterns appearing in bistable media can be modeled by two-component reaction-diffusion systems:

$$\begin{cases} \sigma\varepsilon\, w_t = \varepsilon^2\, \nabla^2 w + H(w - a_0) - w - v, \\ v_t = D\nabla^2 v + \mu w - v, \qquad t > 0, \mathbf{x} \in \mathbb{R}^n, \end{cases} \tag{1}$$

where ε, σ and a_0 are positive constant parameters, and ∇ is the gradient operator. Here u and v measure the levels of two diffusing quantities, and H is a Heaviside function satisfying $H(z) = 1$ for $z > 0$ and $H(z) = 0$ for $z < 0$.

When ε is sufficiently small, the singular limit analysis $\varepsilon \to 0$ is applied to show the existence and the stability of localized radially symmetric equilibrium solutions ([**15, 16**]). In one-dimensional space, such equilibrium solutions should undergo certain instabilities and the loss of stability resulting from a Hopf bifurcation produces a kind of periodic oscillation in the location of the internal layers ([**2, 4, 11, 14, 15**]). As the parameter D varies, the stability of the spherically symmetric solutions and their symmetry-breaking bifurcations into layer solutions which are not spherically symmetric have been examined in [**17, 18**]. Moreover, for $\varepsilon = 0$, a free boundary problem of (1) in one dimensional space has been obtained and a Hopf bifurcation of this problem has been examined in [**6, 7, 10**] as a parameter σ varies.

In the present literature, the free boundary problem of (1) for the case when $\varepsilon = 0$ in two- and three- dimensional space has not been studied. Motivated by these facts, our main purpose of this paper is to study this problem. In order to consider the free boundary in this problem, the equation of interfaces can be derived from (1). Suppose that there is only one $(n-1)$ -dimensional hypersurface $\eta(t)$ which is simply single closed curve given in the whole plane $\mathbb{R}^n$ in such a way that $\mathbb{R}^n = \Omega_1(t) \cup \eta(t) \cup \Omega_0(t)$, where $\Omega_1(t) = \{(\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty)\ : w(\mathbf{x}, t) > a_0\}$ and $\Omega_0(t) = \{(\mathbf{x}, t) \in \mathbb{R}^n \times (0, \infty)\ : w(\mathbf{x}, t) < a_0\}$. Then the equation of $\eta(t)$ is given by (see [**9, 12, 16**]):

$$\frac{d\,\eta(t)}{dt} \cdot \nu = C(v_i),\ \ (\mathbf{x}, t) \in \eta(t), \tag{2}$$

where ν is the outward normal vector on $\eta(t)$, v_i is the value of v on the interface $\eta(t)$, and $C(v)$ is the velocity of the interface. The reaction terms in (1) satisfy the bistable condition, i.e., the nullclines of $H(w - a_0) - w - v = 0$ and $\mu w - v = 0$ must have three intersection points and the nullcline $H(w - a_0) - w - v = 0$ is the triple valued function of w which are called $h^+(v)$, $h^-(v)$ and $h^0(v)$. From [**4, 8, 12**], the trajectory with a unique value of $C = C(v)$ exists which is given by $C(v) = h^+(v) - 2h^0(v) + h^-(v)$. Furthermore, the velocity of the interface $C(v)$ is a continuously differentiable function defined on an interval $I := (-a_0, 1 - a_0)$ and thus the velocity of the interface can be normalized by

$$C(v) = \frac{1}{\sigma} \frac{1 - 2a_0 - 2v}{\sqrt{(v + a_0)(1 - a_0 - v)}}. \tag{3}$$

Hence a free boundary problem of (1) when ε is equal to zero is given by :

$$
(4)\qquad \begin{cases} v_t = \nabla^2 v - (\mu+1)v + \mu, \quad (\mathbf{x},t)\in\Omega_1(t) \\ v_t = \nabla^2 v - (\mu+1)v, \quad (\mathbf{x},t)\in\Omega_0(t) \\ v(\mathbf{x},0) = v_0(\mathbf{x}) \\ v(\eta(t)-0,t) = v(\eta(t)+0,t) \\ \frac{d}{d\nu} v(\eta(t)-0,t) = \frac{d}{d\nu} v(\eta(t)+0,t) \\ \lim_{|\mathbf{x}|\to\infty} v(|\mathbf{x}|,t) = 0,\ t>0. \end{cases}
$$

Our aim is to explore the dynamics of interfaces in the problem (4) in order to investigate the existence of time periodic solutions as the bifurcation parameter σ varies in two and three dimensions. In section 2, a change of variables is given which regularizes problem (4) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show the existence of radially symmetric localized equilibrium solutions for (4) and obtain the linearization of problem (4). In the last section we show the existence of the periodic solutions and the bifurcation of the interface problem as a parameter σ varies in two and three dimensions.

2. Regularization of the interface equation

We look for an existence problem of radially symmetric equilibrium solutions of (4) with $|\mathbf{x}| = r$ where the center and the interface are located at the origin and $r = \eta$, respectively. The problem is given by:

$$
(5)\qquad \begin{cases} v_t = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial v}{\partial r} - (\mu+1)v + \mu\, H(\eta(t)\, - r), \ \ r\in(0,\infty),\ t>0 \\ v(r,0) = v_0(r) \\ \dfrac{\partial v}{\partial r} v(0,t) = 0 = v(\infty,t),\ t>0 \\ \eta'(t)\cdot\nu = C(v(\eta(t),t)),\ t>0. \end{cases}
$$

Let A be a differential operator $A := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \mu + 1$ with the domain $D(A) = \{v \in H^{2,2}(\mathbb{R}^n) : \frac{\partial v}{\partial r} v(0,t) = 0, \lim_{r\to\infty} v(r,t) = 0\}$. For the application of semigroup theory to (4), we choose the space

$$X := L_2(\mathbb{R}^n) \text{ with norm } \|\cdot\|_2\,.$$

We define $g : \mathbb{R}^2 \to \mathbb{R}$,

$$g(r,\eta) := A^{-1}(\mu(H(\eta-\cdot))(r) = \mu\int_\eta^\infty G(r,y)\,dy,$$

where $G : \mathbb{R}^n \to \mathbb{R}$ is a Green's function of A satisfying the boundary conditions:

$$G(r,z) = \begin{cases} zK_0(z\sqrt{1+\mu})\, I_0(r\sqrt{1+\mu})\,, \ 0 < r < z \\ zI_0(z\sqrt{1+\mu})\, K_0(r\sqrt{1+\mu})\,, \ z < r \end{cases} \quad \text{for} \ \ n = 2,$$

$$G(r,z) = \begin{cases} z\, e^{-z\sqrt{1+\mu}}\ \frac{\sinh(r\sqrt{1+\mu})}{r\sqrt{1+\mu}}\,, \ 0 < r < z \\ z \sinh(z\sqrt{1+\mu})\ \frac{e^{-r\sqrt{1+\mu}}}{r\sqrt{1+\mu}}\,, \ z < r \end{cases} \quad \text{for} \ \ n = 3,$$

where I_0 and K_0 are modified Bessel functions.

Applying the transformation $u(t)(r) = v(r,t) - g(r, \eta(t))$ then we obtain an equivalent abstract evolution equation of (5) :

$$\begin{cases} \dfrac{d}{dt}(u,\eta) + \widetilde{A}(u,\eta) = \ f(u,\eta) \\ (u,\eta)(0) = (u_0(r), \eta_0), \end{cases} \tag{6}$$

where $\widetilde{A}$ is a 2×2 matrix whose (1,1)-entry is an operator A and all others are zero. The nonlinear forcing term f is

$$f(u,\eta\,) = \begin{pmatrix} C(u(\eta) + \gamma(\eta))\, G(r,\eta) \\ \\ C(u(\eta) + \gamma(\eta)) \end{pmatrix},$$

where the function $\gamma : \mathbb{R} \to \mathbb{R}$ is defined by $\gamma(\eta) := g(\eta, \eta)$.

The well posedness of solutions of (6) are shown in [**10**] applying the semigroup theory using domains of fractional powers $\alpha \in (3/4, 1]$ of A and $\widetilde{A}$. Moreover, the nonlinear term f is a continuously differentiable function from $W \cap \widetilde{X}^\alpha$ to $\widetilde{X}$ where

$$W := \{(u,\eta) \in C^1(\mathbb{R}) \times \mathbb{R} \,:\, u(\eta) + \gamma(\eta) \in I\} \subset_{\text{open}} C^1(\mathbb{R}) \times \mathbb{R} \times \mathbb{R}\,,$$

$$\widetilde{X} := D(\widetilde{A}) = D(A) \times \mathbb{R}, \ \ X^\alpha := D(A^\alpha) \ \ \text{and} \ \ \widetilde{X}^\alpha := D(\widetilde{A}^\alpha) = X^\alpha \times \mathbb{R}.$$

3. Radially symmetric equilibrium solutions and Linearization of the interface equation

In this section, we shall examine the existence of radially symmetric equilibrium solutions of (6) in $\mathbb{R}^n$ $(n = 2, 3)$. We look for $(u^*, \eta^*) \in D(\widetilde{A}) \cap W$ satisfying the following problem :

$$\begin{cases} \dfrac{d^2u}{dr^2} + \dfrac{n-1}{r}\dfrac{du}{dr} - (\mu+1)u = G(r,\eta)C(u(\eta) + \gamma(\eta)) \\ 0 = C(u(\eta) + \gamma(\eta)) \\ \frac{du}{dr}(0) = 0 = u(\infty). \end{cases} \tag{7}$$

THEOREM 3.1. *Suppose that* $0 < \frac{1}{2} - a_0 < \frac{\mu}{\mu+1}$. *Then equation (6) has at least one radially symmetric equilibrium solutions* $(0, \eta^*)$ *for all* $\sigma \neq 0$.

The linearization of f *at the stationary solution* $(0, \eta^*)$ *is*

$$Df(0,\eta^*)(\hat{u},\hat{\eta}) = \begin{pmatrix} \frac{4}{\sigma}\big(\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta}\,\big)G(\cdot,\eta^*) \\ \\ \frac{4}{\sigma}\big(\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta}\,\big) \end{pmatrix}.$$

The pair $(0,\eta^*)$ corresponds to a unique steady state (v^*,η^*) of (4) for $\sigma \neq 0$ with $v^*(r) = g(r,\eta^*)$.

Proof: System (7) is equivalent to the pair of equations:

$$(8)\qquad \begin{cases} 0 = -u''(r) - \frac{n-1}{r}u'(r) + (1+\mu)u \quad \text{with } u'(r)(0) = 0 = u(\infty) \\ 0 = \frac{1}{2} - a_0 - u(\eta) - \gamma(\eta). \end{cases}$$

For $n = 2$, the general solution of (8) is given by

$$u(r) = c_1 I_0(r\sqrt{1+\mu}) + c_2 K_0(r\sqrt{1+\mu})$$

for some constants c_1 and c_2. The boundary condition $u(\infty) = 0$ implies that $c_1 = 0$ since $\lim_{r\to\infty} K_0(r\sqrt{1+\mu}) = 0$. Moreover, $u'(0) = 0$ implies that $c_2 = 0$. We have $u^* = 0$. For $n = 3$, the general solution of (8) is given by

$$u(r) = c_1 \frac{e^{-r\sqrt{1+\mu}}}{r} + c_2 \frac{e^{r\sqrt{1+\mu}}}{2r\sqrt{1+\mu}}$$

for some constants c_1 and c_2. Applying the boundary conditions, we obtain $u^* = 0$. In order to show an existence of η^* we define $\Gamma(\eta) := \gamma(\eta) - (\frac{1}{2} - a_0)$. Then $\Gamma'(\eta) = \gamma'(\eta) = -\mu\, G(\eta,\eta) + \mu \int_\eta^\infty G_\eta(\eta,y)dy$ which is

$$\begin{aligned} \Gamma'(\eta) &= -\mu\, G(\eta,\eta) + \mu\,\eta\, I_1(\eta\sqrt{1+\mu})\, K_1(\eta\sqrt{1+\mu}) \\ &= \mu\,\eta\,\big(I_1(\eta\sqrt{1+\mu})\, K_1(\eta\sqrt{1+\mu}) - I_0(\eta\sqrt{1+\mu})\, K_0(\eta\sqrt{1+\mu})\big) \quad (n=2), \end{aligned}$$

$$\begin{aligned} \Gamma'(\eta) &= \mu\Big(\frac{1}{\sqrt{1+\mu}} + \frac{1}{(1+\mu)\eta} + \frac{1}{2(1+\mu)^{3/2}\,\eta^2}\Big)e^{-2\eta\sqrt{1+\mu}} \\ &\quad -\tfrac{1}{2(1+\mu)^{3/2}\,\eta^2} \quad (n=3), \end{aligned}$$

where I_i and K_i are the modified Bessel function of the ith order. Since $\gamma'(\eta) < 0$ and $\lim_{\eta\to\infty} \gamma(\eta) = 0$ for $n = 2,3,$ there is a unique $\eta^* \in (0,\infty)$ when $\Gamma(0) > 0$ which means $1/2 - a_0 < \gamma(0) = \frac{\mu}{1+\mu}$.

The formula for $Df(0,\eta^*)$ is obtained from the relation $C'(1/2 - a_0) = 4/\sigma$ and Lemma 4 in [**10**]. The corresponding steady state (v^*,η^*) for (4) is obtained using the transformation and Proposition 7 in [**10**]. ∎

4. A Hopf bifurcation

We shall show that there is a Hopf bifurcation from the curve $\sigma \mapsto (0,\eta^*)$ of radially symmetric stationary solution, and we therefore introduce the following definition.

Definition 4.1. *Under the assumptions of Theorem 3.1, define (for $1 \geq \alpha > 3/4$) the linear operator B from $\widetilde{X}^\alpha$ to $\widetilde{X}$*

$$B := \frac{\sigma}{4}\, Df(0,\eta^*)\,.$$

We then define $(0,\eta^*)$ *to be a* Hopf point *for (6) if and only if there exists an* $\epsilon_0 > 0$ *and a* C^1*-curve*

$$(-\epsilon_0+\tau^*,\tau^*+\epsilon_0)\mapsto(\lambda(\tau),\phi(\tau))\in\mathbb{C}\times\widetilde{X}_{\mathbf{C}}$$

($Y_{\mathbf{C}}$ denotes the complexification of the real space Y) of eigendata for $-\widetilde{A}+\tau B$ *with*

(i) $(-\widetilde{A}+\tau B)(\phi(\tau))=\lambda(\tau)\phi(\tau),\quad (-\widetilde{A}+\tau B)(\overline{\phi(\tau)})=\overline{\lambda(\tau)}\,\overline{\phi(\tau)}$;
(ii) $\lambda(\tau^*)=i\beta$ *with* $\beta>0$;
(iii) $\mathrm{Re}\,(\lambda)\neq 0$ *for all* $\lambda\in\sigma(-\widetilde{A}+\tau^*B)\setminus\{\pm i\beta\}$;
(iv) $\mathrm{Re}\,\lambda'(\tau^*)\neq 0$ *(transversality);*

where $\tau=4/\sigma$.

Next, we have to check (6) for Hopf points. For this we have to solve the eigenvalue problem:

$$-\widetilde{A}(u,\eta)+\tau B(u,\eta)=\lambda(u,\eta)$$

which is equivalent to

$$\begin{cases}(A+\lambda)u &= \tau\,\mu\,(u(\eta^*)+\gamma'(\eta^*)\,\eta)\,G(\cdot,\eta^*)\\ \lambda\,\eta &= \tau(u(\eta^*)+\gamma'(\,\eta^*)\,\eta).\end{cases}\tag{9}$$

Now we shall show that radially symmetric equilibrium solution becomes a Hopf point.

THEOREM 4.2. *Assume* $0<1/2-a_0<\frac{\mu}{\mu+1}$ *and the operator* $-\widetilde{A}+\tau^*B$ *has a unique pair* $\{\pm i\beta\}$ *of purely imaginary eigenvalues for some* $\tau^*>0$. *Then* $(0,\eta^*,\tau^*)$ *is a Hopf point for (6).*

Proof: We assume without loss of generality that $\beta>0$, and ϕ^* is the (normalized) eigenfunction of $-\widetilde{A}+\tau^*B$ with eigenvalue $i\beta$. We have to show that $(\phi^*,i\beta)$ can be extended to a C^1-curve $\tau\mapsto(\phi(\tau),\lambda(\tau))$ of eigendata for $-\widetilde{A}+\tau B$ with $\mathrm{Re}(\lambda'(\tau^*))\neq 0$.

For this let $\phi^*=(\psi_0,\eta_0)\in D(A)\times\mathbb{R}\times\mathbb{R}$. First, we see that $\eta_0\neq 0$, for otherwise, by (9), $(A+i\beta)\psi_0=\mu\,i\beta\,\eta_0\,G(\cdot,\eta^*)=0$, which is not possible because A is symmetric. So without loss of generality, let $\eta_0=1$. Then $E(\psi_0,i\beta,\tau^*)=0$ by (9), where

$$E:D(A)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{R}\longrightarrow X_{\mathbf{C}}\times\mathbb{C},$$

$$E(u,\lambda,\tau):=\begin{pmatrix}(A+\lambda)u-\mu\,\tau\big(u(\eta^*)+\gamma'(\eta^*)\big)\,G(\cdot,\eta^*)\\ \lambda-\tau\big(u(\eta^*)+\gamma'(\eta^*)\big)\end{pmatrix}.$$

The equation $E(u,\lambda,\tau)=0$ is equivalent to λ being an eigenvalue of $-\widetilde{A}+\tau B$ with eigenfunction $(u,1)$. We shall apply here the implicit function theorem to E. For that, it is necessary that E is of C^1 -class and

$$D_{(u,\lambda)}E(\psi_0,i\beta,\tau^*)\in L(D(A)_{\mathbf{C}}\times\mathbb{C}\times\mathbb{R},X_{\mathbf{C}}\times\mathbb{C})\text{ is an isomorphism.}\tag{10}$$

It is easy to see that E is of C^1- class. In addition, the mapping

$$D_{(u,\lambda)}E(\psi_0,i\beta,\tau^*)(\hat{u},\hat{\lambda})=\begin{pmatrix}(A+i\beta)\hat{u}-\mu\,\tau^*\hat{u}(\eta^*)\,G(\cdot,\eta^*)+\hat{\lambda}\,\psi_0\\ \hat{\lambda}-\tau^*\,\hat{u}(\eta^*)\end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \big((A + i\beta)\hat{u},\ \hat{\lambda}\big)$$

which is invertible. Thus $D_{(u,\,\lambda)}E(\psi_0, i\beta, \tau^*)$ is a Fredholm operator of index 0. Therefore in order to verify (10), it suffices to show that the system

$$D_{(u,\,\lambda)}E(\psi_0, i\beta, \tau^*)(\hat{u}, \hat{\lambda}) = 0$$

which is equivalent to

$$\begin{cases} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \mu\tau^*\, \hat{u}(\eta^*)\, G(\cdot, \eta^*) \\ \hat{\lambda} = \tau^*\, \hat{u}(\eta^*) \end{cases} \tag{11}$$

necessarily implies that $\hat{u} = 0$ and $\hat{\lambda} = 0$. We define $\psi_1 := \psi_0 - \mu\, G(\cdot, \eta^*)$ then the first equation of (11) is given by

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0. \tag{12}$$

On the other hand, since $E(\psi_0, i\beta, \tau^*) = 0$, we have

$$(A + i\beta)\psi_0 = i\beta\, \mu G(\cdot, \eta^*)\,.$$

ψ_1 is a solution to the equation

$$(A + i\,\beta)\psi_1 = -\mu\, \delta_{\eta^*} \tag{13}$$

and

$$i\beta = \tau^*\big(\psi_1(\eta^*) + \mu G(\eta^*, \eta^*) + \gamma'(\eta^*)\big). \tag{14}$$

From (13) and (14),

$$\tau^*\, \mathrm{Im}\, \psi_1(\eta^*) = \beta.$$

Equation (13) implies that

$$-\mu\, \overline{\psi_1(\eta^*)} = \int_{\mathbb{R}} |A^{1/2}\psi_1|^2 + i\beta \int_{\mathbb{R}} |\psi_1|^2\,,$$

so that

$$\mu\, \mathrm{Im}\,(\psi_1(\eta^*)) = \beta \int_{\mathbb{R}} |\psi_1|^2\,.$$

Hence we have

$$\int_{\mathbb{R}} |\psi_1|^2 = \frac{\mu}{\tau^*}\,. \tag{15}$$

From (13), we now can calculate $\hat{u}(\eta^*)$ as $\int_{\mathbb{R}} (A + i\beta)\hat{u}\, \psi_1 = -\mu\, \hat{u}(\eta^*)$ which together with (11), (12) and (15) implies that

$$\begin{aligned} \hat{\lambda} \int \psi_1^2 + i\,\beta\,\mu\,\hat{\eta} \int G(x, \eta^*)\psi_1(x)dx\ &= \mu\hat{u}(\eta^*) \\ = \hat{\lambda}\, \frac{\mu}{\tau^*} - \mu\, \psi_0(\eta^*)\hat{\eta} &= \hat{\lambda} \int_{\mathbb{R}} |\psi_1|^2 - \mu\, \psi_0(\eta^*)\hat{\eta}. \end{aligned}$$

Since from (13)

$$i\beta \int G(x, \eta^*)\psi_1(x)dx\ = -\int A\psi_1(x)\, G(x, \eta^*)dx - \mu\, G(\eta^*, \eta^*)\ = -\psi_0(\eta^*)$$

is obtained, we have

$$\hat{\lambda}\Big(\int_{\mathbb{R}} ({\psi_1}^2 - |\psi_1|^2)\Big) = 0$$

which implies that $\hat{\lambda} = 0$ and thus $\hat{u} = 0$. We have shown (10), and thus get a C^1-curve $\tau \mapsto (\phi(\tau), \lambda(\tau))$ of eigendata such that $\phi(\tau^*) = \phi^*$ and $\lambda(\tau^*) = i\beta$.

It remains to be shown that the transversality condition $\mathrm{Re}\,\lambda'(\tau^*) \neq 0$ holds. Implicit differentiation of $E(\psi_0(\tau),\, \lambda(\tau), \tau) = 0$ implies that

$$D_{(u,\lambda)}E(\psi_0, i\beta, \tau^*)(\psi_0'(\tau^*), \lambda'(\tau^*)) = \begin{pmatrix} \mu(\psi_0(\eta^*) + \gamma'(\eta^*))G(\cdot, \eta^*) \\ \psi_0(\eta^*) + \gamma'(\eta^*) \end{pmatrix}.$$

This means that the function $\tilde{u} := \psi'(\tau^*)$, $\tilde{\eta} := \eta'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$\begin{cases} (A + i\beta)\tilde{u} + \tilde{\lambda}\psi_1 = 0 \\ \tilde{\lambda} - \tau^*(\tilde{u}(\eta^*) + \gamma'(\eta^*)) = \psi_0(\eta^*) + \gamma'(\eta^*), \end{cases} \tag{16}$$

where $\psi_1 := \psi_0 - \mu G(\cdot, \eta^*)$. The equations (14) and (16) implies that

$$\tilde{u}(\eta^*) = \frac{\tilde{\lambda}}{\tau^*} - \frac{i\beta}{{\tau^*}^2}. \tag{17}$$

Multiplying $\overline{\psi}_1$ to (16) then

$$\int (A + i\beta)\tilde{u}\overline{\psi}_1 + \tilde{\lambda}\int |\psi_1|^2 = 0$$

which implies that

$$-\tilde{u}(\eta^*) + 2i\,\beta \int \tilde{u}\overline{\psi}_1 + \tilde{\lambda}\int |\psi_1|^2 = 0\,.$$

Multiplying $\overline{\tilde{u}}$ to (16) then

$$\int |A^{1/2}\tilde{u}|^2 - i\beta \int |\tilde{u}|^2 + \overline{\tilde{\lambda}} \int \tilde{u}\overline{\psi}_1 = 0.$$

Comparing the above equations, we have

$$-\frac{1}{\tau^*}|\tilde{\lambda}|^2 + \frac{i\,\beta}{{\tau^*}^2}\overline{\tilde{\lambda}} + |\tilde{\lambda}|^2 \int |\psi_1|^2 - 2i\beta \int |A^{1/2}\tilde{u}|^2 - 2\beta^2 \int |\tilde{u}|^2 = 0$$

which implies that

$$\mathrm{Im}\,\tilde{\lambda} = 2\beta\,{\tau^*}^2 \int |\tilde{u}|^2 \quad \text{and} \quad \mathrm{Re}\,\tilde{\lambda} = 2{\tau^*}^2 \int |A^{1/2}\tilde{u}|^2\,.$$

Hence $\mathrm{Re}\tilde{\lambda} = \mathrm{Re}\,\lambda'(\tau^*) = 2{\tau^*}^2 \int |A^{1/2}\tilde{u}|^2 > 0$ which implies that the transversality condition $\mathrm{Re}\tilde{\lambda} \neq 0$ holds. ∎

We shall show that there exists a unique $\tau^* > 0$ such that $(0, \eta^*, \tau^*)$ is a Hopf point, thus τ^* is the origin of a branch of nontrivial periodic orbits.

LEMMA 4.3. *Let G_β be a Green function of the differential operator $A + i\beta$. Then the expression $\mathrm{Re}\,(G_\beta(\eta^*, \eta^*))$ is strictly decreasing in $\beta \in \mathbb{R}^+$ with*

$$\mathrm{Re}\,G_0(\eta^*, \eta^*) = G(\eta^*, \eta^*), \quad \lim_{\beta\to\infty} \mathrm{Re}\,G_\beta(\eta^*, \eta^*) = 0\,,$$

and $\mathrm{Im}G_\beta(\cdot, \eta^*) < 0$ for any $\beta > 0$.

Proof: First we have $(A+i\beta)^{-1} = (A-i\beta)(A^2+\beta^2)^{-1}$, so if $L(\beta) := \mathrm{Re}\,(A+i\beta)^{-1}$ and $T(\beta) := \mathrm{Im}\,(A+i\beta)^{-1}$, then

$$L(\beta) = A(A^2+\beta^2)^{-1} \text{ and } T(\beta) = -\beta(A^2+\beta^2)^{-1}\,.$$

Since $(A^2+\beta^2)^{-1}$ is a positive operator, it follows that $-T(\beta)$ is positive for $\beta > 0$, which implies that $\mathrm{Im}G_\beta(\cdot,\eta^*) < 0$. Moreover, $L(\beta) \longrightarrow A^{-1}$ as $\beta \to 0$ and $L(\beta) \longrightarrow 0$ as $\beta \to \infty$, which results in the corresponding limiting behavior for $\mathrm{Re}\big(G_\beta(\eta^*,\eta^*)\big)$.

Now to show that $\beta \mapsto \mathrm{Re}(G_\beta(\eta^*,\eta^*))$ is strictly decreasing, define $h(\beta)(x) := G_\beta(x,\eta^*) - G(x,\eta^*)$. Then (in the weak sense at first)

$$(A+i\beta)h(\beta) = -i\beta G(\cdot,\eta^*).$$

As a result $h(\beta) \in D(A)_{\mathbf{C}}$ and $h : \mathbb{R}^+ \to D(A)_{\mathbf{C}}$ is differentiable with $ih(\beta) + (A+i\beta)h'(\beta) = -i\,G(\cdot,\eta^*)$, therefore

$$(A+i\beta)h'(\beta) = -iG_\beta(\cdot,\eta^*).$$

We thus get

$$\begin{aligned} -i\,\overline{h'(\beta)(\eta^*))} &= \int_{\mathbb{R}} (A+i\beta)^2 h'(\beta)\overline{h'(\beta)(x)}\,dx \\ &= \int_{\mathbb{R}} (A+i\beta)h'(\beta)\cdot(A+i\beta)\overline{h'(\beta)}\,dx \\ &= \int_{\mathbb{R}} |Ah'(\beta)|^2 - \beta^2|h'(\beta)|^2\,dx + 2i\beta\int_{\mathbb{R}} Ah'(\beta)\overline{h'(\beta)}\,dx\,. \end{aligned}$$

It follows that

$$\mathrm{Re}\,(h'(\beta)(\eta^*)) = -2\beta\int_{\mathbb{R}} |A^{1/2}h'(\beta)|^2 < 0.$$

Since $\mathrm{Re}\,h'(\beta)$ cannot vanish identically, $\mathrm{Re}(G_\beta(\eta^*,\eta^*)) < 0$. ∎

THEOREM 4.4. *Assume* $0 < 1/2 - a_0 < \frac{\mu}{\mu+1}$, *then for a unique critical point* $\tau^* > 0$, *there exists a unique, purely imaginary eigenvalue* $\lambda = i\beta$ *of (9) with* $\beta > 0$.

Proof: We need to show only that the function $(u,\beta,\tau) \mapsto E(u,i\beta,\tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system (9) with $\lambda = i\beta$ and $u = v - \mu G(\cdot,\eta^*)$,

$$(A+i\beta)v = -\mu\,\delta_{\eta^*} \tag{18}$$

and

$$\frac{i\beta}{\tau^*} = v(\eta^*) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*). \tag{19}$$

The real and imaginary parts of the above equation are given by

$$\begin{cases} \frac{\beta}{\tau^*} = -\,\mu\,\mathrm{Im}\big(G_\beta(\eta^*,\eta^*)\big) \\ 0 = -\mu\,\mathrm{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*). \end{cases} \tag{20}$$

Since $\mu\,\mathrm{Im}\big(G_\beta(\eta^*,\eta^*)\big)$ is negative, there is a critical point τ^* provided the existence of β. We now define

$$K(\beta) := -\mu\,\mathrm{Re}\big(G_\beta(\eta^*,\eta^*)\big) + \mu G(\eta^*,\eta^*) + \gamma'(\eta^*). \tag{21}$$

By Lemma 4.3, then $K'(\beta) > 0$, $K(0) = \gamma'(\eta^*) < 0$ and

$$\lim_{\beta\to\infty} K(\beta) = \mu G(\eta^*,\eta^*) + \gamma'(\eta^*)$$
$$= \begin{cases} \mu\,\eta^*\, I_1(\eta^*\sqrt{\mu+1}\,)\, K_1(\eta^*\sqrt{\mu+1}\,) > 0 \quad (n=2), \\ \mu\,\eta^*\, I_{3/2}(\eta^*\sqrt{\mu+1}\,)\, K_{3/2}(\eta^*\sqrt{\mu+1}\,) > 0 \quad (n=3). \end{cases}$$

Therefore, there exists a unique $\beta > 0$. □

The following theorem summarizes the things we have proved.

THEOREM 4.5. *Assume that* $0 < \frac{1}{2} - a_0 < \frac{\mu}{\mu+1}$. *Then (6), respectively (5), has at least one radially symmetric stationary solutions* (u^*,η^*) *where* $u^* = 0$, *respectively* (v^*,η^*) *for all* σ. *Then there exists a unique* σ^* *such that the linearization* $-\widetilde{A} + \frac{4}{\sigma^*}B$ *has a purely imaginary pair of eigenvalues. The point* $(0,\eta^*,\sigma^*)$ *is then a Hopf point for (6) and there exists a* C^0*-curve of nontrivial periodic orbits for (6), respectively (5), bifurcating from* $(0,\eta^*,\sigma^*)$, *respectively* (v^*,η^*,σ^*).

In two- and three- dimensional systems we have found the same variety of behaviors including Hopf bifurcation appearing in the unidimensional system.

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Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea
E-mail address: sglee@skku.edu

Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea
E-mail address: ymham@kyonggi.ac.kr