Dynamics of PDE, Vol.4, No.2, 167-196, 2007

Global Dynamics of the Brusselator Equations

Yuncheng You

Communicated by Y. Charles Li, received December 29, 2006.

ABSTRACT. In this work the existence of a global attractor for the solution semiflow of the Brusselator equations is proved. A new decomposition method is introduced to overcome the difficulties in proving the asymptotical compactness of the coupled reaction-diffusion equations whose nonlinearity does not possess dissipative property. It is proved that the Hausdorff dimension and the fractal dimension of the global attractor are finite. The existence of a global attractor with finite dimensionality is also shown by the same approach for the Gray-Scott equations, the Glycolysis equations, and the Schnackenberg equations.

CONTENTS

1. Introduction 167
2. Absorbing Property 172
3. Asymptotical Compactness by Decomposition 176
4. κ-Contracting Property 181
5. Finite Dimensionality of Global Attractor 190
References 195

1. Introduction

A generic class of nonlinear reaction-diffusion systems is in the form

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a_1 u + b_1 v + f(u,v) + g_1(x),$$
$$\frac{\partial v}{\partial t} = d_2 \Delta v + a_2 u + b_2 v - f(u,v) + g_2(x),$$

with homogeneous Dirichlet or Neumann boundary condition on a bounded domain $\Omega \subset \Re^n, n \leq 3$, with locally Lipschitz continuous boundary. It is well known that

1991 *Mathematics Subject Classification.* 37L30, 35B40, 35B41, 35K55, 35K57, 35Q80.
Key words and phrases. Brusselator, Gray-Scott equation, global dynamics, global attractor, absorbing set, asymptotic compactness.

reaction and diffusion of chemical or biochemical species can produce a variety of spatial patterns. This class of reaction diffusion systems includes some significant *pattern formation* equations arising from the modeling of kinetics of chemical or biochemical reactions and from the biological pattern formation theory.

In this group, the following four systems are typically important and serve as mathematical models in physical chemistry and in biology:

Brusselator model: $a_1 = -(b+1), b_1 = 0, a_2 = b, b_2 = 0, f = u^2 v, g_1 = a, g_2 = 0$, where a and b are positive constants.

Gray-Scott model: $a_1 = -(F+k), b_1 = 0, a_2 = 0, b_2 = -F, f = u^2 v, g_1 = 0, g_2 = F$, where F and k are positive constants.

Glycolysis model: $a_1 = -1, b_1 = k, a_2 = 0, b_2 = -k, f = u^2 v, g_1 = \rho, g_2 = \delta$, where k, ρ, and δ are positive constants.

Schnackenberg model: $a_1 = -k, b_1 = a_2 = b_2 = 0, f = u^2 v, g_1 = a, g_2 = b$, where k, a, and b are positive constants.

The Gray-Scott model was originated from describing an isothermal autocatalytic, continuously fed, unstirred reaction and diffusion of two chemicals U and V with concentrations $u(t,x)$ and $v(t,x)$, cf. [**11**, **12**]. Since 1993, a variety of spatial patterns generated by the steady state solutions and longtime evolving solutions have been exposed by experiments [**11**, **15**], by numerical simulations [**18**], or by mathematical analysis [**6**, **32**].

The Brusselator is originally a system of two ordinary differential equations as the reaction rate equations for an autocatalytic, oscillating chemical reaction, cf. [**3**, **19**, **26**]. The name is after the hometown of scientists who proposed it. In many autocatalytic systems, even ODEs [**26**], complex dynamics are seen, including multiple steady states, periodic orbits, and bifurcations.

The Belousov-Zhabotinsky reaction [**8**] is a generic chemical reaction in which the concentrations of the reactants exhibit somewhat oscillating behavior. In particular, the Brusselator model describes the case in which the chemical reactions follow the scheme

$$\begin{aligned} \mathrm{A} &\longrightarrow \mathrm{U}, \\ \mathrm{B} + \mathrm{U} &\longrightarrow \mathrm{V} + \mathrm{D}, \\ 2\mathrm{U} + \mathrm{V} &\longrightarrow 3\mathrm{U}, \\ \mathrm{U} &\longrightarrow \mathrm{E}, \end{aligned}$$

where A, B, D, E, U, and V are chemical compounds. Let $u(t,x)$ and $v(t,x)$ be the concentrations of U and V, and assume that the concentrations of the input compounds A and B are held constant during the reaction process, denoted by a and b respectively. Then one obtains the following system of two nonlinearly coupled reaction-diffusion equations,

$$\frac{\partial u}{\partial t} = d_1 \Delta u + u^2 v - (b+1)u + a, \qquad (t,x) \in (0,\infty) \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - u^2 v + bu, \qquad (t,x) \in (0,\infty) \times \Omega,, \tag{1.2}$$

$$u(t,x) = v(t,x) = 0, \qquad t > 0,\ x \in \partial\Omega,$$

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x), \qquad x \in \Omega,$$

where d_1, d_2, a, and b are positive constants. Here it is assumed that the rate coefficients for the intermediate reactions are equal to one. Actually the results of this paper will not be affected by taking different reaction coefficients. The system of equations (1.1) and (1.2) is called the *Brusselator equations*.

Note that there are several known examples of autocatalysis which can be modeled by the Brusselator equations, such as ferrocyanide-iodate-sulphite reaction, chlorite-iodide-malonic acid reaction, arsenite-iodate reaction, some enzyme catalytic reactions, and fungal mycelia growth, cf. [**2**, **3**, **5**, **7**].

Since 1970's there have been some but limited studies on the spatial pattern solutions as steady states, their local stabilities and bifurcations for the Brusselator equations. In [**9**, **16**, **20**, **22**, **33**], some Turing patterns generated by the Brusselator equations were studied numerically or analytically, including spike patterns, stripe patterns, and oscillating instabilities. In [**1**] numerical results were presented, showing mazelike patterns, frustrated hexagonal patterns, and chaotic-looking patterns, for the 2D Brusselator equations and the hyperbolic version with the diffusion flux equations. In [**14**], under the assumptions of slow input and slow diffusion rates, the existence of a mesa-type patterns for the 1D Brusselator is shown along with a threshold for the local stabilities and a Hopf bifurcation to a breather-type instability by using the singular perturbation method.

For the Brusselator equations and the other three model equations of space dimension $n \leq 3$, however, we have not seen substantial research results in the front of global dynamics. In this paper, without any restrictive assumptions, we show the existence of a global attractor in the L^2 phase space for the solution semiflow of the Brusselator equations (1.1) and (1.2) with homogeneous Dirichlet boundary conditions.

The basic theory of global attractors and applications can be found in [**13**, **24**, **29**] and many references therein. Since the years 1980 the existence of a global attractor has been proved for quite a few dissipative parabolic equations and damped nonlinear wave equations. The typical dissipativity for a single reaction-diffusion equation is embodied in the asymptotic sign condition on the right-hand side nonlinear function $f(u)$, i.e.

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \leq 0.$$

As for systems of two or more reaction diffusion equations, the corresponding asymptotic sign condition in vector version is usually not satisfied. A limited results on the existence of a global attractor were proved for partially dissipative reaction-diffusion systems, such as FitzHugh-Nagumo equations. Some results based on the construction of a positively invariant region in $\Re^n$ in general provide local attractors only.

For the Brusselator equations (1.1) and (1.2), the essential difficulties in proving the existence of a global attractor lie in the fact that the oppositely interactive polynomial nonlinearity in the two coupled equations does not possess partial dissipativity or asymptotic dissipativity, which causes certain obstacle in proving the existence of absorbing sets and even more challenge in showing the asymptotical compactness. In this paper, a new decomposition technique is explored and used to show the κ-contraction of the solution semiflow.

We start with the formulation of an evolutionary equation associated with the Brusselator equations. We shall introduce certain concepts and present the main result.

Define the product Hilbert spaces as follows,

$$\begin{aligned} H &= L^2(\Omega) \times L^2(\Omega), \\ E &= H_0^1(\Omega) \times H_0^1(\Omega), \\ W &= (H_0^1(\Omega) \cap H^2(\Omega)) \times (H_0^1(\Omega) \cap H^2(\Omega)). \end{aligned}$$

The norm and inner-product of H or $L^2(\Omega)$ will be denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The norm of $L^p(\Omega)$ will be denoted by $\| \cdot \|_{L^p}$. By the Poincar$\acute{e}$ inequality with the homogeneous Dirichlet boundary condition as specified, there is a constant $\gamma > 0$ such that

$$\|\nabla \varphi\|^2 \geq \gamma \|\varphi\|^2, \quad \text{for } \varphi \in E \text{ or } H_0^1(\Omega),$$

and we shall take $\|\nabla \varphi\|$ for the equivalent norm of the space E and of the space $H_0^1(\Omega)$. We use $| \cdot |$ to denote an absolute value or a vector norm in a Euclidean space.

It is easy to check that, by the Lumer-Phillips theorem and the analytic semigroup generation theorem [**24**], the densely defined, sectorial operator

$$A = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix} : D(A)(= W) \longrightarrow H \tag{1.3}$$

is the generator of an analytic C_0-semigroup on the Hilbert space H. By the fact that $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ is a continuous embedding for $n \leq 3$ and using the generalized Hölder inequality,

$$\|u^2 v\| \leq \|u\|_{L^6}^2 \|v\|_{L^6}, \quad \text{for } u, v \in L^6(\Omega),$$

one can verify that the nonlinear mapping

$$F(u,v) = \begin{pmatrix} u^2 v - (b+1)u + a \\ -u^2 v + bu \end{pmatrix} : E \longrightarrow H \tag{1.4}$$

is well defined on E and locally Lipschitz continuous. Then by the theory of evolutionary equations [**24**], one can easily show the local existence and uniqueness of the strong solution

$$u \in C([0, T_{max}); H) \cap L^2(0, T_{max}; E), \tag{1.5}$$

where $[0, T_{max})$ is the maximal interval of existence, of the Brusselator evolution equation

$$\frac{dw}{dt} = Aw + F(w), \quad t > 0, \tag{1.6}$$

where $w(t) = \text{col}\ (u(t, \cdot), v(t, \cdot))$, or written as $(u(t, \cdot), v(t, \cdot))$, for any initial data $w(0) = w_0 = (u_0, v_0) \in H$.

We refer to [**24**] and [**29**] for the concepts and basic facts in the theory of infinite dimensional dynamical systems, including few given below for clarity.

Definition 1. Let M be a complete metric space. A time-dependent family of maps $\{S(t)\}_{t\geq 0}$ is called a *semiflow* (or *semigroup*) on M, if the following conditions are satisfied:
(i) $S(0)w = w$, for all $w \in M$. Namely, $S(0) = I$, the identity map on M.
(ii) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.
(iii) The map: $(t, w) \mapsto S(t)w$ is continuous from $[0, \infty) \times M$ into M.

Definition 2. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a complete metric space M. A subset B_0 of M is called an absorbing set in M if, for any bounded subset $B \subset M$, there is some finite time $t_0 \geq 0$ depending on B such that $S(t)B \subset B_0$ for all $t \geq t_0$.

Definition 3. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a complete metric space M whose metric is denoted by $d(\cdot, \cdot)$. A subset $\mathcal{A}$ of M is called a global attractor for this semiflow, if $\mathcal{A}$ has the following properties:
(i) $\mathcal{A}$ is a nonempty, compact, and invariant set in the sense that $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$.
(ii) $\mathcal{A}$ attracts any bounded set B of M in the sense that, in terms of the Hausdorff distance,

$$\operatorname{dist}(S(t)B, \mathcal{A}) = \sup_{x\in B} \inf_{y\in\mathcal{A}} d(x, y) \to 0, \quad \text{as } t \to \infty.$$

Definition 4. A semiflow $\{S(t)\}_{t\geq 0}$ on a complete metric space M is called asymptotically compact if for any sequences $\{u_n\}$ which is bounded in M and $\{t_n\} \subset (0, \infty)$ with $t_n \to \infty$, there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$, such that $\lim_{k\to\infty} S(t_{n_k})w_{n_k}$ exists in M.

Here is the main result of this paper.

Theorem 1 (Main Theorem). *For any positive parameters d_1, d_2, a, and b, there exists a global attractor $\mathcal{A}$ in H for the solution semiflow $\{S(t)\}_{t\geq 0}$ generated by the Brusselator equations* (1.1)–(1.2).

By the same approach the existence of a global attractor for the Gray-Scott equations, the Glycolysis equations, and the Schnackenberg equations [**23**, **31**] in space dimension $n \leq 3$ is also proved.

In [**24**, Chapter 2], the basic existence theory of global attractors is provided, which can be concisely stated in the following lemma.

Lemma 1. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X, which has the following two properties*:
(i) *there exists a bounded absorbing set $B_0 \subset X$ for $\{S(t)\}_{t\geq 0}$, and*
(ii) *$\{S(t)\}_{t\geq 0}$ is asymptotically compact on X.*
Then there exists a global attractor $\mathcal{A}$ for $\{S(t)\}_{t\geq 0}$, which is the ω-limit set of B_0,

$$\mathcal{A} = \omega(B_0) \overset{\text{def}}{=} \bigcap_{\tau\geq 0} Cl_X \bigcup_{t\geq\tau} (S(t)B_0).$$

In Section 2 we shall prove the absorbing property of the Brusselator semiflow. A new decomposition approach is presented in Section 3, which is used to prove the κ-contracting property in Section 4. Thus the existence of a global attractor is proved for the Brusselator equations and for the other three model equations. In Section 5 we prove that the global attractors for all these equations have finite Hausdorff dimensions and finite fractal dimensions respectively.

2. Absorbing Property

In the sequal, we shall write $u(t,x)$ and $v(t,x)$ simply as $u(t)$ and $v(t)$, or even as u and v, and similarly for other functions of (t,x). Taking the inner product $\langle(1.2), v(t)\rangle$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 &= \int_\Omega (-u^2v^2 + buv)\,dx \\ &= \int_\Omega -(uv - \frac{b}{2})^2\,dx + \frac{1}{4}b^2|\Omega| \;\le\; \frac{1}{4}b^2|\Omega|.\end{aligned} \tag{2.1}$$

It follows that

$$\frac{d}{dt}\|v\|^2 + 2\gamma d_2\|v\|^2 \le b^2|\Omega|,$$

which yields

$$\|v(t)\|^2 \le e^{-2\gamma d_2 t}\|v_0\|^2 + \frac{b^2|\Omega|}{2\gamma d_2}, \quad \text{for } t \in [0, T_{max}). \tag{2.2}$$

Hence, we find that

$$\limsup_{t\to\infty} \|v(t)\|^2 \le \rho_0 = \frac{b^2|\Omega|}{\gamma d_2}. \tag{2.3}$$

Moreover, for any $t \ge 0$, (2.1) also implies that

$$\begin{aligned}\int_t^{t+1} \|\nabla v(s)\|^2\,ds &\le \frac{1}{d_2}(\|v(t)\|^2 + b^2|\Omega|) \\ &\le \frac{1}{d_2}\left(e^{-2\gamma d_2 t}\|v_0\|^2 + \frac{b^2|\Omega|}{2\gamma d_2}\right) + \frac{b^2|\Omega|}{d_2}.\end{aligned} \tag{2.4}$$

In order to treat the u-component, we add up (1.1) and (1.2) to get an equation for the sum $z(t) = u(t) + v(t)$, which is

$$z_t = d_1\Delta z - z + [(d_2 - d_1)\Delta v + v + a]. \tag{2.5}$$

Taking the inner-product $\langle(2.5), z(t)\rangle$ we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z\|^2 + d_1\|\nabla z\|^2 + \|z\|^2 &= \int_\Omega [(d_2 - d_1)\Delta v + v + a]z\,dx \\ &= (d_1 - d_2)\langle\nabla v, \nabla z\rangle + \langle v, z\rangle + \langle a, z\rangle \\ &\le |d_1 - d_2|\|\nabla v\|\|\nabla z\| + \|v\|\|z\| + a|\Omega|^{1/2}\|z\| \\ &\le \frac{d_1}{2}\|\nabla z\|^2 + \frac{|d_1 - d_2|^2}{2d_1}\|\nabla v\|^2 + \frac{1}{4}\|z\|^2 + \|v\|^2 + \frac{1}{4}\|z\|^2 + a^2|\Omega|,\end{aligned}$$

which implies that

$$\begin{aligned}\frac{d}{dt}\|z\|^2 + d_1\|\nabla z\|^2 + \|z\|^2 &\le \frac{|d_1 - d_2|^2}{d_1}\|\nabla v\|^2 + 2\|v\|^2 + 2a^2|\Omega| \\ &\le \frac{|d_1 - d_2|^2}{d_1}\|\nabla v\|^2 + 2e^{-2\gamma d_2 t}\|v_0\|^2 + \frac{b^2|\Omega|}{\gamma d_2} + 2a^2|\Omega|.\end{aligned}$$

Let

$$C(v_0, t) = 2e^{-2\gamma d_2 t}\|v_0\|^2 + \left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega|.$$

Then we get

$$\frac{d}{dt}\|z\|^2 + d_1\|\nabla z\|^2 + \|z\|^2 \le \frac{|d_1 - d_2|^2}{d_1}\|\nabla v\|^2 + C(v_0, t). \tag{2.6}$$

Integration of inequality (2.6) shows that the strong solution $z(t)$ of equation (2.5) satisfies the estimate

$$\begin{aligned}\|z(t)\|^2 \le \|u_0 + v_0\|^2 &+ \frac{|d_1 - d_2|^2}{d_1}\int_0^t \|\nabla v(s)\|^2\, ds \\ &+ \frac{1}{\gamma d_2}\|v_0\|^2 + t\left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega|.\end{aligned} \tag{2.7}$$

From (2.1) we see that

$$d_2\int_0^t \|\nabla v(s)\|^2\, ds \le \|v_0\|^2 + b^2|\Omega| t$$

so that, for $t \in [0, T_{max})$,

$$\begin{aligned}\|z(t)\|^2 &\le \|u_0 + v_0\|^2 + \frac{|d_1 - d_2|^2}{d_1\, d_2}\left(\|v_0\|^2 + b^2|\Omega| t\right) \\ &\quad + \frac{1}{\gamma d_2}\|v_0\|^2 + \left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega| t \\ &\le \|u_0 + v_0\|^2 + \left(\frac{1}{\gamma d_2} + \frac{|d_1 - d_2|^2}{d_1\, d_2}\right)\|v_0\|^2 \\ &\quad + \left(\left(\frac{1}{\gamma d_2} + \frac{|d_1 - d_2|^2}{d_1\, d_2}\right) b^2 + 2a^2\right)|\Omega| t.\end{aligned} \tag{2.8}$$

Therefore, we have shown the global existence of the strong solutions of the Brusselator evolutionary equation (1.6), as stated in the following lemma.

LEMMA 2. *For any initial data $w_0 = (u_0, v_0) \in H$, there exists a unique, global, strong solution $w(t) = (u(t), v(t))$, $t \in [0, \infty)$, of the Brusselator evolutionary equation* (1.6).

PROOF. Note that

$$\|u(t)\| \le \|z(t)\| + \|v(t)\| \qquad \text{and} \qquad \|w(t)\|^2 = \|u(t)\|^2 + \|v(t)\|^2.$$

The proof is done by (2.2) and (2.8), which means that the strong solution $w(t)$ of the equation (1.6) will never blow up in H at any finite time. □

Due to Lemma 2, the family of global strong solutions $\{w(t; w_0), t \ge 0, w_0 \in H\}$ defines a semiflow on H,

$$S(t) : w_0 \mapsto w(t; w_0), \quad w_0 \in H,\ t \ge 0,$$

which is called the Brusselator semiflow, generated by the Brusselator equations.

LEMMA 3. *There exists a bounded absorbing set* B_0 *in* H *for the Brusselator semiflow* $\{S(t)\}_{t\ge 0}$,

$$B_0 = \left\{\|w\| \in H : \|w\|^2 \le K_0\right\},$$

where K_0 *is a positive constant independent of any initial data.*

PROOF. From (2.6) we can deduce that

$$\frac{d}{dt}\left(e^t\|z(t)\|^2\right) \le \frac{|d_1 - d_2|^2}{d_1} e^t\|\nabla v(t)\|^2 + C(v_0, t)e^t. \tag{2.9}$$

Integrate (2.9) to obtain

$$\|z(t)\|^2 \le e^{-t}\|u_0 + v_0\|^2 + \frac{|d_1 - d_2|^2}{d_1} \int_0^t e^{-(t-\tau)}\|\nabla v(\tau)\|^2\, d\tau + C_1(v_0, t), \tag{2.10}$$

where

$$\begin{aligned} C_1(v_0, t) &= 2e^{-t} \int_0^t e^{(1-2\gamma d_2)\tau}\, d\tau\, \|v_0\|^2 + \left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega|. \\ &\le \alpha(t)\|v_0\|^2 + \left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega| \end{aligned} \tag{2.11}$$

in which

$$\alpha(t) = \begin{cases} \frac{2}{|1-2\gamma d_2|} e^{-2\gamma d_2 t}, & \text{if } 1 - 2\gamma d_2 > 0; \\ 2te^{-t} \le 4e^{-1}e^{-t/2}, & \text{if } 1 - 2\gamma d_2 = 0; \\ \frac{2}{|1-2\gamma d_2|} e^{-t}, & \text{if } 1 - 2\gamma d_2 < 0. \end{cases} \tag{2.12}$$

We now treat the term

$$\int_0^t e^{-(t-\tau)}\|\nabla v(\tau)\|^2\, d\tau.$$

Multiplying (2.1) by e^t and then integrating it, we get

$$\frac{1}{2}\int_0^t \left(e^\tau \frac{d}{d\tau}\|v(\tau)\|^2\right) d\tau + d_2 \int_0^t e^\tau\|\nabla v(\tau)\|^2\, d\tau \le b^2|\Omega|e^t,$$

so that, by integration by parts and using (2.2), we have

$$
\begin{aligned}
(2.13) \qquad d_2 \int_0^t e^\tau \|\nabla v(\tau)\|^2 \, d\tau &\le b^2 |\Omega| e^t - \frac{1}{2} \int_0^t \left(e^\tau \frac{d}{d\tau} \|v(\tau)\|^2 \right) d\tau \\
&= b^2 |\Omega| e^t - \frac{1}{2} \left[e^t \|v(t)\|^2 - \|v_0\|^2 - \int_0^t e^\tau \|v(\tau)\|^2 \, d\tau \right] \\
&\le b^2 |\Omega| e^t + \|v_0\|^2 + \int_0^t e^\tau \|v(\tau)\|^2 \, d\tau \\
&\le b^2 |\Omega| e^t + \|v_0\|^2 + \int_0^t e^{(1-2\gamma d_2)\tau} \|v_0\|^2 \, d\tau + \frac{b^2|\Omega|}{2\gamma d_2} e^t \\
&\le b^2 |\Omega| e^t + \left(1 + \alpha(t) e^t\right) \|v_0\|^2 + \frac{b^2|\Omega|}{2\gamma d_2} e^t \\
&= \left(1 + \frac{1}{2\gamma d_2} \right) b^2 |\Omega| e^t + \left(1 + \alpha(t) e^t\right) \|v_0\|^2, \quad \text{for any } t \ge 0.
\end{aligned}
$$

Substituting (2.13) into (2.10) and using (2.11), we obtain that, for all $t \ge 0$,

$$
\begin{aligned}
(2.14) \qquad \|z(t)\|^2 &\le e^{-t} \|u_0 + v_0\|^2 + C_1(v_0, t) \\
&\quad + \frac{|d_1 - d_2|^2}{d_1 \, d_2} e^{-t} \left[\left(1 + \frac{1}{2\gamma d_2} \right) b^2 |\Omega| e^t + \left(1 + \alpha(t) e^t\right) \|v_0\|^2 \right] \\
&\le e^{-t} \|u_0 + v_0\|^2 + \alpha(t) \|v_0\|^2 + \left(\frac{b^2}{\gamma d_2} + 2a^2 \right) |\Omega| \\
&\quad + \frac{|d_1 - d_2|^2}{d_1 \, d_2} e^{-t} \left[\left(1 + \frac{1}{2\gamma d_2} \right) b^2 |\Omega| e^t + \left(1 + \alpha(t) e^t\right) \|v_0\|^2 \right]
\end{aligned}
$$

Note that (2.12) shows

$$
0 < \alpha(t) \le \max \left\{ \frac{2}{|1 - 2\gamma d_2|} \, , \, 4e^{-1} \right\} e^{-\tilde{\alpha} t} \longrightarrow 0, \quad \text{as } \ t \to \infty,
$$

where

$$
\tilde{\alpha} = \min \left\{ 2\gamma d_2 \, , \, \frac{1}{2} \right\} > 0.
$$

Therefore, we can conclude from (2.14) that

$$
(2.15) \qquad \limsup_{t \to \infty} \|z(t)\|^2 \le \left(\frac{b^2}{\gamma d_2} + 2a^2 \right) |\Omega| + \frac{|d_1 - d_2|^2}{d_1 \, d_2} \left(1 + \frac{1}{2\gamma d_2} \right) b^2 |\Omega|,
$$

The right-hand side of inequality (2.15) is a uniform positive constant, which is independent of any initial data. Let

$$
\rho_1 = 1 + \left(\frac{b^2}{\gamma d_2} + 2a^2 \right) |\Omega| + \frac{|d_1 - d_2|^2}{d_1 \, d_2} \left(1 + \frac{1}{2\gamma d_2} \right) b^2 |\Omega|.
$$

Then we get

$$
(2.16) \qquad \limsup_{t \to \infty} \|z(t)\|^2 < \rho_1.
$$

Putting together (2.3) and (2.16), we end up with

$$\limsup_{t\to\infty} \|w(t)\|^2 = \limsup_{t\to\infty} \left(\|z(t) - v(t)\|^2 + \|v(t)\|^2\right) < K_0 = 2\rho_1 + 3\rho_0. \tag{2.17}$$

Note that K_0 in (2.17) is a uniform positive constant independent of the initial data. Therefore the conclusion of this lemma is valid and we have shown the absorbing property of the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ in the phase space H. □

3. Asymptotical Compactness by Decomposition

For investigation of the asymptotical compactness for the Brusselator semiflow, we shall take the approach of showing the κ-contracting property for this semiflow $\{S(t)\}_{t\geq 0}$. Recall the definition of the Kuratowski measure of noncompactness for bounded sets in a Banach space X,

$$\kappa(B) \stackrel{\text{def}}{=} \inf\{\delta : B \text{ has a finite cover by open sets in } X \text{ of diameters} < \delta\}.$$

If B is an unbounded set, then we define $\kappa(B) = \infty$. The basic properties of the Kuratowski measure are listed in the following lemma, cf. [**24**, Lemma 22.2] and [**17**, Lemma 2.4].

Lemma 4. *Let X be a Banach space and κ be the Kuratowski measure of non-compactness of bounded sets in X. Then κ has the following properties:*
(i) $\kappa(B) = 0$ *if and only if B is precompact in X, i.e. $Cl_X B$ is a compact set in X.*
(ii) $\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2)$, *for any linear sum $B_1 + B_2$.*
(iii) $\kappa(B_1) \leq \kappa(B_2)$ *whenever $B_1 \subset B_2$.*
(iv) *Suppose X is a direct sum of two closed linear subspaces X_1 and X_2,*

$$X = X_1 \oplus X_2, \quad \text{with} \quad \dim X_1 < \infty,$$

and $P : X \to X_1$ and $Q : X \to X_2$ are the canonical projection operators. Let B be a bounded set of X. If

$$\operatorname{diam} Q(B) < \varepsilon,$$

then $\kappa(B) < \varepsilon$.

Definition 5. A semiflow $\{S(t)\}_{t\geq 0}$ on a complete metric space X is called *κ-contracting* if for every bounded subset B in X, one has

$$\lim_{t\to\infty} \kappa(S(t)B) = 0.$$

A semiflow $\{S(t)\}_{t\geq 0}$ on a complete metric space X is called *ω-limit compact*, if for every bounded subset B of X, one has

$$\lim_{t\to\infty} \kappa\left(\bigcup_{\tau\geq t} S(\tau)B\right) = 0.$$

The following lemma provides the connection of the κ-contracting concept to the asymptotical compactness, cf. [**24**, Lemma 23.8] and Lemma 1.

LEMMA 5. *Let $\{S(t)\}_{t\geq 0}$ be a semiflow on a Banach space X. If the following conditions are satisfied*:
(i) $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set in X, and*
(ii) $\{S(t)\}_{t\geq 0}$ *is κ-contracting,*
then $\{S(t)\}_{t\geq 0}$ is asymptotically compact and there exists a global attractor $\mathcal{A}$ in X for this semiflow.

Let us now try to make some *a priori* estimates to see what are the essential difficulties for an attempt to prove the κ-contracting property of the Brusselator semiflow. Taking the inner-product $\langle (1.2), -\Delta v(t)\rangle$, we have

$$-\langle v_t, \Delta v\rangle + d_2\|\Delta v\|^2 = \langle u^2 v, \Delta v\rangle - b\langle u, \Delta v\rangle,$$

By Green's formula and the homogeneous Dirichlet boundary condition, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 &= \int_\Omega u^2 v\Delta v\, dx - b\int_\Omega u\Delta v\, dx \\
&\leq \int_\Omega u^2 v\Delta v\, dx + c\|u\|^2 + \frac{d_2}{2}\|\Delta v\|^2 \\
&= -\int_\Omega u^2|\nabla v|^2\, dx - 2\int_\Omega uv(\nabla u\cdot\nabla v)\, dx + c\|u\|^2 + \frac{d_2}{2}\|\Delta v\|^2 \\
&= -\int_\Omega |u\nabla v + v\nabla u|^2\, dx + \int_\Omega v^2|\nabla u|^2\, dx + c\|u\|^2 + \frac{d_2}{2}\|\Delta v\|^2 \\
&\leq \int_\Omega v^2|\nabla u|^2\, dx + c\|u\|^2 + \frac{d_2}{2}\|\Delta v\|^2,
\end{aligned}$$

where c is a positive constant depending only on b and d_2. Consequently we get

$$\frac{d}{dt}\|\nabla v\|^2 + d_2\|\Delta v\|^2 \leq 2\int_\Omega u^2 v\Delta v\, dx + 2c\|u\|^2 \tag{3.1}$$

$$\leq 2\int_\Omega v^2|\nabla u|^2\, dx + 2c\|u\|^2. \tag{3.2}$$

The troubling terms are the integrals on the right-hand side of (3.1) and (3.2), and much difficulty and complexity occur in this kind of estimation for $u(t)$ or $z(t)$. As commented in Section 1, these difficulties originate from the lack of dissipativity in the structural polynomial nonlinearity of the Brusselator equations.

A generic and good idea in dealing with the issue of proving the asymptotical compactness and the κ-contracting property is through a decomposition approach. We have seen different decomposition methods in different settings.

In many papers addressing the existence of global attractors for nonlinear wave equations and other nonlinear hyperbolic evolutionary equations, the solution semigroup $S(t)$ is decomposed into two parts, $S(t) = S_1(t) + S_2(t)$, where $S_1(t)B$ uniformly converges to zero as $t\to\infty$ and $S_2(t)B$ is precompact or κ-contracting for any bounded set B, cf. [**24**, **27**, **35**] and references therein.

Another decomposition method featuring smallness of tail estimates was introduced in [**4**, **30**] and effectively used in search for global attractors both for the first-order and for the second-order lattice and PDE systems. Combined with the Littlewood-Paley projection techniques of harmonic analysis, such a decomposition with tail estimation was applied to the asymptotic smoothing and the existence of global attractors for the generalized Benjamin-Bona-mahony equations on $\Re^3$ in [**25**].

The spectral decomposition [**17**] and the spatial cut-off decomposition [**28**] were also introduced to address the existence of global attractors for certain classes of scalar nonlinear reaction-diffusion equations in L^p spaces, or on unbounded domains.

Here we present a new decomposition method, which is stated in the next theorem. In the next section, we shall check the κ-contracting property and the asymptotical compactness of the Brusselator semiflow by this method.

We shall use the notation $\Omega(|u| \geq M) = \{x \in \Omega : |u(x)| \geq M\}$ and $\Omega(|u| < M) = \{x \in \Omega : |u(x)| < M\}$, and use $m(\cdot)$ to denote the Lebesgue measure of a subset in Ω.

Theorem 2. *Let $Y = L^2(\Omega)$ or H. Let $\{S(t)\}_{t\geq 0}$ be a semiflow on Y. Then there exists a global attractor $\mathcal{A}$ in Y for this semiflow if and only if the following two conditions are satisfied :*
(i) *There exists a bounded absorbing set B_0 in Y for this semiflow.*
(ii) *For any $\varepsilon > 0$, there are positive constants $M = M(\varepsilon)$ and $T = T(\varepsilon)$ such that*

$$\int_{\Omega(|S(t)w_0|\geq M)} |S(t)w_0|^2 \, dx < \varepsilon, \quad \text{for any } t \geq T, \ w_0 \in B_0. \tag{3.3}$$

and

$$\kappa\left((S(t)B_0)_{\Omega(|S(t)B_0|<M)}\right) \longrightarrow 0, \quad \text{as } t \to \infty, \tag{3.4}$$

where

$$(S(t)B_0)_{\Omega(|S(t)B_0|<M)} \stackrel{\text{def}}{=} \{(S(t)w_0)(\cdot)\theta_M(\cdot\,; t, w_0) : \text{for } w_0 \in B_0\}.$$

and $\theta_M(x; t, w_0)$ is the characteristic function of the subset $\Omega(|S(t)w_0| < M)$.

Proof. First we prove the sufficiency. By the condition (i), according to Lemma 5, it suffices to show that $\{S(t)\}_{t\geq 0}$ is κ-contracting, i.e.

$$\lim_{t\to\infty} \kappa(S(t)B) = 0, \quad \text{for any bounded set } B \subset Y.$$

Since B_0 attracts any bounded set B, there is a time T_B such that $S(t)B \subset B_0$, for $t \geq T_B$. Therefore, it suffices to show that

$$\lim_{t\to\infty} \kappa\left(S(t)B_0\right) = 0, \tag{3.5}$$

for the absorbing set B_0. By a property of the Kuratowski measure,

$$\kappa(B_1 + B_2) \leq \kappa(B_1) + \kappa(B_2),$$

we see that

$$\kappa(S(t)B_0) \leq \kappa(S(t)B_0(1-\theta_M)) + \kappa(S(t)B_0\theta_M), \tag{3.6}$$

where $S(t)B_0\theta_M$ represents the set

$$S(t)B_0\theta_M \stackrel{\text{def}}{=} \{(S(t)w_0) \cdot \theta_M(\cdot; t, w_0) : \text{for all } w_0 \in B_0\},$$

and similarly for $S(t)B_0(1-\theta_M)$. Now (3.3) in the condition (ii) indicates that for any $\varepsilon > 0$, there exist positive constants $M = M(\varepsilon)$ and $T = (\varepsilon)$ such that

$$\int_{\Omega} |S(t)w_0(1-\theta_M(x;t,w_0))|^2\,dx = \int_{\Omega(|S(t)w_0|\geq M)} |S(t)w_0|^2\,dx < \frac{\varepsilon^2}{4}, \tag{3.7}$$

for $t \geq T(\varepsilon)$, $w_0 \in B_0$. Then (3.7) implies that

$$\kappa(S(t)B_0(1-\theta_M)) < \varepsilon, \quad \text{for } t \geq T(\varepsilon).$$

On the other hand, by (3.4), for the same arbitrary ε and the same $M = M(\varepsilon)$, there exists a time $T_0(\varepsilon, B_0) > 0$ such that

$$\kappa(S(t)B_0\theta_M) = \kappa\left(S(t)(B_0)_{\Omega(|S(t)B_0|<M)}\right) < \varepsilon, \quad \text{for } t \geq T_0(\varepsilon, B_0).$$

Substituting the above two inequalities into (3.6), we obtain

$$\kappa(S(t)B_0) < 2\varepsilon, \quad \text{for any } t \geq T(\varepsilon) + T_0(\varepsilon, B_0).$$

Therefore, (3.5) is proved and $\{S(t)\}_{t\geq 0}$ is κ-contracting. By Lemma 5 there exists a global attractor for this semiflow.

Now we prove the necessity. We check each of the two conditions stated in this theorem. Suppose there exists a global attractor $\mathcal{A}$ in Y for this semiflow. Then $\mathcal{A}$ is a compact set. Given a constant $\varepsilon_0 > 0$, then for every bounded set $B \subset Y$ there is a time t_B such that

$$S(t)B \subset N_{\varepsilon_0}(\mathcal{A}), \quad \text{for } t \geq t_B,$$

where the neighborhood $N_{\varepsilon_0}(\mathcal{A})$ of $\mathcal{A}$ is a bounded absorbing set. Thus the condition (i) is satisfied.

To show the condition (ii) is satisfied, we have the following observations. First, since $\operatorname{dist}(S(t)B_0, \mathcal{A}) \to 0$, as $t \to \infty$, for any given $\varepsilon > 0$, there is a $T^* > 0$ such that

$$\operatorname{dist}(S(t)B_0, \mathcal{A}) \leq \left(\frac{\varepsilon}{6}\right)^{1/2}, \quad \text{for } t \geq T^*.$$

Since it is a compact set in Y, there is a finite covering net $\{\varphi_1, \cdots, \varphi_r\} \subset \mathcal{A}$ with the property that for any $\varphi \in \mathcal{A}$, there is a $\varphi_i \in \{\varphi_1, \cdots, \varphi_r\}$ such that

$$\|\varphi - \varphi_i\|^2 < \frac{\varepsilon}{6}.$$

By the absolute continuity of the Lebesgue integral, for this arbitrarily given $\varepsilon > 0$, there is an $\eta = \eta(\varepsilon) > 0$ such that for any measurable set $\Omega_s \subset \Omega$ satisfying $m(\Omega_s) < \eta$, one has

$$\int_{\Omega_s} |\varphi_i(x)|^2\,dx < \frac{\varepsilon}{6}, \quad \text{for } i = 1, \cdots, r.$$

Note that for every $w_0 \in B_0$, there exists a uniform positive constant $T(B_0)$ such that $\|S(t)w_0\|^2 \leq K_0$, for $t \geq T(B_0)$, where K_0 is the uniform constant given by (2.17). Thus there is a large $M = M(\varepsilon) > 0$ such that

$$m(\Omega(|S(t)w_0| \geq M)) \leq \frac{\|S(t)w_0\|^2}{M^2} \leq \frac{K_0}{M^2} < \eta, \quad \text{for any } w_0 \in B_0.$$

Putting all together, we obtain that for this $M = M(\varepsilon)$,

$$\begin{aligned}(3.8)\qquad \int_{\Omega(|S(t)w_0|\geq M)} |S(t)w_0|^2\,dx &\leq 2\int_{\Omega(|S(t)w_0|\geq M)} |S(t)w_0-\varphi|^2\,dx \\ &\quad + 2\int_{\Omega(|S(t)w_0|\geq M)} |\varphi(x)-\varphi_{i_0}|^2\,dx \\ &\quad + 2\int_{\Omega(|S(t)w_0|\geq M)} |\varphi_{i_0}|^2\,dx, \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon, \quad \text{for } t\geq T,\end{aligned}$$

where φ is some point in $\mathcal{A}$, $i_0 \in \{1, \cdots, r\}$, and $T = \max\{T^*, T(B_0)\}$. Therefore, it is proved that (3.3) in the condition (ii) is satisfied by this semiflow.

Now we verify (3.4) in the condition (ii). By [**17**, Theorem 3.9], the existence of a global attractor $\mathcal{A}$ in Y implies that the semiflow $\{S(t)\}_{t\geq 0}$ is ω-limit compact, which by Definition 5 implies that the semiflow $\{S(t)\}_{t\geq 0}$ is κ-contracting. Hence, for the bounded absorbing set B_0 in Y, we have

$$\kappa(S(t)B_0) \to 0, \quad \text{as } t\to\infty.$$

By the definition of the θ_M functions, we have

$$\kappa\left(S(t)B_0\theta_M\right) \leq \kappa\left(S(t)B_0\right) + \kappa\left(S(t)B_0(1-\theta_M)\right).$$

For any $\varepsilon > 0$, there exists $\widetilde{T}(\varepsilon) > 0$ such that $\kappa(S(t)B_0) < \varepsilon$. whenever $t \geq \widetilde{T}(\varepsilon)$. On the other hand, we have shown in the verification of (3.3) that for every $w_0 \in B_0$,

$$\int_\Omega |(S(t)w_0)(x)(1-\theta_M(x;t,w_0))|^2\,dx < \varepsilon, \quad \text{for } t\geq T,$$

where T is shown in (3.8). This implies that

$$\kappa\left(S(t)B_0(1-\theta_M)\right) < 2\varepsilon,$$

and, consequently,

$$\kappa\left(S(t)B_0\theta_M\right) < 3\varepsilon, \quad \text{for } t \geq \widehat{T} = \max\{\widetilde{T}(\varepsilon), T\}.$$

This shows that (3.4) in the condition (ii) is valid. □

Similarly we can prove the following theorem which gives another set of necessary and sufficient conditions for the existence of a global attractor in this case and will be used to show the κ-contracting property of the Brusselator semiflow $\{S(t)\}_{t\geq 0}$.

THEOREM 3. *For the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ on H, there exists a global attractor $\mathcal{A}$ in H if and only if the following two conditions are satisfied* :
(i) *There exists a bounded absorbing set B_0 in H for this semiflow.*
(ii) *For any $\varepsilon > 0$, there are positive constants $M = M(\varepsilon)$ and $T = T(\varepsilon)$ such that*

$$(3.9)\qquad \int_{\Omega(|v(t)|\geq M)} |S(t)w_0|^2\,dx < \varepsilon, \quad \textit{for any } t\geq T,\ w_0\in B_0,$$

and

$$\kappa\left((S(t)B_0)_{\Omega(|v(t)|<M)}\right) \longrightarrow 0, \quad \textit{as } t \to \infty, \tag{3.10}$$

where

$$(S(t)B_0)_{\Omega(|v(t)|<M)} \stackrel{\text{def}}{=} \left\{(S(t)w_0)(\cdot)\zeta_M(\cdot\,; t, w_0) : \text{for } w_0 \in B_0\right\},$$

in which $\zeta_M(x; t, w_0)$ *is the characteristic function of the subset* $\Omega(|v(t)| < M)$*, and* $v(t) = v(t, x, w_0)$ *is the* v*-component of the solution of the Brusselator equations* (1.1)–(1.2).

The proof of Theorem 3 is just parallel to the proof of Theorem 2 and is omitted. Note that in Theorem 2 and in Theorem 3 the condition (ii) corresponds to the κ-contracting property.

4. κ-Contracting Property

In this section, we shall check that the conditions in Theorem 3 are satisfied by the Brusselator semiflow. Through this decomposition approach, the existence of a global attractor for this Brusselator semiflow $\{S(t)\}_{t\geq 0}$ will be shown.

Lemma 6. *For any* $\varepsilon > 0$*, there exist positive constants* $M_1 = M_1(\varepsilon)$ *and* $T_1 = T_1(\varepsilon)$ *such that the* v*-component of the solution of the Brusselator equations* (1.1)–(1.2)*,* $v(t) = v(t, x, w_0)$*, satisfies*

$$\int_{\Omega(|v(t)|\geq M_1)} |v(t)|^2\, dx < \frac{2b^2}{d_2\gamma}\varepsilon, \quad \textit{for } t \geq T_1,\ w_0 = (u_0, v_0) \in B_0. \tag{4.1}$$

Proof. First of all, since B_0 is a bounded absorbing set, there exist uniform constants $T_0 = T_0(B_0) > 0$ and $K_0 > 0$, where K_0 is given in (2.17), such that

$$\|v(t)\|^2 = \|v(t,\cdot,w_0)\|^2 \leq \|S(t)w_0\|^2 \leq K_0, \quad \text{for } t \geq T_0,\ w_0 = (u_0, v_0) \in B_0,$$

Hence we have

$$M^2 m(\Omega(|v(t)| \geq M)) \leq \int_{\Omega(|v(t)|\geq M)} |S(t)w_0|^2\, dx \leq K_0,$$

so that there exists an $M = M(\varepsilon) > 0$ such that

$$m(\Omega(|v(t)| \geq M)) \leq \frac{K_0}{M^2} < \frac{\varepsilon}{2}, \quad \text{for } t \geq T_0,\ w_0 \in B_0. \tag{4.2}$$

Taking the inner-product $\langle (1.2), (v(t) - M)_+ \rangle$, where M is given in (4.2) and

$$(\varphi - M)_+ = \begin{cases} \varphi(x) - M, & \text{if } \varphi(x) \geq M, \\ 0, & \text{if } \varphi(x) < M, \end{cases}$$

we obtain

$$\begin{aligned}(4.3)\qquad &\frac{1}{2}\frac{d}{dt}\|(v-M)_+\|^2 + d_2\int_{\Omega(v(t)\geq M)} |\nabla(v-M)_+|^2\,dx \\ &= -\int_{\Omega(v(t)\geq M)} u^2 v(v-M)_+ dx + \int_{\Omega(v(t)\geq M)} bu(v-M)_+ dx \\ &\leq -\int_{\Omega(v(t)\geq M)} \left[\left(u(v-M)_+ - \frac{b}{2}\right)^2 + u^2 M(v-M)_+\right] dx \\ &+ \frac{b^2}{4} m(\Omega(v(t)\geq M)) \leq \frac{b^2}{4}\cdot\frac{\varepsilon}{2} = \frac{b^2}{8}\varepsilon.\end{aligned}$$

It follows that

$$\frac{d}{dt}\|(v-M)_+\|^2 + 2d_2\gamma\|v(t)-M\|^2 \leq \frac{b^2}{4}\varepsilon$$

and, by the Gronwall inequality,

$$\|(v(t)-M)_+\|^2 \leq e^{-2d_2\gamma t}\|(v_0-M)_+\|^2 + \frac{b^2\varepsilon}{8d_2\gamma}.$$

Thus there exists a time $T_+(\varepsilon, M)$ such that for any $t \geq T_+$ and any $w_0 \in B_0$, one has

$$(4.4)\qquad \|(v(t)-M)_+\|^2 = \int_{\Omega(v(t)\geq M)} |v(t)-M|^2\,dx < \frac{b^2\varepsilon}{4d_2\gamma}.$$

Symmetrically we can prove that there exists a time $T_-(\varepsilon, M)$ such that for any $t \geq T_-$ and any $w_0 \in B_0$, one has

$$(4.5)\qquad \|(v(t)+M)_-\|^2 = \int_{\Omega(v(t)\leq -M)} |v(t)+M|^2\,dx < \frac{b^2\varepsilon}{4d_2\gamma},$$

where

$$(\varphi+M)_- = \begin{cases}\varphi(x)+M, & \text{if } \varphi(x)\leq -M,\\ 0, & \text{if } \varphi(x) > -M.\end{cases}$$

Therefore, for any $t \geq T_1(\varepsilon) = \max\{T_+, T_-\}$, and for any $w_0 \in B_0$, there holds

$$(4.6)\qquad \int_{\Omega(|v(t)|\geq M)} (|v(t)|\ - M)^2\,dx < \frac{b^2\varepsilon}{2d_2\gamma}.$$

Next we can deduce that there exists a positive integer k such that

$$(4.7)\qquad \int_{\Omega(|v(t)|\geq kM)} |v(t)|^2\,dx < \frac{2b^2\varepsilon}{d_2\gamma}.$$

Indeed, referring to (4.2), we see that for any $w_0 \in B_0$,

$$(4.8)\qquad m(\Omega(|v(t)| \geq kM)) \leq \frac{K_0}{k^2M^2},$$

where $K_0/M^2 < \varepsilon/2$. Then it follows that

$$\int_{\Omega(|v(t)|\geq kM)} |v(t)|^2\, dx \leq 2\int_{\Omega(|v(t)|\geq M)} (|v(t)| \, - M)^2\, dx + 2M^2 m(\Omega(|v(t)| \geq kM))$$
$$\leq \frac{b^2\varepsilon}{d_2\gamma} + \frac{2M^2K_0}{k^2M^2} = \frac{b^2\varepsilon}{d_2\gamma} + \frac{2K_0}{k^2} < \frac{2b^2\varepsilon}{d_2\gamma}.$$

for a sufficiently large integer $k(\varepsilon)$. Therefore (4.7) holds and, consequently, (4.1) is proved with $M_1 = M_1(\varepsilon) = kM$, where M is given in (4.2), and $T_1 = T_1(\varepsilon) = \max\{T_+, T_-\}$, for any $w_0 = (u_0, v_0) \in B_0$. □

Lemma 7. *Let $P_v : H \to L^2(\Omega)_v$ be the orthogonal projection from the product Hilbert space H onto the second component space associated with the v-component. Let B_0 be the bounded absorbing set of the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ in H, shown in Lemma 3. Then for any given $M > 0$, it holds that*

$$\kappa\left(P_v(S(t)B_0)_{\Omega(|v(t)|<M)}\right) \longrightarrow 0, \quad \text{as } t \to \infty. \tag{4.9}$$

where $(S(t)B_0)_{\Omega(|v(t)|<M)}$ has been specified in Theorem 3.

Proof. Let us consider the inequality similar to (3.2) but with integrals over the set $\Omega_{v,M} = \Omega(|v(t)| < M)$, namely,

$$\frac{d}{dt}\|\nabla v\|^2_{\Omega_{v,M}} + d_2\|\Delta v\|^2_{\Omega_{v,M}} \leq 2\int_{\Omega_{v,M}} v^2|\nabla u|^2\, dx + 2c\|u\|^2_{\Omega_{v,M}},$$

where the $L^2(\Omega_{v,M})$-norm is denoted by $\|\cdot\|_{\Omega_{v,M}}$. Consequently, we have

$$\frac{d}{dt}\|\nabla v\|^2_{\Omega_{v,M}} + d_2\gamma\,\|\nabla v\|^2_{\Omega_{v,M}} \leq 2M^2\|\nabla u\|^2_{\Omega_{v,M}} + 2cK_0, \tag{4.10}$$

for any $w_0 \in B_0$ and any $t \geq T_0 = T_0(B_0)$, where K_0 is given by (2.17) and T_0 is given in the beginning of the proof of Lemma 6. Inequality (4.10) implies that

$$\frac{dy}{dt} \leq gy + h, \quad t > 0, \tag{4.11}$$

where

$$y(t) = \|\nabla v(t)\|^2_{\Omega_{v,M}}, \quad g(t) = d_2\gamma, \quad \text{and}$$
$$h(t) = 2M^2\|\nabla u\|^2_{\Omega_{v,M}} + 2cK_0.$$

From (2.4) now over $\Omega_{v,M}$ we see that there exists a constant $T_2 = T_2(B_0) > 0$, such that

$$\int_t^{t+1} \|\nabla v(s)\|^2_{\Omega_{v,M}}\, ds \leq C_2(M) = \frac{b^2|\Omega_{v,M}|}{d_2}\left(\frac{1}{\gamma d_2} + 1\right), \tag{4.12}$$

for $t \geq T_2,\ w_0 \in B_0$. By integrating the inequality corresponding to (2.6) now over $\Omega_{v,M}$ on the time interval $[t, t+1]$, and using (4.12), we can deduce that there exists a constant $T_3 = T_3(B_0)$, with $T_3 \geq \max\{T_0(B_0), T_2(B_0)\}$, such that

$$d_1 \int_t^{t+1} \|\nabla z(s)\|^2_{\Omega_{v,M}}\, ds \le \frac{|d_1 - d_2|^2}{d_1} C_2(M) + 2\left(\frac{b^2}{\gamma d_2} + 2a^2\right) |\Omega_{v,M}| + 2K_0,$$

for $t \ge T_3$, which implies that

$$\begin{aligned}\int_t^{t+1} \|\nabla u(s)\|^2_{\Omega_{v,M}}\, ds &= \int_t^{t+1} \|\nabla z(s) - \nabla v(s)\|^2_{\Omega_{v,M}}\, ds \\ &\le 2\int_t^{t+1} \left(\|\nabla z(s)\|^2_{\Omega_{v,M}} + \|\nabla v(s)\|^2_{\Omega_{v,M}}\right) ds \\ &\le C_3(M), \qquad \text{for } t \ge T_3,\ w_0 \in B_0,\end{aligned} \tag{4.13}$$

where

$$C_3(M) = 2\left[C_2(M)\left(1 + \frac{|d_1 - d_2|^2}{d_1^2}\right) + \frac{2}{d_1}\left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega_{v,M}| + \frac{2K_0}{d_1}\right].$$

This yields that

$$\int_t^{t+1} h(s)\, ds \le 2M^2 C_3(M) + 2cK_0, \quad \text{for } t \ge T_3,\ w_0 \in B_0. \tag{4.14}$$

Obviously,

$$\int_t^{t+1} g(s) ds \le d_2 \gamma. \tag{4.15}$$

Finally by inequalities (4.12), (4.14) and (4.15), and applying the uniform Gronwall inequality [**24**, **29**] to (4.11), we can conclude that the following inequality holds,

$$\|\nabla v(t)\|^2_{\Omega_{v,M}} \le \left(2M^2 C_3(M) + 2cK_0 + C_2(M)\right) e^{d_2 \gamma} \quad \text{for any } t \ge T_3 + 1, w_0 \in B_0. \tag{4.16}$$

Note that the right-hand side of (4.16) is a uniform constant depending on the absorbing set B_0 and the arbitrarily given M only.

The inequality (4.16) shows that for any fixed $t \ge T_3 + 1$,

$$P_v(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ is a bounded set in } H_0^1(\Omega).$$

Due to the compact Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for $n \le 3$, it turns out that for any fixed $t \ge T_3 + 1$,

$$P_v(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ is a precompact set in } L^2(\Omega).$$

Therefore, by Lemma 4 it is proved that, in the space $L^2(\Omega)$,

$$\kappa\left(P_v(S(t)B_0)_{\Omega(|v(t)|<M)}\right) = 0, \quad \text{for } t \ge T_3(B_0) + 1.$$

This shows that (4.9) is valid for any given $M > 0$. The lemma is proved. □

Next we treat the u-component through $z(t) = u(t) + v(t)$. We shall use the notation

$$\Omega_M^v = \Omega(|v(t)| \ge M).$$

LEMMA 8. *For any* $\varepsilon > 0$, *there exist positive constants* $M_2 = M_2(\varepsilon)$ *and* $\tau_2 = \tau_2(\varepsilon)$ *such that the u-component of the solution of the Brusselator equations* (1.1)–(1.2) *with the initial datum* w_0 *satisfies*

$$\int_{\Omega^v_{M_2}} |u(t)|^2 \, dx < \Gamma_2 \, \varepsilon, \quad \text{for } t \geq \tau_2, \ w_0 \in B_0, \tag{4.17}$$

where $\Gamma_2 > 0$ *is a uniform constant independent of the initial datum* $w_0 \in B_0$.

PROOF. For any $w_0 \in B_0$, according to (4.2), we have

$$m(\Omega(|v(t)| \geq M)) \leq \frac{K_0}{M^2} < \frac{\varepsilon}{2}, \quad \text{for } t > T_0.$$

Let $z(t) = u(t,x,w_0) + v(t,x,w_0)$, where (u,v) is the solution of the Brusselator equations with the initial datum w_0. Taking the inner-product $\langle (2.5), z(t)\rangle_{\Omega^v_M}$ over the set Ω^v_M, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{\Omega^v_M} + d_1\|\nabla z(t)\|^2_{\Omega^v_M} + \|z(t)\|^2_{\Omega^v_M} \\
&\quad = \int_{\Omega^v_M} (d_2 - d_1) z(t) \Delta v(t) \, dx + \int_{\Omega^v_M} (v(t) + a) z(t) \, dx, \\
&\quad \leq \frac{d_1}{2}\|\nabla z(t)\|^2_{\Omega^v_M} + \frac{|d_1 - d_2|^2}{2d_1}\|\nabla v(t)\|^2_{\Omega^v_M} + \frac{1}{2}\|z(t)\|^2_{\Omega^v_M} + \frac{1}{2}\|v(t) + a\|^2_{\Omega^v_M},
\end{aligned}$$

from which we can get

$$\begin{aligned}
&\frac{d}{dt}\|z(t)\|^2_{\Omega^v_M} + d_1\|\nabla z(t)\|^2_{\Omega^v_M} + \|z(t)\|^2_{\Omega^v_M} \\
&\qquad \leq \frac{|d_1 - d_2|^2}{d_1}\|\nabla v(t)\|^2_{\Omega^v_M} + \|v(t) + a\|^2_{\Omega^v_M} \\
&\qquad \leq \frac{|d_1 - d_2|^2}{d_1}\|\nabla v(t)\|^2_{\Omega^v_M} + 2\|v(t)\|^2_{\Omega^v_M} + 2a^2|\Omega^v_M|.
\end{aligned} \tag{4.18}$$

It follows that

$$\frac{d}{dt}\left(e^t \|z(t)\|^2_{\Omega^v_M}\right) \leq \frac{|d_1 - d_2|^2}{d_1} e^t \|\nabla v(t)\|^2_{\Omega^v_M} + e^t\left(2\|v(t)\|^2_{\Omega^v_M} + 2a^2|\Omega^v_M|\right).$$

Integrating both sides of this inequality on $[0,t]$, we obtain

$$\begin{aligned}
\|z(t)\|^2_{\Omega^v_M} = \int_{\Omega^v_M} |z(t)|^2 \, dx &\leq e^{-t}\|u_0 + v_0\|^2_{\Omega^v_M} \\
&\quad + \frac{|d_1 - d_2|^2}{d_1} e^{-t} \int_0^t e^s \|\nabla v(s)\|^2_{\Omega^v_M} \, ds \\
&\quad + e^{-t} \int_0^t e^s \left(2\|v(s)\|^2_{\Omega^v_M} + 2a^2|\Omega^v_M|\right) ds.
\end{aligned} \tag{4.19}$$

Similar to (2.13), we have

$$d_2 \int_0^t e^s \|\nabla v(s)\|^2_{\Omega^v_M} \, ds \leq \left(1 + \frac{1}{2d_2\gamma}\right) b^2 |\Omega^v_M| e^t + \left(1 + \alpha(t) e^t\right) K_0, \tag{4.20}$$

where K_0 is the uniform constant given in (2.17). Then we can estimate the terms on the right-hand side of the inequality (4.19) as follows. For sufficiently large $t > 0$, there holds

$$e^{-t}\|u_0 + v_0\|^2_{\Omega^v_M} \le 2K_0 e^{-t} < \varepsilon.$$

Since $|\Omega^v_M| < \varepsilon/2$ and $\alpha(t) \to 0$ as $t \to \infty$, (4.20) implies that, for sufficiently large $t > 0$,

$$\begin{aligned} &e^{-t}\frac{|d_1 - d_2|^2}{d_1}\int_0^t e^s \|\nabla v(s)\|^2_{\Omega^v_M}\, ds \\ &\le \frac{|d_1 - d_2|^2}{d_1 d_2}\left[\left(1 + \frac{1}{2d_2\gamma}\right) b^2|\Omega^v_M| + \left(e^{-t} + \alpha(t)\right) K_0\right] \\ &< \frac{|d_1 - d_2|^2}{d_1 d_2}\left(1 + \frac{1}{2d_2\gamma}\right) b^2\varepsilon. \end{aligned}$$

Moreover, for sufficientlt large $t > 0$,

$$e^{-t}\int_0^t e^s\|v(s)\|^2_{\Omega^v_M}\, ds \le e^{-t}\int_0^t \frac{e^s}{\gamma}\|\nabla v(s)\|^2_{\Omega^v_M}\, ds < \frac{1}{d_2\gamma}\left(1 + \frac{1}{2d_2\gamma}\right) b^2\varepsilon,$$

and

$$e^{-t}\int_0^t e^s 2a^2|\Omega^v_M|\, ds < 2a^2|\Omega^v_M| < a^2\varepsilon.$$

Finally, substitute these estimates into (4.19), we find that there exists a time $\tau_1 = \tau_1(\varepsilon, M)$ such that

$$\begin{aligned} \|z(t)\|^2_{\Omega^v_M} &= \int_{\Omega(|v(t)|\ge M)} |z(t)|^2\, dx \\ &\le \varepsilon(1 + a^2) + \left(\frac{|d_1 - d_2|^2}{d_1 d_2} + \frac{1}{d_2\gamma}\right)\left(1 + \frac{1}{2d_2\gamma}\right) b^2\varepsilon \\ &= \Gamma_1\varepsilon, \quad \text{for } t \ge \tau_1,\ w_0 \in B_0, \end{aligned} \tag{4.21}$$

where Γ_1 is a uniform constant. Combining (4.21) with (4.1),

$$\|v(t)\|^2_{\Omega^v_{M_1}} < \frac{2b^2}{d_2\gamma}\varepsilon, \quad \text{for } t \ge T_1,$$

we see that there exist positive constants

$$M_2 = \max\{M, M_1\}, \quad \Gamma_2 = 2\left(\Gamma_1 + \frac{2b^2}{d_2\gamma}\right), \quad \tau_2 = \max\{T_1, \tau_1\},$$

such that

$$\int_{\Omega(|v(t)|\ge M_2)} |u(t)|^2\, dx \le 2\left(\|v(t)\|^2_{\Omega^v_{M_1}} + \|z(t)\|^2_{\Omega^v_M}\right) < \Gamma_2\varepsilon,$$

for $t \ge \tau_2, w_0 = (u_0, v_0) \in B_0$. Therefore, (4.17) is proved. □

LEMMA 9. *Let $P_u : H \to L^2(\Omega)_u$ be the orthogonal projection from H onto the first component space associated with the u-component. Let B_0 be the bounded absorbing set of $\{S(t)\}_{t\ge 0}$ in H, shown in Lemma 3. Then for any given $M > 0$, it holds that*

$$\kappa\left(P_u(S(t)B_0)_{\Omega(|v(t)|<M)}\right) \longrightarrow 0, \quad \textit{as } t \to \infty. \tag{4.22}$$

where $(S(t)B_0)_{\Omega(|v(t)|<M)}$ *has been specified in Theorem* 3.

PROOF. We still use the notation $\Omega_{v,M} = \Omega(|v(t)| < M)$ as in Lemma 7. Taking the inner-product $\langle (1.1), -\Delta u(t)\rangle_{\Omega_{v,M}}$, we can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{\Omega_{v,M}} + d_1\|\Delta u\|^2_{\Omega_{v,M}} &= \int_{\Omega_{v,M}} u^2 v(-\Delta u)\,dx \\ &+ (b+1)\int_{\Omega_{v,M}} u\Delta u\,dx - \int_{\Omega_{v,M}} a\Delta u\,dx \\ &\le M\int_{\Omega_{v,M}} u^2|\Delta u|\,dx + \frac{d_1}{4}\|\Delta u\|^2_{\Omega_{v,M}} + \frac{1}{d_1}\|(b+1)u-a\|^2_{\Omega_{v,M}} \\ &\le \frac{M^2}{d_1}\int_{\Omega_{v,M}} u^4\,dx + \frac{d_1}{2}\|\Delta u\|^2_{\Omega_{v,M}} + \frac{1}{d_1}\|(b+1)u-a\|^2_{\Omega_{v,M}}.\end{aligned}$$

Note that for $n \le 3$ the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is continuous and there exists a uniform constant $\delta > 0$ such that

$$\|\varphi\|^2_{L^4(\Omega)} \le \delta\|\varphi\|^2_{H_0^1(\Omega)}, \quad \text{for } \varphi \in H_0^1(\Omega), \tag{4.23}$$

which holds also for functions on $\Omega_{v,M}$. Thus it follows that

$$\begin{aligned}&\frac{d}{dt}\|\nabla u\|^2_{\Omega_{v,M}} + d_1\gamma\|\nabla u\|^2_{\Omega_{v,M}} \\ &\quad\le \frac{2M^2}{d_1}\|u\|^4_{L^4(\Omega_{v,M})} + \frac{4}{d_1}\left((b+1)^2K_0 + a^2|\Omega_{v,M}|\right) \\ &\quad\le \frac{2M^2\delta^2}{d_1}\|\nabla u\|^4 + \frac{4}{d_1}\left((b+1)^2K_0 + a^2|\Omega_{v,M}|\right)\end{aligned} \tag{4.24}$$

for $t \ge T_0$, where T_0 is given at the beginning of the proof of Lemma 6. The inequality (4.24) implies that

$$\frac{dy}{dt} \le gy + h, \quad \text{for } t \ge T_0, \tag{4.25}$$

where

$$y(t) = \|\nabla u\|^2_{\Omega_{v,M}}, \quad g(t) = \frac{2M^2\delta^2}{d_1}\|\nabla u\|^2_{\Omega_{v,M}}, \quad \text{and}$$

$$h(t) = \frac{4}{d_1}\left((b+1)^2K_0 + a^2|\Omega_{v,M}|\right).$$

In view of (4.13), we can apply the uniform Gronwall inequality to (4.25) to obtain

$$\begin{aligned}&\|\nabla u(t)\|^2_{\Omega_{v,M}} \le \\ &\left[C_3(M) + \frac{4}{d_1}\left((b+1)^2K_0 + a^2|\Omega_{v,M}|\right)\right]\exp\left(\frac{2M^2\delta^2}{d_1}C_3(M)\right), \\ &\quad \text{for } t \ge T_4 = \max\{T_0(B_0), T_3(B_0)\} + 1 = T_3(B_0) + 1,\end{aligned} \tag{4.26}$$

where T_3 is given in the proof of Lemma 7. Inequality (4.26) shows that

$$P_u(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ is a bounded set in } H_0^1(\Omega), \text{ for any } t \ge T_4,$$

so that

$$P_u(S(t)B_0)_{\Omega(|v(t)|<M)} \text{ is a precompact set in } L^2(\Omega).$$

Therefore, by Lemma 4, it is proved that in $L^2(\Omega)$,

$$\kappa\left(P_u(S(t)B_0)_{\Omega(|v(t)|<M)}\right) = 0, \quad \text{for } t \geq T_4,$$

and (4.22) is valid. The proof is completed. □

Assembling the results shown in Lemma 6 through Lemma 9, we have proved that the Brusselator semiflow $\{S(t)\}_{t\geq 0}$ has the κ-contracting property. We now give the proof of Theorem 1 (Main Theorem) as follows.

Proof. By Lemma 3, $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set B_0 in H and the condition (i) in Theorem 3 is satisfied. According to Lemmas 6 through 9, this $\{S(t)\}_{t\geq 0}$ satisfies the conditions (ii) in Theorem 3. Then by Theorem 3, there exists a global attractor $\mathcal{A}$ in H for the Brusselator semiflow $\{S(t)\}_{t\geq 0}$. □

By the same approach, we can prove the existence of a global attractor for all the Gray-Scott equations, the Glycolysis equations, and the Schnackenberg equations mentioned in Section 1 as well. For the Gray-Scott equations,

$$\frac{\partial u}{\partial t} = d_1 \Delta u - (F + k)u + u^2 v, \tag{4.27}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + F(1 - v) - u^2 v, \tag{4.28}$$

we can prove the following result.

Theorem 4. *For any positive parameters d_1, d_2, F and k, there exists a global attractor $\mathcal{A}_Q$ in H for the solution semiflow $\{Q(t)\}_{t\geq 0}$ generated by the Gray-Scott equations* (4.27)–(4.28).

Proof. Very similar to (2.1), we have the following inequality for the v-component,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 &= \int_\Omega (-u^2v^2 - Fv^2)\,dx + \int_\Omega Fv\,dx \\ &\leq \frac{1}{2}\left(d_2\gamma\|v\|^2 + \frac{F^2}{d_2\gamma}|\Omega|\right), \end{aligned} \tag{4.29}$$

so that

$$\frac{d}{dt}\|v\|^2 + d_2\gamma\|v\|^2 \leq \frac{F^2}{d_2\gamma}|\Omega|. \tag{4.30}$$

On the other hand, for the Gray-Scott equations, $z(t) = u(t) + v(t)$ satisfies the following equation which is very similar to (2.5) in the Brusselator case,

$$z_t = d_1\Delta z - (F + k)z + [(d_2 - d_1)\Delta v + kv + F]. \tag{4.31}$$

Starting from (4.29) and (4.31), and exactly parallel to the lemmas shown in Section 2 and Section 4, one can prove the existence of a global attractor in H for the solution semiflow $\{Q(t)\}_{t\geq 0}$ generated by the Gray-Scott equations (4.27)–(4.28). We omit the details. □

For the Glycolysis equations,

$$\frac{\partial u}{\partial t} = d_1 \Delta u - u + kv + u^2 v + \rho, \tag{4.32}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - kv - u^2 v + \delta, \tag{4.33}$$

we can also prove the existence of a global attractor in the same way.

THEOREM 5. *For any positive parameters d_1, d_2, k, ρ, and δ, there exists a global attractor $\mathcal{A}_{\mathcal{G}}$ in H for the solution semiflow $\{G(t)\}_{t\geq 0}$ generated by the Glycolysis equations* (4.32)–(4.33).

PROOF. Correspondingly we have the following inequality for the v-component,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 &= \int_\Omega (-u^2v^2 - kv^2)\,dx + \int_\Omega \delta v\,dx \\ &\leq \frac{1}{2}\left(d_2\gamma\|v\|^2 + \frac{\delta^2}{d_2\gamma}|\Omega|\right),\end{aligned} \tag{4.34}$$

and the following equation satisfied by $z(t) = u(t) + v(t)$,

$$z_t = d_1\Delta z - z + [(d_2 - d_1)\Delta v + v + (\rho + \delta)]. \tag{4.35}$$

Starting from (4.34) and (4.35), exactly parallel to the lemmas shown in Section 2 and Section 4, one can prove the existence of a global attractor in H for the solution semiflow $\{G(t)\}_{t\geq 0}$ generated by the Glycolysis equations (4.32)–(4.33). □

For the Schnackenberg equations [**23**, **31**],

$$\frac{\partial u}{\partial t} = d_1 \Delta u - ku + u^2 v + a, \tag{4.36}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v - u^2 v + b, \tag{4.37}$$

the following theorem also holds.

THEOREM 6. *For any positive parameters d_1, d_2, k, a, and b, there exists a global attractor $\mathcal{A}_{\mathcal{R}}$ in H for the solution semiflow $\{R(t)\}_{t\geq 0}$ generated by the Schnackenberg equations* (4.36)–(4.37).

PROOF. In this case, we have the following inequality for the v-component,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + d_2\|\nabla v\|^2 = \int_\Omega (-u^2v^2 + bv)\,dx \leq \frac{1}{2}\left(d_2\gamma\|v\|^2 + \frac{b^2}{d_2\gamma}|\Omega|\right), \tag{4.38}$$

and the following equation satisfied by $z(t) = u(t) + v(t)$,

$$z_t = d_1\Delta z - kz + [(d_2 - d_1)\Delta v + kv + (a + b)]. \tag{4.39}$$

Therefore, starting from (4.38) and (4.39), and exactly parallel to the lemmas shown in Section 2 and Section 4, one can prove the existence of a global attractor in H for the solution semiflow $\{R(t)\}_{t\geq 0}$ generated by the Schnackenberg equations (4.36)–(4.37). □

5. Finite Dimensionality of Global Attractor

In this section we shall prove that the global attractor $\mathcal{A}$ for the Brusselator semiflow has finite Hausdorff and fractal dimensions and estimate upper bounds for these dimensions.

DEFINITION 6. Let X be a complete metric space and Y be a subset of X. Given $d > 0$, the d-dimensional Hausdorff measure of Y is defined by

$$\mu_H(Y, d) = \lim_{\varepsilon \to 0^+} \mu(Y, d, \varepsilon) = \sup_{\varepsilon > 0} \mu(Y, d, \varepsilon),$$

where

$$\mu(Y, d, \varepsilon) = \inf \sum_{i \in I} r_i^d,$$

where the infimum is taken over all coverings of Y by a family of balls $\{B_i\}_{i \in I}$ in X of radii $r_i \le \varepsilon$. The Hausdorff dimension of Y is defined as the unique number $d_H(Y) \in [0, \infty]$ which satisfies the folllowing two conditions,

$$\begin{aligned} \mu_H(Y, d) &= 0, \quad \text{for } d > d_H(Y), \\ \mu_H(Y, d) &= \infty, \quad \text{for } d < d_H(Y). \end{aligned}$$

DEFINITION 7. The fractal dimension of a subset Y in a complete metric space X is defined as the number

$$d_F(Y) = \limsup_{\varepsilon \to 0^+} \frac{\log n_Y(\varepsilon)}{\log 1/\varepsilon},$$

where $n_Y(\varepsilon)$ is the minimum number of balls in X of radius ε which cover the set Y.

One can use the alternative expression [**29**] for $d_F(Y)$:

$$d_F(Y) = \inf\{d > 0 : \mu_F(Y, d) = \limsup_{\varepsilon \to 0^+} (\varepsilon^d n_Y(\varepsilon)) = 0\}.$$

Note that $\mu_H(Y, d) \le \mu_F(Y, d)$ and $d_H(Y) \le d_F(Y)$.

The following lemma [**29**, Chapter 5] based on the theory of Lyapunov exponents will be used to estimate the upper bounds of the Hausdorff and fractal dimensions of the global attractor $\mathcal{A}$.

LEMMA 10. *Let $\mathcal{A}$ be the global attractor of the Brusselator semiflow $\{S(t)\}_{t \ge 0}$ in H. Define*

$$q_m(t) = \sup_{w_0 \in \mathcal{A}} \sup_{\substack{\xi_i \in H, \|\xi_i\| = 1 \\ i = 1, \cdots, m}} \left(\frac{1}{t} \int_0^t \mathrm{Tr}\,(A + F'(S(\tau)w_0)) \circ Q_m(\tau)\, d\tau \right), \tag{5.1}$$

$$q_m = \limsup_{t \to \infty} q_m(t), \tag{5.2}$$

where $\mathrm{Tr}\,(A + F'(S(\tau)w_0))$ is the trace of the linear operator $A + F'(S(\tau)w_0)$, with $F(w)$ being the nonlinear map in (1.6), and $Q_m(t)$ stands for the orthogonal projection of the space H on the subspace spanned by $y_1(t), \cdots, y_m(t)$, with

$$y_i(t) = L(S(t)w_0)\xi_i, \quad i = 1, \cdots, m. \tag{5.3}$$

Here $F'(S(\tau)w_0)$ is the Fréchet derivative of that map F at $S(\tau)w_0$, and $L(S(t)w_0)$ is the Fréchet derivative of the map $S(t)$ at w_0, with t being fixed. If there is an *integer m such that $q_m < 0$, then the Hausdorff dimension and the fractal dimension of $\mathcal{A}$ satisfy, respectively,*

$$d_H(\mathcal{A}) \le m, \text{ and} \tag{5.4}$$

$$d_F(\mathcal{A}) \le m \max_{1\le j\le m-1}\left(1+\frac{(q_j)_+}{|q_m|}\right) \le 2m. \tag{5.5}$$

First it can be shown that for any given $t > 0$, $S(t)$ is Fréchet differentiable in H and its Fréchet derivative at w_0 is the bounded linear operator $L(S(t)w_0)$ given by

$$L(S(t)w_0)\xi \overset{\text{def}}{=} y(t) = (U(t), V(t)), \quad \text{for any } \xi = (\eta, \zeta) \in H,$$

where $(U(t), V(t))$ is the strong solution of the following Brusselator variational equation

$$\frac{\partial U}{\partial t} = d_1\Delta U + 2u(t)v(t)U + u^2(t)V - (b+1)U, \tag{5.6}$$

$$\frac{\partial V}{\partial t} = d_2\Delta V - 2u(t)v(t)U - u^2(t)V + bU, \tag{5.7}$$

$$U(0) = \eta, \quad V(0) = \zeta. \tag{5.8}$$

Here $w(t) = (u(t), v(t)) = S(t)w_0$ is the solution of the Brusselator evolutionary equation (1.6) with the initial condition $w(0) = w_0$. Due to the space limitation, we omit the detailed verification of this fact. The initial value problem (5.6)–(5.7)–(5.8) can be written as

$$\frac{dy}{dt} = (A + F'(S(t)w_0))y, \quad y(0) = \xi. \tag{5.9}$$

Note that the invariance of $\mathcal{A}$ implies $\mathcal{A} \subset B_0$, where B_0 is the bounded absorbing set given in Lemma 3. Hence we have

$$\sup_{w_0\in\mathcal{A}} \|S(t)w_0\|^2 \le K_0,$$

where K_0 is the constant given in (2.17). Here we need one more property of the global attractor $\mathcal{A}$, stated in the next lemma.

LEMMA 11. *Let $\mathcal{A}$ be the global attractor of the Brusselator semiflow $\{S(t)\}_{t\ge 0}$ in H. There there exists a uniform constant $K_1 > 0$ such that*

$$\|\nabla w\|^2 \le K_1, \quad \text{for any } w \in \mathcal{A}. \tag{5.10}$$

PROOF. In (1.6), $A : D(A)(= W) \to H$ is a positive sectorial operator and $F \in C^{\text{Lip}}_{\text{loc}}(E, H)$. By (1.5), for any $w_0 \in \mathcal{A}$, there is a $t_0 \in (0, \frac{1}{2})$ such that $S(t_0)w_0 \in E$. Since $\mathcal{A} \subset H$ is compact and invariant, $S(t)\mathcal{A} = \mathcal{A}$, by the solution theory in [**24**, Section 4.7], one has

$$S(\cdot)w_0 \in C([t_0,\infty), E) \cap C^{0,\frac{1}{2}}_{\text{loc}}((t_0,\infty), E) \cap C((t_0,\infty), W), \tag{5.11}$$

were $C^{0,\frac{1}{2}}_{\text{loc}}$ stands for the space of Hölder strongly continuous functions with exponent 1/2. Since for any $\widehat{w} \in \mathcal{A}$ and any $t \ge 1$, there is a particular $w_0 \in \mathcal{A}$ such that $\widehat{w} = S(t)w_0$, the global attractor has the regularity $\mathcal{A} \subset E$. But this does not imply in general that $\mathcal{A}$ attracts bounded sets with respect to the norm of E.

Next we prove that $\mathcal{A}$ is a bounded set in E. Suppose the contrary. Then there exist sequences $\{N_\ell\} \subset (0, \infty)$, with $N_\ell \geq \ell$, and $\{w_\ell\} \subset \mathcal{A}$, such that

$$\|\nabla w_\ell\| \geq N_\ell, \quad \ell = 1, 2, \cdots .$$

Let $w_\ell^0 \in \mathcal{A}$ be given such that $w_\ell = S(N_\ell)w_\ell^0$. By (5.11), there is a Hölder constant $c_0 > 0$ such that

$$\|\nabla S(t)w_\ell^0\|^2 \geq \left(N_\ell - \frac{c_0}{\sqrt{2}}\right)_+^2, \quad \text{for } t \in I_\ell = (N_\ell - \frac{1}{2}, N_\ell + \frac{1}{2}).$$

This shows that

$$\int_{N_\ell - \frac{1}{2}}^{N_\ell + \frac{1}{2}} \|\nabla S(\tau)w_\ell^0\|^2 \, d\tau \geq \left(N_\ell - \frac{c_0}{\sqrt{2}}\right)_+^2 \longrightarrow \infty, \quad \text{as } \ell \to \infty.$$

This is a contradiction to the fact that for any $w_0 \in \mathcal{A}$, due to (2.4),

$$\int_t^{t+1} \|\nabla v(s)\|^2 \, ds \leq c_1 = \frac{1}{d_2}\left(K_0 + \frac{b^2|\Omega|}{2\gamma d_2}\right) + \frac{b^2|\Omega|}{d_2}, \quad \text{for any } \ t > 0,$$

and, due to (2.6),

$$\int_t^{t+1} \|\nabla z(s)\|^2 \, ds \leq c_2 = \frac{1}{d_1}\left[3K_0 + \frac{|d_2 - d_1|^2}{d_1} c_1 + \left(\frac{b^2}{\gamma d_2} + 2a^2\right)|\Omega|\right],$$

for any $t > 0$, where c_1 and c_2 are uniform constants. Therefore, the conclusion is valid. □

Now we can prove the finite dimensionality of the global attractors.

THEOREM 7. *The global attractors $\mathcal{A}$ for the Brusselator equations, $\mathcal{A}_\mathcal{Q}$ for the Gray-Scott equations, $\mathcal{A}_\mathcal{G}$ for the Glycolysis equations, and $\mathcal{A}_\mathcal{R}$ for the Schnackenberg equations all have finite Hausdorff dimenions and fractal dimensions, respectively.*

PROOF. By Lemma 10, we shall estimate $\operatorname{Tr}(A + F'(S(\tau)w_0)) \circ Q_m(\tau)$. At any given time $\tau > 0$, let $\{\varphi_j(\tau) : j = 1, \cdots, m\}$ be an H-orthonormal basis for the subspace

$$Q_m(\tau)H = \operatorname{Span}\{y_1(\tau), \cdots, y_,(\tau)\},$$

where $y(t) = (y_1(t), \cdots, y_m(t))$ satisfies (5.9) with $y(0) = \xi = (\xi_1, \cdots, \xi_m)$, and without loss of generality assuming that $\{\xi_1, \cdots, \xi_m\}$ is a linearly independent set in H. By the Gram-Schmidt orthogonalization scheme, $\varphi_j(\tau) = (\varphi_j^1(\tau), \varphi_j^2(\tau)) \in E$, for $j = 1, \cdots, m$, since $y_1(\tau), \cdots, y_m(\tau) \in E$, for $\tau > 0$, and $\varphi_j(\tau)$ are strongly measurable in τ. Let $d_0 = \min\{d_1, d_2\}$. Then we have

$$\begin{aligned}
\text{(5.12)} \qquad & \operatorname{Tr}(A + F'(S(\tau)w_0) \circ Q_m(\tau) \\
&= \sum_{j=1}^m \langle A\varphi_j(\tau), \varphi_j(\tau)\rangle + \sum_{j=1}^m \langle F'(S(\tau)w_0)\varphi_j(\tau), \varphi_j(\tau)\rangle \\
&\leq -d_0 \sum_{j=1}^m \|\nabla \varphi_j(\tau)\|^2 + J_1 + J_2 + J_3,
\end{aligned}$$

where

$$J_1 = \sum_{j=1}^m \int_\Omega 2u(\tau)v(\tau)\left(|\varphi_j^1(\tau)|^2 - \varphi_j^1(\tau)\varphi_j^2(\tau)\right) dx$$
$$\le \sum_{j=1}^m \int_\Omega 2|u(\tau)||v(\tau)|\left(|\varphi_j^1(\tau)|^2 + |\varphi_j^1(\tau)||\varphi_j^2(\tau)|\right) dx,$$

$$J_2 = \sum_{j=1}^m \int_\Omega u^2(\tau)\left(\varphi_j^1(\tau)\varphi_j^2(\tau) - |\varphi_j^2(\tau)|^2\right) dx \le \sum_{j=1}^m \int_\Omega u^2(\tau)|\varphi_j^1(\tau)||\varphi_j^2(\tau)|\, dx,$$

and

$$J_3 = \sum_{j=1}^m \int_\Omega \left(-(b+1)|\varphi_j^1(\tau)|^2 + b\varphi_j^1(\tau)\varphi_j^2(\tau)\right) dx$$
$$\le \sum_{j=1}^m \int_\Omega b\varphi_j^1(\tau)\varphi_j^2(\tau)\, dx.$$

We can estimate each of the three terms as follows. First, by the generalized Hölder inequality and the Sobolev embedding $E \hookrightarrow L^4(\Omega)\times L^4(\Omega)$ for $n \le 3$, and using Lemma 11, we get

$$\begin{aligned} J_1 &\le 2\sum_{j=1}^m \|u(\tau)\|_{L^4}\|v(\tau)\|_{L^4}\left(\|\varphi_j^1(\tau)\|_{L^4}^2 + \|\varphi_j^1(\tau)\|_{L^4}\|\varphi_j^2(\tau)\|_{L^4}\right) \\ &\le 4\sum_{j=1}^m \|S(\tau)w_0\|_{L^4}^2\|\varphi_j(\tau)\|_{L^4}^2 \le 4\delta\sum_{j=1}^m \|\nabla S(\tau)w_0\|^2\|\varphi_j(\tau)\|_{L^4}^2 \\ &\le 4\delta K_1\sum_{j=1}^m \|\varphi_j(\tau)\|_{L^4}^2, \end{aligned} \tag{5.13}$$

where δ is the Sobolev embedding coefficient given in (4.23). Now we apply the Garliardo-Nirenberg interpolation inequality, cf. [**24**, Theorem B.3],

$$\|\varphi\|_{W^{k,p}} \le C\|\varphi\|_{W^{m,q}}^\theta \|\varphi\|_{L^r}^{1-\theta}, \quad \text{for } \varphi \in W^{m,q}(\Omega), \tag{5.14}$$

provided that $p, q, r \ge 1, 0 < \theta \le 1$, and

$$k - \frac{n}{p} \le \theta\left(m - \frac{n}{q}\right) - (1-\theta)\frac{n}{r}, \quad \text{where } n = \dim\Omega.$$

Here with $W^{k,p}(\Omega) = L^4(\Omega), W^{m,q}(\Omega) = H_0^1(\Omega), L^r(\Omega) = L^2(\Omega)$, and $\theta = n/4 \le 3/4$, it follows from (5.14) that

$$\|\varphi_j(\tau)\|_{L^4} \le C\|\nabla\varphi_j(\tau)\|^{\frac{n}{4}}\|\varphi_j(\tau)\|^{1-\frac{n}{4}} = C\|\nabla\varphi_j(\tau)\|^{\frac{n}{4}}, \quad j = 1, \cdots, m, \tag{5.15}$$

since $\|\varphi_j(\tau)\| = 1$, where C is a uniform constant. Substitute (5.15) into (5.13) to obtain

$$J_1 \le 4\delta K_1 C^2 \sum_{j=1}^m \|\nabla\varphi_j(\tau)\|^{\frac{n}{2}}. \tag{5.16}$$

Similarly, by the generalized Hölder inequality, we can get

$$J_2 \le \delta K_1 \sum_{j=1}^{m} \|\varphi_j(\tau)\|_{L^4}^2 \le \delta K_1 C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}}. \tag{5.17}$$

Moreover, we have

$$J_3 \le \sum_{j=1}^{m} b\|\varphi_j(\tau)\|^2 = bm. \tag{5.18}$$

Substituting (5.16), (5.17) and (5.18) into (5.12), we obtain

$$\begin{aligned} \operatorname{Tr}(A + F'(S(\tau)w_0) \circ Q_m(\tau) \le -d_0 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + \\ 5\delta K_1 C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}} + bm. \end{aligned} \tag{5.19}$$

By Young's inequality, for $n \le 3$, we have

$$5\delta K_1 C^2 \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^{\frac{n}{2}} \le \frac{d_0}{2} \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + K_2(n)m,$$

where $K_2(n)$ is a uniform constant depending only on $n = \dim(\Omega)$. Hence,

$$\operatorname{Tr}(A + F'(S(\tau)w_0) \circ Q_m(\tau) \le -\frac{d_0}{2} \sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 + (K_2(n) + b)\, m.$$

According to the generalized Sobolev-Lieb-Thirring inequality [**29**, Appendix, Corollary 4.1], since $\{\varphi_1(\tau), \cdots, \varphi_m(\tau)\}$ is an orthonormal set in H, so there exists a uniform constant $K_3 > 0$ only depending on the shape and dimension of Ω, such that

$$\sum_{j=1}^{m} \|\nabla \varphi_j(\tau)\|^2 \ge K_3 \frac{m^{1+\frac{2}{n}}}{|\Omega|^{\frac{2}{n}}}. \tag{5.20}$$

Therefore, we end up with

$$\operatorname{Tr}(A + F'(S(\tau)w_0) \circ Q_m(\tau) \le -\frac{d_0 K_3}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + (K_2(n) + b)\, m. \tag{5.21}$$

Then we can conclude that

$$\begin{aligned} q_m(t) &= \sup_{w_0 \in \mathcal{A}} \sup_{\substack{\xi_i \in H, \|\xi_i\| = 1 \\ i = 1, \cdots, m}} \left(\frac{1}{t} \int_0^t \operatorname{Tr}(A + F'(S(\tau)w_0)) \circ Q_m(\tau)\, d\tau \right) \\ &\le -\frac{d_0 K_3}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + (K_2(n) + b)\, m, \quad \text{for any } t > 0, \end{aligned} \tag{5.22}$$

so that

$$q_m = \limsup_{t \to \infty} q_m(t) \le -\frac{d_0 K_3}{2|\Omega|^{\frac{2}{n}}} m^{1+\frac{2}{n}} + (K_2(n) + b)\, m < 0, \tag{5.23}$$

if the integer m satisfies the following condition,

$$m - 1 \le \left(\frac{2(K_2(n) + b)}{d_0 K_3} \right)^{n/2} |\Omega| < m. \tag{5.24}$$

According to Lemma 10, we have shown that the Hausdorff dimension and the fractal dimension of the global attractor $\mathcal{A}$ are finite and their upper bounds are given by

$$d_H(\mathcal{A}) \leq m \quad d_F(\mathcal{A}) \leq 2m,$$

where m satisfies (5.24). Thus the theorem is proved for the Brusselator semiflow. The proof for the other three model equations is done similarly. □

References

[1] M. Al-Ghoul and B.C. Eu, *Hyperbolic reaction-diffusion equations, patterns, and phase speeds for the Brusselator*, J. Phys. Chemistry, **100** (1996), 18900-18910.

[2] M. Ashkenazi and H.G. Othmer, *Spatial patterns in coupled biochemical oscillators*, J. Math. Biology, **5** (1978), 305-350.

[3] J.F.G. Auchmuty and G. Nicolis, *Bifurcation analysis of nonlinear reaction-diffusion equations - I: Evolution equations and the steady state solutions*, Bull. Math. Biology, **37** (1975), 323-365.

[4] P.W. Bates, K. Lu, and B. Wang, *Attractors for lattice dynamical systems*, Intern. J. Bifurcation and Chaos, **11** (2001), 143-153.

[5] K.J. Brown and F.A. Davidson, *Global bifurcation in the Brusselator system*, Nonlinear Analysis, **24** (1995), 1713-1725.

[6] A. Doelman, T.J. Kaper, and P.A. Zegeling, *Pattern formation in the one-dimensional Gray-Scott model*, Nonlinearity, **10** (1997), 523-563.

[7] I.R. Epstein, *Complex dynamical behavior in simple chemical systems*, J. Phys. Chemistry, **88** (1984), 187-198.

[8] I.R. Epstein and J.A. Pojman, *An Introduction to Nonlinear Chemical Dynamics*, Oxford Univ. Press, New York, 1998.

[9] T. Erneux and E. Reiss, *Brusselator isolas*, SIAM J. Appl. Math., **43** (1983), 1240-1246.

[10] R.J. Fields and M. Burger, *Oscillation and Traveling Waves in Chemical Systems*, John Wiley and Sons, 1985.

[11] P. Gray and S.K. Scott, *Sustained oscillations and other exotic patterns of behavior in isothermal reactions*, J. Phys. Chemistry, **89** (1985), 22-32.

[12] P. Gray and S.K. Scott, *Chemical Waves and Instabilities*, Clarendon, Oxford, 1990.

[13] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[14] T. Kolokolnikov, T. Erneux, and J. Wei, *Mesa-type patterns in one-dimensional Brusselator and their stability*, Physica D, **214**(1) (2006), 63-77.

[15] K.J. Lee, W.D. McCormick, Q. Ouyang, and H. Swinney, *Pattern formation by interacting chemical fronts*, Science, **261** (1993), 192-194.

[16] R. Lefever and G. Nicolis, *Chemical instabilities and sustained oscillations*, J. Theoretical Biology, **30** (1971), 267-284.

[17] Q. Ma, S. Wang, and C. Zhong, *Necessary and sufficient conditions for the existence of global attractors for semigroups and applications*, Indiana Univ. Math. Journal, **51** (2002), 1541-1559.

[18] J.E. Pearson, *Complex patterns in a simple system*, Science, **261** (1993), 189-192.

[19] I. Prigogine and R. Lefever, *Symmetry-breaking instabilities in dissipative systems*, J. Chem. Physics, **48** (1968), 1665-1700.

[20] B. Peña and C. Pérez-García, *Stability of Turing patterns in the Brusselator model*, Phys. Review E, **64**(5), 2001.

[21] Ya.B. Pesin and A.A. Yurchenko, *Some physical models of the reaction-diffusion equation and coupled map lattices*, Russian Math. Surveys, **59**(3) (2004), 481-513.

[22] T. Rauber and G. Runger, *Aspects of a distributed solution of the Brusselator equation*, Proc. of the First Aizu International Symposium on Parallel Algorithms and Architecture Syntheses, (1995), 114-120.

[23] J. Schnackenberg, *Simple chemical reaction systems with limit cycle behavior*, J. Theor. Biology, **81** (1979), 389-400.

[24] G.R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer, New York, 2002.

[25] M. Stanislavova, A. Stefanov, and B. Wang, *Asymptotic smoothing and attractors for the generalized Benjamin-Bona-Mahony equations on* $\Re^3$, J. Dff. Eqns., **219** (2005), 451-483.
[26] S.H. Strogatz, *Nonlinear Dynamics and Chaos*, Westview Press, 1994.
[27] C. Sun, M. Yang, and C. Zhong, *Global attractors for the wave equation with nonlinear damping*, J. Diff. Eqns., **227** (2006), 427-443.
[28] C. Sun and C. Zhong, *Attractors for the semilinear reaction-diffusion equation with distributed derivatives in unbounded domains*, Nonlinear Analysis, **63** (2005), 49-65.
[29] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1988.
[30] B. Wang, *Attractors for reaction-diffusion equation in unbounded domains*, Physica D, **128** (1999), 41-52.
[31] M.J. Ward and J. Wei, *The existence and stability of asymmetric spike patterns for the Schnackenberg model*, Stud. Appl. Math., **109** (2002), 229-264.
[32] J. Wei and M. Winter, *Asymmetric spotty patterns for the Gray-Scott model in* $\Re^2$, Stud. Appl. math., **110** (2003), 63-102.
[33] A. De Wit, D. Lima, G. Dewel, and P. Borckmans, *Spatiotemporal dynamics near a codimension-two point*, Phys. Review E, **54** (1), 1996.
[34] Y. You, *Global dynamics of nonlinear wave equations with cubic non-monotone damping*, Dynamics of PDE, **1** (2004), 65-86.
[35] Y. You, *Finite dimensional reduction of global dynamics and lattice dynamics of a damped nonlinear wave equation*, in "Control Theory and mathematical Finance", edit. S. Tang and J. Yong, World Scientific, 2005, in press.

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620
E-mail address: you@math.usf.edu