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Existence of Global Attractors for Wave Equation of Kirchhoff Type with Nonlinear Damping and Memory Term at Boundary

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Abstract. In this paper, we prove that the existence of global attractors in the phase space $\mathcal{H}_0 = H^1(\Omega) \times L^2(\Omega)$ for wave equation of Kirchhoff type with nonlinear dissipation and memory term at boundary. To this end, we first obtain an bounded absorbing set by the perturbed energy method (see Zuazua [**9, 22**], combined with techniques from Munoz Revera [**13**]). Then we utilize an especial method of decompose to verity the asymptotic compactness for the problems.

Contents

1. Introduction

In this paper, our main purpose is to study the existence of global attractors for a nonlinear wave equation of Kirchhoff type with nonlinear damping and memory term at boundary. To formalize this problem let us take Ω a open bounded set of R^n with smooth boundary Γ and let us assume that Γ can be divided into two non-null parts

$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \overline{\Gamma_0} \cap \overline{\Gamma_1} = \phi,$$

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and Γ_0, Γ_1 have positive measure. Let us denote by $\nu(x)$ the unit normal vector at $x \in \Gamma$ outside of Ω and let us consider the following initial boundary value problems:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \alpha\Delta u_{tt} - \Delta u_t + f(u) = h(x) \quad in \quad \Omega \times (0,\infty), \tag{1.1}$$

$$u = 0 \quad on \quad \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$M(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_{tt}}{\partial \nu} + \frac{\partial u_t}{\partial \nu} + u + u_t + g(t)|u_t|^\rho u_t = g * |u|^\gamma u \quad on \quad \Gamma_1 \times (0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad in \quad \Omega. \tag{1.4}$$

Here, $\alpha > 0$, $h(x) \in L^2(\Omega)$, $\|\nabla u\|^2 = \sum_{i=1}^{n} \int_\Omega |\frac{\partial u}{\partial x_i}(x)|^2 dx$, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, $g * u = \int_0^t g(t-r)u(r)dr$ and

$$0 < \gamma, \rho \le \frac{1}{n-2} \quad if \quad n \ge 3 \quad or \quad \gamma, \rho > 0 \quad if \quad n = 1, 2. \tag{1.5}$$

Problems (1.1)-1.4) has its origin in the mathematical description of small amplitude vibrations of an elastic string [**15**]. In fact, a mathematical model for the deflection of an elastic string of length $L > 0$ is given by the mixed problem for the nonlinear wave equation

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} \quad for \quad 0 < x < L, t \ge 0, \tag{1.6}$$

where u is the lateral deflection, x the space coordinate, t the time, E the Young's modulus, ρ the mass density, h the cross section area and p_0 the initial axial tension. Kirchhoff was the first to introduce (1.6) in the study of oscillations of stretched strings and plates, so that (1.6) is called the wave equation of Kirchhoff type after him. There is an extensive literature on the study of wave equation of Kirchhoff type. For example, the existence of global solutions and exponential decay to problem (1.1) and (1.2) with $\partial\Omega = \Gamma_0$ and $f(u) = h(x) \equiv 0$ has been investigated by many authors [**12, 14**].

On the other hand, there exists a large body of literature regarding viscoelastic problems with the memory term acting in the domain or in the boundary. Among the numerous works in this direction, we can cite Santos [**19, 20**]. Cavalcanti et al. [**2, 3**] studied the existence and uniform decay of strong solutions of wave equation with nonlinear boundary damping and memory source term. Park et al.[**17**] have been investigated the existence and uniform decay of the solutions of (1.1)-(1.4) with $M(s) = 1 + s$ and $f(u) = h(x) = \alpha \equiv 0$. While Bae et al.[**1**] have been obtained the same results for problems (1.1)-(1.4) with $M(s) = 1 + s$ and $f(u) = h(x) \equiv 0$.

As far as dynamic is concerned the existence of the global attractors for wave equations with nonlinear damping and memory term acting in the domain is study by authors(for example, see[**4, 8, 18**] and the references therein). Also, there exist some literature considering the same questions for nonlinear damping acting on the boundary (for example, see[**5, 6, 7, 10**]). It is important to mention that Papadopoulos and Stavrakakis [**16**] have been investigated the existence of global for nonlinear Kirchhoff equation on R^N.

In the present paper, our main goal is to show the existence of global attractors for problem (1.1)-(1.4). It is well known that the proof of the existence of a compact global attractor for the semigroup will follow the traditional scheme (see, for example, [**11, 21**]). A compact global attractor exists if and only if the semigroup has a bounded absorbing set and is asymptotically compact. To obtain bounded absorbing set we used the perturbed energy method, see Zuazua [**9, 22**], combined with techniques from Munoz Revera [**13**]. Generally, one can obtain asymptotically compact by decompose the solution operator into a compact part and a asymptotically small part. However, there are additional difficulties when proving asymptotically compact because $M(\|\nabla u\|^2)$ is nonlinear. To overcome the difficulty, we utilize a especial method of decompose to verity the asymptotic compactness for problem (1.1)-(1.4).

2. Preliminaries and main result

Now let us introduce the functional spaces. Let

$$V := \{u \in H^1(\Omega); \quad u = 0 \quad on \quad \Gamma_0\},$$

which equipped with the topology given by the norm $\|\nabla \cdot\|_{L^2(\Omega)}$ is a Hilbert subspace of $H^1(\Omega)$. We denote

$$(u, v) := \int_\Omega uv dx, \quad \|u\|^2 = \int_\Omega |u|^2 dx,$$
$$(u, v)_{\Gamma_1} := \int_{\Gamma_1} uv d\Gamma, \quad \|u\|^p_{p,\Gamma_1} = \int_{\Gamma_1} |u|^p d\Gamma, \quad \|u\|^2_{\Gamma_1} = \|u\|^2_{2,\Gamma_1},$$

and let

$$\mathcal{H}_0 = H^1(\Omega) \times H^1(\Omega), \quad \|(u, u_t)\|^2_{\mathcal{H}_0} = \|\nabla u\|^2 + \|\nabla u_t\|^2 + \|u_t\|^2.$$

Let $\lambda_\Omega > 0$ and $\lambda_\Gamma > 0$ are two constants such that for $\forall v \in V$

$$\|v\| \le \lambda_\Omega \|\nabla v\|, \quad \|v\|_{\Gamma_1} \le \lambda_\Gamma \|\nabla v\|.$$

Let us make some hypothesis for function f, M and g:

Hyp.f *The function $f \in C^1(R)$ satisfy*

$$f(0) = 0, \tag{2.1}$$

and f is super-linear, that is

$$f(s)s \ge (2 + \delta)F(s), F(s) + K_0 \ge 0, \quad F(s) = \int_0^s f(z)dz, \quad \forall s \in R, \tag{2.2}$$

for some $\delta > 0$ and $K_0 > 0$ with the following growth condition:

$$|f(x) - f(y)| \le K_1(1 + |x|^{\mu-1} + |y|^{\mu-1})|x - y|, \quad \forall x, y \in R, \tag{2.3}$$

for some $K_1 > 0$ and $\mu \ge 1$ such that $(n - 2)\mu \le n$.

Hyp.M *The function $M \in C^1([0, \infty))$ satisfy*

$$M(\lambda) \ge m_0 > 0, \quad M(\lambda)\lambda \ge \widehat{M}(\lambda), \quad \forall \lambda \ge 0, \tag{2.4}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$.

Hyp.g *The function $g \in W^{1,\infty}(0, \infty) \cap W^{1,1}(0, \infty)$; $g(t) \ge 0$, $\forall t \ge 0$ such that*

$$\begin{aligned} -a_0 g(t) \leq g'(t) \leq -a_1 g(t), \quad \forall t \geq t_0, \\ g(0) = 0, \quad |g'(t)| \leq a_2 g(t), \quad t \in [0, t_0], \\ 1 - \int_0^\infty g(s) ds > 0. \end{aligned} \tag{2.5}$$

for some constants a_0*,* a_1*,* $a_2 > 0$*.*

We define the energy $e(t)$ of problems (1.1)-(1.4):

$$\begin{aligned} e(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{\alpha}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) \\ + \int_\Omega F(u(t)) dx + \frac{1}{2}\|u(t)\|^2_{\Gamma_1}. \end{aligned} \tag{2.6}$$

Applying almost the same argument as that used to in [**1, 2, 3, 17**] we can show the well-posedness of problems (1.1)-(1.4):

Lemma 2.1. *Assume that conditions (2.1)-(2.5) hold, if* $(u_0, u_1) \in H^1(\Omega) \times H^1(\Omega)$ *and* $\rho \geq \gamma$*, then there is only one solution* u *of system (1.1)-(1.4) satisfying*

$$u \in C(0, \infty; H^1(\Omega)), \quad u_t \in C(0, \infty; H^1(\Omega)). \tag{2.7}$$

Lemma 2.1 allows us to define the semigroup S_t. For every $t \geq 0$, we define the operator S_t mapping $\mathcal{H}_0$ into itself by

$$S_t : (u_0, u_1) \to (u(t), u_t(t)).$$

Now we are in position to state our main result:

Theorem 2.2. *Under the hypotheses of Lemma 2.1, if* $\rho = \gamma$*, then the semigroup* S_t *associated with problem (1.1)-(1.4) possesses a global attractor* $\mathcal{A}$ *in* $\mathcal{H}_0$*.*

Our paper is organized as follows: in section 3, we shall show the existence of absorbing set in $\mathcal{H}_0$, and in section 4, we shall show the asymptotic compactness for the semigroup S_t.

3. Absorbing set in $\mathcal{H}_0$

In this section we shall show that the semigroup S_t has a bounded absorbing set, i.e., a bounded set $B \subset \mathcal{H}_0$ satisfying the following condition: for any bounded $A \subset \mathcal{H}_0$ there exists $t(A) > 0$ such that $S_t A \subset B$ for all $t \geq t(A)$.

Theorem 3.1 *There exists a bounded absorbing set* B *for the semigroup* S_t *in* $\mathcal{H}_0$*.*

To this end, let us make some preliminaries. Firstly, let us denote by

$$(g\Box u)(t) := \int_0^t g(t-s)\Big\||u(s)|^\gamma u(s) - u(t)\Big\|_{\Gamma_1}^2 ds, \tag{3.1}$$

a simple computation gives us

$$\begin{aligned} (g\Box u)'(t) : \;&= \int_0^t g'(t-s)\Big\||u(s)|^\gamma u(s) - u(t)\Big\|_{\Gamma_1}^2 ds \\ &\quad + \Big(\frac{d}{dt}\|u(t)\|_{\Gamma_1}^2\Big)\int_0^t g(s)ds \\ &\quad -2\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s), u_t(t)\Big)_{\Gamma_1} ds \\ &= (g'\Box u)(t) + \frac{d}{dt}\Big(\|u(t)\|_{\Gamma_1}^2 \int_0^t g(s)ds\Big) - g(t)\|u(t)\|_{\Gamma_1}^2 \\ &\quad -2\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s), u_t(t)\Big)_{\Gamma_1} ds. \end{aligned}$$

Thus we get

$$\begin{aligned} &\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s), u_t(t)\Big)_{\Gamma_1} ds \\ &= -\frac{1}{2}(g\Box u)'(t) + \frac{1}{2}(g'\Box u)(t) \\ &\quad + \frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{\Gamma_1}^2 \int_0^t g(s)ds\Big) - \frac{1}{2}g(t)\|u(t)\|_{\Gamma_1}^2. \end{aligned} \tag{3.2}$$

Next we introduce the following modified energy:

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|^2 + \frac{\alpha}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\widehat{M}(\|\nabla u(t)\|^2) \\ &\quad + \int_\Omega F(u(t))dx + \frac{1}{2}\Big(1 - \int_0^t g(s)ds\Big)\|u(t)\|_{\Gamma_1}^2 \\ &\quad + \frac{1}{\gamma+2}g(t)\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} + \frac{1}{2}(g\Box u)(t), \end{aligned} \tag{3.3}$$

and let us define the perturbed modified energy, for every $\varepsilon > 0$,

$$E_\varepsilon(t) = E(t) + \varepsilon\psi(t), \tag{3.4}$$

where

$$\psi(t) = \frac{1}{2}\|\nabla u(t)\|^2 + \int_\Omega u(t)u_t(t)dx + \alpha\int_\Omega \nabla u(t)\cdot\nabla u_t(t)dx. \tag{3.5}$$

Applying Cauchy-Schwarz inequality and assumption (2.4), we get

$$\begin{aligned} |\psi(t)| &\leq \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\lambda_\Omega(\|\nabla u(t)\|^2 + \|u_t(t)\|^2) \\ &\quad + \frac{\alpha}{2}\|\nabla u(t)\|^2 + \frac{\alpha}{2}\|\nabla u_t(t)\|^2 \\ &\leq \frac{1}{2}\lambda_\Omega\|u_t(t)\|^2 + \frac{\alpha}{2}\|\nabla u_t(t)\|^2 \\ &\quad + \frac{1+\lambda_\Omega+\alpha}{2m_0}\widehat{M}(\|\nabla u(t)\|^2) \\ &\leq \lambda_0 E(t), \end{aligned}$$

where $\lambda_0 = \max\{\lambda_\Omega, \frac{1+\lambda_\Omega+\alpha}{m_0}, \alpha\}$. Thus we get

$$|E_\varepsilon(t) - E(t)| \leq \lambda_0 \varepsilon E(t), \quad \forall \varepsilon > 0, t \geq 0. \tag{3.6}$$

Lemma 3.2 *If $\rho = \gamma$, then there exist $C_1, C_2 > 0$ and $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1]$,*

$$\frac{d}{dt}E_\varepsilon(t) \leq -\varepsilon C_1 E(t) + C_2\|h\|^2. \tag{3.7}$$

Proof Multiplying equation (1.1) by u_t we get

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_\Omega h(x)u_t(t)dx - \|\nabla u_t(t)\|^2 - g(t)\|u_t(t)\|^{\rho+2}_{\rho+2,\Gamma_1} \\ &\quad + \frac{1}{\gamma+2}g'(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_1} + g(t)\Big(|u(t)|^\gamma u(t), u_t(t)\Big)_{\Gamma_1} \\ &\quad - \frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_1} - \|u_t(t)\|^2_{\Gamma_1} + \frac{1}{2}(g'\Box u)(t). \end{aligned}$$

(3.8)

Differentiating the equation (3.5) with respect to t and from equation (1.1)-(1.3) we get

$$\begin{aligned} \frac{d}{dt}\psi(t) &= \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \alpha\|\nabla u_t(t)\|^2 + \|u_t(t)\|^2 \\ &\quad + \int_\Omega u(t)\Big(\alpha\Delta u_{tt}(t) + M(\|\nabla u(t)\|^2)\Delta u(t) \\ &\quad + \Delta u_t(t) - f(u(t)) + h(x)\Big)dx + \alpha\int_\Omega \nabla u(t)\cdot\nabla u(t)dx \\ &= \|u_t(t)\|^2 + \alpha\|\nabla u_t(t)\|^2 - \int_\Omega u(t)f(u(t))dx \\ &\quad - M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 - \|u(t)\|^2_{\Gamma_1} - (u_t(t), u(t))_{\Gamma_1} \\ &\quad - \Big(g(t)|u_t(t)|^\rho u_t(t), u(t)\Big)_{\Gamma_1} + \int_\Omega h(x)u(t)dx \\ &\quad + \int_0^t g(t-s)\Big(|u(s)|^\gamma u(s), u(t)\Big)_{\Gamma_1} ds. \end{aligned}$$

(3.9)

From (3.4), (3.8) and (3.9) it follows that

$$
\begin{aligned}
\frac{d}{dt}E_\varepsilon(t) \;\; = \;\; & (\alpha\varepsilon - 1)\|\nabla u_t(t)\|^2 + \frac{1}{\gamma+2}g'(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_1} \\
& -g(t)\|u_t(t)\|^{\rho+2}_{\rho+2,\Gamma_1} + g(t)\Big(|u(t)|^\gamma u(t), u_t(t)\Big)_{\Gamma_1} \\
& -\frac{1}{2}g(t)\|u(t)\|^2_{\Gamma_1} - \|u_t(t)\|^2_{\Gamma_1} + \frac{1}{2}(g'\Box u)(t) \\
& + \int_\Omega h(x)u_t(t)dx - \varepsilon\int_\Omega u(t)f(u(t))dx \\
& -\varepsilon M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 + \varepsilon\int_\Omega h(x)u(t)dx \\
& +\varepsilon\|u_t(t)\|^2 - \varepsilon\Big(g(t)|u_t(t)|^\rho u_t(t), u(t)\Big)_{\Gamma_1} \\
& +\varepsilon\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s), u(t)\Big)_{\Gamma_1} ds \\
& -\varepsilon\|u(t)\|^2_{\Gamma_1} - \varepsilon(u_t(t), u(t))_{\Gamma_1}.
\end{aligned}
$$

(3.10)

We now majorize the right-hand side of (3.10). Firstly, using Schwarz's inequality and Young's inequality, we get the following five inequalities:

$$
\begin{aligned}
& g(t)\Big(|u(t)|^\gamma u(t), u_t(t)\Big)_{\Gamma_1} \\
\leq \;\; & g(t)\Big(\int_{\Gamma_1}|u(t)|^{\gamma+2}d\Gamma\Big)^{\frac{\gamma+1}{\gamma+2}}\Big(\int_{\Gamma_1}|u_t(t)|^{\gamma+2}d\Gamma\Big)^{\frac{1}{\gamma+2}} \\
\leq \;\; & \eta g(t)\|u_t(t)\|^{\gamma+2}_{\gamma+2,\Gamma_1} + \eta^{-\frac{1}{\gamma+1}}g(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_1},
\end{aligned}
$$

(3.11)

$$
\begin{aligned}
& -\varepsilon\Big(g(t)|u_t(t)|^\rho u_t(t), u(t)\Big)_{\Gamma_1} \\
\leq \;\; & \varepsilon\theta(\eta)g(t)\|u_t(t)\|^{\rho+2}_{\rho+2,\Gamma_1} + \varepsilon\eta g(t)\|u(t)\|^{\rho+2}_{\rho+2,\Gamma_1},
\end{aligned}
$$

(3.12)

$$
\begin{aligned}
\int_\Omega h(x)u_t(t)dx \;\; & \leq \;\; \lambda_\Omega\|\nabla u_t\|\|h\| \\
& \leq \;\; \alpha\varepsilon\|\nabla u_t\|^2 + \frac{\lambda_\Omega^2}{4\alpha\varepsilon}\|h\|^2,
\end{aligned}
$$

(3.13)

$$
\begin{aligned}
\varepsilon\int_\Omega h(x)u(t)dx \;\; & \leq \;\; \varepsilon m_0\eta\|\nabla u\|^2 + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta}\|h\|^2, \\
& \leq \;\; \varepsilon\eta M(\|\nabla u\|^2)\|\nabla u\|^2 + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta}\|h\|^2,
\end{aligned}
$$

(3.14)

and

$$\begin{aligned}-\varepsilon(u_t(t),u(t))_{\Gamma_1} &\leq \varepsilon m_0\eta\|\nabla u\|^2+\frac{\varepsilon\lambda_\Gamma^2}{4m_0\eta}\|u_t\|_{\Gamma_1}^2,\\ &\leq \varepsilon\eta M(\|\nabla u\|^2)\|\nabla u\|^2+\frac{\varepsilon\lambda_\Gamma^2}{4m_0\eta}\|u_t\|_{\Gamma_1}^2.\end{aligned}$$

(3.15)

On the other hand, applying Young's inequality and (3.2), we get

$$\begin{aligned}&\varepsilon\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s),u(t)\Big)_{\Gamma_1}ds\\ =\ &\varepsilon\int_0^t g(t-s)\Big(|u(s)|^\gamma u(s)-u(t),u(t)\Big)_{\Gamma_1}ds\\ &+\varepsilon\int_0^t g(t-s)ds\|u(t)\|_{\Gamma_1}^2\\ \leq\ &\frac{\varepsilon}{2}\int_0^t g(t-s)\Big\||u(s)|^\gamma u(s)-u(t)\Big\|_{\Gamma_1}^2ds\\ &+\frac{3\varepsilon}{2}\|u(t)\|_{\Gamma_1}^2\int_0^t g(s)ds\\ =\ &\frac{\varepsilon}{2}(g\Box u)(t)+\frac{3\varepsilon}{2}\|u(t)\|_{\Gamma_1}^2\int_0^t g(s)ds.\end{aligned}$$

(3.16)

Thus, from (3.10)-(3.16) and considering $\rho=\gamma$ we get

$$\frac{d}{dt}E_\varepsilon(t)\leq G_1(u), \tag{3.17}$$

where

$$\begin{aligned}G_1(u) =\ &(2\alpha\varepsilon+\varepsilon\lambda_\Omega^2-1)\|\nabla u_t(t)\|^2+(\frac{\varepsilon\lambda_\Gamma^2}{4m_0\eta}-1)\|u_t\|_{\Gamma_1}^2\\ &-\varepsilon\int_\Omega u(t)f(u(t))dx+(\frac{\lambda_\Omega^2}{4\alpha\varepsilon}+\frac{\varepsilon\lambda_\Omega^2}{4m_0\eta})\|h\|^2\\ &+(2\eta-1)\varepsilon M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2\\ &+(\eta+\varepsilon\theta(\eta)-1)g(t)\|u_t(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}\\ &+(\varepsilon\eta+\eta^{-\frac{1}{\gamma+1}}-\frac{a_1}{\gamma+2})g(t)\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2}\\ &+(\frac{\varepsilon}{2}-\frac{a_1}{2})(g\Box u)(t)-\frac{1}{2}g(t)\|u(t)\|_{\Gamma_1}^2\\ &-\varepsilon\|u(t)\|_{\Gamma_1}^2+\frac{3\varepsilon}{2}\int_0^t g(s)ds\|u(t)\|_{\Gamma_1}^2.\end{aligned}$$

(3.18)

On the other hand, for $C_1 = 2(1-2\eta)$, $\eta = 2^{-\frac{1}{\gamma+1}}$, we have

$$\begin{aligned} G_1(u) + C_1\varepsilon E(t) \quad = \quad & (\frac{\lambda_\Omega^2}{4\alpha\varepsilon} + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta})\|h\|^2 + (\frac{\varepsilon\lambda_\Gamma^2}{4m_0\eta} - 1)\|u_t\|_{\Gamma_1}^2 \\ & +(3\alpha\varepsilon + 2\varepsilon\lambda_\Omega^2 - 2\alpha\varepsilon\eta - 2\lambda_\Omega^2\eta\varepsilon - 1)\|\nabla u_t(t)\|^2 \\ & +\varepsilon\int_\Omega \Big(2(1-2\eta)F(u(t)) - u(t)f(u(t))\Big)dx \\ & +\varepsilon(1-2\eta)\widehat{M}(\|\nabla u(t)\|^2) \\ & +(2\eta-1)\varepsilon M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 \\ & +(\eta + \varepsilon\theta(\eta) - 1)g(t)\|u_t(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ & +(\varepsilon\eta + \eta^{-\frac{1}{\gamma+1}} + \frac{2\varepsilon - 4\varepsilon\eta - a_1}{\gamma+2})g(t)\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ & +(\frac{3\varepsilon}{2} - \frac{a_1}{2} - 2\varepsilon\eta)(g\Box u)(t) - \frac{1}{2}g(t)\|u(t)\|_{\Gamma_1}^2 \\ & -2\varepsilon\eta\|u(t)\|_{\Gamma_1}^2 + (\frac{\varepsilon}{2} + 2\varepsilon\eta)\int_0^t g(s)ds\|u(t)\|_{\Gamma_1}^2. \end{aligned}$$

(3.19)

Noting that

$$\varepsilon\int_\Omega \Big(2(1-2\eta)F(u(t)) - u(t)f(u(t))\Big)dx \le 0,$$

$$\varepsilon(1-2\eta)\widehat{M}(\|\nabla u(t)\|^2) + (2\eta-1)\varepsilon M(\|\nabla u(t)\|^2)\|\nabla u(t)\|^2 \le 0,$$

and

$$-2\varepsilon\eta\|u(t)\|_{\Gamma_1}^2 + (\frac{\varepsilon}{2} + 2\varepsilon\eta)\int_0^t g(s)ds\|u(t)\|_{\Gamma_1}^2 \le 0,$$

we obtain

$$\begin{aligned} G_1(u) + C_1\varepsilon E(t) \quad \le \quad & (\frac{\lambda_\Omega^2}{4\alpha\varepsilon} + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta})\|h\|^2 + (\frac{\varepsilon\lambda_\Gamma^2}{4m_0\eta} - 1)\|u_t\|_{\Gamma_1}^2 \\ & +(3\alpha\varepsilon + 2\varepsilon\lambda_\Omega^2 - 2\alpha\varepsilon\eta - 2\lambda_\Omega^2\eta\varepsilon - 1)\|\nabla u_t(t)\|^2 \\ & +(\eta + \varepsilon\theta(\eta) - 1)g(t)\|u_t(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ & +(\varepsilon\eta + \eta^{-\frac{1}{\gamma+1}} + \frac{2\varepsilon - 4\varepsilon\eta - a_1}{\gamma+2})g(t)\|u(t)\|_{\gamma+2,\Gamma_1}^{\gamma+2} \\ & +(\frac{3\varepsilon}{2} - \frac{a_1}{2} - 2\varepsilon\eta)(g\Box u)(t). \end{aligned}$$

(3.20)

Let

(3.21) $$\varepsilon_1 = min\Big\{\frac{4m_0\eta}{\lambda_\Gamma^2}, \frac{1}{\alpha(3-2\eta) + 2\lambda_\Omega^2(1-\eta)}, \frac{1-\eta}{\theta(\eta)}, \frac{a_1 - 2\gamma - 4}{2 - 2\eta + \gamma\eta}, \frac{a_1}{3-4\eta}\Big\}.$$

For each $0 < \varepsilon \le \varepsilon_1$, then (3.20) implies that

(3.22) $$G_1(u) + C_1\varepsilon E(t) \le (\frac{\lambda_\Omega^2}{4\alpha\varepsilon} + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta})\|h\|^2.$$

Thus by (3.17) and (3.22) we get

$$\frac{d}{dt}E_\varepsilon(t) \leq -C_1\varepsilon E(t) + (\frac{\lambda_\Omega^2}{4\alpha\varepsilon} + \frac{\varepsilon\lambda_\Omega^2}{4m_0\eta})\|h\|^2. \tag{3.23}$$

The proof of Lemma 3.2 is completed.

Proof of Theorem 3.1 Let $\varepsilon_0 = min\{\frac{1}{2\lambda_0}, \varepsilon_1\}$ and let us consider $\varepsilon \in (0, \varepsilon_0]$. As we have $\varepsilon < \frac{1}{2\lambda_0}$, we conclude from (3.6)

$$(1-\varepsilon\lambda_0)E(t) < E_\varepsilon(t) < (1+\varepsilon\lambda_0)E(t)$$

and so

$$\frac{1}{2}E(t) < E_\varepsilon(t) < \frac{3}{2}E(t). \tag{3.24}$$

From (3.7) and (3.24) it follows that

$$\frac{d}{dt}E_\varepsilon(t) < -\frac{2}{3}C_1\varepsilon E_\varepsilon(t) + C_2\|h\|^2.$$

By Gronwall's inequality we get

$$E_\varepsilon(t) \leq E_\varepsilon(0)e^{-\frac{2}{3}C_1\varepsilon t} + C_3(1 - e^{-\frac{2}{3}C_1\varepsilon t}). \tag{3.25}$$

Inequality (3.25) implies

$$\|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 + \widehat{M}(\|\nabla u(t)\|^2) \leq C_4e^{-\frac{2}{3}C_1\varepsilon t} + C_5(1 - e^{-\frac{2}{3}C_1\varepsilon t}). \tag{3.26}$$

The proof of Theorem 3.1 is completed.

4. Asymptotic compactness

By definition, the semigroup S_t is asymptotically compact if for any bounded $A \subset \mathcal{H}_0$ and any $\varepsilon > 0$ there exists a precompact set $K \subset \mathcal{H}_0$ and a time t such that $dist(S_tA, K) < \varepsilon$.

To establish the asymptotic compactness of the semigroup generated by problem (1.1)-(1.4) we adopt the general scheme([**11, 21**]). The idea is to decompose the solution operator into two parts:

$$S_t(u_0, u_1) = V_t(u_0, u_1) + W_t(u_0, u_1),$$

where V_t is a contraction in the sense that $V_t(u_0, u_1) \to 0$ as $t \to +\infty$ uniformly in $(u_0, u_1) \in A$, and W_t is a compact mapping for all t. Then choosing t sufficiently large so that $\|V_t(u_0, u_1)\| < \varepsilon$ for all $(u_0, u_1) \in A$, we have $dist(S_tA, W_tA) < \varepsilon$, which proves the asymptotic compactness.

However, there are additional difficulties when proving asymptotically compact because $M(\|\nabla u\|^2)$ is nonlinear. To overcome the difficulty, we utilize a especial method of decompose to verity the asymptotic compactness of problems (1.1)-(1.4).

Firstly, let us define V_t as the solution operator of the following problems:

$$v_{tt} - \Delta v - \alpha\Delta v_{tt} - \Delta v_t = 0 \quad in \quad \Omega \times (0, \infty), \tag{4.1}$$

$$v = 0 \quad on \quad \Gamma_0 \times (0, \infty), \tag{4.2}$$

$$\frac{\partial v}{\partial \nu} + \alpha\frac{\partial v_{tt}}{\partial \nu} + \frac{\partial v_t}{\partial \nu} = 0, \quad on \quad \Gamma_1 \times (0, \infty), \tag{4.3}$$

$$v(x, 0) = u_0(x), v_t(x, 0) = u_1(x) \quad in \quad x \in \Omega. \tag{4.4}$$

Proposition 4.1 *Assume that $(u_0, u_1) \in \mathcal{H}_0$, then the problem (4.1)-(4.4) admits a unique global solution v satisfying*

$$v \in C(0, +\infty; H^1(\Omega)), \quad v_t \in C(0, +\infty; L^2(\Omega)).$$

Moreover, for each bounded $A \subset \mathcal{H}_0$,

$$\sup_{(u_0,u_1)\in A} \|V_t(u_0, u_1)\|_{\mathcal{H}_0} \to 0, \quad as \quad t \to +\infty. \tag{4.5}$$

Proof For

$$\theta \in [0, \theta_0], \quad \theta_0 = \min\{\theta_0 \Big| 1 - \alpha\theta - \theta\lambda_\Omega^2 - \frac{\theta^2\lambda_\Omega^2}{2} > 0, 1 + \alpha\theta^2 - \theta - \frac{\theta\lambda_\Omega^2}{2} > 0\}, \tag{4.6}$$

it is easy to obtain,

$$\begin{aligned} &(v_t + \theta v)_t + (\alpha\theta - 1)\Delta(v_t + \theta v) - \theta(v_t + \theta v) \\ &+(\theta - \alpha\theta^2 - 1)\Delta v - \alpha\Delta(v_t + \theta v)_t + \theta^2 v = 0. \end{aligned}$$

Multiplying above equation by $v_t + \theta v$ we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}G_2(v) + (1 - \alpha\theta)\|\nabla(v_t + \theta v)\|^2 - \theta\|v_t + \theta v\|^2 \\ &+\theta(1 + \alpha\theta^2 - \theta)\|\nabla v\|^2 + \theta^2(v, v_t + \theta v) = 0, \end{aligned} \tag{4.7}$$

where

$$G_2(v) = \|v_t + \theta v\|^2 + (1 + \alpha\theta^2 - \theta)\|\nabla v\|^2 + \alpha\|\nabla(v_t + \theta v)\|^2. \tag{4.8}$$

Here we note that

$$\begin{aligned} &(1 - \alpha\theta)\|\nabla(v_t + \theta v)\|^2 - \theta\|v_t + \theta v\|^2 \\ &+\theta(1 + \alpha\theta^2 - \theta)\|\nabla v\|^2 + \theta^2(v, v_t + \theta v) \\ \geq\ &(1 - \alpha\theta)\|\nabla(v_t + \theta v)\|^2 - \theta\lambda_\Omega^2\|\nabla(v_t + \theta v)\|^2 \\ &+\theta(1 + \alpha\theta^2 - \theta)\|\nabla v\|^2 - \frac{\theta^2\lambda_\Omega^2}{2}\|\nabla v\|^2 \\ &-\frac{\theta^2\lambda_\Omega^2}{2}\|\nabla(v_t + \theta v)\|^2 \\ \geq\ &(1 - \alpha\theta - \theta\lambda_\Omega^2 - \frac{\theta^2\lambda_\Omega^2}{2})\|\nabla(v_t + \theta v)\|^2 \\ &+\theta(1 + \alpha\theta^2 - \theta - \frac{\theta\lambda_\Omega^2}{2})\|\nabla v\|^2. \end{aligned} \tag{4.9}$$

From (4.6)-(4.9), there is a constant $K > 0$ such that

$$\frac{d}{dt}G_2(v) + KG_2(v) \leq 0 \tag{4.10}$$

By Gronwall's inequality we can get

$$G_2(v(t)) \leq G_2(v(0))e^{-Kt}. \tag{4.11}$$

On the other hand, we have

$$\begin{aligned}\|v_t(t)\|^2 &= \|v_t(t)+\theta v(t)-\theta v(t)\|^2\\ &\le \|v_t(t)+\theta v(t)\|^2+\theta^2\|v(t)\|^2\\ &\le \|v_t(t)+\theta v(t)\|^2+\theta^2\lambda_\Omega^2\|\nabla v(t)\|^2,\end{aligned}$$

(4.12)

and

$$\begin{aligned}\|\nabla v_t(t)\|^2 &= \|\nabla(v_t(t)+\theta v(t))-\theta\nabla v(t)\|^2\\ &\le \|\nabla(v_t(t)+\theta v(t))\|^2+\theta^2\|\nabla v(t)\|^2.\end{aligned}$$

(4.13)

From (4.11)-(4.13) we get, for some constant $C>0$

$$\|v_t(t)\|^2+\|\nabla v_t(t)\|^2+\|\nabla v(t)\|^2 \le Ce^{-Kt}. \tag{4.14}$$

The proof of Proposition 4.1 is completed.

Secondly, we now pass to the proof of the compactness of mapping $W_t = S_t - V_t$. Clearly, if w is the first component of $W_t(u_0,u_1)$, then its second component is w_t and the function w satisfies the following problems:

$$\begin{aligned}&w_{tt}-\Delta w-\alpha\Delta w_{tt}-\Delta w_t = M(\|\nabla u\|^2)\Delta u\\ &-\Delta u - f(u)+h(x) \quad in \quad \Omega\times(0,\infty),\end{aligned}$$

(4.15)

$$w=0 \quad on \quad \Gamma_0\times(0,\infty), \tag{4.16}$$

$$\begin{aligned}&\frac{\partial w}{\partial\nu}+\alpha\frac{\partial w_{tt}}{\partial\nu}+\frac{\partial w_t}{\partial\nu} = -M(\|\nabla u\|^2)\frac{\partial u}{\partial\nu}+\frac{\partial u}{\partial\nu}\\ &-u-u_t-g(t)|u_t|^\rho u_t+g*|u|^\gamma u \quad on \quad \Gamma_1\times(0,\infty),\end{aligned}$$

(4.17)

$$w(x,0)=0,\quad w_t(x,0)=0 \quad in \quad \Omega. \tag{4.18}$$

Proposition 4.2 *For each $t\in R_+$ the mapping W_t: $\mathcal{H}_0\to\mathcal{H}_0$ is compact.*

Proof Fix an arbitrary bounded sequence $(u_0^k,u_1^k)\in\mathcal{H}_0$. Let $u^k(x,t)$, $v^k(x,t)$, $w^k(x,t)$ denote the first components of $S_t(u_0^k,u_1^k)$, $V_t(u_0^k,u_1^k)$, and $W_t(u_0^k,u_1^k)$ respectively.

The first step is to show that for any p, $1<p<q\equiv\frac{2n}{n-2}$(in the case $n=1,2$ choose any finite $q>2\mu+2$), the sequences u^k is precompact in $C((0,\infty),L^p(\Omega))$. The energy equation $e(t)$ for (1.1)-(1.4) implies that u^k and u_t^k are bounded in $L^\infty(0,\infty;H^1(\Omega))$. By the Sobolev Embedding Theorem in particular we have u^k is bounded in $L^\infty(0,\infty;L^q(\Omega))$, and $u^k(t,\cdot)$ is precompact in $L^p(\Omega)$ for almost all $t\in(0,\infty)$. By the interpolation inequality for almost all t, $s\in(0,\infty)$ we have

$$\begin{aligned}
& \|u^k(\cdot,t) - u^k(\cdot,s)\|_{L^p} \\
\leq\ & \|u^k(\cdot,t) - u^k(\cdot,s)\|_{L^2}^{\theta} \|u^k(\cdot,t) - u^k(\cdot,s)\|_{L^q}^{1-\theta} \\
\leq\ & \Big(\int_t^s \|u_t^k(\cdot,z)\|_{L^2} dz\Big)^{\theta} \Big(\|u^k(\cdot,t)\|_{L^q} + \|u^k(\cdot,s)\|_{L^q}\Big)^{1-\theta},
\end{aligned} \tag{4.19}$$

where $\theta = \frac{q/p-1}{q/2-1} \in (0,1)$. The second factor at the right-hand side of (4.19) is bounded, and the first one vanishes as $|t-s| \to 0$ uniformly in k. Thus (4.19) show that the functions u^k are equicontinuous in $C(0,\infty; L^p(\Omega))$. So the sequence u^k is precompactness in $C(0,\infty; L^p(\Omega))$. Fix two different integers i, j and observe that the function $U = w^{k_i} - w^{k_j}$ satisfies

$$\begin{aligned}
& U_{tt} - \Delta U - \alpha \Delta U_{tt} - \Delta U_t = M(\|\nabla u^{k_i}\|^2)\Delta u^{k_i} \\
& -M(\|\nabla u^{k_j}\|^2)\Delta u^{k_j} + \Delta u^{k_j} - \Delta u^{k_i} \\
& +f(u^{k_j}) - f(u^{k_i}), \quad in \quad \Omega \times (0,\infty),
\end{aligned}$$

$$U = 0 \quad on \quad \Gamma_0 \times (0,\infty), \tag{4.20}$$

$$\begin{aligned}
& \frac{\partial U}{\partial \nu} + \alpha \frac{\partial U_{tt}}{\partial \nu} + \frac{\partial U_t}{\partial \nu} = -M(\|\nabla u^{k_i}\|^2)\frac{\partial u^{k_i}}{\partial \nu} \\
& +M(\|\nabla u^{k_j}\|^2)\frac{\partial u^{k_j}}{\partial \nu} + \frac{\partial u^{k_i}}{\partial \nu} - \frac{\partial u^{k_j}}{\partial \nu} \\
& +(u^{k_j} - u^{k_i}) + (u_t^{k_j} - u_t^{k_i}) \\
& +g(t)(|u_t^{k_j}|^{\rho} u_t^{k_j} - |u_t^{k_i}|^{\rho} u_t^{k_i}) \\
& +g * (|u^{k_i}|^{\gamma} u^{k_i} - |u^{k_j}|^{\gamma} u^{k_j}), \quad on \quad \Gamma_1 \times (0,\infty),
\end{aligned}$$

$$U(x,0) = 0, U_t(x,0) = 0, \quad in \quad \Omega. \tag{4.21}$$

Multiplying (4.20) by U_t we get

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\{\|U_t\|^2 + \alpha\|\nabla U_t\|^2 + \|\nabla U\|^2\} \\
= \ & -\|\nabla U_t\|^2 - M(\|\nabla u^{k_i}\|^2)\int_\Omega \nabla u^{k_i} \cdot \nabla U_t dx \\
& +M(\|\nabla u^{k_j}\|^2)\int_\Omega \nabla u^{k_j} \cdot \nabla U_t dx \\
& +\int_\Omega (f(u^{k_j}) - f(u^{k_i}))U_t dx \\
& +\int_\Omega \nabla(u^{k_i} - u^{k_j}) \cdot \nabla U_t dx \\
& +\int_{\Gamma_1} (u^{k_j} - u^{k_i})U_t d\Gamma + \int_{\Gamma_1} (u_t^{k_j} - u_t^{k_i})U_t d\Gamma \\
& +\int_{\Gamma_1} g(t)(|u_t^{k_j}|^{\rho} u_t^{k_j} - |u_t^{k_i}|^{\rho} u_t^{k_i})U_t d\Gamma \\
& +\int_{\Gamma_1} g * (|u^{k_i}|^{\gamma} u^{k_i} - |u^{k_j}|^{\gamma} u^{k_j})U_t d\Gamma.
\end{aligned}$$

We now majorize the right-hand side of (4.24). Firstly, using Schwarz's inequality and Young's inequality, we get the following four inequalities:

$$\begin{aligned}
&-M(\|\nabla u^{k_i}\|^2)\int_\Omega \nabla u^{k_i}\cdot\nabla U_t dx\\
&+M(\|\nabla u^{k_j}\|^2)\int_\Omega \nabla u^{k_j}\cdot\nabla U_t dx\\
=\ &\Big(M(\|\nabla u^{k_j}\|^2)-M(\|\nabla u^{k_i}\|^2)\Big)\int_\Omega \nabla u^{k_j}\cdot\nabla U_t dx\\
&+M(\|\nabla u^{k_i}\|^2)\int_\Omega \nabla(u^{k_j}-u^{k_i})\cdot\nabla U_t dx\\
\le\ &K(\eta)\Big(M(\|\nabla u^{k_j}\|^2)-M(\|\nabla u^{k_i}\|^2)\Big)^2\\
&+K(\eta)\Big\|\nabla(u^{k_j}-u^{k_i})\Big\|^2+\eta\|\nabla U_t\|^2,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega (f(u^{k_j})-f(u^{k_i}))U_t dx\\
\le\ &K(\eta)\Big\|f(u^{k_j})-f(u^{k_i})\Big\|^2+\eta\|\nabla U_t\|^2,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \nabla(u^{k_i}-u^{k_j})\cdot\nabla U_t dx+\int_{\Gamma_1}(u^{k_j}-u^{k_i})U_t d\Gamma\\
\le\ &K(\eta)\Big\|\nabla(u^{k_j}-u^{k_i})\Big\|^2+\eta\|\nabla U_t\|^2,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Gamma_1} g(t)(|u_t^{k_j}|^\rho u_t^{k_j}-|u_t^{k_i}|^\rho u_t^{k_i})U_t d\Gamma\\
\le\ &K(\eta)g^2(t)\Big\||u_t^{k_j}|^\rho u_t^{k_j}-|u_t^{k_i}|^\rho u_t^{k_i}\Big\|_{\Gamma_1}^2+\eta\|\nabla U_t\|^2.
\end{aligned}$$

On the other hand, by a simple computation, and taking (2.5) into account, we get

$$\begin{aligned}
&\int_{\Gamma_1} g*(|u^{k_i}|^\gamma u^{k_i}-|u^{k_j}|^\gamma u^{k_j})U_t d\Gamma\\
=\ &\int_0^t g(t-s)\Big(|u^{k_i}(s)|^\gamma u^{k_i}(s)-|u^{k_j}(s)|^\gamma u^{k_j}(s),U_t(t)\Big)_{\Gamma_1}ds\\
=\ &-\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)\Big\||u^{k_i}(s)|^\gamma u^{k_i}(s)-|u^{k_j}(s)|^\gamma u^{k_j}(s)-U(t)\Big\|_{\Gamma_1}^2ds\\
&+\frac{1}{2}\int_0^t g'(t-s)\Big\||u^{k_i}(s)|^\gamma u^{k_i}(s)-|u^{k_j}(s)|^\gamma u^{k_j}(s)-U(t)\Big\|_{\Gamma_1}^2ds\\
&+\frac{1}{2}\frac{d}{dt}\Big(\|U(t)\|_{\Gamma_1}^2\int_0^t g(s)ds\Big)-\frac{1}{2}g(t)\|U(t)\|_{\Gamma_1}^2\\
\le\ &-\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)\Big\||u^{k_i}(s)|^\gamma u^{k_i}(s)-|u^{k_j}(s)|^\gamma u^{k_j}(s)-U(t)\Big\|_{\Gamma_1}^2ds\\
&+\frac{1}{2}\frac{d}{dt}\Big(\|U(t)\|_{\Gamma_1}^2\int_0^t g(s)ds\Big).
\end{aligned}$$

From (4.24)-(4.29), choosing $\eta > 0$ sufficiently small we can get

$$\begin{aligned} &\frac{d}{dt}\Big\{\|U_t\|^2 + \alpha\|\nabla U_t\|^2 + \|\nabla U\|^2 - \|U(t)\|^2_{\Gamma_1}\int_0^t g(s)ds \\ &+\int_0^t g(t-s)\Big\||u^{k_i}(s)|^\gamma u^{k_i}(s) - |u^{k_j}(s)|^\gamma u^{k_j}(s) - U(t)\Big\|^2_{\Gamma_1} ds\Big\} \\ \leq\ & K\Big\{\Big|M(\|\nabla u^{k_j}\|^2) - M(\|\nabla u^{k_i}\|^2)\Big|^2 + \Big\|\nabla(u^{k_i} - u^{k_j})\Big\|^2 \\ &+\Big\|u_t^{k_j} - u_t^{k_i}\Big\|^2_{\Gamma_1} + \Big\|f(u^{k_j}) - f(u^{k_i})\Big\|^2 \\ &+g^2(t)\Big\||u_t^{k_j}|^\rho u_t^{k_j} - |u_t^{k_i}|^\rho u_t^{k_i}\Big\|^2_{\Gamma_1}\Big\}. \end{aligned}$$

Integrating (4.30) over (0,t) and taking (4.23) and (2.5) into account, we obtain

$$\begin{aligned} &\|U_t(t)\|^2 + \alpha\|\nabla U_t(t)\|^2 + \|\nabla U(t)\|^2 \\ \leq\ & K\int_0^t \Big\{\Big|M(\|\nabla u^{k_j}(s)\|^2) - M(\|\nabla u^{k_i}(s)\|^2)\Big|^2 \\ &+\Big\|\nabla(u^{k_i}(s) - u^{k_j}(s))\Big\|^2 + \Big\|u_t^{k_j}(s) - u_t^{k_i}(s)\Big\|^2_{\Gamma_1} \\ &+\Big\|f(u^{k_j}(s)) - f(u^{k_i}(s))\Big\|^2 \\ &+g^2(s)\Big\||u_t^{k_j}(s)|^\rho u_t^{k_j}(s) - |u_t^{k_i}(s)|^\rho u_t^{k_i}(s)\Big\|^2_{\Gamma_1}\Big\}ds. \end{aligned}$$

From (4.31) and (3.26), and noting that $V \hookrightarrow L^{2\rho+2}(\Gamma_1)$ we get

$$\begin{aligned} &\|W_t(u_0^{k_i}, u_1^{k_i}) - W_t(u_0^{k_j}, u_1^{k_j})\|^2_{\mathcal{H}_0} \\ \leq\ & K\int_0^t \Big\{\Big|M(\|\nabla u^{k_j}(s)\|^2) - M(\|\nabla u^{k_i}(s)\|^2)\Big|^2 \\ &+\Big\|\nabla(u^{k_i}(s) - u^{k_j}(s))\Big\|^2 + \Big\|u_t^{k_j}(s) - u_t^{k_i}(s)\Big\|^2_{\Gamma_1} \\ &+\Big\|f(u^{k_j}(s)) - f(u^{k_i}(s))\Big\|^2 \\ &+g^2(s)\Big\||u_t^{k_j}(s)|^\rho u_t^{k_j}(s) - |u_t^{k_i}(s)|^\rho u_t^{k_i}(s)\Big\|^2_{\Gamma_1}\Big\}ds \\ \to\ & 0, \quad i, j \to \infty. \end{aligned}$$

Thus the sequence $W_t(u_0^k, u_1^k)$ contains a convergent subsequence, which completes the proof of the Proposition 4.2.

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