

Large Time Behavior of the Zero Dispersion Limit of the Fifth Order KdV Equation

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ABSTRACT. We study the zero dispersion limit of the fifth order KdV equations when time is sufficiently large. In general, the weak limit may be described by an arbitrary odd number of hyperbolic equations. Unlike the KdV case, these are non-strictly hyperbolic equations. However, we show that the weak limit is governed by three hyperbolic equations in a domain in the space-time for all times bigger than a large time. Outside this domain, the weak limit satisfies a single hyperbolic equation.

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1. Introduction

It is well known that the solution of the KdV equation

$$(1.1) \quad u_t + 6uu_x + \epsilon^2 u_{xxx} = 0$$

has a weak limit as $\epsilon \rightarrow 0$ while the initial values

$$(1.2) \quad u(x, 0; \epsilon) = u_0(x)$$

are fixed [7, 15]. This weak limit satisfies the Burgers equation

$$(1.3) \quad u_t + (3u^2)_x = 0$$

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until its solution develops shocks. Immediately after, the weak limit is governed by the Whitham equations [2, 7, 15, 17]

$$(1.4) \quad u_{it} + \lambda_i(u_1, u_2, u_3)u_{ix} = 0, \quad i = 1, 2, 3,$$

where the λ_i 's are given by formulae (2.30). After the breaking of the solution of (1.4), the weak limit is described by systems of at least five hyperbolic equations similar to (1.4).

The KdV equation (1.1) is just the first member of an infinite sequence of equations, the second of which is the so-called fifth order KdV equation

$$(1.5) \quad u_t + 30u^2u_x + 20\epsilon^2u_xu_{xx} + 10\epsilon^2uu_{xxx} + \epsilon^4u_{xxxxx} = 0.$$

The solution of the fifth order KdV equation (1.5) also has a weak limit as $\epsilon \rightarrow 0$ [8]. As in the KdV case, this weak limit satisfies the Burgers type equation

$$(1.6) \quad u_t + (10u^3)_x = 0$$

until the solution of (1.6) forms a shock. Later, the limit is governed by equations similar to (1.4), namely,

$$(1.7) \quad u_{it} + \mu_i(u_1, u_2, u_3)u_{ix} = 0, \quad i = 1, 2, 3,$$

where μ_i 's are given in (2.29). They will be also be called the Whitham equations [10].

In this paper, we are interested in the large time behavior of the weak limit of the fifth order KdV equation (1.5).

For simplicity, we consider $u_0(x)$ of (1.2) which is a decreasing function and is bounded at $x = \pm\infty$:

$$(1.8) \quad \lim_{x \rightarrow -\infty} u_0(x) = 1, \quad \lim_{x \rightarrow +\infty} u_0(x) = 0.$$

The large time behavior of the weak limit of the KdV equation (1.1) has been studied [13]. For generic initial data of (1.8), the weak limit is governed by the Whitham equations (1.4) in a domain in the space-time when t is sufficiently large. Outside this domain, the weak limit satisfies the Burgers equation (1.3). We note that the weak limit may be described by an arbitrary odd number of hyperbolic equations in the intermedia times.

The strong hyperbolicity of the Whitham equations (1.4) for the KdV plays an important role in the paper of [13]. It is well known that equations (1.4) are strictly hyperbolic:

$$\lambda_i(u_1, u_2, u_3) \neq \lambda_j(u_1, u_2, u_3), \quad i, j = 1, 2, 3; i \neq j,$$

and genuinely nonlinear:

$$\frac{\partial}{\partial u_i} \lambda_i(u_1, u_2, u_3) \neq 0, \quad i = 1, 2, 3,$$

for $u_1 > u_2 > u_3$ [9].

However, in the case of fifth order KdV equation (1.5), the Whitham equations (1.7) are neither strictly hyperbolic nor genuinely nonlinear [10].

In this paper, we use the method developed in [3] and [4] to study the weak limit of the fifth order KdV (1.5) for sufficiently large time t . Our approach does not require the Whitham equations to be strongly hyperbolic. Our result is quite similar to the KdV result. Namely, For generic initial data of (1.8), the weak limit is governed by the Whitham equations (1.7) in a domain $x^-(t) < x < x^+(t)$ in the

space time when t is sufficiently large (see Figure 1.). Outside this domain, the weak limit satisfies the Burgers type equation (1.6).

We also find some difference between the KdV case and the fifth order KdV case. In the latter case, the trailing edge is “lifted”, i.e.,

$$u_1 \approx 1, \quad u_2 = u_3 \approx \frac{1}{4} \quad \text{for large } t > 0$$

at the trailing edge $x = x^-(t)$ (see Figure 1.). In the KdV case, instead we have

$$u_1 \approx 1, \quad u_2 = u_3 \approx 0 \quad \text{for large } t > 0$$

at the trailing edge [5, 7].

This phenomenon of “lifted” trailing edge has also been observed in the weak limit of the fifth order KdV equation when the initial values $u_0(x)$ of (1.2) are given by a step-like function

$$u_0(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}.$$

In this case, we have

$$u_1 = 1, \quad u_2 = u_3 = \frac{1}{4}$$

at the trailing edge $x = -15t$ [10].

The organization of this paper is as follows. In Section 2, we will summarize the method of [3]. In Section 3, we will describe our main theorem. In Section 4, we will study the trailing and leading edges, which separate the Burgers type solutions and the Whitham solutions, when t is large. In Section 5, we will study the Whitham solutions that live between the trailing and leading edges.

2. A Minimization Problem

As in the KdV case, the weak limit of the fifth order KdV equation (1.5) is determined by a minimization problem with constraints [8].

The minimization problem is

$$(2.1) \quad \text{Minimize}_{\{\psi \geq 0, \psi \in L^1\}} \left\{ -\frac{1}{2\pi} \int_0^1 \int_0^1 \log \left| \frac{\eta - \mu}{\eta + \mu} \right| \psi(\eta) \psi(\mu) d\eta d\mu + \int_0^1 a(\eta, x, t) \psi(\eta) d\eta \right\}.$$

Function $a(\eta, x, t)$ is given by

$$(2.2) \quad \begin{aligned} a(\eta, x, t) &= \eta x - 16\eta^5 t - \theta(\eta), \\ \theta(\eta) &= \eta f(\eta^2) + \int_{f(\eta^2)}^{+\infty} [\eta - \sqrt{\eta^2 - u_0(x)}] dx, \end{aligned}$$

where $f(u)$ is the inverse function of the initial data $u_0(x)$. We note that in the KdV case, we have $a(\eta, x, t) = \eta x - 4\eta^3 t - \theta(\eta)$ instead of (2.2) [3, 7, 15].

We now give a brief summary of our approach to the minimization problem (2.1).

We first follow Lax & Levermore [7] and Venakides [15, 16] to make the ansatz that the support of ψ consists of a finite union of disjoint intervals

$$(2.3) \quad S = [0, \sqrt{u_{2g+1}}] \cup [\sqrt{u_{2g}}, \sqrt{u_{2g-1}}] \cup \cdots \cup [\sqrt{u_2}, \sqrt{u_1}],$$

where $0 < u_{2g+1} < \cdots < u_2 < u_1 < 1$. Hence, g is the number of gaps in the support S .

We then introduce

$$(2.4) \quad P(\xi, \vec{u}) = 2R^2(\xi, \vec{u})\Phi_g(\xi, \vec{u}) + Q(\xi, \vec{u}),$$

where \vec{u} denotes $(u_1, u_2, \dots, u_{2g+1})$ and $R(\xi, \vec{u}) = \sqrt{(\xi - u_1) \cdots (\xi - u_{2g+1})}$ with the sign given by $\sqrt{1} = 1$ [3].

The function $\Phi_g(\xi, \vec{u})$ is the unique solution of the boundary value problem for the Euler-Poisson-Darboux equations

$$(2.5) \quad 2(u_i - u_j) \frac{\partial^2 \Phi_g}{\partial u_i \partial u_j} = \frac{\partial \Phi_g}{\partial u_i} - \frac{\partial \Phi_g}{\partial u_j}, \quad i, j = 1, 2, \dots, 2g+1,$$

$$(2.6) \quad 2(\xi - u_i) \frac{\partial^2 \Phi_g}{\partial \xi \partial u_i} = \frac{\partial \Phi_g}{\partial \xi} - 2 \frac{\partial \Phi_g}{\partial u_i}, \quad i = 1, 2, \dots, 2g+1,$$

$$(2.7) \quad \Phi_g(u, u, \dots, u) = \frac{2^g}{(2g+1)!!} \frac{d^{g+1}}{du^{g+1}} [x - 30tu^2 - f(u)].$$

The function $Q(\xi, \vec{u})$ is a polynomial of degree $2g$ in ξ :

- (1) for $g = 0$, $Q(\xi, u_1) = x - 30u_1^2 t - f(u_1)$;
- (2) for $g > 0$,

$$\begin{aligned} Q(\xi, \vec{u}) &= 2 \sum_{i=1}^{2g+1} \left[\prod_{l=1, l \neq i}^{2g+1} (\xi - u_l) \right] \frac{\partial q_{g,g}(\vec{u})}{\partial u_i} \\ &\quad + \sum_{k=1}^g [(2k-1) \sum_{l=0}^{g-k} \Gamma_l(\vec{u}) q_{g,k+l}(\vec{u})] P_{g,k-1}(\xi, \vec{u}). \end{aligned}$$

Here $\Gamma_l(\vec{u})$ are the coefficients of the expansion at $\xi = \infty$

$$R(\xi, \vec{u}) = \xi^{g+\frac{1}{2}} \left[\Gamma_0(\vec{u}) + \frac{\Gamma_1(\vec{u})}{\xi} + \frac{\Gamma_2(\vec{u})}{\xi^2} + \dots \right]$$

and $q_{g,k}(\vec{u})$ is the solution of the boundary value problem for another version of the Euler-Poisson-Darboux equations

$$(2.8) \quad 2(u_i - u_j) \frac{\partial^2 q_{g,k}}{\partial u_i \partial u_j} = \frac{\partial q_{g,k}}{\partial u_i} - \frac{\partial q_{g,k}}{\partial u_j}, \quad i, j = 1, \dots, 2g+1,$$

$$(2.9) \quad q_{g,k}(u, \dots, u) = \frac{2^{g-1}}{(2g-1)!!} u^{-k+\frac{1}{2}} \frac{d^{g-k}}{du^{g-k}} \left\{ u^{g-\frac{1}{2}} \left[\frac{d^{k-1}}{du^{k-1}} (x - 30tu^2 - f(u)) \right] \right\}.$$

The polynomial

$$(2.10) \quad P_{g,n}(\xi, \vec{u}) = \xi^{g+n} + a_{g,1} \xi^{g+n-1} + \dots + a_{g,g+n}$$

is defined such that

$$(2.11) \quad \frac{P_{g,n}(\xi, \vec{u})}{R(\xi, \vec{u})} = \xi^{n-\frac{1}{2}} + O(\xi^{-\frac{3}{2}}) \text{ for large } |\xi|,$$

and

$$(2.12) \quad \int_{u_{2k+1}}^{u_{2k}} \frac{P_{g,n}(\xi, \vec{u})}{R(\xi, \vec{u})} d\xi = 0, \quad k = 1, 2, \dots, g,$$

where $\vec{u} = (u_1, u_2, \dots, u_{2g+1})$.

Solutions of the boundary value problem for the Euler-Poisson-Darboux equations can be constructed using those of the following simpler problem as building blocks [11]

$$(2.13) \quad 2(x_1 - x_2) \frac{\partial^2 v}{\partial x_1 \partial x_2} = \frac{\partial v}{\partial x_1} - \rho \frac{\partial v}{\partial x_2}, \quad \rho > 0 \text{ is a constant},$$

$$(2.14) \quad v(x_1, x_1) = g(x_1).$$

The boundary value problem (2.13-2.14) has one and only one solution for smooth boundary data $g(x)$ [11]. A simple calculation shows that the solution is given by the formula

$$v(x_1, x_2) = C \int_{-1}^1 \frac{g\left(\frac{1+\mu}{2}x_1 + \frac{1-\mu}{2}x_2\right)}{\sqrt{1-\mu^2}} (1+\mu)^{\frac{\rho-1}{2}} d\mu,$$

where

$$C = \frac{1}{\int_{-1}^1 \frac{(1+\mu)^{\frac{\rho-1}{2}}}{\sqrt{1-\mu^2}} d\mu}.$$

A change of integration variable gives another expression for the solution

$$(2.15) \quad v(x_1, x_2) = C \left[\frac{2}{x_1 - x_2} \right]^{\frac{\rho-1}{2}} \int_{x_2}^{x_1} g(x) \frac{(x - x_2)^{\frac{\rho-2}{2}}}{(x_1 - x)^{\frac{1}{2}}} dx,$$

where the square root is set to be positive for x between x_1 and x_2 . In particular, when $g = 0$, the solution of the boundary value problem (2.5-2.7) is

$$(2.16) \quad \Phi_0(\xi, u_1) = \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{H'\left(\frac{1+\mu}{2}\xi + \frac{1-\mu}{2}u_1\right)}{\sqrt{1-\mu^2}} d\mu,$$

where $H(u) = x - 30tu^2 - f(u)$. More generally, the solution of equations (2.5-2.7) can be solved explicitly

$$(2.17) \quad \begin{aligned} \Phi_g(\xi, \vec{u}) &= M \times \\ &\int_{-1}^1 \dots \int_{-1}^1 \frac{H^{(g+1)}\left(\frac{1+\mu_{2g+1}}{2}\left(\dots\left(\frac{1+\mu_2}{2}\left(\frac{1+\mu_1}{2}\xi + \frac{1-\mu_1}{2}u_1\right) + \dots\right) + \frac{1-\mu_{2g+1}}{2}u_{2g+1}\right)}{\sqrt{(1-\mu_1)(1-\mu_2)\dots(1-\mu_{2g+1})}} \\ &\times (1+\mu_2)^{\frac{1}{2}}(1+\mu_3)^{\frac{2}{2}}\dots(1+\mu_{2g+1})^{\frac{2g}{2}} d\mu_1 d\mu_2 \dots d\mu_{2g+1}, \end{aligned}$$

where the constant M is chosen so that the boundary condition (2.7) is satisfied [3].

The solution $q_{g,k}(\vec{u})$ of another boundary value problem (2.8), (2.9) is also given by a multiple integral formula similar to (2.17).

We now list some identities of Φ_g and $q_{g,k}$. They will be useful in the subsequent calculations.

LEMMA 2.1. [3]

$$(2.18) \quad 2(\xi - u_i) \frac{\partial}{\partial u_i} \Phi_g(\xi, \vec{u}) = \Phi_g(\xi, \vec{u}) - \Phi_g(u_i, \vec{u}) ,$$

$$(2.19) \quad \Phi_g(\xi, \vec{u})|_{u_l=u_{l+1}=u^*} = \frac{\Phi_{g-1}(\xi, \vec{u}) - \Phi_{g-1}(u^*, \vec{u})}{\xi - u^*} ,$$

$$(2.20) \quad \Phi_g(u^*, \vec{u})|_{u_l=u_{l+1}=u^*} = \frac{\partial \Phi_{g-1}(\xi, \vec{u})}{\partial \xi} \Big|_{\xi=u^*} ,$$

$$(2.21) \quad \Phi_{g-1}(\xi, \vec{u}) = \left(\sum_{i=1}^{2g+1} \frac{\partial}{\partial u_i} q_{g,g}(\vec{u}) \right) \Big|_{u_l=u_{l+1}=\xi} ,$$

where $\vec{u} = (u_1, \dots, u_{2g+1})$, $\vec{u} = (u_1, \dots, u_{l-1}, u_{l+2}, \dots, u_{2g+1})$ and $1 \leq l \leq 2g$.

Our method of solution to the minimization problem is summarized in the following theorem.

THEOREM 2.2. [3] *If x , t and \vec{u} are connected by the equations*

$$(2.22) \quad P(u_i, \vec{u}) = 0 \text{ for } i = 1, 2, \dots, 2g+1,$$

where P is defined by (2.4), and if the inequalities for $k = 0, 1, \dots, g$, $u_0 = 1$ and $u_{2g+2} = 0$

$$(2.23) \quad \operatorname{Re}\{\sqrt{-1}R(\xi, \vec{u})\}\Phi_g(\xi, \vec{u}) < 0 \text{ for } u_{2k+2} < \xi < u_{2k+1} ,$$

$$(2.24) \quad \int_{u_{2k+1}}^{\xi} R(\mu, \vec{u})\Phi_g(\mu, \vec{u})d\mu > 0 \text{ for } u_{2k+1} < \xi < u_{2k} ,$$

are satisfied, the function

$$(2.25) \quad \psi(\eta) = -2\eta \operatorname{Re}\{\sqrt{-1}R(\eta^2, \vec{u})\}\Phi_g(\eta^2, \vec{u})$$

is the minimizer of (2.1). Its support is given by (2.3).

We now analyze equations (2.22). The boundary conditions (2.7) and (2.9) are linear in x and t , as is the function $P(\xi, \vec{u})$. We then use formulae (2.4-2.9) to write

$$(2.26) \quad P(\xi, \vec{u}) = xP_{g,0}(\xi, \vec{u}) - 80tP_{g,2}(\xi, \vec{u}) - P_f(\xi, \vec{u}) ,$$

where $P_{g,0}$ and $P_{g,2}$ are defined by (2.10-2.12), and P_f is also determined by equations (2.4-2.9) with boundary data (2.7) and (2.9) depending only on f and its derivatives.

Equations (2.22) can then be written as

$$(2.27) \quad x = \lambda_{g,i}(\vec{u})t + w_{g,i}(\vec{u}) , \quad i = 1, 2, \dots, 2g+1 ,$$

where

$$\lambda_{g,i} = 80 \frac{P_{g,2}(u_i, \vec{u})}{P_{g,0}(u_i, \vec{u})} , \quad w_{g,i}(\vec{u}) = \frac{P_f(u_i, \vec{u})}{P_{g,0}(u_i, \vec{u})} .$$

It is well known that the solutions $u_1, u_2, \dots, u_{2g+1}$ of (2.27) or equivalently (2.22), as functions of x and t , satisfy an odd number of hyperbolic partial differential equations [1, 3, 6, 12, 14]

$$(2.28) \quad u_{it} + \lambda_{g,i}(u_1, u_2, \dots, u_{2g+1})u_{ix} = 0 , \quad i = 1, 2, \dots, 2g+1 .$$

In other words, the square roots of the end points of the support (2.3) of the minimizer of the variational problem (2.1) satisfy the PDE (2.28).

The coefficients $\lambda_{g,i}$ of (2.28) involve complete hyperelliptic integrals of genus g because $P_{g,n}$'s do so in view of (2.10-2.12). Equations (2.28) are then called the g -phase Whitham equations and its solution a g -phase solution.

In particular, when $g = 0$, we have $\lambda_{0,1}(u_1) = 30u_1^2$ and equation (2.28) becomes the Burgers type equation (1.6).

When $g = 1$, equations (2.28) turn out to be the Whitham equations (1.7), where

$$(2.29) \quad \mu_i(u_1, u_2, u_3) = \lambda_{1,i}(u_1, u_2, u_3) = 80 \frac{P_{1,2}(u_i, u_1, u_2, u_3)}{P_{1,0}(u_i, u_1, u_2, u_3)}, \quad i = 1, 2, 3.$$

We note that, for the KdV equation, λ_i 's of (1.4) are given by

$$(2.30) \quad \lambda_i(u_1, u_2, u_3) = 12 \frac{P_{1,1}(u_i, u_1, u_2, u_3)}{P_{1,0}(u_i, u_1, u_2, u_3)}, \quad i = 1, 2, 3.$$

To solve the system of algebraic equations (2.22), we rely on the Implicit Function Theorem. Although the system is complicated, its Jacobian matrix is diagonal and its determinant is easily calculated.

THEOREM 2.3. [3] *On the solution \vec{u} of (2.22),*

(1) *the Jacobian matrix of the hodograph transform (2.22) is diagonal, i.e.,*

$$\frac{\partial P(u_i, \vec{u})}{\partial u_j} = 0, \quad i, j = 1, 2, \dots, 2g+1; i \neq j,$$

(2) *the Jacobian is*

$$\det \left(\frac{\partial P(u_i, \vec{u})}{\partial u_j} \right) = (-1)^g \prod_{i>j} (u_i - u_j)^2 \prod_{i=1}^{2g+1} \Phi_g(u_i, \vec{u}).$$

Violation of inequalities (2.23-2.24) gives rise to phase changes. Indeed, when (2.23) first fails, function $\Phi_g(\xi, \vec{u})$ must have a double ξ -zero for $\sqrt{\xi}$ in the interior of S of (2.3), say, the zero u^* where $\sqrt{u^*} \in S$. The resulting equations are

$$(2.31) \quad \Phi_g(u^*, \vec{u}) = 0,$$

$$(2.32) \quad \frac{\partial}{\partial \xi} \Phi_g(\xi, \vec{u})|_{\xi=u^*} = 0,$$

$$(2.33) \quad P(u_i, \vec{u}) = 0, \quad i = 1, 2, \dots, 2g+1.$$

They govern the so-called trailing edge separating g and $g+1$ phases. Similarly, when (2.24) is first violated, the indefinite integral must also have a double ξ -zero for $\sqrt{\xi}$ in the complement of S , say, the zero $u^* \in (u_{2k+1}, u_{2k})$. The resulting equations are

$$(2.34) \quad \Phi_g(u^*, \vec{u}) = 0,$$

$$(2.35) \quad \int_{u_{2k+1}}^{u^*} \Phi_g(\xi, \vec{u}) R(\xi, \vec{u}) d\xi = 0,$$

$$(2.36) \quad P(u_i, \vec{u}) = 0, \quad i = 1, 2, \dots, 2g+1,$$

which govern the so-called leading edge separating g and $g+1$ phases.

We close this section with a remark on the other higher order KdV equations. All the zero dispersion limits of the equations in the KdV hierarchy are determined

by the minimization problem (2.1) with $a(\eta, x, t)$ of (2.2) replaced by [8]

$$(2.37) \quad a(\eta, x, t) = \eta x - 4^m \eta^{2m+1} t - \theta(\eta)$$

for $m = 1, 2, \dots$. The function $x - 30tu^2 - f(u)$ in the boundary conditions (2.7) and (2.9) must then be replaced by $x - C_m t u^m - f(u)$, where [3]

$$C_m = \frac{(2m+1)2^{2m+1}}{\pi} \int_0^1 \frac{t^{2m}}{\sqrt{1-t^2}} dt.$$

In particular, $m = 1$ corresponds to the KdV case and $m = 2$ to the fifth order KdV case. All the analysis of Section 3 can be carried over from the fifth order KdV ($m = 2$) case to all the other equations ($m \neq 2$) in the KdV hierarchy.

In the subsequent sections, we will only study the fifth order KdV ($m = 2$) case. The KdV ($m = 1$) case has been well understood [13]. We will indicate why the $m > 2$ case is technically more difficult than the $m = 2$ case when the opportunity presents itself.

3. Main Theorem

We first make assumption on the initial data $u_0(x)$ or its inverse $f(u)$. Since $u_0(x)$ has the limits (1.8) as x goes to the infinity, its inverse $f(u)$ behaves as

$$(3.1) \quad \lim_{u \rightarrow 0} f(u) = +\infty, \quad \lim_{u \rightarrow 1} f(u) = -\infty.$$

We further assume that

$$(3.2) \quad f'''(u) < 0 \text{ in the neighborhood of } u = 0 \text{ and } u = 1.$$

Conditions (3.1-3.2) immediately imply [13]

$$(3.3) \quad \lim_{u \rightarrow 0} f'(u) = \lim_{u \rightarrow 1} f'(u) = -\infty, \quad \lim_{u \rightarrow 0} f''(u) = +\infty, \quad \lim_{u \rightarrow 1} f''(u) = -\infty.$$

THEOREM 3.1. (see Figure 1.) *Under the conditions (3.1) and (3.2), there exists a $T > 0$ so that when $t > T$, the weak limit of the fifth order KdV equation is governed by the Whitham equations (1.7) for $x^-(t) < x < x^+(t)$, where $x^-(t)$ and $x^+(t)$ are some functions of t . The weak limit satisfies the Burgers type equation (1.6) for $x < x^-(t)$ and $x > x^+(t)$.*

This theorem is a consequence of Theorems 5.1 and 5.5, which will be proved in the subsequent sections.

4. Trailing and Leading Edges

In this section, we will study the trailing and leading edges of a single phase solution when t is sufficiently large.

§ 4.1 The trailing edge

We first analyze the tail of a single phase solution at which $u_2 = u_3$ (see Figure 1.). It is a phase transition boundary between zero and single phases; it corresponds to $g = 0$ in equations (2.31-2.33):

$$(4.1) \quad W_1(u_1, u_3, t) := \Phi_0(u_3, u_1) = 0,$$

$$(4.2) \quad W_2(u_1, u_3, t) := \frac{\partial}{\partial \xi} \Phi_0(\xi, u_1)|_{\xi=u_3} = 0,$$

$$(4.3) \quad x - 30tu_1^2 - f(u_1) = 0.$$

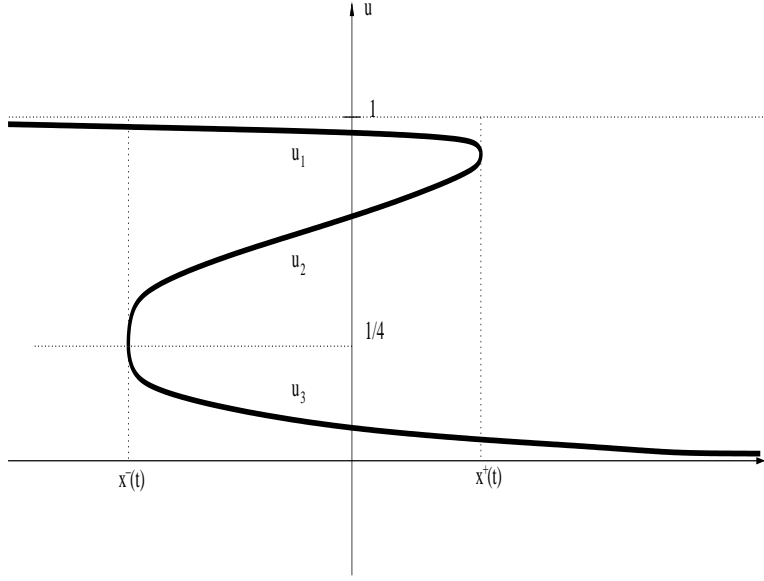


FIGURE 1. Large time behavior of the weak limit of the fifth order KdV for generic initial data. The limit is governed by the Whitham equations (1.7) for $x^-(t) < x < x^+(t)$ and by the Burgers type equation (1.6) otherwise. The trailing edge $x = x^-(t)$ is characterized by $u_1 \approx 1$, $u_2 = u_3 \approx 1/4$.

We split $\Phi_0(\xi, u_1)$ into

$$(4.4) \quad \Phi_0(\xi, u_1) = -30tU_0(\xi, u_1) - F_0(\xi, u_1) ,$$

where U_0 and F_0 are the solutions of the Euler-Poisson-Darboux equations (2.5-2.7) with boundary data $2u$ and $f'(u)$, respectively. In view of formula (2.16), we have

$$(4.5) \quad F_0(\xi, u_1) = \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{f'(\frac{1+\mu}{2}\xi + \frac{1-\mu}{2}u_1)}{\sqrt{1-\mu}} d\mu ,$$

$$(4.6) \quad U_0(\xi, u_1) = \frac{1}{2\sqrt{2}} \int_{-1}^1 \frac{2(\frac{1+\mu}{2}\xi + \frac{1-\mu}{2}u_1)}{\sqrt{1-\mu}} d\mu = \frac{4}{3}\xi + \frac{2}{3}u_1 .$$

Equations (4.1) and (4.2) are then equivalent to

$$(4.7) \quad \frac{F_0}{U_0} + 30t = 0 ,$$

$$(4.8) \quad \frac{\partial}{\partial \xi} \left[\frac{F_0}{U_0} + 30t \right]_{\xi=u_3} = 0 .$$

Equation (4.8) suggests that the point $\xi = u_3$ is a critical point of the function $[F_0/U_0 + 30t]$. We therefore consider an auxiliary minimum problem

$$(4.9) \quad \text{Min}_{0 < \xi < 1} \left[-\frac{F_0(\xi, u_1)}{U_0(\xi, u_1)} \right]$$

for each u_1 close to 1.

To study the minimization problem (4.9), we need the following lemma.

LEMMA 4.1. (1)

$$(4.10) \quad \lim_{\xi \rightarrow 0} F_0(\xi, u_1) = \lim_{\xi \rightarrow 1} F_0(\xi, u_1) = -\infty$$

for each $u_1 \in (0, 1)$.

(2) There exists a $\delta > 0$ such that

$$(4.11) \quad \frac{\partial^2}{\partial \xi^2} F_0(\xi, u_1) < 0 \text{ for } 0 < \xi < 1 \text{ and } 1 - \delta < u_1 < 1 .$$

(3)

$$(4.12) \quad \lim_{u_1 \rightarrow 1} \frac{\frac{\partial F_0(\xi, u_1)}{\partial \xi}}{F_0(\xi, u_1)} = \frac{1}{1 - \xi}$$

uniformly for ξ on every compact subset of the interval $(0, 1)$.

Proof: To prove the first limit of (4.10), we use (4.5) to rewrite $F_0(\xi, u_1)$ as

$$F_0(\xi, u_1) = \frac{1}{2\sqrt{\xi - u_1}} \int_{u_1}^{\xi} \frac{f'(u)}{\sqrt{\xi - u}} du \text{ when } \xi > u_1 .$$

Since $f'(u) < 0$, we estimate the integral when ξ is close to 1:

$$F_0(\xi, u_1) \leq \frac{1}{2(\xi - u_1)} \int_{u_1}^{\xi} f'(u) du = \frac{1}{2(\xi - u_1)} [f(\xi) - f(u_1)] .$$

This together with (3.1) proves the second limit of (4.10). The first limit can be shown in the same way.

To prove inequality (4.11), in view of condition (3.2) we assume $f'''(u) < 0$ for u outside the interval $(\delta_1, 1 - \delta_1)$ where δ_1 is a small positive number. Since $f'''(u) < 0$ for $u > 1 - \delta_1$, it follows from formula (4.5) that it suffices to prove (4.11) for $\xi \leq 1 - \delta_1$.

We use formula (4.5) to write for $u_1 > 1 - \frac{\delta_1}{2} \geq \xi$,

$$(4.13) \quad \begin{aligned} \frac{\partial^2}{\partial \xi^2} F_0(\xi, u_1) &= \frac{1}{2(u_1 - \xi)^{\frac{5}{2}}} \int_{\xi}^{u_1} f'''(u) \frac{(u_1 - u)^2}{\sqrt{u - \xi}} du \\ &= \frac{1}{2(u_1 - \xi)^{\frac{5}{2}}} \left[\int_{1 - \frac{\delta_1}{2}}^{u_1} + \int_{\xi}^{1 - \frac{\delta_1}{2}} \right] . \end{aligned}$$

Since $f'''(u) < 0$ for $u < \delta_1/2$, the second integral in the parenthesis of (4.13) is smaller than

$$\int_{\delta_1}^{1 - \frac{\delta_1}{2}} \frac{|f'''(u)|}{\sqrt{u - \xi}} du ,$$

which, since $f'''(u)$ is bounded on the closed interval $[\delta_1, 1 - \delta_1/2]$, is uniformly bounded for $\xi \leq 1 - \delta_1/2$.

Since $f'''(u) < 0$ for $u > 1 - \delta_1$, the first integral of (4.13) is less than

$$\begin{aligned} &\int_{1 - \frac{\delta_1}{2}}^{u_1} f'''(u)(u_1 - u)^2 du \\ &= 2f(u_1) - f''(1 - \frac{\delta_1}{2})(u_1 - 1 + \frac{\delta_1}{2})^2 - 2f'(1 - \frac{\delta_1}{2})(u_1 - 1 + \frac{\delta_1}{2}) - 2f(1 - \frac{\delta_1}{2}) , \end{aligned}$$

which, in view of (3.1), goes to $-\infty$ as $u_1 \rightarrow 1$. This proves inequality (4.11).

To prove (4.12), we use formula (4.5) to write for $u_1 > \xi$,

$$(4.14) \quad F_0(\xi, u_1) = \frac{1}{2\sqrt{u_1 - \xi}} \int_{\xi}^{u_1} \frac{f'(u)}{\sqrt{u - \xi}} du ,$$

$$(4.15) \quad \frac{\partial}{\partial \xi} F_0(\xi, u_1) = \frac{1}{2(u_1 - \xi)^{\frac{3}{2}}} \int_{\xi}^{u_1} f''(u) \frac{u_1 - u}{\sqrt{u - \xi}} du .$$

We first observe that

$$(4.16) \quad \lim_{u_1 \rightarrow 1} \int_{\xi}^{u_1} \frac{f'(u)}{\sqrt{u - \xi}} du = -\infty$$

uniformly for ξ on every compact subset of the interval $(0, 1)$. Since $f'(u) < 0$, we estimate for $u_1 > \xi$,

$$\int_{\xi}^{u_1} \frac{f'(u)}{\sqrt{u - \xi}} du \leq \int_{\xi}^{u_1} f'(u) du = f(u_1) - f(\xi) .$$

As $u_1 \rightarrow 1$, the right hand side goes to $-\infty$ uniformly for ξ on every compact subset of the interval $(0, 1)$ because $f(u_1) \rightarrow -\infty$ in view of (3.1) and $f(u)$ is bounded on every compact subset. This proves the limit (4.16).

To calculate the limit (4.12), we turn to the following lemma on the uniform convergence version of L'Hospital's rule. The proof of Lemma 4.2 is simply a modification of the standard proof of the classical L'Hospital's rule.

LEMMA 4.2. *Let $g_1(x, y)$ and $g_2(x, y)$ be defined on $(0, 1) \times Y$, where Y is a subset of \mathfrak{R} . If $g_1(x, y)$ and $g_2(x, y)$ satisfy the conditions*

- (1) *for each $x \in (0, 1)$, $g_1(x, y)$ and $g_2(x, y)$ are uniformly bounded for y on Y ,*
- (2) *the partial derivatives $g_{1x}(x, y)$ and $g_{2x}(x, y)$ exist for each $(x, y) \in (0, 1) \times Y$,*
- (3) *$\lim_{x \rightarrow 1} g_2(x, y) = \infty$ uniformly for y on Y ,*
- (4) *$\lim_{x \rightarrow 1} [g_{1x}(x, y)/g_{2x}(x, y)] = L(y)$ uniformly for y on Y ,*
- (5) *$L(y)$ is uniformly bounded on Y .*

then $\lim_{x \rightarrow 1} [g_1(x, y)/g_2(x, y)] = L(y)$ uniformly for y on Y .

Using formulae (4.14) and (4.15), we apply Lemma 4.2 twice to calculate the limit

$$\begin{aligned} \lim_{u_1 \rightarrow 1} \frac{\frac{\partial F_0(\xi, u_1)}{\partial \xi}}{F_0(\xi, u_1)} &= \lim_{u_1 \rightarrow 1} \frac{1}{u_1 - \xi} \frac{\int_{\xi}^{u_1} \frac{f''(u)}{\sqrt{u - \xi}} (u_1 - u) du}{\int_{\xi}^{u_1} \frac{f'(u)}{\sqrt{u - \xi}} du} \\ &= \frac{1}{1 - \xi} \lim_{u_1 \rightarrow 1} \frac{\int_{\xi}^{u_1} \frac{f''(u)}{\sqrt{u - \xi}} du}{\frac{f'(u_1)}{\sqrt{u_1 - \xi}}} \\ &= \frac{1}{\sqrt{1 - \xi}} \lim_{u_1 \rightarrow 1} \frac{\frac{f''(u_1)}{\sqrt{u_1 - \xi}}}{f''(u_1)} \\ &= \frac{1}{1 - \xi} , \end{aligned}$$

uniformly for ξ on every compact subset of $(0, 1)$. Here, the limit (4.16) and the second limit of (3.3) guarantee that Lemma 4.2 applies. This proves the limit (4.12).

The proof of Lemma 4.1 is now complete.

We now use Lemma 4.1 to study the minimum problem (4.9). Because of (4.10), $[-F_0(\xi, u_1)/U_0(\xi, u_1)]$ goes to $+\infty$ as $\xi \rightarrow 0, 1$. Hence, the minimum is reached somewhere in the interval $(0, 1)$.

We next show that the minimum problem (4.9) has only one critical point when u_1 is close to 1. To see this, we calculate the second derivatives of $[-F_0/U_0]$ at any of its critical points

$$(4.17) \quad \frac{\partial^2}{\partial \xi^2} \left[-\frac{F_0(\xi, u_1)}{U_0(\xi, u_1)} \right] = -\frac{F_{0\xi\xi}U_0 - F_0U_{0\xi\xi}}{U_0^2}.$$

Since $U_{0\xi\xi} \equiv 0$ in view of (4.6), it follows from inequality (4.11) that the right hand side is positive if u_1 is close to 1. Hence, all the critical points of $[-F_0/U_0]$ must be local minimizing points. For any two minimizing points, some point between them must be a local maximizing point; an impossibility. This proves the uniqueness of the critical point and hence of the minimizing point in the minimum problem (4.9) when u_1 is close to 1.

Remark: *This is the first place where $m = 2$ in (2.37) is important. It is because of $m = 2$ that we have $U_{0\xi\xi} \equiv 0$ which makes it possible to conclude from formula (4.17) the uniqueness of the critical point in the the minimum problem (4.9).*

We also want to determine the asymptotics of the critical point when t is large. To do this, we write

$$(4.18) \quad \frac{\partial}{\partial \xi} \left[-\frac{F_0(\xi, u_1)}{U_0(\xi, u_1)} \right] = -\frac{F_0}{U_0} \left[\frac{\partial F_0}{\partial \xi} \frac{1}{F_0} - \frac{\partial U_0}{\partial \xi} \frac{1}{U_0} \right].$$

Since F_0 and U_0 are non-zero when $0 < \xi < 1$ and $0 < u_1 < 1$, $[-F_0/U_0]$ has a unique critical point within the interval $(0, 1)$ if and only if the quantity in the parenthesis on the right hand side does so. We now consider a compact set $[\epsilon_0, 1-\epsilon_0]$, where ϵ_0 is a small positive number. According to (4.12), the quantity in the parenthesis of (4.18) has the uniform limit

$$(4.19) \quad \frac{1}{1-\xi} - \frac{\frac{\partial U_0(\xi, 1)}{\partial \xi}}{U_0(\xi, 1)}$$

on this compact set as $u_1 \rightarrow 1$. In view of formula (4.6) for U_0 , we deduce that the function (4.19) has only one zero $\xi = 1/4$ in the interval $0 < \xi < 1$, counting multiplicities. Hence, the quantity in the parenthesis of (4.18) will have a zero near $\xi = 1/4$ when u_1 is close to 1. This zero is exactly the unique critical point of the minimum problem (4.9).

We summarize the above result as a lemma.

LEMMA 4.3. *The minimum problem (4.9) has one and only one critical point when u_1 is close to 1. Furthermore, the critical point approaches $1/4$ as u_1 goes to 1.*

We are now ready to solve equations (4.1-4.3). By Lemma 4.3, equation (4.8) has a unique solution $\xi = u_3$ for each u_1 close to 1. We then use (4.7) to determine t . Hence, equations (4.7-4.8) can be solved to give u_3 and t as functions of u_1 .

We then calculate the Jacobian of equations (4.1-4.2) with respect to u_1 and u_3 on the solution of (4.7-4.8):

$$(4.20) \quad \frac{\partial W_1}{\partial u_1} = \frac{\Phi_0(u_1, u_1)}{2(u_1 - u_3)} > 0 ,$$

$$(4.21) \quad \frac{\partial W_1}{\partial u_3} = 0 ,$$

$$(4.22) \quad \frac{\partial W_2}{\partial u_1} = \frac{\Phi_0(u_1, u_1)}{2(u_1 - u_3)^2} > 0 ,$$

$$(4.23) \quad \frac{\partial W_2}{\partial u_3} = -\frac{\partial^2}{\partial \xi^2} F_0(\xi, u_1)|_{\xi=u_3} > 0 .$$

To prove the equality of (4.20), we use the special case of identity (2.18)

$$2(\xi - u_1) \frac{\partial}{\partial u_1} \Phi_0(\xi, u_1) = \Phi_0(\xi, u_1) - \Phi_0(u_1, u_1) .$$

This together with equation (4.1) proves the equality of (4.20).

Equation (4.21) is the result of (4.2).

The equality of (4.22) follows from (2.6), (4.2) and (4.20).

The equality of (4.23) is the result of the split (4.4) and $U_{0\xi\xi} \equiv 0$.

The sign in (4.23) comes from (4.11).

To determine the signs in (4.20) and (4.22), we split $\Phi_0(u_1, u_1)$ according to (4.4),

$$\begin{aligned} \Phi_0(u_1, u_1) &= -30tU_0(u_1, u_1) - F_0(u_1, u_1) \\ &= U_0(u_1, u_1) \left[\frac{F_0(u_3, u_1)}{U_0(u_3, u_1)} - \frac{F_0(u_1, u_1)}{U_0(u_1, u_1)} \right] > 0 . \end{aligned}$$

Here, we have used (4.7) in the second equality. The inequality follows from the fact that u_3 is the only minimizing point of the minimum problem (4.9).

It now follows from the Implicit Function Theorem that equations (4.1-4.2) can be solved to give u_1 and u_3 as functions of t , respectively, for t larger than a finite time. We have therefore established the following theorem.

THEOREM 4.4. *Under the conditions (3.1) and (3.2), there exists a constant $T^- > 0$ such that for $t > T^-$, equations (4.1-4.3) have a unique solution $u_1^-(t)$, $u_3^-(t)$ and $x^-(t)$. Function $u_1^-(t)$ is an increasing function of t . Furthermore,*

$$\lim_{t \rightarrow +\infty} u_1^-(t) = 1 , \quad \lim_{t \rightarrow +\infty} u_3^-(t) = \frac{1}{4} .$$

§ 4.2 The leading edge

We now turn to the leading edge at which $u_1 = u_2$. It is the head of a single phase solution (see Figure 1.). It is governed by equations (2.34-2.36) for $g = 0$:

$$(4.24) \quad W_3(u_1, u_3, t) =: \Phi_0(u_1, u_3) = 0 ,$$

$$(4.25) \quad W_4(u_1, u_3, t) =: \int_{u_3}^{u_1} \Phi_0(\xi, u_3) \sqrt{\xi - u_3} d\xi = 0 ,$$

$$(4.26) \quad x - 30tu_3^2 - f(u_3) = 0 .$$

We again use the split $\Phi_0(\xi, u_3) = -30tU_0(\xi, u_3) - F_0(\xi, u_3)$, which is similar to (4.4), to rewrite (4.24-4.25) as

$$(4.27) \quad \frac{\int_{u_3}^{u_1} F_0(u, u_3) \sqrt{u - u_3} du}{\int_{u_3}^{u_1} U_0(u, u_3) \sqrt{u - u_3} du} + 30t = 0 ,$$

$$(4.28) \quad \frac{\partial}{\partial u_1} \left[\frac{\int_{u_3}^{u_1} F_0(u, u_3) \sqrt{u - u_3} du}{\int_{u_3}^{u_1} U_0(u, u_3) \sqrt{u - u_3} du} + 30t \right] = 0 .$$

We first simplify equations (4.27-4.28).

LEMMA 4.5.

$$\frac{1}{2(u_1 - u_3)^{\frac{3}{2}}} \int_{u_3}^{u_1} F_0(u, u_3) \sqrt{u - u_3} du = \frac{1}{4\sqrt{2}} \int_{-1}^1 f' \left(\frac{1-\mu}{2} u_3 + \frac{1+\mu}{2} u_1 \right) \sqrt{1-\mu} d\mu .$$

Proof:

$$\begin{aligned} & \frac{\partial}{\partial u_1} \int_{u_3}^{u_1} F_0(u, u_3) \sqrt{u - u_3} du \\ &= F_0(u_1, u_3) \sqrt{u_1 - u_3} \\ &= \frac{1}{2} \int_{u_3}^{u_1} \frac{f'(u)}{\sqrt{u_1 - u}} du \\ &= \frac{\partial}{\partial u_1} \int_{u_3}^{u_1} f'(u) \sqrt{u_1 - u} du \\ &= \frac{\partial}{\partial u_1} \left[\frac{(u_1 - u_3)^{\frac{3}{2}}}{2\sqrt{2}} \int_{-1}^1 f' \left(\frac{1-\mu}{2} u_3 + \frac{1+\mu}{2} u_1 \right) \sqrt{1-\mu} d\mu \right] , \end{aligned}$$

which proves the lemma.

Let

$$(4.29) G_0(\xi, u_3) = \frac{1}{4\sqrt{2}} \int_{-1}^1 f' \left(\frac{1-\mu}{2} u_3 + \frac{1+\mu}{2} \xi \right) \sqrt{1-\mu} d\mu ,$$

$$(4.30) V_0(\xi, u_3) = \frac{1}{4\sqrt{2}} \int_{-1}^1 2 \left(\frac{1-\mu}{2} u_3 + \frac{1+\mu}{2} \xi \right) \sqrt{1-\mu} d\mu = \frac{4}{15} \xi + \frac{2}{5} u_3 .$$

We use Lemma 4.5 to rewrite equations (4.27-4.28) as

$$(4.31) \quad \frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} + 30t = 0 ,$$

$$(4.32) \quad \frac{\partial}{\partial \xi} \left[\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} + 30t \right]_{\xi=u_1} = 0 .$$

As before, equation (4.32) motivates us to consider another auxiliary minimum problem

$$(4.33) \quad \text{Min}_{0 < \xi < 1} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right]$$

for each u_3 close to 0.

LEMMA 4.6. (1) For each $u_3 \in (0, 1)$,

$$(4.34) \quad \lim_{\xi \rightarrow 1} \frac{\partial}{\partial \xi} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right] = +\infty .$$

(2) For any small $\epsilon_1 > 0$, there exists a $\delta^* > 0$ such that

$$(4.35) \quad \frac{\partial}{\partial \xi} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right] < 0$$

for $0 < \xi < 1 - \epsilon_1$ and $0 < u_3 < \delta^*$.

(3)

$$(4.36) \quad \frac{\partial^2}{\partial \xi^2} G_0(\xi, u_3) < 0 \quad \text{uniformly for } 0 < \xi < 1$$

when u_3 is close to 0.

Proof: We omit the proof of (4.36) since it is similar to the proof of (4.11).

To prove (4.34), we write

$$(4.37) \quad \frac{\partial}{\partial \xi} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right] = -\frac{\frac{\partial G_0}{\partial \xi} V_0 - G_0 \frac{\partial V_0}{\partial \xi}}{V_0^2}.$$

We use formula (4.29) to calculate $\partial_\xi G_0$ and integrate by parts to obtain

$$(4.38) \quad \frac{\partial G_0}{\partial \xi} = \frac{F_0(\xi, u_3) - 3G_0(\xi, u_3)}{2(\xi - u_3)},$$

where F_0 is previously defined by (4.5).

Using (4.29) again, we write

$$G_0(\xi, u_3) = \frac{1}{2(\xi - u_3)^{\frac{3}{2}}} \int_{u_3}^{\xi} f'(u) \sqrt{\xi - u} \, du.$$

Since the integral is a decreasing function of ξ because $f'(u) < 0$, $G_0(\xi, u_3)$ goes to either a finite negative number or $-\infty$ as $\xi \rightarrow 1$.

For those initial functions $f(u)$ such that $G_0(\xi, u_3)$ converges as $\xi \rightarrow 1$, the right hand side of (4.38) goes to $-\infty$ because $F_0(\xi, u_3)$ does so by Lemma 4.1; so $G_{0\xi}(\xi, u_3) \rightarrow -\infty$. This together with (4.37) gives the limit (4.34).

For those initial functions $f(u)$ such that $G_0(\xi, u_3) \rightarrow -\infty$ as $\xi \rightarrow 1$, we estimate

$$\frac{F_0(\xi, u_3)}{G_0(\xi, u_3)} = \frac{\int_{u_3}^{\xi} \frac{f'(u)}{\sqrt{\xi - u}} \, du}{(\xi - u_3) \int_{u_3}^{\xi} f'(u) \sqrt{\xi - u} \, du} \geq \frac{\int_{u_3}^{\xi} \frac{f'(u)}{\sqrt{1 - u}} \, du}{\int_{u_3}^{\xi} f'(u) \sqrt{1 - u} \, du}.$$

Using L'Hospital's rule, it is easy to see that the right hand side goes to $+\infty$ as $\xi \rightarrow 1$. This limit, when combined with (4.38), gives

$$\lim_{\xi \rightarrow 1} \frac{\frac{\partial G_0(\xi, u_3)}{\partial \xi}}{G_0(\xi, u_3)} = +\infty,$$

which, in view of (4.37), proves (4.34).

The proof of the first part of Lemma 4.6 is completed.

To prove (4.35), we use (4.37) again. Since V_0 and $\partial_\xi V_0$ are positive functions and G_0 is negative, it suffices to show that

$$(4.39) \quad \frac{\partial G_0(\xi, u_3)}{\partial \xi} > 0$$

for $0 < \xi < 1 - \epsilon_1$ and $0 < u_3 < \delta$.

In view of the third limit of (3.3), there exists a $\delta_2 > 0$ such that

$$(4.40) \quad f''(u) > 0 \quad \text{for } 0 < u < \delta_2.$$

It then follows from formula (4.29) for G_0 that inequality (4.39) is true for $0 < \xi < \delta_2$ and $0 < u_3 < \delta_2$.

For $\delta_2 \leq \xi < 1 - \epsilon_1$ and $0 < u_3 < \delta_2/2$, we use formula (4.29) again to write

$$\begin{aligned} \frac{\partial G_0(\xi, u_3)}{\partial \xi} &= \frac{1}{2(\xi - u_3)^{\frac{5}{2}}} \int_{u_3}^{\xi} f''(u) \sqrt{\xi - u} (u - u_3) du \\ &= \frac{1}{2(\xi - u_3)^{\frac{5}{2}}} \int_{u_3}^{\frac{\delta_2}{2}} f''(u) \sqrt{\xi - u} (u - u_3) du \\ &\quad + \frac{1}{2(\xi - u_3)^{\frac{5}{2}}} \int_{\frac{\delta_2}{2}}^{\xi} f''(u) \sqrt{\xi - u} (u - u_3) du . \end{aligned}$$

The second term is bounded from below by

$$- \frac{2^{\frac{3}{2}}}{\delta_2^{\frac{5}{2}}} \int_{\frac{\delta_2}{2}}^{1 - \epsilon_1} |f''(u)| du ,$$

which is a constant. Since $f'' > 0$ for $0 < u < \delta_2$, the first term is bigger than

$$\frac{\sqrt{\delta_2}}{2\sqrt{2}} \int_{u_3}^{\frac{\delta_2}{2}} f''(u) (u - u_3) du = \frac{\sqrt{\delta_2}}{2\sqrt{2}} \left[\left(\frac{\delta_2}{2} - u_3 \right) f' \left(\frac{\delta_2}{2} \right) - f \left(\frac{\delta_2}{2} \right) + f(u_3) \right] ,$$

which goes to $+\infty$ as $u_3 \rightarrow 0$ because of (3.1). We have therefore proved (4.39) and hence (4.35).

The proof of Lemma 4.6 is complete.

We now use Lemma 4.6 to study the minimization problem (4.33). For each small u_3 , the function $[-G_0(\xi, u_3)/V_0(\xi, u_3)]$ is decreasing when $\xi < 1 - \epsilon_1$ in view of (4.35) and tending to $+\infty$ as $\xi \rightarrow 1$ in view of (4.34). Hence, the minimum is reached somewhere in the interval $(0, 1)$.

We next show that the minimum problem (4.33) has only one critical point when u_3 is close to 0. To see this, we calculate the second derivatives of $[-G_0/V_0]$ at any of its critical points

$$(4.41) \quad \frac{\partial^2}{\partial \xi^2} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right] = -\frac{G_{0\xi\xi}V_0 - G_0V_{0\xi\xi}}{V_0^2} .$$

Since $V_{0\xi\xi} \equiv 0$ from (4.30), the right hand side of (4.41) is positive if u_3 is close to 0, in view of (4.36). Hence, all the critical points of $[-G_0/V_0]$ must be local minimizing points. For any two minimizing points, some point between them must be a local maximizing point; an impossibility. This proves the uniqueness of the critical point and hence of the minimizing point in the minimum problem (4.33) when u_3 is close to 0.

Remark: *This is the second place where $m = 2$ in (2.37) plays an important role. Since $m = 2$, V of (4.30) is a linear function. It then becomes possible to conclude from formula (4.41) that the minimum problem (4.33) has a unique critical point.*

The critical point approaches 1 as u_3 goes to 0. This follows from (4.35).

We summarize the above result as a lemma.

LEMMA 4.7. *The minimum problem (4.33) has one and only one critical point and hence minimizing point when u_3 is close to 0. Furthermore, the critical point tends to 1 as u_3 goes to 0.*

We are now ready to solve equations (4.24-4.26). By Lemma 4.7, equation (4.32) has a unique solution $\xi = u_1$ for each u_3 close to 0. We then use (4.31) to determine t . Hence, equations (4.31-4.32) or equivalently equations (4.24-4.26) can be solved to give u_3 and t as functions of u_1 .

We then calculate the Jacobian of equations (4.24-4.25) with respect to u_1 and u_3 on the solution of (4.31-4.32):

$$(4.42) \quad \frac{\partial W_3}{\partial u_1} = \frac{\partial \Phi_0(\xi, u_3)}{\partial \xi} \Big|_{\xi=u_1} > 0 ,$$

$$(4.43) \quad \frac{\partial W_3}{\partial u_3} = -\frac{\Phi_0(u_3, u_3)}{2(u_1 - u_3)} < 0 ,$$

$$(4.44) \quad \frac{\partial W_4}{\partial u_1} = 0 ,$$

$$(4.45) \quad \frac{\partial W_4}{\partial u_3} = -\Phi_0(u_3, u_3)\sqrt{u_1 - u_3} < 0 .$$

The equalities in (4.43) and (4.45) follow from the related identity (2.18) for $\Phi_0(\xi, u_3)$.

Equation (4.44) is the result of (4.2).

The signs in (4.43) and (4.45) are determined as follows. Since the minimum problem (4.33) has only one minimizing point at $\xi = u_1$, we must have

$$-\frac{G_0(u_3, u_3)}{V_0(u_3, u_3)} > -\frac{G_0(u_1, u_3)}{V_0(u_1, u_3)} .$$

The right hand side is equal to $30t$ in view of (4.31). Hence, $-30tV_0(u_3, u_3) - G_0(u_3, u_3) > 0$, which by formulae (4.29-4.30), reduces to

$$-60tu_3 - f'(u_3) > 0 .$$

In view of the boundary condition (2.7) for Φ_0 , this is exactly $\Phi_0(u_3, u_3) > 0$, which proves the inequalities in (4.43) and (4.45).

To determine the sign in (4.42), since $\xi = u_1$ is the minimizing point in the minimum problem (4.33), we have

$$\frac{\partial^2}{\partial \xi^2} \left[-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \right] \Big|_{\xi=u_1} > 0 ,$$

which, in view of Lemma 4.5, is equivalent to

$$\frac{\partial^2}{\partial \xi^2} \left[-\frac{\int_{u_3}^{\xi} F_0(u, u_3)\sqrt{u - u_3} du}{\int_{u_3}^{\xi} U_0(u, u_3)\sqrt{u - u_3} du} \right] \Big|_{\xi=u_1} > 0 .$$

This together with (4.27-4.28) gives

$$\frac{\partial^2}{\partial \xi^2} \int_{u_3}^{\xi} [-30tU_0(u, u_3) - F_0(u, u_3)]\sqrt{u - u_3} du > 0$$

at $\xi = u_1$. Since $\Phi_0 = -30tU_0 - F_0$, the last inequality together with (4.24) implies that

$$\frac{\partial \Phi_0(\xi, u_3)}{\partial \xi} \Big|_{\xi=u_1} > 0 ,$$

which proves the inequality in (4.42).

We have therefore established the following theorem.

THEOREM 4.8. *Under the conditions (3.1) and (3.2), there exists a constant $T^+ > 0$ such that equations (4.24-4.26) have a unique solution $u_1^+(t)$, $u_3^+(t)$ and $x^+(t)$ for $t > T^+$. Functions $u_1^+(t)$ and $u_3^+(t)$ are increasing and decreasing functions of t , respectively. They have the limits:*

$$\lim_{t \rightarrow +\infty} u_1^+(t) = 1, \quad \lim_{t \rightarrow +\infty} u_3^+(t) = 0.$$

Furthermore,

$$1 - \delta < u_1^+(t) < 1, \quad 0 < u_3^+(t) < \delta_2 \quad \text{for } t > T^+,$$

where δ and δ_2 are defined in (4.11) and (4.40), respectively.

The second part of Theorem 4.8 is only for technical purpose. It will be needed in the next section.

5. Analysis of Zero and Single Phase Solutions

In this section, we shall show that the hodograph transform (2.22) $g = 0, 1$, can be inverted to give solutions $u_1(x, t), \dots, u_{2g+1}(x, t)$ for $-\infty < x < \infty$ and $t > \max\{T^-, T^+\}$, where T^- and T^+ are given in Theorems 4.4 and 4.8, respectively. We will also verify inequalities (2.23-2.24). By theorem 2.2, the function ψ of (2.25) will be the minimizer of (2.1).

For each $t > \max\{T^-, T^+\}$, we split the space into three pieces $(-\infty, x^-(t))$, $[x^-(t), x^+(t)]$ and $[x^+(t), +\infty)$. Here, $x^-(t)$ and $x^+(t)$ are given in Theorems 4.4 and 4.8, respectively.

§ 5.1 Zero phases: $x \leq x^-(t)$ and $x \geq x^+(t)$

We shall focus on the zero phase over the interval $x \geq x^+(t)$. The other zero phase over $x \leq x^-(t)$ can be handled in the same way.

For $x \geq x^+(t)$, we shall use the auxiliary minimum problem (4.33) to show that the minimizer of the minimization problem (2.1) has a compact support $[0, \sqrt{u_3(x, t)}]$. The function $u_3(x, t)$ is governed by equation (2.22), which, when $g = 0$, becomes

$$(5.1) \quad x - 30tu_3^2 - f(u_3) = 0.$$

For each $0 < u_3 < u_3^+(t)$ where $u_3^+(t)$ is given in Theorem 4.8, since $u_3^+(t)$ is a continuous function of t and $u_3^+(t) \rightarrow 0$ as $t \rightarrow +\infty$, there is a $t^* > t$ such that $u_3 = u_3^+(t^*)$. In view of the minimum problem (4.33), we have for $0 < \xi < 1$,

$$-\frac{G_0(\xi, u_3)}{V_0(\xi, u_3)} \geq -\frac{G_0(u_1^+(t^*), u_3)}{V_0(u_1^+(t^*), u_3)} = 30t^* > 30t,$$

where the equality follows from equation (4.31). Hence,

$$(5.2) \quad -30tV_0(\xi, u_3) - G_0(\xi, u_3) > 0$$

for all $0 < \xi < 1$. In particular, when $\xi = u_3$, inequality (5.2) reduces to $-60tu_3 - f'(u_3) > 0$ in view of formulae (4.29) and (4.30). This shows that equation (5.1) can be inverted to give u_3 as a decreasing function of x for $x > x^+(t)$.

We next verify inequalities (2.23-2.24), which now take the form

$$(5.3) \quad \operatorname{Re}\{\sqrt{-1}\sqrt{\xi - u_3}\}\Phi_0(\xi, u_3) < 0 \quad \text{for } 0 < \xi < u_3,$$

$$(5.4) \quad \int_{u_3}^{\xi} \sqrt{u - u_3}\Phi_0(u, u_3)du > 0 \quad \text{for } \xi > u_3.$$

Inequality (5.4) follows easily from (5.2) and Lemma 4.5.

To prove (5.3), we use $u_3 < u_3^+(t)$ again. Since $u_3^+(t) < \delta_2$ according to Theorem 4.8, we have $u_3 < \delta_2$. We then use (4.40) in formula (4.5) for F_0 to obtain that

$$\frac{\partial F_0(\xi, u_3)}{\partial \xi} > 0$$

for $\xi \leq u_3 < \delta_2$. Hence,

$$\frac{\partial \Phi_0(\xi, u_3)}{\partial \xi} = -30t \frac{\partial U_0(\xi, u_3)}{\partial \xi} - \frac{\partial F_0(\xi, u_3)}{\partial \xi} < 0$$

for all $\xi \leq u_3$. This together with $\Phi_0(u_3, u_3) = -60tu_3 - f'(u_3) > 0$ proves $\Phi_0(\xi, u_3) > 0$ for $\xi \leq u_3$ and hence inequality (5.3).

THEOREM 5.1. *Under the conditions (3.1) and (3.2), for each $t > \max\{T^-, T^+\}$, the minimizer of the minimization problem (2.1) is supported on a single interval when $x \leq x^-(t)$ and $x \geq x^+(t)$.*

§ 5.2 Single phase: $x^-(t) < x < x^+(t)$

For $t > \max\{T^-, T^+\}$ and $x^-(t) < x < x^+(t)$, we shall show that the minimizer of (2.1) is supported on two disjoint intervals $[0, \sqrt{u_3(x, t)}]$ and $[\sqrt{u_2(x, t)}, \sqrt{u_1(x, t)}]$, where $u_1(x, t)$, $u_2(x, t)$ and $u_3(x, t)$ are governed by equations (2.22) when $g = 1$

$$(5.5) \quad P(u_1, u_1, u_2, u_3) = 0, \quad P(u_2, u_1, u_2, u_3) = 0, \quad P(u_3, u_1, u_2, u_3) = 0.$$

We will first solve equations (5.5) near the leading edge $x = x^+(t)$. We will then extend the solution over the whole interval $x^-(t) < x < x^+(t)$.

The first two equations of (5.5) are degenerate near the leading edge at which $u_1 = u_2$. We replace them by

$$(5.6) \quad W_5(u_1, u_2, u_3, t) =: \frac{\partial q_{1,1}}{\partial u_1} + \frac{\partial q_{1,1}}{\partial u_2} + \frac{\partial q_{1,1}}{\partial u_3} = 0,$$

$$(5.7) \quad \mathbb{W}_6(u_1, u_2, u_3, t) =: \int_{u_3}^{u_2} \sqrt{(u_1 - \xi)(u_2 - \xi)(\xi - u_3)} \Phi_1(\xi, u_1, u_2, u_3) d\xi = 0,$$

where Φ_1 and $q_{1,1}$ are given by (2.5-2.7) and (2.8-2.9), respectively. It can be shown that equations (5.6) and (5.7) together with $P(u_3, \vec{u}) = 0$ are equivalent to equations (5.5) when $u_1 > u_2 > u_3$ [3].

It is also known that equations (5.6) and (5.7) transform into equations (4.24-4.25) when $u_1 = u_2$ and that equation $P(u_3, \vec{u}) = 0$ into (4.26) [3].

For each fixed t , we now solve equations (5.6) and (5.7) for u_1 and u_3 in terms of u_2 near the leading edge. We use the identities of Lemma 2.1 to calculate the

derivatives of W_5 and W_6 at the point $(u_1^+(t), u_1^+(t), u_3^+(t))$:

$$(5.8) \quad \frac{\partial W_5}{\partial u_1} = \frac{\partial W_5}{\partial u_2} = \frac{1}{2} \frac{\partial \Phi_0(\xi, u_3)}{\partial \xi} \Big|_{\xi=u_1} > 0 ,$$

$$(5.9) \quad \frac{\partial W_5}{\partial u_3} = \frac{\Phi_0(u_1, u_3) - \Phi_0(u_3, u_3)}{2(u_1 - u_3)} = -\frac{\Phi_0(u_3, u_3)}{2(u_1 - u_3)} < 0 ,$$

$$(5.10) \quad \begin{aligned} \frac{\partial W_6}{\partial u_1} &= \frac{\partial W_6}{\partial u_2} = \frac{1}{2} \int_{u_3}^{u_2} \sqrt{\xi - u_3} [2(u_1 - \xi) \frac{\partial \Phi_1(\xi, \vec{u})}{\partial u_1} + \Phi_1(\xi, \vec{u})] d\xi \\ &= \frac{(u_1 - u_3)^{\frac{3}{2}}}{3} \Phi_1(u_1, \vec{u}) = \frac{(u_1 - u_3)^{\frac{3}{2}}}{3} \frac{\partial \Phi_0(\xi, u_3)}{\partial \xi} \Big|_{\xi=u_1} > 0 , \end{aligned}$$

$$(5.11) \quad \begin{aligned} \frac{\partial W_6}{\partial u_3} &= - \int_{u_3}^{u_2} \frac{u_1 - \xi}{2\sqrt{\xi - u_3}} [2(u_3 - \xi) \frac{\partial \Phi_1(\xi, \vec{u})}{\partial u_3} + \Phi_1(\xi, \vec{u})] d\xi \\ &= - \int_{u_3}^{u_2} \frac{u_1 - \xi}{2\sqrt{\xi - u_3}} \Phi_1(u_3, \vec{u}) d\xi \\ &= - \frac{2(u_1 - u_3)^{\frac{3}{2}}}{3} \left[\frac{\Phi_0(u_3, u_3) - \Phi_0(u_1, u_3)}{u_3 - u_1} \right] \\ &= \frac{2\sqrt{u_1 - u_3}}{3} \Phi_0(u_3, u_3) > 0 , \end{aligned}$$

where $\vec{u} = (u_1, u_2, u_3)$.

The second equality of (5.8) follows from identity (2.21).

The first equality of (5.9) is a consequence of (2.18) and (2.21). The second equality follows from equation (4.24).

The third and last equalities of (5.10) follows from identities (2.18) and (2.20), respectively.

The second, third and last equalities of (5.11) follows from (2.18), (2.19) and (4.24), respectively.

All the signs are easily determined according to (4.42-4.45).

Therefore, equations (5.6) and (5.7) can be inverted to give u_1 and u_3 as functions of u_2 when u_2 is close to and less than $u_1^+(t)$. Moreover, u_1 decreases as u_2 increases. We then use the last equation of (5.5) to give x as a function of u_2 . We summarize these result in the lemma.

LEMMA 5.2. *For each $t > T$, equations (5.5) can be inverted to give x , u_1 and u_3 as functions of u_2 when u_2 is close to and less than $u_1^+(t)$.*

Having solved equations (5.5) for x , u_1 and u_3 as functions of u_2 near the leading edge $u_2 = u_2^+(t)$, we now extend the solution by decreasing u_2 . To calculate the Jacobian of (5.5), we turn to Lemma 2.3, which says that the zeros of the Jacobian are determined by the ξ -zeros of $\Phi_1(\xi, u_1, u_2, u_3)$. We again split

$$(5.12) \quad \Phi_1(\xi, u_1, u_2, u_3) = -30tU_1(\xi, u_1, u_2, u_3) - F_1(\xi, u_1, u_2, u_3) ,$$

where U_1 and F_1 satisfy the Euler-Poisson-Darboux equations (2.5-2.7) with the boundary values $4/3$ and $2f''(u)/3$, respectively. Hence, U_1 is a constant; indeed, $U_1 = 4/3$.

LEMMA 5.3. *For each t , $u_1 > u_2 > u_3$,*

$$\lim_{\xi \rightarrow 0} \Phi_1(\xi, u_1, u_2, u_3) = -\infty , \quad \lim_{\xi \rightarrow 1} \Phi_1(\xi, u_1, u_2, u_3) = +\infty .$$

Proof: Since U_1 is a constant, we see from (5.12) that it suffices to prove that

$$(5.13) \quad \lim_{\xi \rightarrow 0} F_1(\xi, u_1, u_2, u_3) = +\infty, \quad \lim_{\xi \rightarrow 1} F_1(\xi, u_1, u_2, u_3) = -\infty.$$

We first write $F_1(\xi, u_1, u_2, u_3)$ in terms of $F_0(\xi, u_1)$. We see from (2.5) that F_1 satisfies

$$2(u_2 - u_3) \frac{\partial^2 F_1}{\partial u_2 \partial u_3} = \frac{\partial F_1}{\partial u_2} - \frac{\partial F_1}{\partial u_3}$$

and that it has the boundary values $F_1(\xi, u_1, u, u) = \frac{F_0(\xi, u_1) - F_0(u, u_1)}{\xi - u}$ on account of (2.19). It then follows from formula (2.15) at $\rho = 1$ that

$$F_1(\xi, u_1, u_2, u_3) = \frac{1}{\pi} \int_{u_3}^{u_2} \frac{F_0(\xi, u_1) - F_0(u, u_1)}{(\xi - u) \sqrt{(u_2 - u)(u - u_3)}} du,$$

which together with (4.10) gives (5.13). This proves Lemma 5.3.

LEMMA 5.4. *Under the conditions (3.1) and (3.2), $\Phi_1(\xi, u_1, u_2, u_3)$ has at most one ξ -zero for all t, u_1, u_2 and u_3 whenever $1 - \delta < u_1 < 1$, where δ is given by Theorem 4.8.*

Proof: We first observe that

$$(5.14) \quad \frac{\partial}{\partial \xi} F_1(\xi, u_1, u_2, u_3) < 0$$

for all ξ, u_1, u_2 and u_3 whenever $1 - \delta < u_1 < 1$. To see this, we will derive a new formula for F_1 in terms of F_0 . For each fixed u_1 , we view $F_1(\xi, u_1, u_2, u_3)$ as a function of ξ, u_2 and u_3 only. Hence, equations (2.5) and (2.6) for F_1 reduce to

$$(5.15) \quad 2(u_2 - u_3) \frac{\partial^2 F_1}{\partial u_2 \partial u_3} = \frac{\partial F_1}{\partial u_2} - \frac{\partial F_1}{\partial u_3}, \quad 2(\xi - u_i) \frac{\partial^2 F_1}{\partial \xi \partial u_i} = \frac{\partial F_1}{\partial \xi} - 2 \frac{\partial F_1}{\partial u_i}, \quad i = 2, 3.$$

The new boundary condition is

$$(5.16) \quad F_1(u, u_1, u, u) = \frac{\partial F_0(\xi, u_1)}{\partial \xi} \Big|_{\xi=u}$$

on account of (2.20). One can then use the method of Section 2 to derive a double integral formula, similar to (2.17), for the solution of equations (5.15-5.16). Indeed, we have

$$F_1(\xi, u_1, u_2, u_3) = \frac{1}{2\sqrt{2}\pi} \int_{-1}^1 \int_{-1}^1 \frac{F_0 \xi \left(\frac{1+\mu}{2} \frac{1+\nu}{2} \xi + \frac{1+\mu}{2} \frac{1-\nu}{2} u_2 + \frac{1-\mu}{2} u_3, u_1 \right)}{\sqrt{(1-\mu)(1-\nu)}} \sqrt{1+\mu} d\mu d\nu,$$

which, when combined with (4.11), gives (5.14).

Since U_1 is a constant, we deduce from the split (5.12) that

$$(5.17) \quad \frac{\partial}{\partial \xi} \Phi_1(\xi, u_1, u_2, u_3) = -\frac{\partial}{\partial \xi} F_1(\xi, u_1, u_2, u_3) > 0 \quad \text{for } 1 - \delta < u_1 < 1,$$

where the inequality follows from (5.14). This proves the uniqueness of the ξ -zero of $\Phi_1(\xi, u_1, u_2, u_3)$ and hence Lemma 5.4.

Remark: *This is the third and also the last place where $m = 2$ is important. If $m > 2$, (5.17) is not true.*

We now continue to extend the solution of equations (5.5) by decreasing u_2 .

By Lemma 5.3, the function $\Phi_1(\xi, u_1, u_2, u_3)$ must have at least a ξ -zero. Indeed, it has a zero, denoted by u^* , between u_3 and u_2 in view of equation (5.7). According to Lemma 5.4, u^* is the only zero of Φ_1 . Hence,

$$(5.18) \quad \Phi_1(\xi, u_1, u_2, u_3) > 0 \quad \text{for } \xi > u^* \text{ and } \Phi_1(\xi, u_1, u_2, u_3) < 0 \quad \text{for } \xi < u^* .$$

Since $u_3 < u_2 < u_1$ and $u_3 < u^* < u_2$, we have

$$(5.19) \quad \Phi_1(u_1, u_1, u_2, u_3) > 0 , \quad \Phi_1(u_2, u_1, u_2, u_3) > 0 , \quad \Phi_1(u_3, u_1, u_2, u_3) < 0 .$$

For $P_{1,0}$ of (2.10-2.12), we also have

$$(5.20) \quad P_{1,0}(u_1, u_1, u_2, u_3) > 0 , \quad P_{1,0}(u_2, u_1, u_2, u_3) > 0 , \quad P_{1,0}(u_3, u_1, u_2, u_3) < 0 .$$

To see this, we observe from (2.10) that $P_{1,0}$ is a linear function of ξ and that it thus has only one ξ -zero. This zero is between u_3 and u_2 on account of equation (2.12). Inequalities (5.20) are then justified.

By Lemma 2.3, we deduce from inequalities (5.19) that the Jacobian of (5.5) is always non-zero. It then follows from the decomposition (2.26) for P and inequalities (5.20) that equations (5.5) can be inverted to give x , u_1 , and u_3 as increasing, decreasing, and decreasing functions of u_2 , respectively, provided that $u_3 < u_2 < u_1$ and $1 - \delta < u_1 < 1$. We now extend the solution as far as possible by decreasing u_2 ; so u_1 always stays in the interval $(1 - \delta, 1)$ because u_1 increases as u_2 decreases. Since both u_1 and u_3 increase as u_2 decreases, the solution will stop at $u_2 = u_3$. At this point, x , u_1 , $u_2 = u_3$ satisfy the trailing edge equations (4.1-4.3). According to Lemma 4.1, the equations have a unique solution for $t > T$. We thus have $x = x^-(t)$, $u_1 = u_1^-(t)$, $u_2 = u_3 = u_3^-(t)$.

We have therefore proved that equations (5.5) can be inverted to give u_1 , u_2 and u_3 as functions of x for $x^-(t) < x < x^+(t)$.

Inequalities (2.23-2.24) are immediate consequences of (5.7) and (5.18). We have therefore established the following theorem.

THEOREM 5.5. *Under the conditions (3.1) and (3.2), for each $t > \max\{T^-, T^+\}$, the minimizer of the minimization problem (2.1) is supported on two disjoint intervals when $x^-(t) < x < x^+(t)$.*

The main theorem of this paper, Theorem 3.1, follows from Theorems 5.1 and 5.5.

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