Large Time Decay and Scattering for Wave Maps

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ABSTRACT. Let $u : \mathbf{R}^{3+1} \longrightarrow \mathbf{H}^2$ be a Wave Map with smooth compactly supported initial data satisfying the smallness condition $||u[0]||_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}} < \epsilon$ for a sufficiently small $\epsilon > 0$. In particular, the Wave Map exists globally in time and is smooth. Then denoting $u(0,x) = u_\infty \in \mathbf{H}$ for $|x|$ large enough, we have

$$||u(t,x) - u_\infty||_{L_x^\infty} \lesssim t^{-\delta}$$

for $|t| >> 1$ and $\delta > 0$ some universal constant. Here the implied constant depends on weighted norms of the the initial data (which, however, need not be small). Furthermore, there exist $(f,g) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ such that

$$u(t,x) = S(t)(f,g) + o_{\dot{H}^{\frac{3}{2}}}(1)$$

where $S(t)$ denotes the free wave propagator.

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1. Introduction

A wave map $u : \mathbb{R}^{n+1} \longrightarrow M$ with (M,g) a Riemannian manifold, and $\mathbb{R}^{n+1}$ the $(n+1)$-dimensional Minkowski space with Minkowski metric $m_{\mu\nu} = \text{diag}(-1, 1, \ldots 1)$,

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is critical with respect to the functional[1]

$$u \to \mathcal{L}(u) := \int_{\mathbb{R}^{n+1}} \langle \partial_\alpha u, \partial^\alpha u \rangle_g d\sigma, \ \partial^\alpha = m^{\alpha\beta} \partial_\beta$$

The Euler Lagrange equations associated with this functional in local coordinate are of the form

$$\Box u^i + \Gamma^i_{jk}(u) \partial_\nu u^j \partial^\nu u^k = 0, \ \Box = \partial_\nu \partial^\nu$$

Although the global well-posedness of the Wave Maps problem for general smooth large data, generic target, and spatial dimension $n \geq 2$ is far from understood (for a recent survey of results and conjectures see e. g. [12]), global existence results have been obtained with increasingly weaker smallness conditions on the data. The first such results follow from the classical work of Klainerman, which uses vector field methods and in particular imposes a very strong smallness condition on the data, in terms of certain weighted H^s-norms. These conditions in particular imply that the Wave Map will be confined to a single coordinate chart inside the target. Further, they immediately yield strong asymptotic results for the corresponding solutions, including pointwise decay estimates as well as scattering in terms of the local coordinate functions.

Recent work by Tataru, Tao and others [25], [26], [23], [24], [6], [13], [9], [11], [27] based on new techniques introduced in [26], [23], [24], led to significantly stronger global existence results. In particular, one only needs to assume that the initial data $u[0] = (u, \partial_t u)_{t=0}$ are small in the critical Sobolev norm $||.||_{\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}}$ in a suitable sense. Note that the latter condition no longer implies that the data are confined to a single chart: indeed, the Wave Map can now move all over the target. The works [23], [24], [6], [13], [9], [11], [27] are based on a geometric trick, introduced in [23], [24], namely exploiting the inherent Gauge invariance of the equations to pass to a more convenient Gauge (e. g. the Coulomb Gauge). Furthermore, they reveal that the cases $n = 3$ and especially $n = 2$ appear significantly more complicated than the cases $n \geq 4$ since the Strichartz estimates by themselves appear no longer strong enough to close the estimates, even using the inherent null-structure. Instead, the only method thus far establishing the well-posedness at the critical level hinges on a sophisticated framework from harmonic analysis, blending $X^{s,\theta}$-type or Bourgain spaces with Tataru's null-frame spaces, both of which are based on localizations of the space-time Fourier transform of the unknown function. Both the fact that one needs to pass into a new Gauge as well as the use of complex spaces based on the space-time Fourier transform render the question of global asymptotic behavior as well as scattering *in the original coordinates* highly non-trivial in the dimensions $n = 2, 3$. The present paper has the goal of answering these questions for $3 + 1$-dimensional[2] Wave Maps with target $\mathbf{H}^2$, the hyperbolic plane. This target appears quite natural, as it occurs in the context of General Relativity, and is the natural counterpoint to Wave Maps with target S^2. As already observed in [9], and exploited in [11], Wave Maps with target $\mathbf{H}^2$ enjoy the remarkable property that the derivative components in the Coulomb Gauge satisfy an autonomous first order div-curl system, which no longer involves the local coordinate functions.

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[1]Throughout this paper, the Einstein summation convention is in force. This means that we sum over repeated raised and lowered indices.

[2]Our argument can obviously be modified to also handle the higher dimensional cases $n \geq 4$. Indeed, the argument becomes much simpler then, since one has stronger estimates for the nonlinearity.

This allows one to conveniently disentangle the global behavior of these derivative components from the global behavior of the actual coordinate functions, and to pass from the former to the latter. The main result of this paper is the following

THEOREM 1.1. *Let $u : \mathbf{R}^{3+1} \longrightarrow \mathbf{H}^2$ be a Wave Map with smooth compactly supported[3] initial data satisfying the smallness condition (suitably interpreted as explained below) $\|u[0]\|_{\dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}} < \epsilon$ for a sufficiently small $\epsilon > 0$. In particular, the Wave Map exists globally in time and is smooth. Then denoting $u(0, x) = u_\infty \in \mathbf{H}$ for $|x|$ large enough, we have*

$$\|u(t, x) - u_\infty\|_{L_x^\infty} \lesssim t^{-\delta}$$

for $|t| >> 1$ and $\delta > 0$ some universal constant. Here the implied constant depends on weighted norms of the the initial data (which, however, need not be small). Furthermore, there exist $(f, g) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$ such that

$$u(t, x) = S(t)(f, g) + o_{\dot{H}^{\frac{3}{2}}}(1)$$

where $S(t)$ denotes the free wave propagator.

REMARK 1.2. The critical case $n = 2$ appears much more technically involved, although the same strategy should work in principle, see the estimates in [**11**]. Furthermore, a similar method should in principle work for more general targets, although the fact that one can no longer formulate an autonomous system of wave equations for the derivative components (they now also depend on the local coordinates) introduces additional technical difficulties.

2. Wave Maps with target $\mathbf{H}^2$: the basic setup

We use the setup in [**11**] : identify $\mathbf{H}^2 = \{(\mathbf{x}, \mathbf{y}) | \mathbf{y} \geq 0\}$ equipped with metric $dg = \frac{d\mathbf{x}^2 + d\mathbf{y}^2}{\mathbf{y}^2}$. Then introduce the derivative components $\phi_\alpha^1 = \frac{\partial_\alpha \mathbf{x}}{\mathbf{y}}$, $\phi_\alpha^2 = \frac{\partial_\alpha \mathbf{y}}{\mathbf{y}}$, $\alpha = 0, 1, 2, 3$. Further, introduce the derivative components in the *Coulomb Gauge* in complex notation

$$\psi_\alpha = \psi_\alpha^1 + i\psi_\alpha^2 = (\phi_\alpha^1 + i\phi_\alpha^2)e^{-i\triangle^{-1}\sum_{j=1}^3 \partial_j \phi_j^1}$$

We recall that we then obtain the following divergence curl system

$$(2.1) \quad \partial_\alpha \psi_\beta - \partial_\beta \psi_\alpha = i\psi_\beta \triangle^{-1} \sum_{j=1}^3 \partial_j(\psi_\alpha^1 \psi_j^2 - \psi_\alpha^2 \psi_j^1) - i\psi_\alpha \triangle^{-1} \sum_{j=1}^3 \partial_j(\psi_\beta^1 \psi_j^2 - \psi_\beta^2 \psi_j^1)$$

$$(2.2) \quad \partial^\nu \psi_\nu = i\psi_\nu \triangle^{-1} \sum_{j=1}^3 \partial_j(\psi_\nu^1 \psi_j^2 - \psi_\nu^2 \psi_j^1)$$

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[3]This means that $u(0, x) = u_\infty$ is a fixed point for $|x|$ sufficiently large, as well as $\partial_t u(0, x) = 0$. One may weaken this condition to sufficiently fast decay at infinity.

From here one infers the wave equations

$$\Box\psi_\alpha = i\partial^\beta[\psi_\alpha \triangle^{-1} \sum_{j=1}^{3} \partial_j(\psi_\beta^1\psi_j^2 - \psi_\beta^2\psi_j^1)]$$

(2.3)
$$- i\partial^\beta[\psi_\beta \triangle^{-1} \sum_{j=1}^{3} \partial_j(\psi_\alpha^1\psi_j^2 - \psi_\alpha^2\psi_j^1)]$$

$$+ i\partial_\alpha[\psi_\nu \triangle^{-1} \sum_{j=1}^{3} \partial_j(\psi^{1\nu}\psi_j^2 - \psi^{2\nu}\psi_j^1)]$$

Schematically speaking, the terms on the right are of the form $\nabla_{x,t}[\psi\nabla^{-1}[\psi^2]]$. In dimensions $n = 2, 3$, the right hand side of (2.3) is not amenable to estimates as is, and as in [**11**], we need to exploit the underlying divergence-curl structure to decompose it into a null-form plus a better error term. For this, we split

(2.4) $$\psi_\alpha = R_\alpha\psi + \chi_\alpha, \ \alpha = 0, 1, 2, 3,$$

where we impose the *vanishing divergence condition* $\sum_{j=1}^{3} \partial_j\chi_j = 0$. Here, the symbols $R_\alpha = \nabla^{-1}\partial_\alpha$ with $\widehat{\nabla^{-1}f}(\xi) = |\xi|^{-1}\hat{f}(\xi)$ denote Riesz type operators. The χ_α can then be obtained as solutions of an elliptic divergence curl system, and are schematically of the form

$$\chi_\alpha = \nabla^{-1}[\psi\nabla^{-1}(\psi^2)]$$

If we now insert the splitting (2.4) into (2.3), we can replace the right hand side by

$$\Box\psi_\alpha = i\partial^\beta[R_\alpha\psi \triangle^{-1} \sum_{j=1}^{3} \partial_j(R_\beta\psi^1 R_j\psi^2 - R_\beta\psi^2 R_j\psi^1)]$$

$$- i\partial^\beta[R_\beta\psi \triangle^{-1} \sum_{j=1}^{3} \partial_j(R_\alpha\psi^1 R_j\psi^2 - R_\alpha\psi^2 R_j\psi^1)]$$

(2.5)
$$+ i\partial_\alpha[R_\nu\psi \triangle^{-1} \sum_{j=1}^{3} \partial_j(R^\nu\psi^1 R_j\psi^2 - R^\nu\psi^2 R_j\psi^1)]$$

$$+ \nabla_{x,t}[\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]\nabla^{-1}(\psi^2)] + \nabla_{x,t}[\psi\nabla^{-1}[\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]\psi]]$$
$$+ \nabla_{x,t}[\psi\nabla^{-1}[\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]\nabla^{-1}[\psi\nabla^{-1}(\psi^2)]]]$$

Here the last three expressions are of course recorded schematically, with each ∇^{-1} denoting operators of the form $\sum_{k=1}^{3} \triangle^{-1}\partial_k$. It is this complicated system of wave equations which shall be at the heart of our analysis, similarly to [**11**].
Recall that the basic paradigm for establishing scattering for a wave equation of the form

$$\Box u = F(u, \nabla u)$$

is to establish $\lim_{|t|\to\infty} \int_t^\infty U(t-s)F(u, \nabla u)(s)ds = 0$ in the underlying Sobolev space $\dot{H}^s$ (which is dictated by scaling reasons, for example). This follows by establishing $L_t^1\dot{H}^s$-bounds on the source $F(u, \nabla u)$. For our system (2.3), such estimates are not available in dimensions $n = 2, 3$.

2.1. The method to obtain scattering for the ψ_α. Fixing a large dyadic time interval $I_l := [-2^{l+2}, 2^{l+2}]$, $l >> 1$, we shall split the components ψ_α into two parts $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$. Here $\psi_{\alpha,1}$ approximates the moderate-frequency part of ψ_α (where 'moderate' is in relation to the scale 2^l), while $\psi_{\alpha,2}$ approximates the very large/small frequency part of $\psi_{\alpha,2}$. While $\psi_{\alpha,2}$ will turn out to be small since the extreme frequencies of ψ_α carry only little energy, $\psi_{\alpha,1}$ will be shown to be small (albeit in a different sense) since we shall be able to apply commutating vector fields to it. Thus the simple basic premise of the present approach is to split ψ_α into a moderate frequency part, obtained by filtering out the extreme frequencies in the nonlinearity, which is basically amenable to classical commutating vector field methods (although of course $\psi_{\alpha,1}$ is not small with respect to the weighted norms), as well as an error term which accounts for the remaining errors due to extreme frequencies. Scattering of ψ_α will then follow by splitting the right hand side of (2.3) into two parts, upon writing $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$. Indeed, crudely denoting the right hand side of (2.3) as $F(\psi)$ (it being understood that it is not a locally defined function evaluated at ψ), we can decompose

$$F(\psi_\alpha) = \sum_l \chi_{t \sim 2^l}[F(\psi_{\alpha,1}) + \text{error}],$$

where the splitting $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$ is the one on the interval $[-2^{l+2}, 2^{l+2}]$. We shall then show that $\psi_{\alpha,1}$ can be placed into $L_t^2 L_x^\infty$, which allows us to estimate $F(\psi_{\alpha,1})$ in $L_1^1 \dot{H}^{-\frac{1}{2}}$, while the error is exponentially decaying in l (but with respect to a more complicated norm!). This shall then imply scattering for ψ_α, as well as pointwise decay. The latter facts allow one to retrace the steps from local coordinates to the ψ_α to obtain decay and scattering for the coordinates $(\mathbf{x}, \mathbf{y})$.

2.2. Tools from harmonic analysis. In order to precisely define $\psi_{\alpha,1}$, we introduce the Littlewood-Paley localizers P_k, $k \in \mathbb{Z}$, as well as the space-time localizers Q_k, as follows: choose a function $\phi \in C_0^\infty(\mathbb{R}_+)$ with the property $\sum_{j \in \mathbb{Z}} \phi(\frac{x}{2^j}) = 1$, $x > 0$. See e.g. [20]. The we define P_k, Q_k via

$$\widehat{P_k f}(\xi) = \phi(\frac{\xi}{2^k})\hat{f}(\xi), \quad \widetilde{Q_k f}(\tau, \xi) = \phi(\frac{||\tau| - |\xi||}{2^k})\tilde{f}(\tau, \xi)$$

Here $\hat{\ }$ denotes the spatial Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-2\pi i x \cdot \xi}dx$, while we denote the space-time Fourier transform by $\tilde{f}(\tau, \xi) = \int_{\mathbb{R}^{3+1}} f(t, x)e^{-2\pi i(t\tau + x \cdot \xi)}dtdx$. We can then also introduce the operators

$$P_{<l} = \sum_{k<l} P_k, \quad P_{[a,b]} = \sum_{k \in [a,b]} P_k, \text{ etc}$$

Further, let $F(\psi)$ denote any of the multilinear expressions on the right hand side of (2.3). Then introduce the operator $\mathbf{P}_{[-\delta l, \delta l]}$, which acts by restricting the frequencies and modulations (i. e. the distance of the space-time Fourier support to the light cone, measured by $||\tau| - |\xi||$) of all the inputs, as follows:

Definition: *the function $\mathbf{P}_{[-\delta l, \delta l]}F(\psi)$ is obtained from $F(\psi)$ by*
(i) replacing the ith input ψ by $P_{k_i}Q_{<k_i+\delta l}\psi$, and summing over $k_i \in [-\delta l, \delta l]$ for each i.
(ii) replacing the ith operator $\triangle^{-1}\partial_j$ by $P_{r_i}\triangle^{-1}\partial_j$ and summing over $r_i \in [-\delta l, \delta l]$ for each i.

(iii) Applying an operator $P_k Q_{<k+\delta l}$ to the output and summing over $k \in [-\delta l, \delta l]$.

Example: the trilinear expression

$$i\partial^\beta[\psi_\alpha \triangle^{-1} \sum_{j=1}^3 \partial_j(\psi_\beta^1 \psi_j^2 - \psi_\beta^2 \psi_j^1)]$$

is replaced by

$$\mathbf{P}_{[-\delta l, \delta l]}\big[i\partial^\beta[\psi_\alpha \triangle^{-1} \sum_{j=1}^3 \partial_j(\psi_\beta^1 \psi_j^2 - \psi_\beta^2 \psi_j^1)]\big]$$

$$= \sum_{k,k_{1,2,3},r \in [-\delta l, \delta l]} P_k Q_{<k+\delta l}[i\partial^\beta[P_{k_1} Q_{<k_1+\delta l}\psi_\alpha \triangle^{-1} P_r$$

$$\sum_{j=1}^3 \partial_j(P_{k_2} Q_{<k_2+\delta l}\psi_\beta^1 P_{k_3} Q_{<k_3+\delta l}\psi_j^2$$

$$- P_{k_3} Q_{<k_3+\delta l}\psi_\beta^2 P_{k_2} Q_{<k_2+\delta l}\psi_j^1)]$$

2.3. Definition of $\psi_{\alpha,1}$, $\psi_{\alpha,2}$ on some interval $[-2^{l+2}, 2^{l+2}]$. We can now define the equation which defines $\psi_{\alpha,1}$, as follows:

(2.6)

$$\Box\psi_{\alpha,1} = \mathbf{P}_{[-\delta l, \delta l]}\big[i\partial^\beta[R_\alpha \psi_1 \triangle^{-1} \sum_{j=1}^3 \partial_j(R_\beta \psi_1^1 R_j \psi_1^2 - R_\beta \psi_1^2 R_j \psi_1^1)]$$

$$- i\partial^\beta[R_\beta \psi_1 \triangle^{-1} \sum_{j=1}^3 \partial_j(R_\alpha \psi_1^1 R_j \psi_1^2 - R_\alpha \psi_1^2 R_j \psi_1^1)]$$

$$+ i\partial_\alpha[R_\nu \psi_1 \triangle^{-1} \sum_{j=1}^3 \partial_j(R^\nu \psi_1^1 R_j \psi_1^2 - R^\nu \psi_1^2 R_j \psi_1^1)]$$

$$+ \nabla_{x,t}[\nabla^{-1}[\psi_1 \nabla^{-1}(\psi_1^2)]\nabla^{-1}(\psi_1^2)] + \nabla_{x,t}[\psi_1 \nabla^{-1}[\nabla^{-1}[\psi_1 \nabla^{-1}(\psi_1^2)]\psi_1]]$$

$$+ \nabla_{x,t}[\psi_1 \nabla^{-1}[\nabla^{-1}[\psi_1 \nabla^{-1}(\psi_1^2)]\nabla^{-1}[\psi_1 \nabla^{-1}(\psi_1^2)]]]\big],$$

$$\psi_{\alpha,1}[0] = \psi_\alpha[0] := (\psi_\alpha(0), \partial_t \psi_\alpha(0))$$

Here we use as before $\psi_1 = -\sum_{j=1}^3 R_j \psi_{j,1}$ for the first three trilinear terms on the right. For the schematic higher order terms, it is understood that ψ_α is replaced by $\psi_{\alpha,1}$ $\forall \alpha$.

We immediately observe that this is not a standard wave equation, since it involves nonlocal operators in its source term, *even in the space-time sense*. Thus it certainly does not satisfy Huyghen's principle! Nevertheless, we shall be able to construct solutions on some interval $[-T, T]$ which contains $[-2^{l+2}, 2^{l+2}]$ and also matches the initial data, via Banach iteration in a suitable space. Indeed, the solution will be smooth. We note here that this iteration is qualitatively different than the procedure used in [**11**]. There the equation (2.3) is only used to deduce *a priori estimates*, while the local existence of a solution is ensured by the classical local existence theory in local coordinates. For the problem (2.6), such a step is not possible, since it is not a geometrically motivated problem. Thus the existence

of local solutions needs to be demonstrated from scratch via iteration.

We can now also define $\psi_{\alpha,2}$ via

$$\psi_{\alpha,2} = \psi_\alpha - \psi_{\alpha,1}$$

2.4. Analytical preliminaries. Here we recall the functional framework underpinning [**26**], [**24**], [**11**], in Tao's formulation. We shall use the following homogeneous $X^{s,\theta}$-type norm:

$$||\psi||_{\dot{X}_k^{p,q,r}} := 2^{kp}(\sum_{j\in\mathbb{Z}}[2^{qj}||Q_j\psi||_{L_t^2 L_x^2}]^r)^{\frac{1}{r}},$$

provided $r < \infty$, as well as the obvious modification when $r = \infty$. In order to estimate the frequency localized components of ψ_α, we have the family of norms $||.||_{S[k]}$, $k \in \mathbb{Z}$, defined as follows: let

$$||\psi||_{S[k]} := ||\nabla_{x,t}\psi||_{L_t^\infty \dot{H}^{-\frac{1}{2}}} + ||\nabla_{x,t}\psi||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},\infty}} +$$
$$\sup_{l<-10}(\sum_{\kappa\in K_l}||P_{k,\pm\kappa}Q_{<k+2l}^\pm\psi||_{S[k,\pm\kappa]}^2)^{\frac{1}{2}}$$

Here we have chosen for each integer $l < -10$ a finitely overlapping cover K_l (with the overlapping being uniform in l) of caps κ of size $\sim 2^l$ of the sphere S^2. The superscript $\pm$ in $Q_{<k+2l}^\pm$ indicates that we further localize (sharply) to the upper or lower half-space $\pm\tau > 0$, repsectively, and finally, the norms $||.||_{S[k,\kappa]}$ are defined as follows: first, let

$$||\psi||_{NFA[\kappa]^*} := \sup_{\omega\notin 2\kappa}\text{dist}(\omega,\kappa)||\psi||_{L_{t_\omega}^\infty L_{x_\omega}^2}$$

where ω ranges over S^2 and we use the *null-coordinates*

$$t_\omega := (t,x)\cdot\frac{1}{\sqrt{2}}(1,\omega),\ x_\omega = (t,x) - t_\omega\frac{1}{\sqrt{2}}(1,\omega)$$

We then also have the *dual norm*, i. e.

$$||\psi||_{NFA[\kappa]} := \inf_{\int_{\omega\in S^2}\psi_\omega=\psi}\int_{\omega\notin 2\kappa}\frac{1}{\text{dist}(\omega,\kappa)}||\psi_\omega||_{L_{t_\omega}^1 L_{x_\omega}^2}$$

Further, we introduce

$$||\psi||_{PW[\kappa]} := \inf_{\int_{\omega\in\kappa}\psi_\omega=\psi}\int_{\omega\in\kappa}||\psi_\omega||_{L_{t_\omega}^2 L_{x_\omega}^\infty}$$

Then we put

$$||\psi||_{S[k,\kappa]} := 2^{\frac{k}{2}}||\psi||_{NFA[\kappa]^*} + |\kappa|^{-\frac{1}{2}}2^{-\frac{k}{2}}||\psi||_{PW[\kappa]} + 2^{\frac{k}{2}}||\psi||_{L_t^\infty L_x^2}$$

Further, the frequency localized components of the source term, i. e. the right hand side of (2.3) etc, shall be evaluated with respect to the following norm:

$$||\psi||_{N[k]} := \inf_{\psi_1+\psi_2+\psi_3+\psi_4=P_k Q_{<k+10}\psi}||\psi_1||_{L_t^1 \dot{H}^{-\frac{1}{2}}} + ||\psi_2||_{\dot{X}^{-\frac{1}{2},-\frac{1}{2},1}}$$
$$+ \inf_{\sum_l \psi_{3l}=\psi_3}(\inf_{\sum'_{\kappa\in K_l}\psi_\kappa=\psi_{3l}}\sum_{\kappa\in K_l}||\psi_\kappa||_{NFA[\kappa]}^2)^{\frac{1}{2}}$$
$$+ ||P_k Q_{>k+10}\psi||_{\dot{X}^{-\frac{1}{2},-\frac{1}{2},\infty}\cap\Box|\nabla_{x,t}|^{-1}L_t^\infty L_x^{-\frac{1}{2}}}$$

Here in the last term but one upon fixing $l < -10$ we only consider those ψ_κ with Fourier satisfying $||\tau|-|\xi|| \leq 2^{k-2l-100}$, $2^{k-4} \leq |\xi| \leq 2^{k+4}$, $\Theta := \frac{\xi}{|\xi|}\frac{\tau}{|\tau|} \in \frac{1}{2}\kappa$. We

note that the norm $||.||_{N[k]}$ we use here is slightly different than the one in [**11**], as the *elliptic portion* $P_k Q_{>k+10}\psi$ is estimate with a weaker norm, namely

$$||P_k Q_{>k+10}\psi||_{\dot{X}_k^{-\frac{1}{2},-\frac{1}{2},\infty}} + ||\nabla_{x,t}\Box^{-1} P_k Q_{>k+10}\psi||_{L_t^\infty \dot{H}_x^{-\frac{1}{2}}}$$

This change barely affects the estimates, though.

Finally, we need time localized versions of these norms: for $T > 0$, we introduce

$$||\psi||_{N[k]([-T,T]\times\mathbb{R}^3)} := \inf_{f\in\mathcal{S}(\mathbb{R}^{3+1})\big|f|_{[-T,T]\times\mathbb{R}^3}=\psi|_{[-T,T]\times\mathbb{R}^3}} ||f||_{N[k]}$$

and similarly for $||.||_{N[k]([-T,T]\times\mathbb{R}^3)}$ etc.

2.5. Some geometric preliminaries. Here we quickly recall the infinitesimal generators of the Poincare group on $\mathbb{R}^{3+1}$, namely the vector fields Γ which are given by $\partial_t, \partial_{x_i}$, $i = 1, 2, 3$ (translations), $t\partial_t + \sum_{i=1}^3 x_i\partial_{x_i}$ (scaling), $x_i\partial_{x_j} - x_j\partial_{x_i}$, $i, j, = 1, 2, 3$ (rotations), $t\partial_{x_i} + x_i\partial_t$, $i = 1, 2, 3$ (Lorentz boosts). We shall denote these by Γ^α where $\alpha = 1, 2, \ldots, 11$ for some ordering. We shall denote products of these by $\Gamma^{\alpha_1}\Gamma^{\alpha_2} =: \Gamma^{(\alpha_1,\alpha_2)}$. In particular, the notation Γ^β, $|\beta| = 2$, refers to a product of two such vector fields (as an operator, with β a pair of indices, hence of length two), while $\Gamma^\beta, |\beta| = 1$, refers to a single such vector field. We recall that we then have the relations

$$[\Box, \Gamma^\beta] = c_\beta\Box, \ |\beta| = 1$$

$$[\Box, \Gamma^\beta] = c_{\beta\gamma}\Gamma^\gamma\Box + d_\beta\Box, \ |\beta| = 2, |\gamma| = 1$$

2.6. The core estimates. With the above setup, we can now formulate

PROPOSITION 2.1. *Make the same assumptions as in Theorem 1.1. Specifically, we assume that the coordinate functions at time zero* $(\mathbf{x}, \mathbf{y})(0, x) : \mathbb{R}^3 \longrightarrow (\mathbb{R}\times\mathbb{R}_+)$, $(\partial_t\mathbf{x}, \partial_t\mathbf{y})(0, x) : \mathbb{R}^3 \longrightarrow (\mathbb{R}\times\mathbb{R})$ *satisfy the condition*

$$\int_{\mathbb{R}^3}[(|\frac{\nabla_x\mathbf{x}|}{\mathbf{y}})^2 + (\frac{|\nabla_x\mathbf{x}|}{\mathbf{y}})^2]dx + \int_{\mathbb{R}^3}[(\frac{\partial_t\mathbf{x}}{\mathbf{y}})^2 + (\frac{\partial_t\mathbf{x}}{\mathbf{y}})^2]dx < \epsilon$$

for sufficiently small $\epsilon > 0$. *Further, fix a time scale* 2^l, $l >> 1$. *Then there exists a smooth solution* $\psi_{\alpha,1}$ *solving* (2.6) *on* $[-2^{l+2}, 2^{l+2}]$. *Further, the following estimates hold: Introducing the families of numbers*

$$c_k := \sum_{0\leq|\beta|\leq 3}(\sum_{l\in\mathbb{Z}} 2^{-\sigma|k-l|}||P_l\Gamma^\beta\psi[0]||^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}})^{\frac{1}{2}},$$

$$d_k := \sum_{l\in\mathbb{Z}} 2^{-\sigma|l-k|}||P_l\psi[0]||^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}})^{\frac{1}{2}}$$

for a sufficiently small $\sigma > 0$, *we have*

$$\sum_{0\leq|\beta|\leq 2}||P_k\Gamma^\beta\psi_{\alpha,1}||_{S[k]} \lesssim 2^{C\delta l}c_k$$

for some fixed C. *We also have*

$$||P_k\psi_{\alpha,1}||_{S[k]} \lesssim d_k$$

PROPOSITION 2.2. *Under the same assumptions as in the previous Proposition, we have*

$$||P_k\psi_{\alpha,2}||_{S[k]} \lesssim \min\{d_k, 2^{\mu\max\{k-\delta l,-k-\delta l\}}d_k\} := e_k$$

for a sufficiently small $\mu > 0$.

These Propositions will follow essentially from estimates established in [**9**], [**10**], as well as some elementary observations concerning commutators of the Γ^β and Fourier localizers. They will be deferred to the end of the paper. In a similar vein, we have the following

PROPOSITION 2.3. *Let $F(\psi)$ denote the right hand side of (2.3). Then for $T \sim 2^l$ we have*

$$||P_k(1 - \mathbf{P}_{[-\delta l, \delta l]})F(\psi)||_{N[k]([-T,T] \times \mathbb{R}^3)} \lesssim e_k$$

where e_k is as in Proposition 2.2. The same estimate applies if we replace ψ by ψ_1 (i. e. we replace each ψ_α by $\psi_{\alpha,1}$). Also, without any further localization, we have

$$||P_k F(\psi_1, \psi_2)||_{N[k]([-T,T] \times \mathbb{R}^3)} \lesssim e_k$$

where $F(\psi_1, \psi_2)$ is any of the multilinear expressions on the right hand side of (2.3) with at least one factor $\psi_{\alpha,\nu,\beta}$ replaced by $\psi_{\alpha,\nu,\beta,2}$.

REMARK 2.4. The numbers d_k, e_k form *frequency envelopes*, a notation borrowed from [**23**]. A frequency envelope is a sequence $\{c_k\}_{k \in Z}$ of non-negative numbers such that for some $\sigma > 0$ we have

$$2^{-\sigma|k-l|}c_l \le c_k \le 2^{\sigma|k-l|}c_l$$

for each k, l.

3. Deducing asymptotic decay and scattering from the core estimates

Assuming the above Propositions, we now deduce asymptotic decay for the $\psi_{\alpha,1}$, $\psi_{\alpha,2}$, as well as scattering for ψ_α. Combining these ingredients, we then obtain Theorem 1.1. We commence with the following crucial

LEMMA 3.1. *Let $\psi_{\alpha,1}$ be as above, and assume $t \in [-2^{l+2}, 2^{l+2}]$ satisfies $t \sim 2^l$. Then we have*

$$|\psi_{\alpha,1}(t,x)| \lesssim 2^{C\delta l - \frac{2}{3}l}$$

In particular, if we choose δ small enough, we can achieve $|\psi_{\alpha,1}(t,x)| \lesssim 2^{-\frac{7l}{12}}$.

PROOF. (Lemma 3.1) This is a consequence of the Klainerman-Sobolev inequalities: recall (see e. g. [**22**]) that we have

$$|\psi_{\alpha,1}(t,x)| \lesssim (1 + |t| + |x|)^{-1}(1 + \big||t| - |x|\big|)^{-\frac{1}{2}} \sum_{|\beta| \le 2} ||\Gamma^\beta \psi_{\alpha,1}||_{L_x^2}$$

Assume first that $|x| \gtrsim 2^l$. Using a smooth cutoff function $\chi_{\big||t|-|x|\big| \sim 2^s}$, we localize this to

$$|\chi_{|x| \gtrsim 2^l} \chi_{\big||t|-|x|\big| \sim 2^s} \psi_{\alpha,1}(t,x)| \lesssim (1 + |t| + |x|)^{-1}(1 + \big||t| - |x|\big|)^{-\frac{1}{2}}$$

$$\times \sum_{|\beta| \le 2} ||\Gamma^\beta[\chi_{|x| \gtrsim 2^l} \chi_{\big||t|-|x|\big| \sim 2^s} \psi_{\alpha,1}]||_{L_x^2}$$

for $s = 1, 2, 3, \ldots$. It is straightforward to verify that the expressions

$$\Gamma^\beta[\chi_{|x| \gtrsim 2^l} \chi_{\big||t|-|x|\big| \sim 2^s}], |\beta| \le 2,$$

are uniformly bounded for $|x| \gtrsim 2^l$. Now we distinguish between the cases $2^s >> |t| \sim 2^l$, $2^s \lesssim |t| \sim 2^l$. In the former, we have using Proposition 2.1 as well as the Sobolev embedding and Holder's inequality

$$||\Gamma^\beta[\chi_{|x| \gtrsim 2^l} \chi_{||t|-|x|| \sim 2^s} \psi_{\alpha,1}]||_{L_x^2}$$

$$\lesssim ||\Gamma^\beta[\chi_{|x| \gtrsim 2^l} \chi_{||t|-|x|| \sim 2^s} \psi_{\alpha,1}]||_{\dot{H}_x^{\frac{1}{2}}} ||\chi_{||t|-|x|| \sim 2^s} \chi_{|t| \sim 2^l}||_{L_x^6} \lesssim 2^{C\delta l} 2^{\frac{s}{2}},$$

whence we obtain

$$|\chi_{||t|-|x|| \sim 2^s} \psi_{\alpha,1}(t,x)| \lesssim 2^{C\delta l} 2^{-s}, \ |x| \gtrsim 2^l$$

Summing over $2^s >> 2^l$ yields the bound $\lesssim 2^{C\delta l} 2^{-l}$, which better than what we need. Next, assuming $2^s \lesssim 2^l$ and arguing in the same way, we get the bound

$$||\Gamma^\beta[\chi_{|x| \gtrsim 2^l} \chi_{||t|-|x|| \sim 2^s} \psi_{\alpha,1}]||_{L_x^2} \lesssim 2^{C\delta l} [2^{2l} 2^s]^{\frac{1}{6}} 2^{-\frac{s}{2}} 2^{-l} \lesssim 2^{-\frac{s}{3}} 2^{C\delta l - \frac{2}{3} l}$$

Summing over $s \geq 1$ yields the desired estimate in this range. Finally, if $|x| << 2^l$, apply the above reasoning to $||\Gamma^\beta[\chi_{|x| << 2^l} \psi_{\alpha,1}]||_{L_x^2}$. □

COROLLARY 3.2. *Under the assumptions of the preceding lemma, we have the estimate*

$$||\chi_{|t| \sim 2^l} \psi_{\alpha,1}||_{L_t^2 L_x^\infty} \lesssim 2^{-\frac{l}{12}},$$

provided $\delta > 0$ is chosen small enough.

We can now prove

PROPOSITION 3.3. *Under the assumptions of Theorem 1.1 and using the preceding terminology, there exists $(\tilde{\psi}_{\alpha 1}, \tilde{\psi}_{\alpha 2}) \in \dot{H}_x^{\frac{1}{2}} \times \dot{H}_x^{-\frac{1}{2}}$ with the property*

$$\lim_{|t| \to \infty} ||\psi_\alpha(t,.) - S(t)(\tilde{\psi}_{\alpha 1}, \tilde{\psi}_{\alpha 2})||_{\dot{H}^{\frac{1}{2}}} = 0, \forall \alpha$$

REMARK 3.4. The proof actually shows that the convergence occurs at rate $|t|^{-\nu}$ for small enough $\nu > 0$.

PROOF. (Proposition 3.3) We decompose $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$ for some fixed time scale 2^l, viz. preceding discussion. Accordingly, denoting the right hand side of (2.3) as $F(\psi)$, we write it as $F(\psi_1 + \psi_2)$. Exploiting multilinearity, we write it as

$$F(\psi) = F(\psi_1) + \text{error},$$

where error is a sum of multilinear expressions each of which contains at least one power of $\psi_{\alpha,2}$ (for some α). Then we further decompose

$$F(\psi) = \mathbf{P}_{[-\delta l, \delta l]} F(\psi_1) + (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1) + \text{error}$$

Finally, we time localize , i. e. write
(3.1)
$$F(\psi) = \sum_{l \geq 1} \phi(\frac{t}{2^l}) F(\psi) = \sum_{l \geq 1} \phi(\frac{t}{2^l}) [\mathbf{P}_{[-\delta l, \delta l]} F(\psi_1) + (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1) + \text{error}]$$

where ψ_1, ψ_2 in each decomposition of course depends on l. Now the frequency localizations implied in $\mathbf{P}_{[-\delta l, \delta l]} F(\psi_1)$, together with lemma 3.1 easily imply

$$||\phi(\frac{t}{2^l}) \mathbf{P}_{[-\delta l, \delta l]} F(\psi_1)||_{L_t^1 \dot{H}^{-\frac{1}{2}}} \lesssim 2^{-\frac{l}{12}},$$

provided we choose $\delta > 0$ small enough. Further, on account of Proposition 2.3, we obtain

$$||P_k(1 - \mathbf{P}_{[-\delta l, \delta l]})F(\psi_1)||_{N[k]([-T,T]\times\mathbb{R}^3)} + ||P_k\text{error}||_{N[k]([-T,T]\times\mathbb{R}^3)} \lesssim e_k$$

Note that error is of the form $F(\psi_1, \psi_2)$ as in Proposition 2.3. Further, due to a lemma in [**24**], we also have the same estimates for $P_k\phi(\frac{t}{2^l})(1 - \mathbf{P}_{[-\delta l, \delta l]})F(\psi_1)$ etc. Now fix some $T_0 >> 1$ as well as $T >> T_0$ and write

$$\psi_\alpha(t, .) = S(t)(\psi_\alpha[0]) + \int_0^t \eta_T^+(t - s)U(t - s)F(\psi)(s)ds$$

where $t \in [T_0, 2T_0]$. Here $\eta_T^+(t)$ vanishes for $t < 0$, equals 1 on $[0, T]$, and is compactly supported and smooth on $(0, \infty]$. For example, upon choosing such a $\eta_1^+(t)$, one can set $\eta_T^+(t) = \eta_1^+(\frac{t}{T})$. Then decompose

$$\int_0^t \eta_T^+(t - s)U(t - s)F(\psi)(s)ds = \int_0^{T_0} \eta_T^+(t - s)U(t - s)F(\psi)(s)ds$$
$$+ \int_{T_0}^t \eta_T^+(t - s)U(t - s)F(\psi)(s)ds$$

We need to show (A) that

$$\lim_{T_0\to\infty} \int_0^{T_0} \eta_T^+(t - s)U(t - s)F(\psi)(s)ds = S(t)(\tilde{\psi}_{\alpha1}, \tilde{\psi}_{\alpha2})$$

in the $\dot{H}^{\frac{1}{2}}$-sense for suitable $(\tilde{\psi}_{\alpha1}, \tilde{\psi}_{\alpha2}) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, as well as (B)

$$\lim_{T_0\to\infty} ||\int_{T_0}^t \eta_T^+(t - s)U(t - s)F(\psi)(s)ds||_{\dot{H}^{\frac{1}{2}}} = 0$$

Use that $U(t - s) = \sqrt{-\triangle}^{-1}\sin(\sqrt{-\triangle}(t - s)) = \sqrt{-\triangle}^{-1}\frac{e^{i(t-s)\sqrt{-\triangle}} - e^{-i(t-s)\sqrt{-\triangle}}}{2i}$. Hence it suffices to consider

$$\int_0^{T_0} \eta_T^+(t - s)\sqrt{-\triangle}^{-1}e^{\pm i(t-s)\sqrt{-\triangle}}F(\psi)(s)ds$$
$$= \sqrt{-\triangle}^{-1}e^{\pm it\sqrt{-\triangle}}\int_0^{T_0} e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds$$

Here we use that $\eta_T^+(t - s) = 1$ for $t \in [T_0, 2T_0]$, $0 \le s \le t$, since $T >> T_0$. We will show that the limit

$$\lim_{T_0\to\infty} \int_0^{T_0} e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds$$

exists in $\dot{H}^{-\frac{1}{2}}$, from which (A) follows. Consider

$$\int_{T_0}^{T_0'} e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds = \sum_{2^l \gtrsim T_0} \int_{T_0}^{T_0'} \phi(\frac{s}{2^l})e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds$$

Then note that

$$||\int_{T_0}^{T_0'} \phi(\frac{s}{2^l})e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds||_{\dot{H}^{-\frac{1}{2}}} = \sup_{||g||_{\dot{H}^{\frac{1}{2}}}=1} \langle \int_{T_0}^{T_0'} \phi(\frac{s}{2^l})e^{\mp is\sqrt{-\triangle}}F(\psi)(s)ds, g\rangle$$

Now, using (3.1), we get

$$|\langle \int_{T_0}^{T_0'} \phi(\frac{s}{2^l}) e^{\mp is\sqrt{-\triangle}} \mathbf{P}_{[-\delta l, \delta l]} F(\psi_1)(s) ds, g \rangle| \lesssim 2^{-\frac{l}{12}} \lesssim T_0^{-\frac{1}{12}}$$

Further, write

$$\langle \int_{T_0}^{T_0'} \phi(\frac{s}{2^l}) e^{\mp is\sqrt{-\triangle}} (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)(s) ds, g \rangle$$

$$= \sum_k \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \phi(\frac{s}{2^l}) P_k (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)(s, x) e^{\pm is\sqrt{-\triangle}} \tilde{P}_k g(x) dx ds$$

Here $\tilde{P}_k$ is a multiplier like P_k but satisfying $\tilde{P}_k P_k = P_k$. Then use that [**24**] $||\phi(\frac{s}{2^l}) e^{\pm is\sqrt{-\triangle}} \tilde{P}_k g||_{S[k]} \lesssim ||g||_{\dot{H}^{\frac{1}{2}}}$, as well as $|<P_k F, \psi>| \lesssim ||F||_{N[k]} ||\psi||_{S[k]}$. Using Cauchy-Schwarz, we conclude that

$$|\langle \int_{T_0}^{T_0'} \phi(\frac{s}{2^l}) e^{\mp is\sqrt{-\triangle}} (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)(s) ds, g \rangle|$$

$$\lesssim \sum_{k \in \mathbb{Z}} ||P_k \phi(\frac{s}{2^l}) (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)(s)||_{N[k]} ||\tilde{P}_k g||_{\dot{H}^{\frac{1}{2}}} \lesssim (\sum_{k \in \mathbb{Z}} e_k^2)^{\frac{1}{2}} \lesssim 2^{-\nu l} \lesssim T_0^{-\nu}$$

for suitable $\nu > 0$. Of course the same argument applies to the contribution of error in (3.1), whence summing over l with $2^l \sim T_0$ and letting $T_0 \to \infty$, we obtain (A), i. e. the $\dot{H}^{\frac{1}{2}}$ convergence of $\int_0^{T_0} e^{\mp is\sqrt{-\triangle}} F(\psi)(s) ds$. The argument for establishing (B), i. e.

$$\lim_{T_0 \to \infty} || \int_{T_0}^{t} \eta_T^+(t-s) U(t-s) F(\psi)(s) ds ||_{\dot{H}^{\frac{1}{2}}} = 0,$$

is no different, using Theorem 4.4 as well as the fact (see e. g. [**24**]) that

$$|| \int_{T_0}^{t} \eta_T^+(t-s) U(t-s) F(\psi)(s) ds ||_{S[k]([-T,T] \times \mathbb{R}^3)} \lesssim ||F(\psi)||_{N[k]([-T,T] \times \mathbb{R}^3)},$$

and applying the same reasoning to the term "error", the proof of Proposition 3.3 is complete. $\square$

We now show how to deduce scattering of the original derivative components from Proposition 3.3. We have

PROPOSITION 3.5. *Under the assumptions of the preceding Proposition, we have*

$$\lim_{|t| \to \infty} ||\phi_\alpha - S(t)(\tilde{\psi}_{\alpha 1}, \tilde{\psi}_{\alpha 2})||_{\dot{H}^{\frac{1}{2}}} = 0$$

Furthermore, there exist pairs $(f_1, g_1) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, $(f_2, g_2) \in \dot{H}^{\frac{3}{2}} \times \dot{H}^{\frac{1}{2}}$, *such that*

$$\lim_{|t| \to \infty} ||\mathbf{x}(t,.) - S(t)(f_1, g_1)||_{\dot{H}^{\frac{3}{2}}} = 0, \lim_{|t| \to \infty} ||\mathbf{y}(t,.) - S(t)(f_2, g_2)||_{\dot{H}^{\frac{3}{2}}} = 0$$

PROOF. (Proposition 3.5). Recall that $\phi_\alpha = \psi_\alpha e^{i\triangle^{-1} \sum_{j=1}^{3} \partial_j \phi_j^1}$. We shall need the following

LEMMA 3.6. *We have*

$$||P_k e^{i\triangle^{-1} \sum_{j=1}^{3} \partial_j \phi_j^1}||_{\dot{H}^{\frac{3}{2}}} + ||P_k \phi_\alpha||_{\dot{H}^{\frac{1}{2}}} \lesssim d_k$$

PROOF. (Lemma 3.6) Expand

$$P_k[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j \phi_j^1}||_{\dot{H}^{\frac{3}{2}}}]$$

$$= P_k \sum_{l=1}^3 \partial_l \triangle^{-1}[P_{<k-5}(\sum_{j=1}^3 \triangle^{-1}\partial_l\partial_j\phi_j^1)P_{[k-2,k+2]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}}$$

$$+ P_k \sum_{l=1}^3 \partial_l \triangle^{-1}[P_{[k-5,k+5]}(\sum_{j=1}^3 \triangle^{-1}\partial_l\partial_j\phi_j^1)P_{<k+10}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}}]$$

$$+ P_k \sum_{l=1}^3 \partial_l \triangle^{-1}[P_{>k+5}(\sum_{j=1}^3 \triangle^{-1}\partial_l\partial_j\phi_j^1)P_{>k+2}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}}]$$

Similarly, we have the identity

$$P_k\phi_\alpha = P_k[P_{<k-5}\psi_\alpha P_{[k-2,k+2]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]$$

$$+ P_k[P_{[k-5,k+5]}\psi_\alpha P_{<k+10}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]$$

$$+ P_k[P_{>k+5}\psi_\alpha P_{>k+2}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]$$

We infer from the first identity above that

$$||P_k[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}} \lesssim ||P_{[k-2,k+2]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}} \max_j\{||\phi_j^1||_{\dot{H}^{\frac{1}{2}}}\}$$

$$+ ||P_{k-5,k+5}\phi_j^1||_{\dot{H}^{\frac{1}{2}}} + \sum_{l>k+5} 2^{\frac{k-l}{2}}||P_l\phi_j^1||_{\dot{H}^{\frac{1}{2}}}$$

Further, from the 2nd equality above we infer

$$||P_k\phi_\alpha||_{\dot{H}^{\frac{1}{2}}} \lesssim ||P_{[k-2,k+2]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}}||P_{<k-5}\psi_\alpha||_{\dot{H}^{\frac{1}{2}}}$$

$$+ ||P_{[k-5,k+5]}\psi_\alpha||_{\dot{H}^{\frac{1}{2}}} + \sum_{l>k+5} 2^{\frac{k-l}{2}}||P_l\psi_\alpha||_{\dot{H}^{\frac{1}{2}}}$$

Substituting the latter inequality into the former, we obtain

$$||P_k[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]||_{\dot{H}^{\frac{3}{2}}} \lesssim ||P_{[k-2,k+2]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}} \max_j\{||\phi_j^1||_{\dot{H}^{\frac{1}{2}}}\}$$

$$+ ||P_{[k-7,k+7]}e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}||_{\dot{H}^{\frac{3}{2}}}||P_{<k}\psi_\alpha||_{\dot{H}^{\frac{1}{2}}}$$

$$+ ||P_{[k-10,k+10]}\psi_\alpha||_{\dot{H}^{\frac{1}{2}}} + \sum_{l>k} 2^{\frac{k-l}{2}}||P_l\psi_\alpha||_{\dot{H}^{\frac{1}{2}}}$$

$$+ \sum_{l>k} 2^{\frac{k-l}{2}}||P_l\psi_j||_{\dot{H}^{\frac{1}{2}}}$$

Using the definition of d_k, this implies

$$||P_k[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]||_{\dot{H}^{\frac{3}{2}}} \lesssim \epsilon||P_{[k-7,k+7]}[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]||_{\dot{H}^{\frac{3}{2}}} + d_k$$

Iterating and choosing ϵ small enough, we obtain the desired bound

$$||P_k[e^{i\triangle^{-1}\sum_{j=1}^3 \partial_j\phi_j^1}]||_{\dot{H}^{\frac{3}{2}}} \lesssim d_k,$$

which in conjunction with the above easily implies $||P_k\phi_\alpha||_{L_x^2} \lesssim d_k$. $\square$

Returning to the proof of the Proposition, fix a time scale $|t| \sim 2^l$, $l >> 1$, and correspondingly decompose $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$. Then we can write $\phi_\alpha = \phi_{\alpha,1} + \phi_{\alpha,2}$ with

$$\phi_{\alpha,1} = \psi_{\alpha,1} e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1}, \; \phi_{\alpha,2} = \psi_{\alpha,2} e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1}$$

Using reasoning as above, it is then clear that

$$||\phi_{\alpha,1}||_{L_x^\infty} \lesssim 2^{-\frac{7}{12}l}, \; ||\phi_{\alpha,2}||_{\dot H^{\frac{1}{2}}} \lesssim 2^{-\nu l},$$

for some $\nu > 0$. Indeed, we can strengthen the latter to $||\phi_{\alpha,2}||_{\dot B^{\frac{1}{2},1}} \lesssim 2^{-\nu l}$ and similarly $||P_{[-\nu l,\nu l]^c} \phi_{\alpha,1}||_{\dot B^{\frac{1}{2},1}} \lesssim 2^{-\nu' l}$ for some $\nu' > 0$. By splitting

$$\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1 = \triangle^{-1} \sum_{j=1}^3 P_{[-\nu l,\nu l]} \partial_j \phi_j^1 + \triangle^{-1} \sum_{j=1}^3 P_{[-\nu l,\nu l]^c} \partial_j \phi_j^1,$$

and using the decomposition $\phi_j^1 = \phi_{j1}^1 + \phi_{j2}^1$ from above, we then deduce

$$||\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1(t,.)||_{L_x^\infty} \lesssim 2^{-\nu' l}, \; |t| \sim 2^l$$

for some $\nu' > 0$. Now write

$$\phi_\alpha - S(t)(\tilde\psi_{\alpha 1}, \tilde\psi_{\alpha 2}) = (e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)\psi_\alpha + \psi_\alpha - S(t)(\tilde\psi_{\alpha 1}, \tilde\psi_{\alpha 2})$$

We need to show that the first expression on the right converges to zero as $t \to \infty$, with respect to $\dot H^{\frac{1}{2}}$. Once again decompose

$$P_k[(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)\psi_\alpha] = P_k[P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{<k-5}\psi_\alpha]$$
$$+ P_k[P_{<k+10}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{[k-5,k+5]}\psi_\alpha]$$
$$+ P_k[P_{>k+2}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{>k+5}\psi_\alpha]$$

For the first term on the right, we can bound

$$||P_k[P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{<k-5}\psi_\alpha]||_{\dot H^{\frac{1}{2}}}$$
$$\lesssim ||P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)||_{\dot H^{\frac{3}{2}}} ||P_{<k-5}\psi_\alpha||_{\dot H^{\frac{1}{2}}}$$

Thus, using Lemma 3.6, if we restrict to $|k| > \nu l$, we obtain exponential decay in l. Hence we can restrict to $|k| < \nu l$. Then use the splitting $\psi_\alpha = \psi_{\alpha,1} + \psi_{\alpha,2}$ and estimate

$$||P_k[P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{<k-5}\psi_{\alpha,1}]||_{\dot H^{\frac{1}{2}}}$$
$$\lesssim ||P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)||_{\dot H^{\frac{3}{2}}} 2^{-k} ||\psi_{\alpha,1}||_{L_x^\infty} \lesssim 2^{\nu l - \frac{7}{12}l} 2^{-k} d_k$$

Choosing $\nu > 0$ small enough and summing over $|k| < \nu l$ still results in exponential decay in l. Further, we have

$$||P_k[P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)P_{<k-5}\psi_{\alpha,2}]||_{\dot H^{\frac{1}{2}}}$$
$$\lesssim ||P_{[k-10,k+10]}(e^{i\triangle^{-1} \sum_{j=1}^3 \partial_j \phi_j^1} - 1)||_{\dot H^{\frac{3}{2}}} 2^{-k} ||P_{<k-5}\psi_{\alpha,2}||_{L_x^\infty} \lesssim e_k,$$

where in the last step we have used Bernstein's inequality and the Sobolev embedding as well as the definition of e_k. Square-summing over k results in an exponential gain in l.

Next, consider the term $P_k[P_{<k+10}(e^{i\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^1}-1)P_{[k-5,k+5]}\psi_\alpha]$. Here we obtain exponential decay from

$$||(e^{i\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^1(t,.)}-1)||_{L_x^\infty} \lesssim 2^{-\nu'l}, \ |t| \sim 2^l,$$

which follows from the bound on the exponent obtained further above. The term $P_k[P_{>k+2}(e^{i\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^1}-1)P_{>k+5}\psi_\alpha]$ is handled similarly.

Finally, we also need to demonstrate scattering for the original coordinate functions $\mathbf{x}$, $\mathbf{y}$. Recall that we have

$$\log[\frac{\mathbf{y}}{\mathbf{y}_\infty}] = \triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^2$$

Reasoning as above for $\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^1$, we obtain

(3.2)
$$\lim_{|t|\to\infty}||\triangle^{-1}\sum_{j=1}^{3}\partial_j\phi_j^2(t,.)||_{L_x^\infty} = 0$$

with a small polynomial decay rate in t. The already proved fact that $\phi_j(t,.) = S(t)(\tilde\psi_{\alpha 1}, \tilde\psi_{\alpha 2}) + o_{\dot H^{\frac{1}{2}}}(1)$ yields

$$\log[\frac{\mathbf{y}}{\mathbf{y}_\infty}](t,.) = S(t)(\triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j1}^2, \triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j2}^2) + o_{\dot H^{\frac{3}{2}}}(1)$$

Here we of course write $\tilde\psi_{\alpha 1,2} = \tilde\psi_{\alpha 1,2}^1 + i\tilde\psi_{\alpha 1,2}^2$. Further, invoking (3.2) and reasoning as above, we obtain from here

$$\frac{\mathbf{y}}{\mathbf{y}_\infty}(t,.) = S(t)(\triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j1}^2, \triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j2}^2) + o_{\dot H^{\frac{3}{2}}}(1)$$

From the relation $\phi_j^1 = \frac{\partial_j\mathbf{x}}{\mathbf{y}}$, we obtain in the same fashion that

$$\frac{\mathbf{x}}{\mathbf{y}_\infty} = S(t)(\triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j1}^1, \triangle^{-1}\sum_{j=1}^{3}\partial_j\tilde\psi_{j2}^1) + o_{\dot H^{\frac{3}{2}}}(1)$$

$\square$

We further have the following

COROLLARY 3.7. *(of preceding proof) We have the bound*

$$||((\mathbf{x},\mathbf{y}) - (\mathbf{x}_\infty,\mathbf{y}_\infty))(t,.)||_{L_x^\infty} \lesssim |t|^{-\nu}$$

for $\nu > 0$ sufficiently small and large $|t|$. Here $(\mathbf{x}_\infty,\mathbf{y}_\infty))$ are the values of $u(0,x)$ for large $|x|$, i. e. the "data at infinity".

4. The core propositions

We now outline the proofs of Proposition 2.1, Proposition 2.2, as well as Proposition 2.3. We observe that these are essentially contained in [**9**], [**10**], the only new ingredient being the presence of the vector fields Γ^β. We shall refer some details to these papers. We begin by collecting some

4.1. Basic facts concerning the spaces $S[k]$, $N[k]$. .

LEMMA 4.1. ([**9**])*We have*

$$||R_\nu P_k\psi||_{L_t^p L_x^q} \lesssim 2^{[(\frac{1}{4}-\frac{1}{p})+(\frac{3}{4}-\frac{3}{p})]k}||P_k\psi||_{S[k]},\ \nu = 0,1,2,3$$

provided $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, *with implicit constant possibly depending on p,q. Also, we have*

$$||P_k\psi||_{S[k]} \lesssim ||P_k\nabla_{x,t}\psi||_{\dot{X}_k^{-\frac{1}{2},\frac{1}{2},1}}$$

The first part of this lemma allows us to control some Strichartz type norms. This shall be especially handy when estimating terms of high degree of multilinearity.

LEMMA 4.2. *(e. g. [**9**]) The following estimate holds for suitable $\delta > 0$:*

$$||P_k Q_j[P_{k_1}\psi_1 P_{k_2}\psi_2]||_{\dot{X}^{-\frac{1}{2},\frac{1}{2},\infty}} \lesssim 2^{\delta\min\{j-\min\{k,k_1,k_2\},0\}}2^{-|k_1-k_2|}\prod_{j=1,2}||P_{k_j}\psi_j||_{S[k_j]}$$

Furthermore, for any $\mu > 0$ we have

$$||P_k[P_{k_1}\psi_1 P_{k_2}\psi_2]||_{L_t^2 L_x^{2+\mu}} \lesssim 2^{\frac{\mu}{4\mu+2}k}2^{-\frac{|k_1-k_2|}{2}}\prod_{j=1,2}||P_{k_j}\psi_j||_{S[k_j]}$$

Finally, the following bound obtains[4] *for $p > 2$:*

$$||P_k[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]||_{L_t^2 L_x^p}$$
$$\lesssim 2^{(\frac{3}{2}-\frac{3}{p})k}2^{\frac{\min\{k,k_{1,2}\}-\max\{k,k_{1,2}\}}{2}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$$

PROOF. (Lemma 4.2) Only the last part requires a justification, and this only in the case $k_1 >> k$. We decompose

$$P_k[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]$$
$$= P_k Q_{<k+20}[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]$$
$$+ P_k Q_{\geq k+20}[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]$$

For the first term, use a simple algebraic identity and estimate

$$||P_k Q_{<k+20}[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]||_{L_t^2 L_x^p}$$
$$\leq ||P_k Q_{<k+20}\partial_\nu[\nabla^{-1}P_{k_1}\psi_1 R_j P_{k_2}\psi_2]||_{L_t^2 L_x^p} + ||P_k Q_{<k+20}\partial_j[R_\nu P_{k_1}\psi_1 \nabla^{-1}P_{k_2}\psi_2]||_{L_t^2 L_x^p}$$
$$\lesssim 2^{-k_1}||P_{k_1}\psi_1||_{L_t^4 L_x^{2p}}||P_{k_2}\psi_2||_{L_t^4 L_x^{2p}}$$

and from here the claimed inequality follows easily. Further, we decompose

$$P_k Q_{\geq k+20}[R_\nu P_{k_1}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1}\psi_1 R_\nu P_{k_2}\psi_2]$$
$$= P_k Q_{\geq k+20}[R_\nu P_{k_1} Q_{\geq k+10}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1} Q_{\geq k+10}\psi_1 R_\nu P_{k_2}\psi_2]$$
$$+ P_k Q_{\geq k+20}[R_\nu P_{k_1} Q_{<k+10}\psi_1 R_j P_{k_2} Q_{\geq k+10}\psi_2 - R_j P_{k_1} Q_{<k+10}\psi_1 R_\nu P_{k_2} Q_{\geq k+10}\psi_2]$$
$$+ P_k Q_{\geq k+20}[R_\nu P_{k_1} Q_{<k+10}\psi_1 R_j P_{k_2} Q_{<k+10}\psi_2 - R_j P_{k_1} Q_{<k+10}\psi_1 R_\nu P_{k_2} Q_{<k+10}\psi_2]$$

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[4]One can significantly strengthen this estimate and also include the case $p = 2$, see e. g. [**11**], but we don't need this here.

The first two terms are estimated similarly: for example, we have

$$||P_k Q_{\geq k+20}[R_\nu P_{k_1} Q_{\geq k+10}\psi_1 R_j P_{k_2}\psi_2 - R_j P_{k_1} Q_{\geq k+10}\psi_1 R_\nu P_{k_2}\psi_2]||_{L_t^2 L_x^2}$$

$$\lesssim 2^{\frac{3k}{2}} ||\nabla_{x,t}\nabla^{-1}P_{k_1}Q_{\geq k+10}\psi_1||_{L_t^2 L_x^2} ||\nabla_{x,t}\nabla^{-1}P_{k_1}Q_{\geq k+10}\psi_1||_{L_t^\infty L_x^2}$$

$$\lesssim 2^{\frac{3k}{2}-\frac{k}{2}-k_1} \prod_{i=1,2} ||P_{k_i}\psi_i||_{S[k_i]},$$

and the inequality of the Lemma follows from Bernstein's inequality. Finally, for the last term above, upon freezing the output to modulation 2^l, $l \geq k + 10$ (i. e. apply an operator Q_l to the expression), use that we may assume $k_1 = l + O(1)$. Then the inequality follows from the same calculation as at the end the of the proof of Lemma 4.8 below, after summing over l. □

Of fundamental importance is furthermore the following

LEMMA 4.3. ([**24**], [**10**])*Let* $j \leq \min\{k_1, k_2\} + O(1)$. *Also, let* F *and* ψ *be Schwartz functions, the former at frequency* $\sim 2^{k_1}$ *and modulation (distance of the space-time Fourier support to the light cone)* $\sim 2^j$, *the latter at frequency* 2^{k_2}. *Then the following inequalities hold for suitably small* $\delta_{1,2} > 0$:

$$||P_k(F\psi)||_{N[k]} \lesssim 2^{-\delta_1|k-\max\{k_1,k_2\}|}2^{-\delta_2|j-\min\{k_1,k_2\}|}||F||_{\dot{X}_{k_1}^{\frac{1}{2},-\frac{1}{2},\infty}}||\psi||_{S[k_2]}$$

$$||\nabla_x P_k(F\psi)||_{N[k]} \lesssim 2^{-\delta_1|k-\max\{k_1,k_2\}|}2^{-\delta_2|j-\min\{k_1,k_2\}|}||F||_{\dot{X}_{k_1}^{\frac{1}{2},-\frac{1}{2},\infty}}||\nabla_x\psi||_{S[k_2]}$$

Finally, the relation between $S[k]$ ad $N[k]$ is obtained via the following

THEOREM 4.4. *The following inequality holds:*

$$||P_k\phi||_{S[k]([-T,T]\times\mathbb{R}^3)} \lesssim ||\Box P_k\phi||_{N[k]([-T,T]\times\mathbb{R}^3)} + ||\phi[0]||_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}}$$

The proof of this follows from simple modifications of the one given in [**24**]. We now give the proof of Propositon 2.1

PROOF. (Proposition 2.1). We fix a number $l >> 1$, and construct a solution $\psi_{\alpha,1}$ (for all α) on the time interval $[-2^{l+2}, 2^{l+2}]$. This solution is obtained via simple Banach iteration: specifically, we require that the iterates $\psi_{\alpha,1,j}$, $j \geq 1$, all be smooth functions which are supported on a compact time interval $[-T', T']$, $T' >> 2^l$, and furthermore solve (2.6) on the interval $[-2^{l+2}, 2^{l+2}]$. The iterative step is given by the following:

$$\psi_{\alpha,1,j+1}(t,.) := \eta_T(t)S(t)(\psi_\alpha[0]) + \int_0^t \eta_T(t-s)\mathbf{P}_{[-\delta l,\delta l]}F(\psi_{1,j})(s)ds, \ j \geq 1$$

where $T' >> T >> 2^l$. Furthermore, we start the iteration with

$$\psi_{\alpha,1,1}(t,.) := \eta_T(t)S(t)(\psi_\alpha[0])$$

Here $\eta_T(t)$ equals 1 on $[-T, T]$ and smoothly truncates to a dilate of this interval, contained within $[-T', T']$. Proposition 2.1 now follows from theorem 4.4 and the following

PROPOSITION 4.5. *Assume we have the bounds*

$$||P_k\psi_{\alpha,1,j}||_{S[k]([-T,T]\times\mathbb{R}^3)} \leq Md_k, \ \forall k \in \mathbb{Z}$$

Further, assume we have

$$\sum_{0 \leq |\beta| \leq 2} ||P_k \Gamma^\beta \psi_{\alpha,1,j}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq \tilde{M} 2^{C\delta l} c_k, \, \forall k \in \mathbb{Z}$$

$$||P_k \nabla_x^N \psi_{\alpha,1,j}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq M_N d_k, \, \forall k \in \mathbb{Z}$$

Here the numbers c_k, d_k are as in Proposition 2.1, and $N \geq 1$ is arbitrary. Then, provided $M, \tilde{M}$ are large enough absolute constants and $\epsilon > 0$ is small enough (again with ϵ as in Proposition 2.1), and $M_N, M_{N-1} \ldots$ are large enough constants depending on the initial data (and N), we infer the bounds

$$||P_k \psi_{\alpha,1,j+1}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq [\epsilon M + C]d_k, \, \forall k \in \mathbb{Z}$$

for some C independent of M, as well as

$$\sum_{0 \leq |\beta| \leq 2} ||P_k \Gamma^\beta \psi_{\alpha,1,j+1}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq [\epsilon \tilde{M} 2^{C\delta l} + C 2^{C\delta l}]c_k, \, \forall k \in \mathbb{Z}$$

$$||P_k \nabla_x^N \psi_{\alpha,1,j+1}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq [\epsilon M_N + M_{N-1}]d_k, \, \forall k \in \mathbb{Z}$$

Furthermore, we obtain for the differences

$$\sum_{0 \leq |\beta| \leq 2} \sum_{0 \leq k \leq N} ||\nabla_x^k \Gamma^\beta P_k[\psi_{\alpha,1,j+1} - \psi_{\alpha,1,j}]||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq C_N \epsilon^j$$

To see how the proof of Proposition 2.1 follows from this, note that each $\nabla_x^k \Gamma^\beta P_k[\psi_{\alpha,1,j}]$ converges with respect to $L_t^\infty \dot{H}^{\frac{1}{2}}$, whence the limit is smooth and satisfies the same bounds. $\qquad \square$

Hence we now direct our efforts to proving Proposition 4.5. This will be achieved via multilinear estimates much in the spirit of [**9**], [**11**], the only new ingredient being the Γ^β:

PROOF. (Proposition 4.5) We shall first establish the inequality

$$||P_k \psi_{\alpha,1,j+1}||_{S[k]([-T,T] \times \mathbb{R}^3)} \leq [\epsilon M + C]d_k, \, \forall k \in \mathbb{Z}$$

This will follow from the energy inequality Theorem 4.4 as well as the following two fundamental Propositions 4.6, 4.9. The first deals with estimating the trilinear null-forms on the right hand side of (2.6), while the 2nd deals with the higher order terms

PROPOSITION 4.6. *The following trilinear null-form estimates hold:*

$$||P_k \partial^\beta [R_\alpha P_{k_1} \psi P_{k_4} \triangle^{-1} \sum_{j=1}^3 \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]||_{N[k]}$$

$$\lesssim 2^{-\delta_1 |k_2 - k_3|} 2^{\delta_2 [k_4 - \max\{k_2,k_3\}]} 2^{-\delta_3 |k - k_1|} \prod_{i=1}^3 ||P_{k_i} \psi_i||_{S[k_i]}$$

$$||P_k \partial^\beta [R_\beta P_{k_1} \psi P_{k_4} \triangle^{-1} \sum_{j=1}^3 \partial_j (R_\alpha P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\alpha P_{k_3} \psi_3)]||_{N[k]}$$

$$\lesssim 2^{-\delta_1 |k_2 - k_3|} 2^{\delta_2 [k_4 - \max\{k_2,k_3\}]} 2^{-\delta_3 |k - k_1|} \prod_{i=1}^3 ||P_{k_i} \psi_i||_{S[k_i]}$$

$$||P_k\partial^\alpha[R_\nu P_{k_1}\psi P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R^\nu P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R^\nu P_{k_3}\psi_3)]||_{N[k]}$$

$$\lesssim 2^{-\delta_1|k_2-k_3|}2^{\delta_2[k_4-\max\{k_2,k_3\}]}2^{-\delta_3|k-k_1|}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

for suitable positive $\delta_{1,2,3}$.

REMARK 4.7. We observe that these estimates are very similar to Theorem 4.2 contained in [**10**], and indeed essentially implicitly contained in the proofs of [**10**], see also [**11**] in the 2-dmensional context. The only extra feature here is an exponential gain in the difference $k_4 - \max\{k_2, k_3\}$. Our treatment shall be correspondingly brief.

PROOF. (Proposition 4.6) We shall treat the first inequality, the other two following from identical reasoning. By scaling invariance we may assume $k = 0$. We note that the cases $k_4 = k_2 + O(1)$, $k_1 \in [-10, 10]^c$ follow from Theorem 4.2 in [**10**]. Hence we now assume $k_2 = k_3 + O(1) >> k_4$, $k_1 \in [-10, 10]$, $k_4 < 15$. One distinguishes between the following cases:

(1): *output in elliptic regime.* This is the expression

$$P_0 Q_{>20}\partial^\beta[R_\alpha P_{k_1}\psi P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]$$

Recalling the definition of $||.||_{N[0]}$, we need to estimate this with respect to

$$||.||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},\infty}\cap\square|\nabla_{x,t}|^{-1}L_t^\infty \dot{H}_x^{-\frac{1}{2}}}$$

First, we observe easily that

$$||\square^{-1}|\nabla_{x,t}|P_0 Q_{>20}\partial^\beta[R_\alpha P_{k_1}\psi$$

$$P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]||_{L_t^\infty \dot{H}^{-\frac{1}{2}}}$$

$$\lesssim ||R_\alpha P_{k_1}\psi||_{L_t^\infty L_x^2}||P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)||_{L_t^\infty L_x^\infty}$$

which can be bounded by $\lesssim 2^{k_4-k_2}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$, which is as desired. Next, freeze the modulation of the output to dyadic size 2^l, $l > 20$. Then we can write (using that $k_1 \in [-10, 10]$)

$$P_0 Q_l\partial^\beta[R_\alpha P_{k_1}\psi P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]$$

$$= P_0 Q_l\partial^\beta[R_\alpha P_{k_1}Q_{\geq l-10}\psi P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]$$

$$+ P_0 Q_l\partial^\beta[R_\alpha P_{k_1}Q_{<l-10}\psi$$

$$P_{k_4}Q_{>l-10}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]$$

The first term on the right is estimated by

$$||P_0 Q_l \partial^\beta [R_\alpha P_{k_1} Q_{\geq l-10} \psi$$
$$P_{k_4} \triangle^{-1} \sum_{j=1}^3 \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},\infty}}$$
$$= \sum_{l_1 \geq l-10} ||P_0 Q_l \partial^\beta [R_\alpha P_{k_1} Q_{l_1} \psi$$
$$P_{k_4} \triangle^{-1} \sum_{j=1}^3 \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},\infty}}$$
$$\lesssim \sum_{l_1 > 10} 2^{\frac{1}{2}(l-l_1)} 2^{k_4-k_2} \prod_{i=1}^3 ||P_{k_i} \psi_i||_{S[k_i]} \lesssim 2^{k_4-k_2} \prod_{i=1}^3 ||P_{k_i} \psi_i||_{S[k_i]}$$

For the 2nd term, we use the following

LEMMA 4.8. *If $l >> k$, then the following bound holds for suitable $\delta > 0$:*

$$||P_k Q_l [R_\nu P_{k_1} \psi_1 P_{k_2} \psi_2]||_{L_t^2 L_x^2} \lesssim 2^{\frac{k-l}{2}} 2^{\delta[\min\{k,k_1,k_2\}-\max\{k,k_1,k_2\}]} \prod_{i=1,2} ||P_{k_i} \psi_i||_{S[k_i]}$$

PROOF. It is very similar to that of the last part of lemma 4.2: we can write

$$P_k Q_l [R_\nu P_{k_1} \psi_1 P_{k_2} \psi_2] =$$
$$P_k Q_l [R_\nu P_{k_1} Q_{\geq l-10} \psi_1 P_{k_2} \psi_2]$$
$$+ P_k Q_l [R_\nu P_{k_1} Q_{<l-10} \psi_1 P_{k_2} Q_{\geq l-10} \psi_2]$$
$$+ P_k Q_l [R_\nu P_{k_1} Q_{<l-10} \psi_1 P_{k_2} Q_{<l-10} \psi_2]$$

Here the last term on the right is nonzero only if $k_1 = l + O(1)$. Then when $k_1 = k_2 + O(1)$ we estimate

$$||P_k Q_l [R_\nu P_{k_1} Q_{\geq l-10} \psi_1 P_{k_2} \psi_2]||_{L_t^2 L_x^2} \lesssim 2^{\frac{3k}{2}} ||R_\nu P_{k_1} Q_{\geq l-10} \psi_1||_{L_t^2 L_x^2} ||P_{k_2} \psi_2||_{L_t^\infty L_x^2}$$
$$\lesssim 2^{\frac{3k}{2}-\frac{1}{2}l-k_1} \prod_{i=,2} ||P_{k_i} \psi_i||_{S[k_i]},$$

which is as desired. The cases $k = k_1 + O(1)$, $k = k_2 + O(1)$ are handled analogously, as is the expression $P_k Q_l [R_\nu P_{k_1} Q_{<l-10} \psi_1 P_{k_2} Q_{\geq l-10} \psi_2]$. Now for the last term above, we may assume $k_1 \in [l-5, l+5]$. Furthermore, we may microlocalize the two inputs to the same half-space $\pm \tau > 0$, i. e.

$$P_k Q_l [R_\nu P_{k_1} Q_{<l-10} \psi_1 P_{k_2} Q_{<l-10} \psi_2] = \sum_\pm P_k Q_l^\pm [R_\nu P_{k_1} Q_{<l-10}^\pm \psi_1 P_{k_2} Q_{<l-10}^\pm \psi_2]$$

We split this into three terms as follows:

$$\sum_\pm P_k Q_l^\pm [R_\nu P_{k_1} Q_{<l-10}^\pm \psi_1 P_{k_2} Q_{<l-10}^\pm \psi_2]$$
$$= \sum_\pm P_k Q_l^\pm [R_\nu P_{k_1} Q_{[\frac{3}{2}k-\frac{1}{2}k_1,l-10]}^\pm \psi_1 P_{k_2} Q_{<l-10}^\pm \psi_2]$$
$$+ \sum_\pm P_k Q_l^\pm [R_\nu P_{k_1} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^\pm \psi_1 P_{k_2} Q_{[\frac{3}{2}k-\frac{1}{2}k_1,l-10]}^\pm \psi_2]$$
$$+ \sum_\pm P_k Q_l^\pm [R_\nu P_{k_1} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^\pm \psi_1 P_{k_2} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^\pm \psi_2]$$

The first term is estimated by

$$||\sum_{\pm} P_k Q_l^{\pm}[R_\nu P_{k_1} Q_{[\frac{3}{2}k-\frac{1}{2}k_1,l-10]}^{\pm}\psi_1 P_{k_2} Q_{<l-10}^{\pm}\psi_2]||_{L_t^2 L_x^2}$$

$$\lesssim \sum_{\pm} 2^{\frac{3}{2}k} 2^{-\frac{k_1}{2}-\frac{\frac{3}{2}k-\frac{1}{2}k_1}{2}}||R_\nu P_{k_1} Q_{[\frac{3}{2}k-\frac{1}{2}k_1,l-10]}^{\pm}\psi_1||_{\dot{X}_{k_1}^{\frac{1}{2},\frac{1}{2},\infty}}||P_{k_2} Q_{<l-10}^{\pm}\psi_2||_{L_t^\infty L_x^2}$$

$$\lesssim 2^{\frac{3}{2}k-k_1-\frac{\frac{3}{2}k-\frac{1}{2}k_1}{2}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]} = 2^{\frac{3}{4}(k-k_1)}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$$

The 2nd term above can be estimated similarly. Finally, for the third term, we can decompose it into

$$\sum_{\pm} P_k Q_l^{\pm}[R_\nu P_{k_1} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_1 P_{k_2} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_2]$$

$$= \sum_{\pm} \sum_{\kappa_{1,2}\in K_{\frac{3}{4}(k-k_1)}|\text{dist}(\kappa_1,-\kappa_2)\lesssim 2^{\frac{3}{4}(k-k_1)}}$$

$$P_k Q_l^{\pm}[R_\nu P_{k_1,\kappa_1} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_1 P_{k_2,\kappa_2} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_2]$$

Using the definition of $||.||_{S[k,\kappa]}$ and the Cauchy-Schwarz inequality, we can estimate this by

$$||\sum_{\pm} \sum_{\kappa_{1,2}\in K_{\frac{3}{4}(k-k_1)}|\text{dist}(\kappa_1,-\kappa_2)\lesssim 2^{\frac{3}{4}(k-k_1)}}$$

$$P_k Q_l^{\pm}[R_\nu P_{k_1,\kappa_1} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_1 P_{k_2,\kappa_2} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_2]||_{L_t^2 L_x^2}$$

$$\lesssim 2^{\frac{3}{4}(k-k_1)} \sum_{\pm} \big(\sum_{\kappa\in K_{\frac{3}{4}(k-k_1)}} ||P_{k_1,\kappa} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_1||_{S[k_1,\kappa_1]}^2 \big)^{\frac{1}{2}}$$

$$\times \big(\sum_{\kappa\in K_{\frac{3}{4}(k-k_1)}} ||P_{k_2,\kappa} Q_{<\frac{3}{2}k-\frac{1}{2}k_1}^{\pm}\psi_1||_{S[k_1,\kappa_1]}^2 \big)^{\frac{1}{2}}$$

This in turn is bounded by $\lesssim 2^{\frac{3}{4}(k-k_1)} \prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$, which is as desired. $\square$

Then we can estimate

$$||P_0 Q_l \partial^\beta [R_\alpha P_{k_1} Q_{<l-10}\psi$$

$$P_{k_4} Q_{>l-10}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},\infty}}$$

$$\lesssim 2^{\frac{l}{2}}||R_\alpha P_{k_1} Q_{<l-10}\psi||_{L_t^\infty L_x^2}$$

$$||P_{k_4} Q_{>l-10}\triangle^{-1}\sum_{j=1}^3 \partial_j(R_\beta P_{k_2}\psi_2 R_j P_{k_3}\psi_3 - R_j P_{k_2}\psi_2 R_\beta P_{k_3}\psi_3)]||_{L_t^2 L_x^\infty}$$

$$\lesssim 2^{\frac{k_4}{2}} 2^{\delta[k_4-k_2]}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

This concludes estimating the contribution of the output in the elliptic regime.

(2): *Output in hyperbolic regime.* This is the expression

$$P_0 Q_{<20} \partial^\beta [R_\alpha P_{k_1} \psi P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]$$

Here, we first reduce $R_\alpha P_{k_1} \psi$ as well as the output further to modulation $< 2^{k_4 - 10}$. To achieve this, estimate

$$||P_0 Q_{20 >. \geq k_4 - 10} \partial^\beta [R_\alpha P_{k_1} \psi_1$$
$$P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]||_{\dot{X}_0^{-\frac{1}{2}, -\frac{1}{2}, 1}}$$
$$\lesssim 2^{-\frac{k_4}{2}} ||R_\alpha P_{k_1} \psi_1||_{L_t^\infty L_x^2}$$
$$||P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)||_{L_t^2 L_x^\infty}$$

Using Lemma 4.2, the right hand factor can be estimated by

$$||P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)||_{L_t^2 L_x^\infty}$$
$$\lesssim 2^{\frac{k_4}{2}} 2^{\frac{k_4 - k_2}{2}} \prod_{i=2,3} ||P_{k_i} \psi_i||_{S[k_i]}$$

and the desired estimate follows. The expression

$$P_0 Q_{<20} \partial^\beta [R_\alpha P_{k_1} Q_{\geq k_4 - 10} \psi_1 P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)]$$

is estimated similarly(place the output into $L_t^1 \dot{H}^{-\frac{1}{2}}$), hence we now need to estimate

$$P_0 Q_{<k_4 - 10} \partial^\beta [R_\alpha P_{k_1} Q_{<k_4 - 10} \psi_1 P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} \psi_2 R_\beta P_{k_3} \psi_3)].$$

Note that we may include an operator $Q_{<k_4 + 10}$ in front of $P_{k_4} \triangle^{-1} \dots$. We further reduce the inner inputs $P_{k_{2,3}} \psi_{2,3}$ to modulation $< 2^{k_4 + 20}$ as follows: for example, consider

$$P_0 Q_{<k_4 - 10} \partial^\beta [R_\alpha P_{k_1} Q_{<k_4 - 10} \psi_1$$
$$P_{k_4} \triangle^{-1} \sum_{j=1}^{3} \partial_j (R_\beta P_{k_2} Q_{\geq k_4 + 20} \psi_2 R_j P_{k_3} \psi_3 - R_j P_{k_2} Q_{\geq k_4 + 20} \psi_2 R_\beta P_{k_3} \psi_3)]$$

Simple geometric reasoning then allows one to include a multiplier $Q_{>k_4+15}$ in front of each $P_{k_3}\psi_3$. Then one estimates

$$||P_0 Q_{<k_4-10}\partial^\beta [R_\alpha P_{k_1} Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1} \sum_{j=1}^3 \partial_j (R_\beta P_{k_2} Q_{\geq k_4+20}\psi_2 R_j P_{k_3} Q_{>k_4+15}\psi_3$$
$$- R_j P_{k_2} Q_{\geq k_4+20}\psi_2 R_\beta P_{k_3} Q_{>k_4+15}\psi_3)]||_{L_t^1 \dot{H}_x^{-\frac{1}{2}}}$$

$$\lesssim 2^{2k_4}||R_\alpha P_{k_1} Q_{<k_4-10}\psi_1||_{L_t^\infty L_x^2}$$
$$||\nabla_{x,t}\nabla^{-1} P_{k_2} Q_{\geq k_4+20}\psi_2||_{L_t^2 L_x^2}||\nabla_{x,t}\nabla^{-1} P_{k_3} Q_{\geq k_4+20}\psi_3||_{L_t^2 L_x^2}$$

$$\lesssim 2^{k_4-k_2}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

We now let the outer derivative fall inside and obtain two terms, each of which admits a null-form expansion: the terms are

$$P_0 Q_{<k_4-10}[R_\alpha \partial^\beta P_{k_1} Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1} \sum_{j=1}^3 \partial_j (R_\beta P_{k_2} Q_{<k_4+20}\psi_2 R_j P_{k_3} Q_{<k_4+20}\psi_3$$
$$- R_j P_{k_2} Q_{<k_4+20}\psi_2 R_\beta P_{k_3} Q_{<k_4+20}\psi_3)]$$

$$P_0 Q_{<k_4-10}[R_\alpha P_{k_1} Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1} \sum_{j=1}^3 \partial_j \partial^\beta (R_\beta P_{k_2} Q_{<k_4+20}\psi_2 R_j P_{k_3} Q_{<k_4+20}\psi_3$$
$$- R_j P_{k_2} Q_{<k_4+20}\psi_2 R_\beta P_{k_3} Q_{<k_4+20}\psi_3)]$$

We treat these separately. *In order to streamline the formulae a bit, we shall omit the localizers $Q_{<k_4+20}$, it being understood that the inputs $P_{k_{2,3}}\psi_{2,3}$ have Fourier support at distance $< 2^{k_4+20}$ from the light cone.*

(3): *The first null-form.* We use the expansion

$$2\sum_{j=1}^3 \triangle^{-1}\partial_j [R_\nu f R_j g - R_j f R_\nu f]\partial^\nu h$$

(4.1)
$$= \sum_{j=1}^3 \square[\triangle^{-1}\partial_j [\nabla^{-1} f R_j g]h] - \sum_{j=1}^3 \square\triangle^{-1}\partial_j [\nabla^{-1} f R_j g]h$$

$$- \sum_{j=1}^3 \triangle^{-1}\partial_j [\nabla^{-1} f R_j g]\square h - \nabla^{-1} f\square[(\nabla^{-1} g)h]$$

$$+ \nabla^{-1} f\square(\nabla^{-1} g)h + \nabla^{-1} f\nabla^{-1} g\square h$$

Hence we need to estimate the following terms: **first assume $k_2 < -20$.**

(i): $P_0 Q_{<k_4-10} \Box[R_\alpha P_{k_1} Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$. Here we use lemma 4.1 to estimate it by

$$||P_0 Q_{<k_4-10}\Box[R_\alpha P_{k_1}Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}$$

$$\lesssim 2^{\frac{k_4}{2}}||R_\alpha P_{k_1}Q_{<k_4-10}\psi_1||_{L_t^\infty L_x^2}||| P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)||_{L_t^2 L_x^\infty}$$

$$\lesssim 2^{-\frac{k_4}{2}-k_2}2^{\frac{6}{p}k_4}||R_\alpha P_{k_1}Q_{<k_4-10}\psi_1||_{L_t^\infty L_x^2}||P_{k_2}\psi_2||_{L_t^4 L_x^p}||P_{k_3}\psi_3||_{L_t^4 L_x^p}$$

This is seen to be bounded by $\lesssim 2^{\delta[k_4-k_2]}\prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$ for some $\delta > 0$, provided we choose $p > 4$ sufficiently close to 4.

(ii): $P_0 Q_{<k_4-10}[R_\alpha P_{k_1}Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$. This we estimate by using Lemma 4.3 as well as Lemma 4.2: First, we have

$$P_0 Q_{<k_4-10}[R_\alpha P_{k_1}Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$$

$$= P_0 Q_{<k_4-10}[R_\alpha P_{k_1}Q_{<k_4-10}\psi_1 P_{k_4}Q_{<k_4+10}\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$$

Then we first remove the localization operators $Q_{<k_4-10}$, which is simple (estimating as before) and omitted. Thus we now need to estimate (including an operator $Q_{<k_1}$ to render the R_α harmless)

$$P_0[R_\alpha P_{k_1}Q_{<k_1}\psi_1 P_{k_4}Q_{<k_4+10}\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$$

$$= \sum_{j<k_4+10} P_0[R_\alpha P_{k_1}Q_{<k_1}\psi_1 P_{k_4}Q_j\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$$

Using Lemma 4.3 we have

$$||P_0[R_\alpha P_{k_1}Q_{<k_1}\psi_1 P_{k_4}Q_j\triangle^{-1}\sum_{j=1}^3 \partial_j\Box(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]||_{N[0]}$$

$$\lesssim 2^{\delta(j-k_4)}||P_{k_1}\psi_1||_{S[k_1]}||\Box\nabla^{-1}P_{k_4}Q_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)||_{\dot{X}_{k_4}^{\frac{1}{2},-\frac{1}{2},1}}$$

for suitable $\delta > 0$. Using Lemma 4.2, we estimate this by

$$2^{\delta(j-k_4)}||P_{k_1}\psi_1||_{S[k_1]}||\Box\nabla^{-1}P_{k_4}Q_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)||_{\dot{X}_{k_4}^{\frac{1}{2},-\frac{1}{2},1}}$$

$$\lesssim 2^{\delta(j-k_4)}||P_{k_1}\psi_1||_{S[k_1]}2^{k_4-k_2}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$$

which is as desired upon summing over j.

(iii): $P_0 Q_{<k_4-10}[R_\alpha\Box P_{k_1}Q_{<k_4-10}\psi_1 P_{k_4}\triangle^{-1}\sum_{j=1}^3 \partial_j(\nabla^{-1}P_{k_2}\psi_2 R_j P_{k_3}\psi_3)]$. This is estimated similarly to (i). Simply place the output into $L_t^1\dot{H}^{-\frac{1}{2}}$.

(iv): the term corresponding to $\nabla^{-1}f\square(\nabla^{-1}gh)$ in the above expansion is the most difficult to handle; we use that P_{k_4} acts via convolution with a function $a_{k_4}(x)$ of bounded L^1-mass. Further, we recall here the precise microlocalization of $P_{k_{2,3}}\psi_{2,3}$. Thus we write this term as

$$\int_{\mathbb{R}^3} a_{k_4}(y)P_0Q_{<k_4-10}[\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2(.-y)]dy$$

Note that the restriction that $P_{k_2}\psi_2 P_{k_3}\psi_3$ is reduced to frequency 2^{k_4} allows us to simultaneously localize them to angular sectors $\kappa_{2,3}$ such that $\operatorname{dist}(\kappa_2,-\kappa_3)\lesssim 2^{\frac{k_4-k_2}{2}}$. Thus we can write

$$\int_{\mathbb{R}^3} a_{k_4}(y)P_0Q_{<k_4-10}[\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2(.-y)]dy$$
$$= \sum_{\kappa_{2,3}\in K_{k_4-k_2}|\operatorname{dist}(\kappa_2,-\kappa_3)\lesssim 2^{k_4-k_2}}$$
$$\int_{\mathbb{R}^3} a_{k_4}(y)P_0Q_{<k_4-10}[\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)]dy$$

Here we first abolish the outer localizer $Q_{<k_4-10}$, and modify $\square$ to $\square Q_{<k_3+O(1)}$: clearly, we have

$$P_0Q_{<k_4-10}[\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)]$$
$$= P_0Q_{<k_4-10}[P_{k_1+O(1)}Q_{<k_3+O(1)}\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)]$$

Then, replacing the outer $Q_{<k_4-10}$ by $Q_{\geq k_4-10}$, say, we can estimate

$$||P_0Q_{\geq k_4-10}[P_{k_1+O(1)}Q_{<k_3+O(1)}\square[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1$$
$$\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)]||_{\dot{X}_0^{-\frac{1}{2},-\frac{1}{2},1}}$$
$$\lesssim 2^{-\frac{k_4}{2}}2^{k_3}||P_{k_1}Q_{<k_4-10}R_\alpha\psi_1||_{L_t^\infty L_x^2}||\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)||_{L_t^4 L_x^\infty}$$
$$||\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)||_{L_t^4 L_x^\infty}$$

We have taken advantage of the fact (see e. g. [**24**]) that the operator $P_{k_1}Q_{<k_3+O(1)}$ acts boundedly on spaces of the form $L_t^p L_x^2$, $1\leq p\leq\infty$. Then we use Lemma 4.2 as well as the fact that

$$\sum_{\kappa\in K_l}||P_{k,\kappa}Q_{<k+2l}\psi||_{S[k]}^2\lesssim ||P_k\psi||_{S[k]}^2$$

This is straightforward for the first two components defining $||.||_{S[k]}$. For the complicated null-frame part, use that $||.||_{S[k,\kappa']}\lesssim ||.||_{S[k,\kappa]}$ for $\kappa\subset\kappa'$. Hence we have

for $-10 \geq l' \geq l$

$$\sum_{\kappa \in K_l} \sum_{\kappa' \in K_{l'}} ||P_{k,\kappa'} Q^{\pm}_{<k+2l'} P_{k,\kappa} Q_{<k+2l}\psi||^2_{S[k,\pm\kappa']}$$

$$\lesssim \sum_{\kappa' \in K_{l'}} \sum_{\kappa \in K_l,\, \kappa \subset 2\kappa'} ||P_{k,\kappa'} Q^{\pm}_{<k+2l'} P_{k,\kappa} Q^{\pm}_{<k+2l}\psi||^2_{S[k,\pm\kappa]}$$

$$\lesssim \sum_{\kappa \in K_l} \sum_{\kappa' \in K_{l'}|\kappa \in 2\kappa'} ||P_{k,\kappa} Q^{\pm}_{<k+2l}\psi||^2_{S[k,\pm\kappa]} \lesssim \sum_{\kappa \in K_l} ||P_{k,\kappa} Q^{\pm}_{<k+2l}\psi||^2_{S[k,\pm\kappa]}$$

The case when $l' < l$ is handled similarly. Hence we now see that (using Bernstein's inequality)

$$\sum_{\kappa_{1,2} \in K_{k_4-k_2}|\operatorname{dist}(\kappa_2,-\kappa_3)\lesssim 2^{k_4-k_2}}$$

$$2^{-\frac{k_4}{2}} 2^{k_3}||P_{k_1} Q_{<k_4-10} R_\alpha \psi_1||_{L_t^\infty L_x^2} ||\nabla^{-1} P_{k_3,\kappa_3} Q_{<k_4+20}\psi_3(.-y)||_{L_t^4 L_x^\infty}$$

$$||\nabla^{-1} P_{k_2,\kappa_2} Q_{<k_4+20}\psi_2(.-y)||_{L_t^4 L_x^\infty}$$

$$\lesssim 2^{(1-)(k_4-k_3)} \prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$$

We have now reduced ourselves to estimating

$$\int_{\mathbb{R}^3} a_{k_4}(y) P_0[\Box Q_{<k_3+O(1)}[P_{k_1} Q_{<k_4-10} R_\alpha \psi_1 P_{k_3} Q_{<k_4+20}\psi_3(.-y)]$$

$$\nabla^{-1} P_{k_2} Q_{<k_4+20}\psi_2(.-y)]dy$$

We first write

$$||\Box P_{k_1+O(1)} Q_{<k_3+O(1)}[P_{k_1} Q_{<k_4-10} R_\alpha \psi_1 \nabla^{-1} P_{k_3,\kappa_3} Q_{<k_4+20}\psi_3(.-y)]||_{\dot{X}_{k_1}^{\frac{1}{2},-\frac{1}{2},1}}$$

$$\lesssim \sum_{l<k_3+O(1)} ||P_{k_1+O(1)} Q_l[P_{k_1} Q_{<k_4-10} R_\alpha \psi_1 \nabla^{-1} P_{k_3,\kappa_3} Q_{<k_4+20}\psi_3(.-y)]||_{\dot{X}_{k_1}^{\frac{1}{2},\frac{1}{2},1}}$$

We need to estimate this expression for fixed l first, the point being to eke out a small gain in $|\kappa_3|$. One distinguishes between different ranges for l: first, assume $l \geq k_4 + 30$ whence $\frac{l-k_3}{2} \geq \frac{k_4-k_3}{2} + 15$. Then write (here we have to go into full detail)

$$P_{k_1+O(1)} Q_l[P_{k_1} Q_{<k_4-10} R_\alpha \psi_1 \nabla^{-1} P_{k_3,\kappa_3} Q_{<k_4+20}\psi_3(.-y)]$$

$$= \sum_{\pm,\pm} \sum_{\kappa'_{1,2} \in K_{\frac{l-k_3}{2}}|\operatorname{dist}(\pm\kappa'_1,\pm\kappa'_2)\sim 2^{\frac{l-k_3}{2}}}$$

$$P_{k_1+O(1)} Q_l[P_{k_1,\kappa'_1} Q^{\pm}_{<k_4-10} R_\alpha \psi_1 \nabla^{-1} P_{k_3,\kappa'_2} P_{k_3,\kappa_3} Q^{\pm}_{<k_4+20}\psi_3(.-y)],$$

At this point, the operator $Q^{\pm}_{<k_4-10}$ becomes harmful, as it is applied to a large frequency input. However, we can easily abolish it, by estimating the contribution from $P_{k_1,\kappa'_1} Q^{\pm}_{\geq k_4-10} R_\alpha \psi_1$ as above, where the operator $Q_{\geq k_4-10}$ was applied to the output (here one places the output into $L_t^1 \dot{H}^{-\frac{1}{2}}$). Now we estimate (Here $Q_\pm$ is

the operator which localizes to $\pm\tau > 0$)

$$||\sum_{\pm,\pm} \sum_{\kappa'_{1,2}\in K_{\frac{l-k_3}{2}}|\mathrm{dist}(\pm\kappa'_1,\pm\kappa'_2)\sim 2^{\frac{l-k_3}{2}}}$$

$$P_{k_1+O(1)}Q_l[P_{k_1,\kappa'_1}Q^{\pm}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q^{\pm}_{<k_4+20}\psi_3(.-y)]||_{\dot{X}^{\frac{1}{2},\frac{1}{2},\infty}_{k_1}}$$

$$\lesssim \sum_{\kappa'_{1,2}\in K_{\frac{l-k_3}{2}}|\mathrm{dist}(\pm\kappa'_1,\pm\kappa'_2)\sim 2^{\frac{l-k_3}{2}}}$$

$$2^{\frac{l}{2}}||P_{k_1,\kappa'_1}Q^{\pm}R_\alpha\psi_1||_{NFA[\pm\kappa'_2]}||\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q^{\pm}_{<k_4+20}\psi_3||_{PW[\pm\kappa_3]}$$

Since κ'_2 is much larger than κ_3 in the present case, we only sum for finitely many $\kappa'_{1,2}$ for fixed κ_3. Now we use the estimate

$$||P_{k_1}Q^{\pm}R_\alpha\psi_1||_{NFA[\pm\kappa'_2]^*} \lesssim |l-k_3|2^{-\frac{l-k_3}{2}}||P_{k_1}\psi_1||_{S[k_1]}$$

Hence we get

$$\sum_{\kappa'_{1,2}\in K_{\frac{l-k_3}{2}}|\mathrm{dist}(\pm\kappa'_1,\pm\kappa'_2)\sim 2^{\frac{l-k_3}{2}}}$$

$$2^{\frac{l}{2}}||P_{k_1,\kappa'_1}Q^{\pm}R_\alpha\psi_1||_{NFA[\pm\kappa'_2]}||\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q^{\pm}_{<k_4+20}\psi_3||_{PW[\pm\kappa_3]}$$
$$\lesssim 2^{\frac{k_4-k_3}{2}}2^{\frac{l-k_3}{2}}2^{-\frac{l-k_3}{2}}|k_3-l|||P_{k_1}\psi_1||_{S[k_1]}||P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3||_{S[k_3]}$$

Note that since $k_4 - k_3 << l - k_3$, the factor $|l-k_3|$ can be absorbed by $2^{\frac{k_4-k_3}{2}}$. Finally, we can wrap up case (iv) under the assumption $l > k_4 + 30$: we have

$$\sum_{\kappa_{2,3}\in K_{k_4-k_2}|\mathrm{dist}(\kappa_2,-\kappa_3)\lesssim 2^{k_4-k_2}}$$
$$||P_0[\Box P_{k_1+O(1)}Q_l[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]$$
$$\nabla^{-1}P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2(.-y)]||_{N[0]}$$

$$\lesssim 2^{\delta[l-k_3]}\sum_{\kappa_{1,2}\in K_{k_4-k_2}|\mathrm{dist}(\kappa_2,-\kappa_3)\lesssim 2^{k_4-k_2}}$$
$$||\Box P_{k_1+O(1)}Q_l[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]||_{\dot{X}^{\frac{1}{2},-\frac{1}{2},1}_{k_1}}$$
$$||P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2||_{S[k_2]}$$

By the preceding, we can estimate this by

$$\sum_{\kappa_{2,3}\in K_{k_4-k_2}|\mathrm{dist}(\kappa_2,-\kappa_3)\lesssim 2^{k_4-k_2}}$$
$$2^{\frac{k_4-k_3}{2}}|k_3-l|||P_{k_1}\psi_1||_{S[k_1]}||P_{k_2,\kappa_2}Q_{<k_4+20}\psi_2||_{S[k_3]}||P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3||_{S[k_3]}$$

Using Cauchy-Schwartz as well as the observation from further above, we obtain the desired estimate by summing over $k_4 + 30 \le l < k_3 + O(1)$.

Next, assume $l < k_4 + 30$. Here we decompose

$$P_{k_1+O(1)}Q_l[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<k_4+20}\psi_3(.-y)]$$

$$= P_{k_1+O(1)}Q_l[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{l-10\leq.<k_4+20}\psi_3(.-y)]$$

$$+ P_{k_1+O(1)}Q_l[P_{k_1}Q_{l-10\leq.<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

$$+ P_{k_1+O(1)}Q_l[P_{k_1}Q_{<l-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

For the first term, estimate

$$||\Box P_{k_1+O(1)}Q_l[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{l-10\leq.<k_4+20}\psi_3(.-y)]||_{\dot{X}^{\frac{1}{2},-\frac{1}{2},\infty}}$$

$$\lesssim 2^{\frac{l}{2}}\sum_{j\in[l-10,k_4+20]}||P_{k_1}Q_{<k_4-10}R_\alpha\psi_1||_{L_t^\infty L_x^2}2^{\frac{3k_3}{2}}2^{\frac{j-k_3}{2+}}||\nabla^{-1}P_{k_3,\kappa_3}Q_j\psi_3||_{L_t^2 L_x^2}$$

We have used here the 'improved Bernstein's inequality', see e. g. [24], [10]. This in turn we can bound by $\lesssim \sum_{j\in[l-10,k_4-10]} 2^{\frac{l-j}{2}}2^{\frac{j-k_3}{2+}}\prod_{i=1,2}||P_{k_i}\psi_i||_{S[k_i]}$. Summing over j and then over $l < k_4 + 20$, one easily obtains the gain $2^{\frac{k_4-k_2}{2+}}$, and from here the argument proceeds just as before. Next, we have

$$P_{k_1+O(1)}Q_l[P_{k_1}Q_{l-10\leq.<k_4-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

$$= \sum_{j\in[l-10,k_4-10]} P_{k_1+O(1)}Q_l[P_{k_1}Q_j R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

Now we can simultaneously localize both $P_{k_1}Q_j R_\alpha\psi_1$, $\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)$ to caps $\kappa'_{1,2}$ of size $\sim 2^{\frac{j-k_3}{2}}$, such that $\pm\kappa'_1, \pm\kappa'_2$ have angular separation $\lesssim 2^{\frac{j-k_3}{2}}$. Hence we can write

$$P_{k_1+O(1)}Q_l[P_{k_1}Q_j R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

$$= \sum_{\pm,\pm}\sum_{\kappa'_{1,2}\in K_{\frac{j-k_3}{2}}|\mathrm{dist}(\pm\kappa'_1,\pm\kappa'_2)\lesssim 2^{\frac{j-k_3}{2}}}$$

$$P_{k_1+O(1)}Q_l[P_{k_1,\kappa'_1}Q_j^\pm R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q_{<l-10}^\pm\psi_3(.-y)]$$

Of course for fixed $\kappa'_{1,2}$, there are only finitely many κ_3 for which this expression does not vanish. Then we can estimate

$$||\Box P_{k_1+O(1)}Q_l[P_{k_1,\kappa'_1}Q_j^\pm R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q_{<l-10}^\pm\psi_3(.-y)]||_{\dot{X}_{k_1}^{\frac{1}{2},-\frac{1}{2},\infty}}$$

$$\lesssim 2^{\frac{l}{2}}||P_{k_1,\kappa'_1}Q_j^\pm R_\alpha\psi_1||_{L_t^2 L_x^2}||\nabla^{-1}P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q_{<l-10}^\pm\psi_3||_{L_t^\infty L_x^\infty}$$

Using Bernstein's inequality for the 2nd factor, we estimate this by

$$\lesssim 2^{\frac{l-j}{2}}2^{\frac{j-k_3}{2}}|l-j|\prod_{i=1,2}||P_{k_1,\kappa'_1}\psi_1||_{S[k_1]}||P_{k_3,\kappa'_2}\psi_3||_{S[k_3]}$$

Now, for fixed j,l, one sums over $\kappa'_{1,2}$ as well as $\kappa_{2,3}$ (of which there are only finitely many for fixed $\kappa'_{1,2}$), and finally sums over j,l in the appropriate ranges, to obtain the desired estimate, just as in the case $l \geq k_4 + 30$.

The expression

$$P_{k_1+O(1)}Q_l[P_{k_1}Q_{<l-10}R_\alpha\psi_1\nabla^{-1}P_{k_3,\kappa_3}Q_{<l-10}\psi_3(.-y)]$$

is handled similarly. Here one localizes the inputs to caps $\kappa'_{1,2}$ which are separated, and reasons as in the case $l \geq k_4$.

This concludes case (iv).

(v): $P_{k_4}(\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2 \Box [\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3] P_{k_1} R_\alpha Q_{<k_4-10}\psi_1$. This is much simpler to estimate: use

$$||P_{k_4}(\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2 \Box [\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3] P_{k_1} R_\alpha Q_{<k_4-10}\psi_1||_{L_t^1 L_x^2}$$
$$\lesssim ||\Box[\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3]||_{L_t^2 L_x^2} ||\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2||_{L_t^4 L_x^\infty} ||P_{k_1}R_\alpha Q_{<k_4-10}\psi_1||_{L_t^4 L_x^\infty}$$

This can be estimated by $2^{\frac{k_4}{2}-\frac{k_2}{4}} \prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$, which is better than what we need.

(vi): This is the expression

$$\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2 \nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3 \Box P_{k_1}R_\alpha Q_{<k_4-10}\psi_1.$$

This is again straightforward: we estimate it by

$$||\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2 \nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3 \Box P_{k_1}R_\alpha Q_{<k_4-10}\psi_1||_{L_t^1 L_x^2}$$
$$\lesssim ||\nabla^{-1}P_{k_2}Q_{<k_4+20}\psi_2||_{L_t^4 L_x^\infty} ||\nabla^{-1}P_{k_3}Q_{<k_4+20}\psi_3||_{L_t^4 L_x^\infty} ||\Box P_{k_1}R_\alpha Q_{<k_4-10}\psi_1||_{L_t^2 L_x^2},$$

and the last expression is easily estimated by $\lesssim 2^{\frac{k_4-k_2}{2}} \prod_{i=1}^3 ||P_{k_i}\psi_i||_{S[k_i]}$.

Now, we assume $k_2 > 20$. Here cases (i), (ii), (iii), (v), (vi) are handled in identical fashion. For (iv), we now have the identity

$$[P_{k_1}Q_{k_4-10}R_\alpha\psi_1 P_{k_3}Q_{<k_4+20}\psi_3(.-y)]]$$
$$= P_{k_3+O(1)}Q_{<k_1+O(1)}[P_{k_1}Q_{k_4-10}R_\alpha\psi_1 P_{k_3,\kappa_3}Q_{<k_4+20}\nabla^{-1}\psi_3(.-y)]]$$

Freeze the modulation to size 2^l. Then we need to distinguish between the cases $l \geq k_4$, $l < k_4$. The calculations are essentially the same, so we outline how to deal wit the first case. Now one writes

$$P_{k_3+O(1)}Q_l[P_{k_1}Q_{k_4-10}R_\alpha\psi_1 P_{k_3}Q_{<k_4+20}\nabla^{-1}\psi_3(.-y)]]$$
$$= \sum_{\substack{\kappa'_{1,2}\in K_{\frac{l-k_1}{2}} |\text{dist}(\pm\kappa'_1,\pm\kappa'_2)\sim 2^{\frac{l-k_1}{2}}}}$$
$$P_{k_3+O(1)}Q_l[P_{k_1,\kappa'_1}Q^{\pm}_{k_4-10}R_\alpha\psi_1 P_{k_3,\kappa'_2}P_{k_3,\kappa_3}Q^{\pm}_{<k_4+20}\nabla^{-1}\psi_3(.-y)]]$$

From this point on the proof is an exact replica of the case $k_3 < -20$.
The case $k_2 \in [-20, 20]$ is also essentially the same, the only difference being that now in case (iv), the term $[P_{k_1}Q_{<k_4-10}R_\alpha\psi_1 P_{k_3}Q_{<k_4+20}\nabla^{-1}\psi_3(.-y)]]$ may have very small frequency. This is handled similarly, see e. g. [11], where such an argument is carried out in detail.

(4) *The 2nd null-form.* This is handled just like the first null-form, the only difference being that one uses the identity

$$\sum_{j=1}^{3} \triangle^{-1}\partial_j\partial^{\nu}[R_{\nu}fR_jg - R_jfR_{\nu}g]h$$

$$= \Box[\sum_{j=1}^{3}\triangle^{-1}\partial_j(\nabla^{-1}fR_jg)]h - \frac{1}{2}\Box(\nabla^{-1}f)\nabla^{-1}gh$$

$$- \frac{1}{2}\Box[\nabla^{-1}f\nabla^{-1}g]h + \frac{1}{2}\Box(\nabla^{-1}g)\nabla^{-1}fh$$

These can be handled just as the terms in the identity underlying **(3)**.

The remaining expressions in the statement of Proposition 4.6 can be estimated in the same way. $\qquad\square$

We next state the following Proposition needed to handle the quintilinear and higher order terms:

PROPOSITION 4.9. *We have the bounds*

$$||P_k\nabla_{x,t}[P_{k_1}\psi_1 P_{r_1}\nabla^{-1}[P_{r_2}\nabla^{-1}[P_{k_2}\psi_2\nabla^{-1}P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]P_{k_5}R_{\nu}\psi_5]]||_{N[k]}$$

$$\lesssim 2^{-\delta[|k-k_1|+|k_3-k_4|]}2^{\delta[\min\{k_{2,5},r_{1,2,3}\}-\max\{k_{2,5},r_{1,2,3}\}]}\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]}$$

for suitable $\delta > 0$. Analogous bounds hold for the other higher order terms in (2.3):

$$||P_k\nabla_{x,t}[P_{r_1}\nabla^{-1}[P_{k_1}\psi_1 P_{r_2}\nabla^{-1}[P_{k_2}\psi_2 P_{k_3}\psi_3]]P_{r_3}\nabla^{-1}[P_{k_4}\psi_4 P_{k_5}\psi_5]]||_{N[k]}$$

$$\lesssim 2^{-\delta[|k_4-k_5|+|k_2-k_3|]}2^{\delta[\min\{r_{1,3},k\}-\max\{r_{1,3},k\}]}2^{\delta[\min\{r_{1,2},k_1\}-\max\{r_{1,2},k_1\}]}$$

$$\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]}$$

PROOF. We show the first estimate, the 2nd following in a similar vein. This is in principle straightforward given Lemma 4.1, the only difficulty being the operator $\nabla_{x,t}$ in front. To deal with it, assume the output is at frequency ~ 1 (as we may by scale invariance) as well as modulation 2^l, $l >> 1$. Thus this is the expression

$$P_0 Q_l \nabla_{x,t}[P_{k_1}\psi_1 P_{r_1}\nabla^{-1}[P_{r_2}\nabla^{-1}[P_{k_2}\psi_2\nabla^{-1}P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]P_{k_5}R_{\nu}\psi_5]]$$

Then either $k_1 \geq l - 10$, or $P_{k_1}\psi_1$ has modulation $\geq 2^{l-10}$, or else $P_{r_1}\nabla^{-1}\dots$ has modulation $\geq 2^{l-10}$. In the last case, repeat the same argument with

$$P_{r_1}\nabla^{-1}[P_{r_2}\nabla^{-1}[P_{k_2}\psi_2\nabla^{-1}P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]P_{k_5}R_{\nu}\psi_5]$$

The conclusion is that at least one of the inputs needs to have frequency at least comparable to 2^l, or else at least one input needs to have modulation at least comparable to 2^l. Note that we can organize the expression as

$$\nabla_{x,t}P_0 Q_l[P_{k_1}\psi_1 P_{r_1}\nabla^{-1}[XP_{k_5}R_{\nu}\psi_5]]$$

with

$$X = P_{r_2}\nabla^{-1}[P_{k_2}\psi_2\nabla^{-1}P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]$$

Then use the decomposition

$$\nabla_{x,t} P_0 Q_l [P_{k_1} \psi_1 P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]$$

$$= \chi_{k_1 \geq l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} \psi_1 P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]$$

$$+ \chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} Q_{\geq l-10} \psi_1 P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]$$

$$+ \chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} Q_{<l-10} \psi_1 P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]$$

To estimate the first and 2nd terms, we use the following

LEMMA 4.10. *We have the bounds*

$$||X||_{L_t^2 \dot{H}_x^1 \cap L_t^2 \dot{W}_x^{\infty, -\frac{1}{2}} \cap L_t^\infty \dot{H}_x^{\frac{1}{2}}} \lesssim 2^{\delta[\min\{r_{2,3}, k_{2,3,4}\} - \max\{r_{2,3}, k_{2,3,4}\}]} \prod_{i=2,3,4} ||P_{k_i} \psi_i||_{S[k_i]}$$

PROOF. (Lemma 4.10) simple application of Bernstein's inequality. □

Then we compute (for suitable $\delta > 0$)

$$||\chi_{k_1 \geq l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} \psi_1 P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]||_{\dot{X}_0^{-\frac{1}{2}, -\frac{1}{2}, \infty}}$$

$$\lesssim 2^{\frac{1}{2} l} ||P_{k_1} \psi_1||_{L_t^\infty L_x^2} ||P_{k_1 + O(1)} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]||_{L_t^2 L_x^2}$$

$$\lesssim 2^{\delta[\min\{r_{2,3}, k_{2,3,4}\} - \max\{r_{2,3}, k_{2,3,4}\}]} \prod_{i=2,3,4} ||P_{k_i} \psi_i||_{S[k_i]}$$

$$\times 2^{\frac{l}{2}} 2^{-k_1} 2^{\delta[\min\{k_1, r_2, k_5\} - \max\{k_1, r_2, k_5\}]} \prod_{i=1,5} ||P_{k_i} \psi_i||_{S[k_i]}$$

where we have again invoked Bernstein's inequality a number of times in the last step. This is of course more than enough to get the estimate of Proposition 4.9 in this case. Next, estimate

$$||\chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} Q_{\geq l-10} \psi_1 P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]||_{N[0]}$$

$$\lesssim 2^{\frac{l}{2}} \min\{||P_{k_1} Q_{\geq l-10} \psi_1||_{L_t^2 L_x^2} ||P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]||_{L_t^\infty L_x^\infty + L_t^\infty L_x^2},$$

$$||P_{k_1} Q_{\geq l-10} \psi_1||_{L_t^2 L_x^\infty} |P_{r_1} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]||_{L_t^\infty L_x^2}\}$$

Again, one checks this is bounded by an expression as in the Proposition. Finally, for the last term, use a similar decomposition

$$P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]$$

$$= \chi_{k_5 \geq l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]$$

$$+ \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu Q_{\geq l-20} \psi_5]$$

$$+ \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1} [Q_{\geq l-20} X P_{k_5} R_\nu Q_{<l-20} \psi_5]$$

For the first term here, use that

$$||\chi_{k_5 \geq l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]||_{L_t^2 L_x^2} \lesssim 2^{\frac{r_1}{2}} ||X||_{L_t^2 L_x^2} ||R_\nu \psi_5||_{L_t^\infty L_x^2}$$

and from here one concludes

$$||\chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l [P_{k_1} Q_{<l-10} \psi_1 \chi_{k_5 \geq l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1} [X P_{k_5} R_\nu \psi_5]]||_{N[0]}$$

$$\lesssim 2^{\frac{l}{2}} ||P_{k_1} Q_{<l-10} \psi_1||_{L_t^\infty L_x^\infty} ||P_{k_5 + O(1)} X||_{L_t^2 L_x^2} ||P_{k_5} R_\nu \psi_5||_{L_t^\infty L_x^2}$$

$$\lesssim 2^{\frac{l}{2} + k_1 - \frac{3}{2} k_5} 2^{\delta(\min\{r_{2,3}, k_{2,3,4}\} - \max\{r_{2,3}, k_{2,3,4}\})} \prod_{i=i}^{5} ||P_{k_i} \psi_i||_{S[k_i]}$$

Further, write

$$\chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1}[X P_{k_5} R_\nu Q_{\geq l-20} \psi_5]$$
$$= \sum_{l_1 \geq l-20} \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1}[X P_{k_5} R_\nu Q_{l_1} \psi_5]$$

Then we get

$$||\sum_{l_1 \geq l-20} \chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l[P_{k_1} Q_{<l-10}\psi_1 \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10}$$
$$\nabla^{-1}[X P_{k_5} Q_{l_1} R_\nu \psi_5]]||_{N[0]}$$
$$\lesssim \sum_{l_1 \geq l-20} 2^{\frac{1}{2}(l-l_1)} 2^{\delta(\min\{k_{1,2,3,4,5}, r_{1,2,3}\} - \max\{k_{1,2,3,4,5}, r_{1,2,3}\})} \prod_{i=1}^5 ||P_{k_i} \psi_i||_{S[k_i]}$$

for suitable $\delta > 0$. Finally, consider the contribution of

$$\chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1}[Q_{\geq l-20} X P_{k_5} R_\nu Q_{<l-20} \psi_5]$$

Here we use

LEMMA 4.11. *The following estimate holds:*

$$||Q_{l_1} X||_{L_t^2 L_x^2} \lesssim 2^{-\frac{1}{2}l_1} 2^{-\frac{1}{2}r_2} 2^{\delta(\min\{r_2, r_3, k_{2,3,4}\} - \max\{r_2, r_3, k_{2,3,4}\})} \prod_{i=2,3,4} ||P_{k_i} \psi_i||_{S[k_i]}$$

for suitable $\delta > 0$.

PROOF. (Lemma 4.11) Use reasoning as in Lemma 4.8, as well as above.   □

Using this, we can estimate

$$||\chi_{k_1 < l-100} \nabla_{x,t} P_0 Q_l[P_{k_1} \psi_1 \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10} \nabla^{-1}[Q_{\geq l-20} X P_{k_5} R_\nu Q_{<l-20} \psi_5]||_{N[0]}$$
$$\leq \sum_{l_1 \geq l-20} 2^{\frac{l}{2}} ||\chi_{k_1 < l-100} P_0 Q_l[P_{k_1} \psi_1 \chi_{k_5 < l-100} P_{r_1} Q_{\geq l-10}$$
$$\nabla^{-1}[Q_{l_1} X P_{k_5} R_\nu Q_{<l-20} \psi_5]||_{L_t^2 L_x^2}$$
$$\lesssim \sum_{l_1 \geq l-20} 2^{\frac{1}{2}(l-l_1)} 2^{\delta[\min\{k_{1,2,3,4,5}, r_{1,2,3}\} - \max\{k_{1,2,3,4,5}, r_{1,2,3}\}]} \prod_{i=1}^5 ||P_{k_i} \psi_i||_{S[k_i]}$$

for suitable $\delta > 0$. This follows as usual via various frequency trichotomies as well as Bernstein's inequality. This concludes dealing with the large output-at-large-modulation-case, and we now need to focus on

$$\nabla_{x,t} P_0 Q_{<20}[P_{k_1}\psi_1 P_{r_1} \nabla^{-1}[P_{r_2} \nabla^{-1}[P_{k_2}\psi_2 \nabla^{-1} P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]P_{k_5} R_\nu \psi_5]]]$$

This we do via the following

LEMMA 4.12. *We have the estimate*

$$2^{(\frac{3}{4} + \frac{3}{p} - 1)r_2} ||P_{r_2} \nabla^{-1}[P_{k_2}\psi_2 \nabla^{-1} P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]||_{L_t^{\frac{4}{3}} L_x^p}$$
$$\lesssim 2^{\delta[\min\{k_2, r_{2,3}\} - \max\{k_2, r_{2,3}\}]} 2^{-\delta|k_3 - k_4|} \prod_{i=2,3,4} ||P_{k_i} \psi_i||_{S[k_i]}$$

for suitable $\delta > 0$ and $p \geq 2$.

PROOF. (Lemma 4.12) Follows from Lemma 4.2 and some simple frequency trichotomies and Bernstein's inequality.   □

From here one easily concludes

$$||\nabla_{x,t}P_0Q_{<20}[P_{k_1}\psi_1 P_{r_1}\nabla^{-1}[P_{r_2}\nabla^{-1}[P_{k_2}\psi_2\nabla^{-1}P_{r_3}[P_{k_3}\psi_3 P_{k_4}\psi_4]]P_{k_5}R_\nu\psi_5]]]||_{L_t^1\dot{H}_x^{-\frac{1}{2}}}$$

$$\lesssim 2^{-\delta[|k_1|+|k_3-k_4|]}2^{\delta[\min\{k_{2,5},r_{1,2,3}\}-\max\{k_{2,5},r_{1,2,3}\}]}\prod_{i=1}^{5}||P_{k_i}\psi_i||_{S[k_i]},$$

again choosing $\delta > 0$ small enough. This concludes the proof of Proposition 4.9 for the first inequality. The 2nd follows similar reasoning. $\square$

We now continue with the proof of Proposition 4.5, where we need to establish the last 3 inequalities for the iterates $\psi_{\alpha,1,j}$. This only requires some additional work for the terms involving the vector fields Γ^β. To get estimates on $\Gamma^\beta\psi_{\alpha,1,j}$, we need commute the vector fields Γ^β inside the source terms on the right hand side of (2.6). To do this, we require the following simple

LEMMA 4.13. *The following commutator bounds obtain for any translation invariant norm* $||.||_S$:

$$||[\Gamma^\beta,\square^{-1}P_kQ_{\geq k}]\psi||_S \lesssim 2^{-2k}2^{-|\beta|k}||\nabla_{x,t}^{|\beta|}P_k\psi||_S + (|\beta|-1)$$
$$\sum_{|\gamma|=|\beta|-1}2^{-3k}||\nabla_{x,t}\Gamma^\gamma P_k\psi||_S$$

$$||[\Gamma^\beta,P_kQ_{<k+\delta l}]\psi||_S \lesssim 2^{-|\beta|k}||\nabla_{x,t}^{|\beta|}P_k\psi||_S + (|\beta|-1)\sum_{|\gamma|=|\beta|-1}2^{-k}||\nabla_{x,t}\Gamma^\gamma P_k\psi||_S$$

First considering the case $|\beta|=1$, note that $\square^{-1}P_kQ_{\geq k}$ is given by convolution with a function $a(t,x)$ with the property $||a(t,x)||_{L_{t,x}^1(\mathbb{R}^{3+1})} \lesssim 2^{-2k}$. Then we have

$$[\Gamma,\square^{-1}P_kQ_{\geq k}] = \Gamma_{t,x}\int_{\mathbb{R}^{3+1}}a(t-s,x-y)\psi(s,y)dsdy$$

$$-\int_{\mathbb{R}^{3+1}}a(t-s,x-y)\Gamma_{s,y}\psi(s,y)dsdy$$

Taking into account that Γ is of the form $\partial_t, \partial_{x_i}, x_i\partial_{x_j}-x_j\partial_{x_i}, t\partial_t+\sum_{i=1}^3 x_i\partial_{x_i}, x_i\partial_t+t\partial_{x_i}$, we see the above integral can be written in the form

$$\int_{\mathbb{R}^{3+1}}sa(s,y)\nabla_{t,x}\psi(t-s,x-y)dsdy + \int_{\mathbb{R}^{3+1}}ya(s,y)\nabla_{t,x}\psi(t-s,x-y)dsdy$$

Then conclude by observing

$$||sa(s,y)||_{L_{s,y}^1} + |||y|(s,y)||_{L_{s,y}^1} \lesssim 2^{-3k}$$

For higher order operators, use

$$[\Gamma_1\Gamma_2,\square^{-1}P_kQ_{\geq k}] = \Gamma_1[\Gamma_2,\square^{-1}P_kQ_{\geq k}] + [\Gamma_1,\square^{-1}P_kQ_{\geq k}]\Gamma_2,$$

as well as the fact that $[\Gamma_2,\square^{-1}P_kQ_{\geq k}]$ can be written as a convolution operator whose kernel enjoys similar properties to the one associated with $\square^{-1}P_kQ_{\geq k}$. The proof or $[\Gamma^\beta,P_kQ_{<k+\delta l}]$ is similar. $\square$

With this in hand, observe that we have

$$\square[\Gamma\psi_{\alpha,1,j+1}] = [\square,\Gamma]\psi_{\alpha,1,j+1} + \Gamma\square[\eta_T(t)S(t)(\psi_\alpha[0])]$$
$$+\Gamma\square[\int_0^t \eta_T(t-s)\mathbf{P}_{[-\delta l,\delta l]}F(\psi_{1,j})(s)ds]$$

Then clearly we have

$$\Box[\Gamma\psi_{\alpha,1,j+1}]|_{[-T,T]} = [\Box,\Gamma]\psi_{\alpha,1,j+1}|_{[-T,T]} + \Gamma\mathbf{P}_{[-\delta l,\delta l]}F(\psi_{1,j})|_{[-T,T]}$$

In order to estimate the right hand side, and in particular the right most term, observe that application of Γ does leave the null-structure (4.1) intact: indeed, $[\Box,\Gamma] = c\Box$ for each vector field Γ and a suitable number $c = c_\Gamma$. Re-iterating the proofs of Proposition 4.6, 4.9, and applying the preceding Lemma, we can conclude that

$$||\Gamma\mathbf{P}_{[-\delta l,\delta l]}F(\psi_{1,j})|_{[-T,T]}||_{N[k]([-T,T]\times\mathbb{R}^3)} \lesssim \epsilon\tilde{M}2^{C\delta l}c_k,$$

provided we make the bootstrap assumption $||P_k\Gamma\psi_1||_{S[k]([-T,T]\times\mathbb{R}^3)} \lesssim \tilde{M}2^{C\delta l}c_k$. The claim of the Proposition follows when $|\beta| = 1$ for $\psi_{\alpha,1,j+1}$, by using the energy inequality Theorem 4.4, and one argues similarly for $\Gamma^\beta\psi_{\alpha,1}$, $|\beta| = 2$, etc. This concludes the proof of Proposition 4.5.

Next, we explain the proof of Proposition 2.2:

PROOF. (Proposition 2.2) We shall again use Proposition 4.6, Proposition 4.9, as well as some elementary additional estimates. Fix a time scale $l >> 1$ and consider $t\psi_{\alpha,2}$ defined on $[-2^{-l+2}, 2^{l+2}]$ as in the preceding. The equations satisfied by the $\psi_{\alpha,2} = \psi_\alpha - \psi_{\alpha,1}$ is as follows:

$$\Box\psi_{\alpha,2} = F(\psi_1,\psi_2) + (1 - \mathbf{P}_{[-\delta l,\delta l]})F(\psi_1),\ \psi_{\alpha,2}$$

Here $F(\psi_1,\psi_2)$ is the right hand side of (2.3) but with ψ replaced by $\psi_{1,2}$, and the additional requirement that at least one copy of ψ_2 is present. We need to establish the following

Bootstrapping step: *Let $0 \le T \le 2^{l+2}$ and assume that we have*

$$||P_k\psi_{\alpha,2}||_{S[k]([-T,T]\times\mathbb{R}^3)} \le Me_k,\ \alpha = 0, 1, 2, 3$$

for some M larger than some $M_0 = M_0(\psi[0])$. Then we have

$$||P_k\psi_{\alpha,2}||_{S[k]([-T,T]\times\mathbb{R}^3)} \le \frac{M}{2}e_k,\ \alpha = 0, 1, 2, 3$$

We recall the definition of $e_k := \min\{d_k, 2^{\mu(\max\{k-\delta l, -k-\delta l\})}d_k\}$ for $\mu > 0$ sufficiently small. Having established the bootstrapping, Proposition 2.2 will then follow from the fact that (see e. g. [**11**])

$$\lim_{T\to 0}\sup_{k\in\mathbb{Z}}||P_k\psi_{\alpha,2}||_{S[k]([-T,T]\times\mathbb{R}^3)} = 0$$

We will show that the assumption above implies

$$||P_kF(\psi_1,\psi_2)||_{N[k]([-T,T]\times\mathbb{R}^3)} + ||P_k(1 - \mathbf{P}_{[-\delta l,\delta l]})F(\psi_1)||_{N[k]([-T,T]\times\mathbb{R}^3)} \lesssim \epsilon M^9 e_k,$$

which implies the bootstrapping step by invoking Theorem 4.4. We shall first prove the estimate

$$||P_kF(\psi_1,\psi_2)||_{N[k]([-T,T]\times\mathbb{R}^3)} \lesssim \epsilon M^9 e_k,$$

Recall that $P_k F(\psi_1, \psi_2)$ consists of trilinear null-forms as well as quintilinear and higher order terms. From Proposition 4.6, we have the schematic relation

$$||"P_k \nabla_{x,t} [P_{k_1} \psi_1 \nabla^{-1} P_{k_4} [P_{k_2} \psi_2 P_{k_3} \psi_3]]"||_{N[k]([-T,T] \times \mathbb{R}^3)}$$

$$\lesssim 2^{-\delta_1 |k_2 - k_3|} 2^{-\delta_2 [k_4 - \max\{k_2, k_3\}]} 2^{-\delta_3 |k-k_1|} \prod_{i=1}^{3} ||P_{k_i} \psi_i||_{S[k_i]([-T,T] \times \mathbb{R}^3)}$$

Here the term $"P_k \nabla_{x,t} \ldots"$ of course stands for one of the trilinear null-forms in (2.3). Now, we we think of the entries ψ_1, ψ_2, ψ_3 as either given by a $\psi_{\alpha,1}$ or a $\psi_{\alpha,2}$, with at least one occurence of $\psi_{\alpha,2}$. Then we have two possible cases:

(i): *at least one copy of $\psi_{\alpha,2}$ inside $\nabla^{-1} P_{k_4}[\ldots]$. Let this be $P_{k_2} \psi_{\alpha,2}$.*

This we estimate by

$$\lesssim 2^{-\delta_3 |k-k_1|} 2^{-\delta_1 |k_2 - k_3|} 2^{-\delta_2 |k_4 - \max\{k_2, k_3\}|} d_{k_1} e_{k_2} d_{k_3}$$

Now use that

$$e_{k_2} = \min\{d_{k_2}, 2^{\mu \max\{k_2 - \delta l, -k_2 - \delta l\}} d_{k_2}\} \lesssim 2^{-\mu \delta l}$$

since we have $d_k \lesssim 2^{-\sigma |k|}$ for some $\sigma > 0$, and we assume that μ is chosen small enough. Then we have

$$\sum_{k_1 \in \mathbb{Z}} 2^{-\mu \delta l} 2^{-\delta_3 |k-k_1|} d_{k_1} \lesssim e_k$$

We can now infer that

$$\sum_{k_{1,2,3} \in \mathbb{Z}} 2^{-\delta_3 |k-k_1|} 2^{-\delta_1 |k_2 - k_3|} 2^{-\delta_2 |k_4 - \max\{k_2, k_3\}|} d_{k_1} e_{k_2} d_{k_3} \lesssim \epsilon e_k$$

(ii): *The first input $P_{k_1} \psi_1$ is $P_{k_1} \psi_{\alpha,2}$.* Here we infer the desired bound directly since the e_k form a frequency envelope.

The remaining terms in $F(\psi_1, \psi_2)$ if higher degree of multilinearity are treated similarly and omitted here.

Now we turn to estimating the term

$$P_k (1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)$$

Recalling the definition of $-\mathbf{P}_{[-\delta l, \delta l]}$, we see that in the expressions constituting $(1 - \mathbf{P}_{[-\delta l, \delta l]}) F(\psi_1)$ either at least one frequency 2^k satisfies $|k| \geq \delta l$, or else at least one operator $P_k Q_{\geq k+\delta l}$ occurs. We need to show that this implies an extra exponential gain in l. We do this here for the trilinear terms, the higher order terms being treated similarly: first, again using the schematic notation

$$"P_k \nabla_{x,t} [P_{k_1} \psi_1 \nabla^{-1} P_{k_4} [P_{k_2} \psi_2 P_{k_3} \psi_3]]",$$

assume that one of $k_{1,2,3,4}$ is of large absolute size. Then reasoning exactly as before in (i), (ii), implies

$$||"P_k \nabla_{x,t} [P_{k_1} \psi_1 \nabla^{-1} P_{k_4} [P_{k_2} \psi_2 P_{k_3} \psi_3]]"||_{N[k]} \lesssim e_k$$

Further, we have

Lemma 4.14. *The following estimates hold, where we again use schematic notation:*

$$||"P_kQ_{>k+\delta l}\nabla_{x,t}[P_{k_1}\psi_1\nabla^{-1}P_{k_4}[P_{k_2}\psi_2P_{k_3}\psi_3]]"||_{N[k]} \lesssim 2^{-\nu l}d_k$$

$$||"P_k\nabla_{x,t}[P_{k_1}Q_{>k_1+\delta l}\psi_1\nabla^{-1}P_{k_4}[P_{k_2}\psi_2P_{k_3}\psi_3]]"||_{N[k]} \lesssim 2^{-\nu l}d_k$$

$$||"P_k\nabla_{x,t}[P_{k_1}\psi_1\nabla^{-1}P_{k_4}[P_{k_2}Q_{>k_2+\delta l}\psi_2P_{k_3}\psi_3]]"||_{N[k]} \lesssim 2^{-\nu l}d_k$$

$$||"P_k\nabla_{x,t}[P_{k_1}\psi_1\nabla^{-1}P_{k_4}Q_{>k_4+\delta l}[P_{k_2}\psi_2P_{k_3}\psi_3]]"||_{N[k]} \lesssim 2^{-\nu l}d_k$$

Here $\nu > 0$ is a constant depending on δ but not on μ.

Proof. (Lemma 4.14). All of these follow essentially directly from the proof of Proposition 4.6. □

The preceding Lemma completes our treatment of the trilinear terms in $(1 - \mathbf{P}_{[-\delta l,\delta l]})F(\psi_1)$ since $2^{-\nu l}d_k \leq e_k$ for μ (in the definition of e_k) small enough. The quintilinear etc terms in $(1 - \mathbf{P}_{[-\delta l,\delta l]})F(\psi_1)$ are treated similarly. □

Proof. (Proposition 2.3) This is contained in the preceding proof. □

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