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Blow Up of the Solutions of Nonlinear Wave Equation in Reissner-Nordström Metric

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ABSTRACT. In this paper we study the properties of the solutions to the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0,1], x \in \mathcal{R}^3, \tag{1}$$

$$u(1,x) = u_0 \in \dot{B}^{\gamma}_{p,q}(\mathcal{R}^3), \quad u_t(1,x) = u_1 \in \dot{B}^{\gamma-1}_{p,q}(\mathcal{R}^3), \tag{2}$$

where g_s is the Reissner-Nordström metric (see [2]); $p > 1$, $q \geq 1$, $\gamma \in (0,1)$ are fixed constants, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable choosen.

When $g(r) \equiv 0$ we prove that the Cauchy problem (1), (2) has nontrivial solution $u \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+))$ in the form

$$u(t,r) = \begin{cases} v(t)\omega(r) & for \quad r \leq r_1, \quad t \in [0,1], \\ 0 \quad for & r \geq r_1, \quad t \in [0,1], \end{cases}$$

where $r = |x|$, for which $\lim_{t \longrightarrow 0} ||u||_{\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)} = \infty$.

When $g(r) \neq 0$ we prove that the Cauchy problem (1), (2) has nontrivial solution $u \in \mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+))$ in the form

$$u(t,r) = \begin{cases} v(t)\omega(r) & for \quad r \leq r_1, \quad t \in [0,1], \\ 0 \quad for & r \geq r_1, \quad t \in [0,1], \end{cases}$$

for which $\lim_{t \longrightarrow 0} ||u||_{\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)} = \infty$.

1. Introduction

In this paper we study the properties of the solutions to the Cachy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad t \in [0,1], x \in \mathcal{R}^3, \tag{1}$$

$$u(1,x) = u_0 \in \dot{B}^{\gamma}_{p,q}(\mathcal{R}^3), \quad u_t(1,x) = u_1 \in \dot{B}^{\gamma-1}_{p,q}(\mathcal{R}^3), \tag{2}$$

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where g_s is the Reissner-Nordstr$\ddot{o}$m metric (see [2])

$$g_s = \frac{r^2 - Kr + Q^2}{r^2} dt^2 - \frac{r^2}{r^2 - Kr + Q^2} dr^2 - r^2 d\phi^2 - r^2 sin^2\phi d\theta^2,$$

K and Q are positive constants, $p \in (1, \infty)$, $q \geq 1$ and $\gamma \in (0, 1)$ are fixed, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(|x|) \geq 0$, $g(|x|) = 0$ for $|x| \geq r_1$, a and b are positive constants, $r_1 > 0$ is suitable choosen.

The Cauchy problem (1), (2) we may rewrite in the form

$$\frac{r^2}{r^2 - Kr + Q^2} u_{tt} - \frac{1}{r^2} \partial_r ((r^2 - Kr + Q^2) u_r) -$$

$$\frac{1}{r^2 sin\phi} \partial_\phi (sin\phi u_\phi) - \frac{1}{r^2 sin^2\phi} u_{\theta\theta} = f(u) + g(r), \tag{1}$$

$$u(1, r, \phi, \theta) = u_0 \in \dot{B}^\gamma_{p,q}(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi]),$$

$$u_t(1, r, \phi, \theta) = u_1 \in \dot{B}^{\gamma-1}_{p,q}(\mathcal{R}^+ \times [0, 2\pi] \times [0, \pi]). \tag{2}$$

When g_s is the Minkowski metric; $u_0, u_1 \in \mathcal{C}^\infty_0(\mathcal{R}^3)$ in [6](see and [1], section 6.3) is proved that there exists $T > 0$ and a unique local solution $u \in \mathcal{C}^2([0, T) \times \mathcal{R}^3)$ for the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad f \in \mathcal{C}^2(\mathcal{R}), \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

for which

$$\sup_{t<T, x\in\mathcal{R}^3} |u(t, x)| = \infty.$$

When g_s is the Minkowski metric, $1 \leq p < 5$ and initial data are in $\mathcal{C}^\infty_0(\mathcal{R}^3)$, in [6](see and [1], section 6.3) is proved that the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u|u|^{p-1}, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Minkowski metric and initial data are in $\mathcal{C}^\infty_0(\mathcal{R}^3)$, in [5](see and [1], section 6.3) is proved that there exists a number $\epsilon_0 > 0$ such that for any data $(u_0, u_1) \in \mathcal{C}^\infty_0(\mathcal{R}^3)$ with $E(u(0)) < \epsilon_0$, the initial value problem

$$(u_{tt} - \Delta u)_{g_s} = u^5, \quad t \in [0, T], x \in \mathcal{R}^3,$$

$$u|_{t=0} = u_0, u_t|_{t=0} = u_1,$$

admits a global smooth solution.

When g_s is the Reissner-Nordstr$\ddot{o}$m metrics in [4] is proved that the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} + m^2 u = f(u), \quad t \in [0, 1], x \in \mathcal{R}^3,$$

$$u(1, x) = u_0 \in \dot{B}^\gamma_{p,p}(\mathcal{R}^3), \quad u_t(1, x) = u_1 \in \dot{B}^{\gamma-1}_{p,p}(\mathcal{R}^3),$$

where $m \neq 0$ is constant and $f \in \mathcal{C}^2(\mathcal{R}^2)$, $a|u| \leq |f^{(l)}(u)| \leq b|u|$, $l = 0, 1$, a and b are positive constants, has unique nontrivial solution $u(t, r) \in \mathcal{C}((0, 1] \dot{B}^\gamma_{p,p}(\mathcal{R}^+))$, $r = |x|$, $p > 1$, for which

$$\lim_{t \longrightarrow 0} ||u||_{\dot{B}^\gamma_{p,p}(\mathcal{R}^+)} = \infty.$$

When g_s is the Minkowski metric in [7] is proved that the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u), \quad t \in [0,1], x \in \mathcal{R}^3,$$

$$u(1,x) = u_0, \quad u_t(1,x) = u_1,$$

has global solution. Here $f \in \mathcal{C}^2(\mathcal{R})$, $f(0) = f'(0) = f''(0) = 0$,

$$|f''(u) - f''(v)| \le B|u - v|^{q_1}$$

for $|u| \le 1$, $|v| \le 1$, $B > 0$, $\sqrt{2} - 1 < q_1 \le 1$, $u_0 \in \mathcal{C}^5_\circ(\mathcal{R}^3)$, $u_1 \in \mathcal{C}^4_\circ(\mathcal{R}^3)$, $u_0(x) = u_1(x) = 0$ for $|x - x_0| > \rho$, x_0 and ρ are suitable choosen.

Our main results are

Theorem 1. *Let $p > 1$, $q \ge 1$, $\gamma \in (0,1)$ and K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, \quad \frac{1}{1-K+Q^2} \ge 1, \\ 1 - K + Q^2 > 0 \text{ is small enough such that} \\ \frac{K-\sqrt{K^2-4Q^2}}{2} - 2\sqrt{1-K+Q^2} > 0. \end{cases}$$

Let also $g \equiv 0$, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \le f'(u) \le b|u|$, a and b are positive constants. Then the homogeneous Cauchy problem (1), (2) has nontrivial solution $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{B}^\gamma_{p,q}(\mathcal{R}^+))$ for which

$$\lim_{t \longrightarrow 0} ||u||_{\dot{B}^\gamma_{p,q}(\mathcal{R}^+)} = \infty.$$

Theorem 2. *Let $p > 1$, $q \ge 1$, $\gamma \in (0,1)$ and K, Q are positive constants for which*

$$\begin{cases} K^2 > 4Q^2, \quad \frac{1}{1-K+Q^2} \ge 1, \\ 1 - K + Q^2 > 0 \text{ is small enough such that} \\ \frac{K-\sqrt{K^2-4Q^2}}{2} - 2\sqrt{1-K+Q^2} > 0. \end{cases}$$

Let also $g \ne 0$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \ge 0$ for $r \ge 0$, $g(r) = 0$ for $r \ge r_1$, $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \le f'(u) \le b|u|$, a and b are positive constants. Then the nonhomogeneous Cauchy problem (1), (2) has nontrivial solution $u(t,r) = v(t)\omega(r) \in \mathcal{C}((0,1]\dot{B}^\gamma_{p,q}(\mathcal{R}^+))$ for which

$$\lim_{t \longrightarrow 0} ||u||_{\dot{B}^\gamma_{p,q}(\mathcal{R}^+)} = \infty.$$

The paper is organized as follows. In section 2 we will prove some preliminary results. In section 3 we prove theorem 1. In section 4 we prove theorem 2. In appendix we prove some results which are used in the proof of theorem 1 and theorem 2.

2. Preliminary results

For fixed $q \ge 1$ and $\gamma \in (0,1)$ we put

$$C = \Big(\frac{q\gamma 2^{q\gamma}}{2^{q\gamma} - 1}\Big)^{\frac{1}{q}}$$

For fixed $p > 1$, $q \ge 1$, $\gamma \in (0,1)$ and $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \ge 0$ for $r \ge 0$ we suppose that the constants $A > 0$, $Q > 0$, $a > 0$, $b > 0$, $B > 0$, $K > 0$, $1 < \beta < \alpha$ satisfy the conditions

$$i1)\frac{1}{1-K+Q^2}\left(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\right)\leq 1,\quad A>1,\quad \frac{A^2}{a}>1;$$

$$i2)\begin{cases}\frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\frac{a}{2A^4}-\frac{br_1^2}{AB}\geq 0,\\ \frac{a}{2A^6\alpha^2(1-\alpha K+\alpha^2Q^2)}-\frac{2br_1^2}{A^2B^2}\geq 0,\\ \frac{a}{2A^4(1-\alpha K+\alpha^2Q^2)}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)-\frac{4ar_1}{A^3B(1-K+Q^2)^2}\geq 0,\\ \left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^2\frac{1}{(1-\alpha K+\alpha^2Q^2)^2}\frac{a}{4A^6}\\ -r_1^2\frac{2b}{(1-K+Q^2)A^2B^2}-r_1^2\frac{1}{1-K+Q^2}\max_{r\in[0,r_1]}g(r)\geq\frac{1}{A^2},\\ \left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^2\frac{1}{(1-\alpha K+\alpha^2Q^2)^2}\frac{a}{2A^4}-\frac{2ar_1^4}{AB(1-K+Q^2)}>0,\end{cases}$$

$$i3)C\left(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}+\frac{2b}{AB(1-K+Q^2)}+\frac{1}{A^2}\right)\frac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}+$$
$$C\frac{2^{1-\gamma}}{A^2(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}}<1,$$

$$i4)\begin{cases}\left(\frac{1}{\alpha^2}\frac{1}{1-\alpha K+\alpha^2Q^2}\frac{a}{4A^6}-\frac{1}{\beta^2}\frac{a}{4A^4}\right)>0,\\ \left(\frac{1}{\alpha^2}\frac{1}{1-\alpha K+\alpha^2Q^2}\frac{a}{2A^6}-\frac{1}{\beta^2}\frac{a}{2A^4}\right)>0,\end{cases}$$

$$i5)\begin{cases}K^2>4Q^2,\quad A\geq\frac{1}{1-K+Q^2}>1,\\ 1>\frac{2Q^2}{K}>\frac{K-\sqrt{K^2-4Q^2}}{2},\\ 1-K+Q^2>0\quad is\quad enough\quad small\quad such\quad that\\ 1>\frac{K-\sqrt{K^2-4Q^2}}{2}-2\sqrt{1-K+Q^2}>0,\\ \frac{2}{K-\sqrt{K^2-4Q^2}-2(1-K+Q^2)}<\beta<\alpha\leq 3,\end{cases}$$

$$i6)\frac{1}{1-K+Q^2}\left(\frac{2}{AB}\left(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\right)+r_1^2\max_{s\in[0,r_1]}g(s)\right)\leq\frac{2}{AB};$$

$$i7)\quad \max_{s\in[0,r_1]}g(s)\leq\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)^2\frac{1}{A^4},$$

where

$$r_1=\frac{K-\sqrt{K^2-4Q^2}}{2}-\sqrt{1-K+Q^2}.$$

Example. Let $0<\epsilon<<\frac{1}{3}$ is enough small,

$$\begin{array}{l}A=\frac{1}{\epsilon^4},\quad B=\frac{1}{\epsilon},\quad p=\frac{3}{2},\quad q=\frac{3}{2},\quad \gamma=\frac{1}{3},\quad \alpha=3,\\ \frac{1}{\beta}=\frac{K-\sqrt{K^2-4Q^2}}{2}-2\sqrt{1-K+Q^2},\\ g(r)=\begin{cases}\epsilon^{11}(r-r_1)^2\quad for\quad r\leq r_1,\\ 0\quad for\quad r\geq r_1,\end{cases}\\ K=\frac{4}{3}+\frac{1}{6}\epsilon^{20}-\frac{3}{2}\epsilon^2,\\ Q^2=\frac{1}{3}+\frac{1}{6}\epsilon^{20}-\frac{1}{2}\epsilon^2,\\ a=\epsilon^4,\quad b=\epsilon^3.\end{array}$$

Then

$$\begin{array}{l}1-\alpha K+\alpha^2Q^2=1-3K+9Q^2=\epsilon^{20},\\ 1-K+Q^2=\epsilon^2\bullet.\end{array}$$

When $g(r)\equiv 0$ we put

$$(1')\qquad u_0:=v(1)\omega(r)=$$

$$= \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2-Ks+Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) \right) ds d\tau \\ \quad for \quad r \le r_1, \\ 0 \quad for \quad r \ge r_1, \end{cases}$$

and $u_1 \equiv 0$. Here $v(t)$ is fixed function which satisfies the conditions

$$(H1) \quad v(t) \in \mathcal{C}^3[0,\infty), \quad v(t) > 0 \quad for \quad \forall t \in [0,1];$$
$$(H2) \quad v''(t) > 0 \quad for \quad \forall t \in [0,1], \quad v'(1) = v'''(1) = 0, \quad v(1) \ne 0;$$
$$(H3) \begin{cases} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{a}{2A^4}, \quad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2a}{A^2}; \\ \lim_{t\longrightarrow 0}[v''(t) - \frac{a}{2A^4}v(t)] = +0, \quad v''(t) - \frac{a}{2A^4}v(t) \ge 0 \quad for \quad t \in [0,1]. \end{cases}$$

In section 3 we will prove that the equation $(1')$ has unique nontrivial solution $\omega(r)$ for which $\omega(r) \in \mathcal{C}^2[0,r_1]$, $\omega(r) \in \dot{B}^\gamma_{p,q}[0,r_1]$, $|\omega(r)| \le \frac{2}{AB}$ for $r \in [0,r_1]$, $\omega(r) \ge \frac{1}{A^2}$ for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\omega(r_1) = \omega'(r_1) = \omega''(r_1) = 0$.

When $g(r) \ne 0$ we put

$$(1'') \qquad u_0 = v(1)\omega(r) =$$

$$= \begin{cases} \int_r^{r_1} \frac{1}{\tau^2-K\tau+Q^2} \int_\tau^{r_1} \left(\frac{s^4}{s^2-Ks+Q^2} v''(1)\omega(s) - s^2 f(v(1)\omega(s)) - s^2 g(s) \right) ds d\tau \\ for \quad r \le r_1, \\ 0 \quad for \quad r \ge r_1, \end{cases}$$

$u_1 \equiv 0$. Here $v(t)$ is fixed function which satisfies the hypothesis $(H1)$, $(H2)$ and

$$(H4) \begin{cases} \min_{t\in[0,1]} \frac{v''(t)}{v(t)} \ge \frac{a}{4A^4}, \quad \max_{t\in[0,1]} \frac{v''(t)}{v(t)} \le \frac{2a}{A^2}; \\ \lim_{t\longrightarrow 0}[v''(t) - \frac{a}{4A^4}v(t)] = +0, \quad v''(t) - \frac{a}{4A^4}v(t) \ge 0 \quad for \quad t \in [0,1]. \end{cases}$$

In section 4 we will prove that the equation $(1'')$ has unique nontrivial solution $\omega(r)$ for which $\omega(r) \in \mathcal{C}^2[0,r_1]$, $\omega(r) \in \dot{B}^\gamma_{p,q}[0,r_1]$, $|\omega(r)| \le \frac{2}{AB}$ for $r \in [0,r_1]$, $\omega(r) \ge \frac{1}{A^2}$ for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, $\omega(r_1) = \omega'(r_1) = \omega''(r_1) = 0$.

Example.

1) There exists function $v(t)$ for which (H1)-(H3) are hold. Really, let us consider the function

$$(3) \qquad v(t) = \frac{(t-1)^2 + \frac{4A^4}{a} - 1}{\frac{A^3}{a}},$$

where the constants A and a satisfy the conditions $A > 1$, $\frac{A^2}{a} > 1$. Then

1) $v(t) \in \mathcal{C}^3[0,\infty)$ and $v(t) > 0$ for all $t \in [0,1]$, i.e. (H1) is hold.

2)

$$v'(t) = \frac{2(t-1)}{\frac{A^3}{a}}, \quad v'(1) = 0,$$
$$v''(t) = \frac{2}{\frac{A^3}{a}} \ge 0 \quad \forall \quad t \in [0,1],$$
$$v'''(t) = 0, \quad v'''(1) = 0,$$

consequently (H2) is hold. On the other hand we have

$$\frac{v''(t)}{v(t)} = \frac{2}{(t-1)^2 + \frac{4A^4}{a} - 1}.$$

From here

$$\begin{aligned}
&\min_{t\in[0,1]} \tfrac{v''(t)}{v(t)} \geq \tfrac{a}{2A^4},\\
&\max_{t\in[0,1]} \tfrac{v''(t)}{v(t)} \leq \tfrac{2a}{A^2},\\
&v''(t) - \tfrac{a}{2A^4}v(t) = \tfrac{1}{\frac{2A^7}{a^2}}(2-t)t,\\
&\lim_{t\longrightarrow 0}[v''(t) - \tfrac{a}{2A^4}v(t)] = +0,
\end{aligned}$$

i.e. (H3) is hold.

2) The function

$$v(t) = \frac{(t-1)^2 + \frac{8A^4}{a} - 1}{\frac{A^3}{a}}, \tag{3'}$$

satisfies the hypothesis (H1), (H2) and (H4).

Remark. Here we will use the following definition of the $\dot{B}^{\gamma}_{p,q}(M)$-norm ($\gamma \in (0,1)$, $p>1$, $q\geq 1$) (see [3, p.94, def. 2], [1])

$$||u||_{\dot{B}^{\gamma}_{p,q}(M)} = \left(\int_0^2 h^{-1-q\gamma}||\Delta_h u||^q_{L^p(M)}dh\right)^{\frac{1}{q}},$$

where

$$\Delta_h u = u(x+h) - u(x).$$

Lemma 1. *Let $u(x)\in\mathcal{C}^2([0,r_1])$, $u(x)=0$ for $x\geq r_1$, $0<r_1<1$. Then for $\gamma\in(0,1)$, $p>1$, $q\geq 1$ we have*

$$C||u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \geq ||u||_{L^p([0,r_1])}.$$

Proof. We have

$$\begin{aligned}
||u||^q_{\dot{B}^{\gamma}_{p,q}([0,r_1])} &= \int_0^2 h^{-1-q\gamma}||\Delta_h u||^q_{L^p([0,r_1])}dh =\\
&= \int_0^2 h^{-1-q\gamma}||u(x+h)-u(x)||^q_{L^p([0,r_1])}dh \geq\\
&\geq \int_1^2 h^{-1-q\gamma}||u(x+h)-u(x)||^q_{L^p([0,r_1])}dh =\\
&= \int_1^2 h^{-1-q\gamma}||u(x)||^q_{L^p([0,r_1])}dh =\\
&= ||u(x)||^q_{L^p([0,r_1])}\int_1^2 h^{-1-q\gamma}dh =\\
&= ||u(x)||^q_{L^p([0,r_1])}\frac{2^{q\gamma}-1}{q\gamma 2^{q\gamma}},
\end{aligned}$$

i.e.

$$||u||^q_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \geq \frac{2^{q\gamma}-1}{q\gamma 2^{q\gamma}}||u(x)||^q_{L^p([0,r_1])},$$

from where we get

$$C||u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \geq ||u(x)||_{L^p([0,r_1])}.\bullet$$

Remark. We note that from i6) we have $g(r) = r^2 - Kr + Q^2 > 0$ for $r\in[0,r_1]$, $g(r)$ is decrease function for $r\in[0,r_1]$. Also (for $r\in[0,r_1]$) we have

$$\frac{r^2}{r^2-Kr+Q^2} \leq \frac{1}{1-K+Q^2}.$$

Really, let

$$\epsilon = \sqrt{1 - K + Q^2}.$$

Then, if $\tilde{r} = \frac{K - \sqrt{K^2 - 4Q^2}}{2}$, we have $r_1 = \tilde{r} - \epsilon$. We note that

$$\frac{r^2}{r^2 - Kr + Q^2}$$

is increase function for $r \in [0, r_1]$. Therefore

$$\frac{r^2}{r^2 - Kr + Q^2} \leq \frac{r_1^2}{r_1^2 - Kr_1 + Q^2} \leq \frac{1}{\epsilon^2} = \frac{1}{1 - K + Q^2}.$$

Also we have

$$\frac{1}{r^2 - Kr + Q^2} \leq \frac{1}{1 - K + Q^2}$$

for $r \in [0, r_1]$.

Proposition 2.1. Let $g(r) \equiv 0$, $f \in \mathcal{C}(\mathcal{R})$. If for every fixed $t \in [0, 1]$ $u(t, r) = v(t)\omega(r)$ satisfies the integral equation

$$(1^\star) \quad u(t, r) = \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_r^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 f(u) \Big) ds d\tau$$

then $u(t, r) = v(t)\omega(r)$ satisfies the equation (1) for every fixed $t \in [0, 1]$. Here $v(t)$ is function which satisfies the hypothesis (H1)-(H3) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$.

Proof. Let $t \in [0, 1]$ is fixed. Let also $u(t, r) = v(t)\omega(r)$ satisfies the integral equation $(1^\star)$ for every fixed $t \in [0, 1]$. Then for fixed $t \in [0, 1]$ we have

$$u_r(t, r) = \tfrac{1}{r^2 - Kr + Q^2} \int_{r_1}^r \Big(\tfrac{s^4}{s^2 - Ks + Q^2} \tfrac{v''(t)}{v(t)} u(t, s) - s^2 f(u) \Big) ds,$$
$$(r^2 - Kr + Q^2) u_r(t, r) = \int_{r_1}^r \Big(\tfrac{s^4}{s^2 - Ks + Q^2} \tfrac{v''(t)}{v(t)} u(t, s) - s^2 f(u) \Big) ds,$$
$$\partial_r((r^2 - Kr + Q^2) u_r(t, r)) = \tfrac{r^4}{r^2 - Kr + Q^2} \tfrac{v''(t)}{v(t)} u(t, r) - r^2 f(u),$$

Since for every fixed $t \in [0, 1]$ we have

$$u_{tt}(t, r) = v''(t)\omega(r)$$

we have

$$\partial_r((r^2 - Kr + Q^2) u_r(t, r)) = \frac{r^4}{r^2 - Kr + Q^2} u_{tt}(t, r) - r^2 f(u)$$

for every fixed $t \in [0, 1]$. From here for every fixed $t \in [0, 1]$ we have

$$\frac{r^2}{r^2 - Kr + Q^2} u_{tt}(t, r) - \frac{1}{r^2} \partial_r((r^2 - Kr + Q^2) u_r) = f(u),$$

i.e. for every fixed $t \in [0, 1]$ if $u(t, r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1)-(H3) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, satisfies the integral equation $(1^\star)$ we have that $u(t, r) = v(t)\omega(r)$ satisfies the equation (1) for every fixed $t \in [0, 1]$.

Proposition 2.2. Let $g(r) \equiv 0$, $f \in \mathcal{C}(\mathcal{R})$. If for every fixed $t \in [0, 1]$ $u(t, r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1)-(H3) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, $\omega(r_1) = \omega'(r_1) = 0$, satisfies the equation (1) then $u(t, r) = v(t)\omega(r)$ satisfies the equation $(1^\star)$ for every fixed $t \in [0, 1]$.

Proof. Let $t \in [0,1]$ is fixed. Let also $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1)-(H3) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, satisfies the equation (1). Then

$$\begin{array}{l} \frac{r^2}{r^2-Kr+Q^2}u_{tt}(t,r) - \frac{1}{r^2}\partial_r((r^2-Kr+Q^2)u_r) = f(u), \\ \partial_r((r^2-Kr+Q^2)u_r(t,r)) = \frac{r^4}{r^2-Kr+Q^2}u_{tt}(t,r) - r^2 f(u). \end{array}$$

Now we integrate the last equation with respect the variable r and we use that $\omega'(r_1) = 0$, $u_r(t,r_1) = 0$ for every fixed $t \in [0,1]$

$$(r^2-Kr+Q^2)u_r(t,r) = \int_{r_1}^{r}\Big(\frac{s^4}{s^2-Ks+Q^2}u_{tt}(t,s) - s^2 f(u)\Big)ds.$$

Again we integrate with respect the variable r and we use that $\omega(r_1) = 0$, $u(t,r_1) = 0$ for every fixed $t \in [0,1]$. Then we get

$$u(t,r) = \int_{r}^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_{\tau}^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}u_{tt}(t,s) - s^2 f(u)\Big)dsd\tau$$

for every fixed $t \in [0,1]$. Since for every fixed $t \in [0,1]$ we have

$$u_{tt}(t,r) = v''(t)\omega(r)$$

we get

$$u(t,r) = \int_{r}^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_{\tau}^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s) - s^2 f(u)\Big)dsd\tau$$

for every fixed $t \in [0,1]$. Therefore, if for every fixed $t \in [0,1]$ $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1)-(H3) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, $\omega(r_1) = \omega'(r_1) = 0$, satisfies the equation (1) then $u(t,r) = v(t)\omega(r)$ satisfies the equation ($1^\star$) for every fixed $t \in [0,1]$.

Proposition 2.3. Let $g(r) \in \mathcal{C}(\mathcal{R})$, $f \in \mathcal{C}(\mathcal{R})$. If for every fixed $t \in [0,1]$ $u(t,r) = v(t)\omega(r)$ satisfies the integral equation

($1^{\star\star}$)

$$u(t,r) = \int_{r}^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_{r}^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s) - s^2 f(u) - s^2 g(s)\Big)dsd\tau$$

then $u(t,r) = v(t)\omega(r)$ satisfies the equation (1) for every fixed $t \in [0,1]$. Here $v(t)$ is function which satisfies the hypothesis (H1), (H2),(H4) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$.

Proof. Let $t \in [0,1]$ is fixed. Let also $u(t,r) = v(t)\omega(r)$ satisfies the integral equation ($1^{\star\star}$) for every fixed $t \in [0,1]$. Then for fixed $t \in [0,1]$ we have

$$\begin{array}{l} u_r(t,r) = \frac{1}{r^2-Kr+Q^2}\int_{r_1}^{r}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s) - s^2 f(u) - s^2 g(s)\Big)ds, \\ (r^2-Kr+Q^2)u_r(t,r) = \int_{r_1}^{r}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s) - s^2 f(u) - s^2 g(s)\Big)ds, \\ \partial_r((r^2-Kr+Q^2)u_r(t,r)) = \frac{r^4}{r^2-Kr+Q^2}\frac{v''(t)}{v(t)}u(t,r) - r^2 f(u) - r^2 g(r), \end{array}$$

Since for every fixed $t \in [0,1]$ we have

$$u_{tt}(t,r) = v''(t)\omega(r)$$

we have

$$\partial_r((r^2-Kr+Q^2)u_r(t,r)) = \frac{r^4}{r^2-Kr+Q^2}u_{tt}(t,r) - r^2 f(u) - r^2 g(r)$$

for every fixed $t \in [0,1]$. From here for every fixed $t \in [0,1]$ we have

$$\frac{r^2}{r^2 - Kr + Q^2} u_{tt}(t,r) - \frac{1}{r^2}\partial_r((r^2 - Kr + Q^2)u_r) = f(u) + g(r),$$

i.e. for every fixed $t \in [0,1]$ if $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1), (H2),(H4) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, satisfies the integral equation ($1^{\star\star}$) we have that $u(t,r) = v(t)\omega(r)$ satisfies the equation (1) for every fixed $t \in [0,1]$.

Proposition 2.4. Let $g(r) \in \mathcal{C}(\mathcal{R})$, $f \in \mathcal{C}(\mathcal{R})$. If for every fixed $t \in [0,1]$ $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1), (H2),(H4) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, $\omega(r_1) = \omega'(r_1) = 0$, satisfies the equation (1) then $u(t,r) = v(t)\omega(r)$ satisfies the equation ($1^{\star\star}$) for every fixed $t \in [0,1]$.

Proof. Let $t \in [0,1]$ is fixed. Let also $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1), (H2),(H4) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, $\omega(r_1) = \omega'(r_1) = 0$, satisfies the equation (1). Then

$$\begin{array}{l} \frac{r^2}{r^2-Kr+Q^2} u_{tt}(t,r) - \frac{1}{r^2}\partial_r((r^2 - Kr + Q^2)u_r) = f(u) + g(r), \\ \partial_r((r^2 - Kr + Q^2)u_r(t,r)) = \frac{r^4}{r^2-Kr+Q^2} u_{tt}(t,r) - r^2 f(u) - r^2 g(r). \end{array}$$

Now we integrate the last equation with respect the variable r and we use that $\omega'(r_1) = 0$, $u_r(t,r_1) = 0$ for every fixed $t \in [0,1]$,

$$(r^2 - Kr + Q^2)u_r(t,r) = \int_{r_1}^{r} \Big(\frac{s^4}{s^2 - Ks + Q^2} u_{tt}(t,s) - s^2 f(u) - s^2 g(s)\Big) ds$$

for every fixed $t \in [0,1]$. Again we integrate with respect the variable r and we use that $\omega(r_1) = 0$, $u(t,r_1) = 0$ for every fixed $t \in [0,1]$. Then we get

$$u(t,r) = \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} u_{tt}(t,s) - s^2 f(u) - s^2 g(s)\Big) ds d\tau$$

for every fixed $t \in [0,1]$. Since for every fixed $t \in [0,1]$ we have

$$u_{tt}(t,r) = v''(t)\omega(r)$$

we get

$$u(t,r) = \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) - s^2 f(u) - s^2 g(s)\Big) ds d\tau$$

for every fixed $t \in [0,1]$. Therefore, if for every fixed $t \in [0,1]$ $u(t,r) = v(t)\omega(r)$, where $v(t)$ is function which satisfies the hypothesis (H1), (H2), (H4) and $\omega(r) \in \mathcal{C}^2(\mathcal{R})$, $\omega(r_1) = \omega'(r_1) = 0$, satisfies the equation (1) then $u(t,r) = v(t)\omega(r)$ satisfies the equation ($1^{\star\star}$) for every fixed $t \in [0,1]$.

3. Proof of Theorem 1

3.1. Local existence of nontrivial solutions of homogeneous Cauchy problem (1), (2). Let $v(t)$ is fixed function which satisfies the hypothesis $(H1) - (H3)$.

In this section we will prove that the homogeneous Cauchy problem (1), (2) has nontrivial solution in the form

$$u(t,r) = \begin{cases} v(t)\omega(r) \quad for \quad r \le r_1, \quad t \in [0,1], \\ 0 \quad for \quad r \ge r_1, \quad t \in [0,1]. \end{cases}$$

Let us consider the integral equation

$$(\star)\quad u(t,r)=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u(t,s))\Big)dsd\tau,\\ \quad 0\le r\le r_1,\quad t\in[0,1],\\ 0\quad for\quad r\ge r_1,\quad t\in[0,1],\end{cases}$$

where $u(t,r)=v(t)\omega(r)$.

Theorem 3.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1)-(H3). Let also $p>1$, $q\in[1,\infty)$ and $\gamma\in(0,1)$ are fixed and the positive constants A, a, b, B, Q,K, $\alpha>\beta>1$ satisfy the conditions i1)-i5) and $f\in\mathcal{C}^1(\mathcal{R}^1)$, $f(0)=0$, $a|u|\le f'(u)\le b|u|$. Then the equation $(\star)$ has unique nontrivial solution $u(t,r)=v(t)\omega(r)$ for which $w\in\mathcal{C}^2[0,r_1]$, $u(t,r)=u_r(t,r)=u_{rr}(t,r)=0$ for $r\ge r_1$, $u(t,r)\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}[0,r_1])$, for $r\in\Big[\frac{1}{\alpha},\frac{1}{\beta}\Big]$ and $t\in[0,1]$ $u(t,r)\ge\frac{1}{A^2}$, for $r\in\Big[0,r_1\Big]$ and $t\in[0,1]$ $|u(t,r)|\le\frac{2}{AB}$.*

Proof. Let N is the set

$$N=\{u(t,r)\in\mathcal{C}([0,1]\times[0,r_1]);u(t,r)=u_r(t,r)=u_{rr}(t,r)=0\quad for\quad r\ge r_1$$

$$and\quad t\in[0,1],u(t,r)\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}[0,r_1]);$$

$$for\quad r\in\Big[\frac{1}{\alpha},\frac{1}{\beta}\Big]\quad and\quad t\in[0,1]\quad u(t,r)\ge\frac{1}{A^2};\quad u(t,r)\ge0\quad for\quad t\in[0,1]$$

$$and\quad r\in\Big[\frac{1}{\alpha},r_1\Big];\ for\ r\in\Big[0,r_1\Big]\quad and\quad t\in[0,1]\quad |u(t,r)|\le\frac{2}{AB}\}.$$

Let also $t\in[0,1]$ is fixed.

We define the operator R as follow

$$R(u)=\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u)\Big)dsd\tau,0\le r\le r_1.$$

First we will show that $R:N\longrightarrow N$. We have

1) $R(u)\in\mathcal{C}^2([0,r_1])$ because $u(t,r)\in\mathcal{C}([0,r_1])$, $R(u)\in\mathcal{C}([0,1]\times[0,r_1])$, $R(u)|_{r=r_1}=0$,

$$\frac{\partial}{\partial r}R(u)=\frac{1}{r^2-Kr+Q^2}\int_{r_1}^r\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u-s^2f(u)\Big]ds,\ \frac{\partial}{\partial r}R(u)|_{r=r_1}=0,$$

$$\frac{\partial^2}{\partial r^2}R(u)=\frac{K-2r}{(r^2-Kr+Q^2)^2}\int_{r_1}^r\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u-s^2f(u)\Big]ds+$$

$$+\frac{r^4}{(r^2-Kr+Q^2)^2}\frac{v''(t)}{v(t)}u(t,r)-\frac{r^2}{r^2-Kr+Q^2}f(u).$$

Since $u(t,r_1)=0$, $f(u(t,r_1))=f(0)=0$ we get that

$$\frac{\partial^2}{\partial r^2}R(u)|_{r=r_1}=0.$$

2) For $r\in[0,r_1]$ and $t\in[0,1]$ we have $|u(t,r)|\le\frac{2}{AB}$. Then

$$|R(u)|=\Big|\int_{r_1}^r\frac{1}{\tau^2-K\tau+Q^2}\int_{r_1}^\tau\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u-s^2f(u)\Big)dsd\tau\Big|\le$$

$$\le\int_{r_1}^r\frac{1}{\tau^2-K\tau+Q^2}\int_{r_1}^\tau\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}|u|+s^2|f(u)|\Big)dsd\tau\le$$

(since $a|u| \le f'(u) \le b|u|$ and $f(0)=0$ we have $|f(u)| \le \frac{b}{2}|u|^2$)

$$\le \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + \frac{b}{2} s^2 |u|^2 \Big) ds d\tau =$$

$$\int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + \frac{b}{2} s^2 |u| \Big) |u| ds d\tau \le$$

$$\le \frac{2}{AB} \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + \frac{b}{AB} s^2 \Big) ds d\tau \le$$

(here we use that $\frac{r^2}{r^2 - Kr + Q^2} \le \frac{1}{1 - K + Q^2}$, $\frac{1}{r^2 - Kr + Q^2} \le \frac{1}{1 - K + Q^2}$ for $r \in [0, r_1]$)

$$\le \frac{2}{AB} \frac{1}{1 - K + Q^2} \Big(\frac{1}{1 - K + Q^2} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + \frac{b}{AB} \Big) \le$$

$$\le \frac{2}{AB} \frac{1}{1 - K + Q^2} \Big(\frac{1}{1 - K + Q^2} \frac{2a}{A^2} + \frac{b}{AB} \Big) \le \frac{2}{AB}.$$

(in the last inequality we use i1)). Consequently

$$|R(u)| \le \frac{2}{AB} \quad for \quad r \in [0, r_1], \quad t \in [0,1].$$

3) For $r \in \left[\frac{1}{\alpha}, r_1\right]$ and $t \in [0,1]$ we have $u(t,r) \ge 0$. Then $au \le f'(u) \le bu$ for $r \in \left[\frac{1}{\alpha}, r_1\right]$ and $t \in [0,1]$. Since $f(0) = 0$ we conclude that $f(u) \le \frac{b}{2} u^2$ for $r \in \left[\frac{1}{\alpha}, r_1\right]$ and $t \in [0,1]$. Then for $t \in [0,1]$ and $r \in \left[\frac{1}{\alpha}, r_1\right]$ we have

$$R(u) = \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) - s^2 f(u) \Big) ds d\tau \ge$$

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) - s^2 \frac{b}{2} u^2 \Big) ds d\tau =$$

$$= \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} - s^2 \frac{b}{2} u \Big) u ds d\tau \ge$$

(now we use that $u(t,r) \le \frac{2}{AB}$ for $r \in \left[\frac{1}{\alpha}, r_1\right]$ and $t \in [0,1]$)

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} - s^2 \frac{b}{AB} \Big) u ds d\tau \ge$$

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} - s^2 \frac{b}{AB} \Big) u ds d\tau \ge$$

(here we use (H3))

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} - s^2 \frac{b}{AB} \Big) u ds d\tau \ge$$

(Here we use that for $s \in [0, r_1]$ the function $\frac{s^2}{s^2 - Ks + Q^2}$ is increase function. Therefore for $s \in \left[\frac{1}{\alpha}, r_1\right]$ we have $\frac{s^4}{s^2 - Ks + Q^2} \ge \frac{1}{\alpha^2 (1 - \alpha K + \alpha^2 Q^2)}$)

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{1}{\alpha^2 (1 - \alpha K + \alpha^2 Q^2)} \frac{a}{2A^4} - s^2 \frac{b}{AB} \Big) u ds d\tau \ge$$

$$\ge \int_{r}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{1}{\alpha^2 (1 - \alpha K + \alpha^2 Q^2)} \frac{a}{2A^4} - r_1^2 \frac{b}{AB} \Big) u ds d\tau,$$

i.e.

$$(4)\qquad R(u)\geq\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\frac{a}{2A^4}-r_1^2\frac{b}{AB}\Big)udsd\tau$$

for $r\in\left[\frac{1}{\alpha},r_1\right]$ and $t\in[0,1]$. From the first inequality of i2) we have

$$\frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\frac{a}{2A^4}-r_1^2\frac{b}{AB}\geq 0.$$

Since $u(t,r)\geq 0$ for $r\in\left[\frac{1}{\alpha},r_1\right]$, $t\in[0,1]$ and $\frac{1}{\tau^2-K\tau+Q^2}>0$ for $\tau\in[0,r_1]$ from (4) we conclude that $R(u)\geq 0$ for $r\in\left[\frac{1}{\alpha},r_1\right]$ and $t\in[0,1]$.

On the other hand, for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$,

$$\frac{\partial}{\partial r}R(u)=\frac{1}{r^2-Kr+Q^2}\int_r^{r_1}\Big(s^2f(u)-\frac{v''(t)}{v(t)}\frac{s^4}{s^2-Ks+Q^2}u\Big)ds.$$

Let

$$l(r)=\int_r^{r_1}\Big(s^2f(u)-\frac{v''(t)}{v(t)}\frac{s^4}{s^2-Ks+Q^2}u\Big)ds.$$

Then

$$l'(r)=-r^2f(u)+\frac{v''(t)}{v(t)}\frac{r^4}{r^2-Kr+Q^2}u.$$

Then for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ we have

$$l'(r)\geq\frac{a}{2A^4}\frac{1}{\alpha^2(1-\alpha K+\alpha^2Q^2)}\frac{1}{A^2}-2r_1^2\frac{b}{A^2B^2}\geq 0.$$

(see second inequality of i2)). Consequently $l(r)$ is increase function for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$. From here

$$l(r)\geq\int_{\frac{1}{\alpha}}^{r_1}\Big(s^2f(u)-\frac{v''(t)}{v(t)}\frac{s^4}{s^2-Ks+Q^2}u\Big)ds$$

for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$. Since $\frac{1}{r^2-Kr+Q^2}\geq 0$ for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ we have for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$

$$\frac{\partial}{\partial r}R(u)\geq\frac{1}{r^2-Kr+Q^2}\int_{\frac{1}{\alpha}}^{r_1}\Big(s^2f(u)-\frac{v''(t)}{v(t)}\frac{s^4}{s^2-Ks+Q^2}u\Big)ds\geq$$

$$\geq\frac{1}{r^2-Kr+Q^2}\Big(\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}s^2f(u)ds-\int_0^{r_1}\frac{v''(t)}{v(t)}\frac{s^4}{s^2-Ks+Q^2}uds\Big)\geq$$

$$\geq\frac{a}{2A^4(1-\alpha K+\alpha^2Q^2)}\Big(\frac{1}{\beta}-\frac{1}{\alpha}\Big)-\frac{4ar_1}{A^3B(1-K+Q^2)^2}\geq 0.$$

(see third inequality of i2)) Therefore $R(u)$ is increase function for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$. Consequently, for $r\in\left[\frac{1}{\alpha},\frac{1}{\beta}\right]$,

$$R(u)\geq\int_{\frac{1}{\alpha}}^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u-s^2f(u)\Big)dsd\tau\geq$$

$$\geq\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}udsd\tau-$$

$$\int_{\frac{1}{\alpha}}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} s^2 f(u) ds d\tau \geq$$

$$\geq \Big(\frac{1}{\beta} - \frac{1}{\alpha}\Big)^2 \frac{1}{(1-\alpha K + \alpha^2 Q^2)^2} \frac{a}{2A^6} - r_1^2 \frac{2b}{(1-K+Q^2)A^2B^2} \geq \frac{1}{A^2}.$$

(in the last inequality we use the fourth inequality of i2)) Consequently, for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$ and $t \in [0,1]$ we have

$$R(u) \geq \frac{1}{A^2}.$$

4)

$$||\Delta_h R(u)||_{L^p}^q =$$

$$\Big(\int_0^{r_1} \Big(\Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s) - s^2 f(u)\Big) ds d\tau\Big|\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

$$\leq \Big(\int_0^{r_1} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + \frac{b}{2}|u|^2 s^2\Big) ds d\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

(here we use that for $s \in [0, r_1]$ we have $\frac{s^4}{s^2-Ks+Q^2} \leq \frac{1}{1-K+Q^2}$, also for $r \in [0, r_1]$ and $t \in [0,1]$ we have $|u(t,r)| \leq \frac{2}{AB}$)

$$\leq \Big(\int_0^{r_1} \Big(\int_r^{r+h} \frac{1}{1-K+Q^2} \int_\tau^{r_1} \Big(\frac{1}{1-K+Q^2} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} |u| +$$

$$+ \frac{b}{AB}|u|\Big) ds d\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

(here we use (H3))

$$\leq \Big(\int_0^{r_1} \Big(\int_r^{r+h} \frac{1}{1-K+Q^2} \int_\tau^{r_1} \Big(\frac{1}{1-K+Q^2} \frac{2a}{A^2}|u| + \frac{b}{AB}|u|\Big) ds d\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

$$\leq \Big(\int_0^{r_1} \Big(\int_r^{r+h} \frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} \int_0^{r_1} |u| ds + \frac{1}{1-K+Q^2} \frac{b}{AB} \int_0^{r_1} |u| ds\Big) d\tau\Big)^p dr\Big)^{\frac{q}{p}}$$

$$\leq h^q \Big(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} ||u||_{L^p} + \frac{1}{1-K+Q^2} \frac{b}{AB} ||u||_{L^p}\Big)^q,$$

i.e.

$$||\Delta_h R(u)||_{L^p}^q \leq h^q \Big(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} ||u||_{L^p} + \frac{1}{1-K+Q^2} \frac{b}{AB} ||u||_{L^p}\Big)^q.$$

Consequently

$$||R(u)||_{\dot{B}^\gamma_{p,q}[0,r_1]}^q = \int_0^2 h^{-1-q\gamma} ||\Delta_h R(u)||_{L^p}^q dh \leq$$

$$\leq \Big(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} ||u||_{L^p} + \frac{b}{AB(1-K+Q^2)} ||u||_{L^p}\Big)^q \int_0^2 h^{-1+q(1-\gamma)} dh =$$

$$= \Big(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} ||u||_{L^p} + \frac{b}{AB(1-K+Q^2)} ||u||_{L^p}\Big)^q \frac{2^{q(1-\gamma)}}{q(1-\gamma)}.$$

Therefore

$$||R(u)||_{\dot{B}^\gamma_{p,q}[0,r_1]} \leq \Big(\frac{1}{(1-K+Q^2)^2} \frac{2a}{A^2} ||u||_{L^p} + \frac{b}{AB(1-K+Q^2)} ||u||_{L^p}\Big) \frac{2^{(1-\gamma)}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From lemma 1 we get

$$||R(u)||_{\dot{B}^{\gamma}_{p,q}[0,r_1]} \leq$$

$$C\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}+\frac{b}{AB(1-K+Q^2)}||u||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From the last inequality, if $u \in \dot{B}^{\gamma}_{p,q}[0,r_1]$ we get $R(u) \in \dot{B}^{\gamma}_{p,q}[0,r_1]$. From 1), 2), 3), 4) we get $R : N \longrightarrow N$.

Let $u, u_1 \in N$. Then

$$||\Delta_h(R(u)-R(u_1))||^q_{L^p} =$$

$$= \Big(\int_0^{r_1}\Big(\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}(u(t,s)-u_1(t,s))-$$

$$-s^2(f(u)-f(u_1))\Big)dsd\tau\Big|\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}|u-u_1|+$$

$$|f(u)-f(u_1)|s^2\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

(from the midle point theorem we have $|f(u)-f(u_1)| = |f'(\xi)||u-u_1|$, also we have $|f'(\xi)| \leq b|\xi| \leq b\max\{|u|,|u_1|\} \leq \frac{2b}{AB}$, therefore $|f(u)-f(u_1)| \leq \frac{2b}{AB}|u-u_1|$)

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}|u-u_1|+$$

$$\frac{2b}{AB}|u-u_1|s^2\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

(here we use that for $s \in (0,r_1)$ we have $\frac{s^4}{s^2-Ks+Q^2} \leq \frac{1}{1-K+Q^2}$, also for $r \in [0,r_1]$ we have $\frac{1}{r^2-Kr+Q^2} \leq \frac{1}{1-K+Q^2}$)

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{1-K+Q^2}\int_\tau^{r_1}\Big(\frac{1}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u-u_1|+$$

$$+\frac{2b}{AB}|u-u_1|\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

(here we use (H3))

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{1-K+Q^2}\int_\tau^{r_1}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}|u-u_1|+$$

$$\frac{2b}{AB}|u-u_1|\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}\int_0^{r_1}|u-u_1|ds$$

$$+\frac{1}{1-K+Q^2}\frac{2b}{AB}\int_0^{r_1}|u-u_1|ds\Big)d\tau\Big)^p dr\Big)^{\frac{q}{p}} \leq$$

$$\leq h^q\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{L^p}+\frac{1}{1-K+Q^2}\frac{2b}{AB}||u-u_1||_{L^p}\Big)^q,$$

i.e.

$$||\Delta_h(R(u)-R(u_1))||^q_{L^p} \leq h^q\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{L^p}+$$

$$\frac{1}{1-K+Q^2}\frac{2b}{AB}||u-u_1||_{L^p}\Big)^q.$$

Consequently

$$||R(u)-R(u_1)||^q_{\dot{B}^{\gamma}_{p,q}[0,r_1]}=\int_0^2 h^{-1-q\gamma}||\Delta_h(R(u)-R(u_1))||^q_{L^p}dh\leq$$

$$\leq\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{L^p}+$$

$$\frac{2b}{AB(1-K+Q^2)}||u-u_1||_{L^p}\Big)^q\int_0^2 h^{-1+q(1-\gamma)}dh=$$

$$=\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{L^p}+\frac{2b}{AB(1-K+Q^2)}||u-u_1||_{L^p}\Big)^q\frac{2^{q(1-\gamma)}}{q(1-\gamma)}.$$

Therefore

$$||R(u)-R(u_1)||^q_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\leq$$

$$\leq\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{L^p}+\frac{2b}{AB(1-K+Q^2)}||u-u_1||_{L^p}\Big)^q\frac{2^{q(1-\gamma)}}{(q(1-\gamma))}.$$

From lemma 1 we get

$$||R(u)-R(u_1)||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\leq$$

$$\leq C\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}+$$

$$\frac{2b}{AB(1-K+Q^2)}||u-u_1||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From i3) we have

$$C\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}+\frac{2b}{AB(1-K+Q^2)}\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}<1.$$

therefore

$$||R(u)-R(u_1)||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}<||u-u_1||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}.$$

Consequently the operator $R:N\longrightarrow N$ is contractive operator. We note that N is closed subset of $\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}[0,r_1])$(for the proof see lemma 5.1 in the appendix of this paper). Therefore the equation $(\star)$ has unique nontrivial solution in the set N. •

Let $\tilde{u}$ is the solution from the theorem 3. 1, i.e $\tilde{u}$ is the solution to the equation $(\star)$. From proposition 2.1 $\tilde{u}$ satisfies the equation (1). Then $\tilde{u}$ is solution to the Cauchy problem (1), (2) with initial data

$$u_0=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v''(1)\omega(s)-s^2f(v(1)\omega(s))\Big)dsd\tau\\ \quad for\quad r\leq r_1,\\ 0\quad for\ r\geq r_1,\end{cases}$$

$$u_1=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v'''(1)\omega(s)-s^2f'(u)v'(1)\omega(s)\Big)dsd\tau=0\\ \quad for\quad r\leq r_1,\\ 0\quad for\quad r\geq r_1,\end{cases}$$

$u_0\in\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)$, $u_1\in\dot{B}^{\gamma-1}_{p,q}(\mathcal{R}^+)$, $\tilde{u}\in\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}[0,r_1])$.

3.2. Blow up of the solutions of the homogeneous Cauchy problem (1), (2). Let $v(t)$ is same function as in Theorem 3.1.

Theorem 3.2. *Let $p>1$, $q\geq 1$ and $\gamma\in(0,1)$ are fixed and the positive constants a, b, A, B, Q, K, $1<\beta<\alpha$ satisfy the conditions i1)-i5). Let $f\in\mathcal{C}^1(\mathcal{R}^1)$, $f(0)=0$, $a|u|\leq f'(u)\leq b|u|$. Then for the solution $\tilde{u}$ to the Cauchy problem (1), (2) we have*

$$\lim_{t\longrightarrow 0}||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}=\infty.$$

Proof. We suppose that $t\in(0,1]$. Then we have

$$||\Delta_h R(\tilde{u})||^q_{L^p}=\Big(\int_0^{r_1}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}$$
$$-s^2f(\tilde{u})\Big]dsd\tau\Big|^p dr\Big)^{\frac{q}{p}}=$$
$$=\Big(\int_0^{\frac{1}{\alpha}}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|^p dr+$$
$$+\int_{\frac{1}{\alpha}}^{r_1}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|^p dr\Big)^{\frac{q}{p}}.$$

Let

$$I_1=\int_0^{\frac{1}{\alpha}}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|^p dr,$$
$$I_2=\int_{\frac{1}{\alpha}}^{r_1}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u}))\Big]dsd\tau\Big|^p dr.$$

For I_1 we have the following estimate

$$I_1\leq C^p\frac{1}{(1-K+Q^2)^p}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big)^p||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}}h^p.$$

(in the last inequality we use that $|f(u)|\leq\frac{b}{2}|u|^2\leq\frac{b}{AB}|u|$) For I_2 we have

$$I_2=\int_{\frac{1}{\alpha}}^{r_1}\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau+$$
$$+\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\beta}}^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|^p dr\leq$$
$$\leq\int_{\frac{1}{\alpha}}^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau+$$
$$+\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\beta}}^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|\Big)^p dr.$$

Let

$$I_{21}=\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau,$$
$$I_{22}=\Big|\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_{\frac{1}{\beta}}^{r_1}\Big[\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}\tilde{u}-s^2f(\tilde{u})\Big]dsd\tau\Big|.$$

Then

$$I_2 \leq \int_{\frac{1}{\alpha}}^{r_1} \Big(I_{21} + I_{22}\Big)^p dr.$$

For I_{21} we have the following estimate (here we use that for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ $\tilde{u} \geq 0$, $f(\tilde{u}) \geq \frac{a}{2}\tilde{u}^2 \geq \frac{a}{2A^2}\tilde{u}$, therefore $-f(\tilde{u}) \leq -\frac{a}{2A^2}\tilde{u}$)

$$I_{21} \leq \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}\tilde{u} - \frac{a}{2A^2}s^2\tilde{u}\Big) ds d\tau =$$

$$= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}\frac{A^2\tilde{u}}{A^2} - \frac{a}{2A^2}s^2\frac{A^2\tilde{u}}{A^2}\Big) ds d\tau =$$

$$= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}\frac{1}{A^2} - s^2\frac{a}{2A^4}\Big) A^2\tilde{u} ds d\tau.$$

From i4) we have that (for $s \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$)

$$\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}\frac{1}{A^2} - \frac{a}{2A^2}\frac{s^2}{A^2} \geq$$

$$\geq \frac{1}{\alpha^2}\frac{1}{1 - \alpha K + \alpha^2 Q^2}\frac{a}{2A^6} - \frac{1}{\beta^2}\frac{a}{2A^4} > 0.$$

On the other hand we have $\tilde{u} \geq \frac{1}{A^2}$ for every $t \in [0,1]$ and every $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$, therefore $A^2\tilde{u} \geq 1$ and $A^2\tilde{u} \leq A^{2p}\tilde{u}^p$. Consequently

$$I_{21} \leq \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}\frac{1}{A^2} - as^2\frac{1}{2A^4}\Big) A^{2p}\tilde{u}^p ds d\tau.$$

From i5) we have that

$$\frac{1}{1 - K + Q^2} \leq A \leq A^2.$$

From here we get

$$I_{21} \leq \int_r^{r+h} \frac{1}{1 - K + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big[\frac{v''(t)}{v(t)} - \frac{a}{2A^4}\Big] A^{2p}\tilde{u}^p ds d\tau \leq$$

$$\leq \frac{1}{1 - K + Q^2}\frac{v''(t) - \frac{a}{2A^4}v}{v} A^{2p}\int_0^1 \tilde{u}^p ds h =$$

$$= \frac{1}{1 - K + Q^2}\frac{v''(t) - \frac{a}{2A^4}v}{v} A^{2p}||\tilde{u}||_{L^p}^p h \leq$$

$$\leq \frac{1}{(1 - K + Q^2)^2}\frac{v''(t) - \frac{a}{2A^4}v}{v} A^{2p}||\tilde{u}||_{L^p}^p h.$$

After we use the lemma 1 we get

$$I_{21} \leq C^p \frac{1}{(1 - K + Q^2)^2}\frac{v''(t) - \frac{a}{2A^4}v}{v} A^{2p}||\tilde{u}||_{\dot{B}^\gamma_{p,q}}^p h.$$

For I_{22} we have

$$I_{22} \leq C\frac{1}{(1 - K + Q^2)}\Big(\frac{1}{1 - K + Q^2}\frac{2a}{A^2} + \frac{b}{AB}\Big)||\tilde{u}||_{\dot{B}^\gamma_{p,q}} h.$$

Consequently

$$I_2 \leq \Big[C\frac{1}{(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big)||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}}h+$$

$$+C^p\frac{1}{(1-K+Q^2)^2}\frac{v''(t)-\frac{a}{2A^4}v}{v}A^{2p}||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}}h\Big]^p,$$

$$||\Delta_h R(\tilde{u})||^q_{L^p} \leq \Big[\Big[C\frac{1}{(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big)||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}}h+$$

$$+C^p\frac{1}{(1-K+Q^2)^2}\frac{v''(t)-\frac{a}{2A^4}v}{v}A^{2p}||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}}h\Big]^p+$$

$$C^p\frac{1}{(1-K+Q^2)^p}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big)^p||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}}h^p\Big]^{\frac{q}{p}}.$$

Then

$$||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}} \leq C\frac{2^{2-\gamma}}{(1-K+Q^2)(q(1-\gamma))^{\frac{1}{q}}}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big)||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}}+$$

$$+C^p\frac{2^{1-\gamma}}{(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}}\frac{v''(t)-\frac{a}{2A^4}v}{v}A^{2p}||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}}.$$

Let

$$C_1 = C\frac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}+\frac{b}{AB}\Big),$$

$$D = v(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}\frac{1}{C^p2^{1-\gamma}A^{2p}}.$$

From i3) we have that $C_1 < 1$. Then

$$||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}} \leq C_1||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}}+\frac{v''(t)-\frac{a}{2A^4}v}{D}||\tilde{u}||^p_{\dot{B}^{\gamma}_{p,q}},$$

from here

$$\frac{(1-C_1)D}{v''(t)-\frac{a}{2A^4}v} \leq ||\tilde{u}||^{p-1}_{\dot{B}^{\gamma}_{p,q}}.$$

Since

$$\lim_{t\longrightarrow 0}\Big[v''(t)-\frac{a}{2A^4}v\Big] = +0,$$

we have

$$\lim_{t\longrightarrow 0}||\tilde{u}||_{\dot{B}^{\gamma}_{p,q}} = \infty.$$

4. Proof of Theorem 2.

4.1. Local existence of nontrivial solutions for nonhomogenious Cauchy problem (1), (2). Let $v(t)$ is fixed function which satisfies the conditions (H1), (H2) and (H4).

Let us consider the equation

$(\star')$

$$u(t,r) = \begin{cases} \int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}u(t,s)-s^2f(u(t,s))-s^2g(s)\Big)dsd\tau, \\ \qquad 0\leq r\leq r_1, \\ 0 \quad for \quad r\geq r_1, \end{cases}$$

$t\in[0,1]$, where $u(t,r) = v(t)\omega(r)$.

Theorem 4.1. *Let $v(t)$ is fixed function which satisfies the hypothesis (H1), (H2), (H4). Let also $p > 1$, $q \in [1, \infty)$ and $\gamma \in (0, 1)$ are fixed and the positive constants A, a, b, B, Q,K, $\alpha > \beta > 1$ satisfy the conditions i2), i3), i5)-i7) and $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \le f'(u) \le b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \ge 0$ for $\forall r \in \mathcal{R}^+$, $g(r) = 0$ for $r \ge r_1$. Then the equation $(\star')$ has unique nontrivial solution $u(t, r) = v(t)\omega(r)$ for which $w \in \mathcal{C}^2[0, r_1]$, $u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0$ for $r \ge r_1$, $u(t, r) \in \mathcal{C}((0, 1]\dot{B}^{\gamma}_{p,q}[0, r_1])$, for $r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right]$ and $t \in [0, 1]$ $u(t, r) \ge \frac{1}{A^2}$, for $r \in \left[0, r_1\right]$ and $t \in [0, 1]$ $|u(t, r)| \le \frac{2}{AB}$.*

Proof. Let N_1 is the set

$$N_1 = \{u(t, r) \in \mathcal{C}([0, 1] \times [0, r_1]); u(t, r) = u_r(t, r) = u_{rr}(t, r) = 0$$

$$for \quad r \ge r_1 \quad and \quad t \in [0, 1],$$

$$u(t, r) \in \mathcal{C}((0, 1]\dot{B}^{\gamma}_{p,q}[0, r_1]); for \quad r \in \left[\frac{1}{\alpha}, \frac{1}{\beta}\right] \quad and \quad t \in [0, 1] \quad u(t, r) \ge \frac{1}{A^2};$$

$$for \quad r \in \left[0, r_1\right] \quad and \quad t \in [0, 1] \quad |u(t, r)| \le \frac{2}{AB}\}.$$

Let also $t \in [0, 1]$ is fixed.

We define the operator R_1 as follow

$$R_1(u) = \int_r^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t, s) - s^2 f(u) - s^2 g(s)\Big) ds d\tau,$$

where $0 \le r \le r_1$, and $t \in [0, 1]$. First we will show that $R_1 : N_1 \longrightarrow N_1$. We have

1) $R_1(u) \in \mathcal{C}^2([0, r_1])$, $R_1(u)|_{r=r_1} = 0$,

$$\frac{\partial}{\partial r} R_1(u) = \frac{1}{r^2 - Kr + Q^2} \int_{r_1}^{r} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u$$

$$-s^2 f(u) - s^2 g(s)\Big] ds, \frac{\partial}{\partial r} R_1(u)|_{r=r_1} = 0,$$

$$\frac{\partial^2}{\partial r^2} R_1(u) = \frac{K - 2r}{(r^2 - Kr + Q^2)^2} \int_{r_1}^{r} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u - s^2 f(u) - s^2 g(s)\Big] ds +$$

$$+ \frac{r^4}{(r^2 - Kr + Q^2)^2} \frac{v''(t)}{v(t)} u(t, r) - \frac{r^2}{r^2 - Kr + Q^2} (f(u) + g(r)).$$

Since $u(t, r_1) = 0$, $f(u(t, r_1)) = f(0) = 0$ and $g(r_1) = 0$ we get that

$$\frac{\partial^2}{\partial r^2} R_1(u)|_{r=r_1} = 0.$$

2) For $r \in [0, r_1]$ and $t \in [0, 1]$ we have $|u(t, r)| \le \frac{2}{AB}$. Then

$$|R_1(u)| = \Big|\int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u - s^2 f(u) - s^2 g(s)\Big) ds d\tau\Big| \le$$

$$\le \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + s^2 |f(u)| + s^2 |g(s)|\Big) ds d\tau \le$$

(since $|f'(u)| \le b|u|$ and $f(0) = 0$ we have $|f(u)| \le \frac{b}{2}|u|^2$)

$$\le \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |u| + \frac{b}{2} s^2 |u|^2 + s^2 |g(s)|\Big) ds d\tau =$$

$$\int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + \frac{b}{2} s^2 |u|\Big) |u| + s^2 |g(s)|\Big) ds d\tau \le$$

$$\leq \int_{r_1}^{r} \frac{1}{\tau^2 - K\tau + Q^2} \int_{r_1}^{\tau} \Big(\frac{2}{AB} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} + \frac{b}{2} \frac{2}{AB} s^2 \Big)$$

$$+ s^2 \max_{s \in [0, r_1]} |g(s)| \Big) ds d\tau \leq$$

(here we use that $\frac{r^2}{r^2 - Kr + Q^2} \leq \frac{1}{1-K+Q^2}$, $\frac{1}{r^2-Kr+Q^2} \leq \frac{1}{1-K+Q^2}$ for $r \in [0, r_1]$)

$$\leq \frac{1}{1-K+Q^2} \Big(\frac{2}{AB} \Big(\frac{1}{1-K+Q^2} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + \frac{b}{AB} \Big) + s^2 \max_{s \in [0, r_1]} |g(s)| \Big) \leq$$

$$\leq \frac{1}{1-K+Q^2} \Big(\frac{2}{AB} \Big(\frac{1}{1-K+Q^2} \frac{2a}{A^2} + \frac{b}{AB} \Big) + r_1^2 \max_{s \in [0, r_1]} |g(s)| \Big) \leq \frac{2}{AB}$$

(in the last inequality we use i6)). Consequently

$$|R_1(u)| \leq \frac{2}{AB} \quad for \quad r \in [0, r_1], \quad t \in [0, 1].$$

3) For $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$,

$$(5) \quad \frac{\partial}{\partial r} R_1(u) = \frac{1}{r^2 - Kr + Q^2} \int_{r}^{r_1} \Big(s^2 f(u) + s^2 g(s) - \frac{v''(t)}{v(t)} \frac{s^4}{s^2 - Ks + Q^2} u \Big) ds \geq$$

(as in point 3) of the proof of theorem 2.1)

$$\geq \frac{1}{r^2 - Kr + Q^2} \int_{r}^{r_1} \Big(s^2 f(u) - \frac{v''(t)}{v(t)} \frac{s^4}{s^2 - Ks + Q^2} u \Big) ds \geq 0.$$

From (5) we conclude that $R_1(u)$ is increase function for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$. Consequently, for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$,

$$R_1(u) \geq \int_{\frac{1}{\alpha}}^{r_1} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \Big(\frac{v''(t)}{v(t)} \frac{s^4}{s^2 - Ks + Q^2} u - s^2 f(u) - s^2 g(s) \Big) ds d\tau \geq$$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \frac{\alpha^2}{1 - \alpha K + \alpha^2 Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \frac{1}{\alpha^2 (1 - K\alpha + \alpha^2 Q^2)} \frac{a}{4A^6} ds d\tau -$$

$$- r_1^2 \frac{2b}{(1-K+Q^2) A^2 B^2} - r_1^2 \max_{r \in [0, r_1]} g(r) \frac{1}{1-K+Q^2} \geq$$

$$\geq \Big(\frac{1}{\beta} - \frac{1}{\alpha} \Big)^2 \frac{1}{(1 - \alpha K + \alpha^2 Q^2)^2} \frac{a}{4A^6} -$$

$$- r_1^2 \frac{2b}{(1-K+Q^2) A^2 B^2} - r_1^2 \frac{1}{1-K+Q^2} \max_{r \in [0, r_1]} g(r) \geq \frac{1}{A^2}.$$

(see fourth inequality of i2)) Consequently for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$ and $t \in [0, 1]$ we have

$$R_1(u) \geq \frac{1}{A^2}.$$

4)

$$||\Delta_h R_1(u)||_{L^p}^q =$$

$$= \Big(\int_0^{r_1} \Big(\Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} u(t,s)$$

$$- s^2 f(u) - s^2 g(s) \Big) ds d\tau \Big| \Big)^p dr \Big)^{\frac{q}{p}} \leq$$

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}\frac{v''(t)}{v(t)}|u|+ \frac{b}{2}|u|^2s^2+s^2g(s)\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}}\leq$$

(here we use that for $s\in[0,r_1]$ we have $\frac{s^4}{s^2-Ks+Q^2}\leq\frac{1}{1-K+Q^2}$, also for $r\in[0,r_1]$ and $t\in[0,1]$ we have $|u(t,r)|\leq\frac{2}{AB}$)

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{1-K+Q^2}\int_\tau^{r_1}\Big(\frac{1}{1-K+Q^2}\max_{t\in[0,1]}\frac{v''(t)}{v(t)}|u|+ +\frac{b}{AB}|u|+\max_{s\in[0,r_1]}|g(s)|\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}}\leq$$

(here we use (H4))

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{1-K+Q^2}\int_\tau^{r_1}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2}|u|+ \frac{b}{AB}|u|+\max_{s\in[0,r_1]}|g(s)|\Big)dsd\tau\Big)^p dr\Big)^{\frac{q}{p}}\leq$$

$$\leq \Big(\int_0^{r_1}\Big(\int_r^{r+h}\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}\int_0^{r_1}|u|ds+\frac{1}{1-K+Q^2}\frac{b}{AB}\int_0^{r_1}|u|ds+ +\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)d\tau\Big)^p dr\Big)^{\frac{q}{p}}\leq$$

$$\leq h^q\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{L^p}+ \frac{1}{1-K+Q^2}\frac{b}{AB}||u||_{L^p}+\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)^q,$$

i.e.

$$||\Delta_h R_1(u)||^q_{L^p}\leq h^q\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{L^p}+\frac{1}{1-K+Q^2}\frac{b}{AB}||u||_{L^p}+ +\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)^q.$$

Consequently

$$||R_1(u)||^q_{\dot{B}^\gamma_{p,q}(0,r_1)}=\int_0^2 h^{-1-q\gamma}||\Delta_h R_1(u)||^q_{L^p}dh\leq$$

$$\leq\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{L^p}+\frac{b}{AB(1-K+Q^2)}||u||_{L^p}+ +\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)^q\int_0^2 h^{-1+q(1-\gamma)}dh=$$

$$=\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{L^p}+\frac{b}{AB(1-K+Q^2)}||u||_{L^p} +\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)^q\frac{2^{q(1-\gamma)}}{q(1-\gamma)}.$$

Therefore

$$||R_1(u)||_{\dot{B}^\gamma_{p,q}[0,r_1]}\leq\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{L^p}+\frac{b}{AB(1-K+Q^2)}||u||_{L^p}+$$

$$+\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)\frac{2^{(1-\gamma)}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From lemma 1 we get

$$||R_1(u)||_{\dot{B}^\gamma_{p,q}[0,r_1]}\leq\Big(C\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u||_{\dot{B}^\gamma_{p,q}[0,r_1]}$$

$$+C\frac{b}{AB(1-K+Q^2)}||u||_{\dot{B}^\gamma_{p,q}[0,r_1]}+$$

$$+\frac{1}{1-K+Q^2}\max_{s\in[0,r_1]}|g(s)|\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From the last inequality, if $u\in\dot{B}^\gamma_{p,q}[0,r_1]$ for $t\in[0,1]$ we get $R_1(u)\in\dot{B}^\gamma_{p,q}[0,r_1]$ for $t\in[0,1]$. Also $R_1(u)\in\mathcal{C}((0,1]\dot{B}^\gamma_{p,q}[0,r_1])$.

From 1), 2), 3), 4) we get $R_1:N_1\longrightarrow N_1$.

Let $u,u_1\in N_1$. Then

$$R_1(u)-R_1(u_1)=R(u)-R(u_1).$$

From here and from the proof of theorem 2.1 we get

$$||R_1(u)-R_1(u_1)||_{\dot{B}^\gamma_{p,q}[0,r_1]}\leq$$

$$\leq C\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}||u-u_1||_{\dot{B}^\gamma_{p,q}[0,r_1]}+$$

$$\frac{2b}{AB(1-K+Q^2)}||u-u_1||_{\dot{B}^\gamma_{p,q}[0,r_1]}\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}.$$

From i3) we have

$$C\Big(\frac{1}{(1-K+Q^2)^2}\frac{2a}{A^2}+\frac{2b}{AB(1-K+Q^2)}\Big)\frac{2^{1-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}<1.$$

Therefore

$$||R_1(u)-R_1(u_1)||_{\dot{B}^\gamma_{p,q}[0,r_1]}<||u-u_1||_{\dot{B}^\gamma_{p,q}[0,r_1]}.$$

Consequently the operator $R_1:N_1\longrightarrow N_1$ is contractive operator. Since N_1 is closed subset of $\mathcal{C}((0,1]\dot{B}^\gamma_{p,q}[0,r_1])$(see appendix of this paper) we conclude that $(\star')$ has unique nontrivial solution in the set N. •

Let $\bar{u}$ is the solution from the theorem 4. 1. , i.e $\bar{\bar{u}}$ is the solution to the equation $(\star')$. From proposition 2.3 we have that $\bar{u}$ satisfies the equation (1). Then $\bar{u}$ is solution to the Cauchy problem (1), (2) with initial data

$$\bar{\bar{u}}_0=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v''(1)\omega(s)-s^2f(v(1)\omega(s))-s^2g(s)\Big)dsd\tau\\ \quad for\quad r\leq r_1,\\ 0\quad for\quad r\geq r_1,\end{cases}$$

$$\bar{\bar{u}}_1=\begin{cases}\int_r^{r_1}\frac{1}{\tau^2-K\tau+Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2-Ks+Q^2}v'''(1)\omega(s)-s^2f'(u)v'(1)\omega(s)\Big)dsd\tau\equiv0\\ for\quad r\leq r_1,\\ 0\quad for\quad r\geq r_1,\end{cases}$$

$\bar{\bar{u}}_0\in\dot{B}^\gamma_{p,q}(\mathcal{R}^+)$, $\bar{\bar{u}}_1\in\dot{B}^{\gamma-1}_{p,q}(\mathcal{R}^+)$, $\bar{\bar{u}}\in\mathcal{C}((0,1]\dot{B}^\gamma_{p,q}[0,r_1])$.

4.2. Blow up of the solutions to the nonhomogeneous Cauchy problem (1), (2). Let $v(t)$ is same function as in Theorem 4.1.

Theorem 4.2. *Let $p > 1$, $q \geq 1$ and $\gamma \in (0,1)$ are fixed and the positive constants a, b, A, B, Q, K, $1 < \beta < \alpha$ satisfy the conditions i2)-i7). Let $f \in \mathcal{C}^1(\mathcal{R}^1)$, $f(0) = 0$, $a|u| \leq f'(u) \leq b|u|$, $g \in \mathcal{C}(\mathcal{R}^+)$, $g(r) \geq 0$ for $r \geq 0$, $g(r) = 0$ for $r \geq r_1$. Then for the solution $\bar{\bar{u}}$ to the Cauchy problem (1), (2) we have*

$$\lim_{t\longrightarrow 0} ||\bar{\bar{u}}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]} = \infty.$$

Proof. We suppose that $t \in (0,1]$. Then we have

$$||\Delta_h \bar{u}||^q_{L^p} =$$
$$= \Big(\int_0^{r_1} \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|^p dr\Big)^{\frac{q}{p}} =$$
$$= \Big(\int_0^{\frac{1}{\alpha}} \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|^p dr +$$
$$+ \int_{\frac{1}{\alpha}}^{r_1} \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|^p dr\Big)^{\frac{q}{p}}.$$

Let

$$I_1 = \int_0^{\frac{1}{\alpha}} \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|^p dr,$$
$$I_2 = \int_{\frac{1}{\alpha}}^{r_1} \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|^p dr.$$

Then

$$||\Delta_h \bar{u}||^q_{L^p[0,r_1]} \leq (I_1 + I_2)^{\frac{q}{p}}. \tag{6}$$

For I_1 we have the following estimate

$$I_1 \leq \int_0^{\frac{1}{\alpha}} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |\bar{u}|$$
$$+s^2 f(\bar{u}) + s^2 g(s)\Big] ds d\tau\Big)^p dr \leq$$

(here we use that $r_1 < 1$)

$$\leq \int_0^{\frac{1}{\alpha}} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$+s^2 f(\bar{u}) + g(s)\Big] ds d\tau\Big)^p dr \leq$$

$$\leq \int_0^{\frac{1}{\alpha}} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |\bar{u}|$$
$$+ s^2 f(\bar{u}) + \max_{s\in[0,r_1]} g(s) \Big] ds d\tau \Big)^p dr.$$

From i7) we have

$$\max_{s\in[0,r_1]} g(s) \leq \Big(\frac{1}{\beta} - \frac{1}{\alpha} \Big)^2 \frac{1}{A^4}.$$

Therefore

$$\max_{s\in[0,r_1]} g(s) \leq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \frac{1}{A^4} ds d\tau.$$

Also $\bar{u} \geq \frac{1}{A^2}$ for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta} \Big]$. Consequently

$$\max_{s\in[0,r_1]} g(s) \leq \frac{1}{A^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} |\bar{u}| ds d\tau \leq \frac{1}{A^2} \int_0^{r_1} \int_0^{r_1} |\bar{u}| ds d\tau \leq$$

now we use Hölder's inequality

$$\leq \frac{1}{A^2} ||\bar{u}||_{L^p} \leq$$

now we use lemma 1

$$\leq \frac{1}{A^2} C ||\bar{u}||_{\dot{B}^\gamma_{p,q}[0,r_1]}.$$

From the last inequality we get

$$I_1 \leq \int_0^{\frac{1}{\alpha}} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_\tau^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} |\bar{u}|$$
$$+ s^2 f(\bar{u}) + C \frac{1}{A^2} ||\bar{u}||_{\dot{B}^\gamma_{p,q}[0,r_1]} \Big] ds d\tau \Big)^p dr.$$

From here and as in the proof of theorem 3.1 we have

$$I_1 \leq \Big(C \frac{1}{1 - K + Q^2} \Big(\frac{1}{1 - K + Q^2} \frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2} \Big) ||\bar{u}||_{\dot{B}^\gamma_{p,q}} \Big)^p h^p.$$

For I_2 we have

$$I_2 = \int_{\frac{1}{\alpha}}^{r_1} \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$- s^2 f(\bar{u}) - s^2 g(s) \Big] ds d\tau +$$
$$+ \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$- s^2 f(\bar{u}) - s^2 g(s) \Big] ds d\tau \Big|^p dr \leq$$
$$\leq \int_{\frac{1}{\alpha}}^{r_1} \Big(\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$
$$- s^2 f(\bar{u}) - s^2 g(s) \Big] ds d\tau +$$
$$+ \Big| \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$

$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|\Big)^p dr.$$

Let

$$I_{21} = \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$

$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau,$$

$$I_{22} = \Big|\int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\beta}}^{r_1} \Big[\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$

$$-s^2 f(\bar{u}) - s^2 g(s)\Big] ds d\tau\Big|.$$

Then

$$I_2 \le \int_{\frac{1}{\alpha}}^{r_1} \Big(I_{21} + I_{22}\Big)^p dr.$$

For I_{21} we have the following estimate (here we use that $f(\bar{u}) \ge \frac{a}{2}|\bar{u}|^2 \ge \frac{a}{4A^2}|\bar{u}|$ for $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$)

$$I_{21} \le \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \bar{u}$$

$$-\frac{a}{4A^2} s^2 \bar{u} + C\frac{1}{A^2} ||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) ds d\tau =$$

$$= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{A^2\bar{u}}{A^2}$$

$$-\frac{a}{4A^2} s^2 \frac{A^2\bar{u}}{A^2} + C\frac{1}{A^2} ||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) ds d\tau =$$

$$= \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2}$$

$$-s^2 \frac{a}{4A^4}\Big) A^2\bar{u} + C\frac{1}{A^2} ||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) ds d\tau.$$

From i4) we have that (for $s \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$)

$$\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2} - \frac{a}{4A^2} \frac{s^2}{A^2} \ge$$

$$\ge \frac{1}{\alpha^2} \frac{1}{1 - \alpha K + \alpha^2 Q^2} \frac{a}{4A^6} - \frac{1}{\beta^2} \frac{a}{4A^4} > 0.$$

On the other hand we have $\bar{u} \ge \frac{1}{A^2}$ for every $t \in [0,1]$ and every $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$, therefore $A^2\bar{u} \ge 1$ and $A^2\bar{u} \le A^{2p}\bar{u}^p$. Consequently

$$I_{21} \le \int_r^{r+h} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big(\Big(\frac{s^4}{s^2 - Ks + Q^2} \frac{v''(t)}{v(t)} \frac{1}{A^2}$$

$$-as^2 \frac{1}{4A^4}\Big) A^{2p}\bar{u}^p + C\frac{1}{A^2} ||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) ds d\tau.$$

From i5) we have that

$$\frac{1}{1 - K + Q^2} \le A \le A^2.$$

From here we get

$$I_{21} \le \int_r^{r+h} \frac{1}{1-K+Q^2} \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \Big[\Big[\frac{v''(t)}{v(t)} - \frac{a}{4A^4}\Big] A^{2p} \bar{u}^p + C\frac{1}{A^2}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big] ds d\tau \le$$

$$\le \frac{1}{1-K+Q^2}\Big(\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} \int_0^1 \bar{u}^p ds + C\frac{1}{A^2}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h \le$$

$$\le \frac{1}{1-K+Q^2}\Big(\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{L^p} + C\frac{1}{A^2}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h \le$$

$$\le \frac{1}{(1-K+Q^2)^2}\Big(\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{L^p} + C\frac{1}{A^2}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h.$$

After we use the lemma 1 we get

$$I_{21} \le \frac{1}{(1-K+Q^2)^2}\Big(C^p\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{\dot{B}^{\gamma}_{p,q}} + C\frac{1}{A^2}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h.$$

For I_{22} the following estimate

$$I_{22} \le (C\frac{1}{(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)||\bar{u}||_{\dot{B}^{\gamma}_{p,q}}\Big) h.$$

Consequently

$$I_2 \le \Big[\Big(\frac{C}{(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)||\bar{u}||_{\dot{B}^{\gamma}_{p,q}}\Big) h +$$

$$+ \frac{1}{(1-K+Q^2)^2}\Big(C^p\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{\dot{B}^{\gamma}_{p,q}} + \frac{1}{A^2}C||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h\Big]^p,$$

From (6) we get

$$||\Delta_h \bar{u}||^q_{L^p} \le \Big[\Big[\Big(C\frac{1}{(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)||\bar{u}||_{\dot{B}^{\gamma}_{p,q}}\Big) h +$$

$$+ \frac{1}{(1-K+Q^2)^2}\Big(C^p\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{\dot{B}^{\gamma}_{p,q}} + \frac{1}{A^2}C||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big) h\Big]^p +$$

$$\Big(C\frac{1}{1-K+Q^2}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)||\bar{u}||_{\dot{B}^{\gamma}_{p,q}} h\Big)^p\Big]^{\frac{q}{p}}.$$

Then

$$||\bar{u}||_{\dot{B}^{\gamma}_{p,q}} \le \frac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}}\Big(C\frac{1}{1-K+Q^2}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)||\bar{u}||_{\dot{B}^{\gamma}_{p,q}}\Big) +$$

$$+ \frac{2^{1-\gamma}}{(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}}\Big(C^p\frac{v''(t)-\frac{a}{4A^4}v}{v} A^{2p} ||\bar{u}||^p_{\dot{B}^{\gamma}_{p,q}} + \frac{1}{A^2}C||\bar{u}||_{\dot{B}^{\gamma}_{p,q}[0,r_1]}\Big).$$

Let

$$C_2 = C\frac{2^{2-\gamma}}{(q(1-\gamma))^{\frac{1}{q}}(1-K+Q^2)}\Big(\frac{1}{1-K+Q^2}\frac{2a}{A^2} + \frac{2b}{AB} + \frac{1}{A^2}\Big)$$

$$+ C\frac{2^{1-\gamma}}{A^2(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}},$$

$$D_1 = v(1-K+Q^2)^2(q(1-\gamma))^{\frac{1}{q}}\frac{1}{2^{1-\gamma}C^pA^{2p}},$$

From i3) we have that $C_2 < 1$. Then

$$||\bar{u}||_{\dot{B}^{\gamma}_{p,q}} \leq C_2||\bar{u}||_{\dot{B}^{\gamma}_{p,q}} + \frac{v''(t) - \frac{a}{4A^4}v}{D_1}||\bar{u}||^p_{\dot{B}^{\gamma}_{p,q}},$$

from here

$$\frac{(1-C_2)D_1}{v''(t) - \frac{a}{4A^4}v} \leq ||\bar{u}||^{p-1}_{\dot{B}^{\gamma}_{p,q}}. \tag{7}$$

Since

$$\lim_{t\longrightarrow 0}\Big[v''(t) - \frac{a}{4A^4}v\Big] = +0,$$

we have (from (7))

$$\lim_{t\longrightarrow 0}||\bar{u}||_{\dot{B}^{\gamma}_{p,q}} = \infty.$$

5. Appendix

Lemma 5.1. *Let $\gamma \in (0,1)$, $p > 1$, $q \geq 1$. Then the set N is closed subset of $\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+))$.*

Proof. Let $t \in (0,1]$ is fixed. Let $\{u_n\}$ is a sequence of elements of the set N for which

$$\lim_{n\longrightarrow\infty}||u_n - \tilde{\tilde{u}}||_{\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)} = 0,$$

where $\tilde{\tilde{u}} \in \dot{B}^{\gamma}_{p,q}(\mathcal{R}^+)$. We have

$$\lim_{n\longrightarrow\infty}||u_n - \tilde{\tilde{u}}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = 0.$$

We define $\tilde{u}$ as follows:

$$\tilde{u} = \begin{cases} \tilde{\tilde{u}} & for \quad r \in [0,r_1], \\ 0 & for \quad r > r_1. \end{cases}$$

We have

$$\lim_{n\longrightarrow\infty}||u_n - \tilde{u}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = 0.$$

First we note that for $u \in N$ $R(u)$ is continuous function of u and there exists $R'(u)$ because $f(u) \in \mathcal{C}^2(\mathcal{R}^1)$. For $R'(u)$ we have

$$R'(u) = \int_r^{r_1}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)} - s^2 f'(\xi)\bar{u}\Big)dsd\tau.$$

From here

$$|R'(u)| \geq \int_r^{r_1}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)} - s^2|f'(u)|\Big)dsd\tau \geq$$

$$\geq \int_r^{r_1}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)} - as^2|u|\Big)dsd\tau \geq$$

$$\geq \int_r^{r_1}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)} - \frac{2a}{AB}s^2\Big)dsd\tau \geq$$

$$\geq \int_r^{r_1}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\Big(\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)} - \frac{2a}{AB}r_1^2\Big)dsd\tau \geq$$

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}}\frac{1}{\tau^2 - K\tau + Q^2}\int_\tau^{r_1}\frac{s^4}{s^2 - Ks + Q^2}\frac{v''(t)}{v(t)}dsd\tau - \frac{2a}{(1-K+Q^2)AB}r_1^4 \geq$$

(here we use (H3))

$$\geq \int_{\frac{1}{\alpha}}^{\frac{1}{\beta}} \frac{1}{\tau^2 - K\tau + Q^2} \int_{\tau}^{r_1} \frac{s^4}{s^2 - Ks + Q^2} \frac{a}{2A^4} ds d\tau - \frac{2a}{AB(1-K+Q^2)} r_1^4 \geq$$

$$\geq \Big(\frac{1}{\alpha} - \frac{1}{\beta}\Big)^2 \frac{1}{(1-\alpha K + \alpha^2 Q^2)^2} \frac{a}{2A^4} - \frac{2a}{(1-K+Q^2)AB} r_1^4 > 0.$$

(in the last inequality we use the fifth inequality of i2)) From here for $u \in N$ exists

$$M := \min_{x \in [0, r_1]} |R'(u)(x)| > 0$$

because $R'(u)(x)$ is continuous function of $x \in [0, r_1]$.

Let

$$M_1 = \max_{r \in [0, r_1]} \Big| \frac{\partial}{\partial r} (R'(u))(r) \Big|.$$

Now we will prove that

$$for \quad \forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) > 0 \quad : \quad from \quad |x - y| < \delta$$
$$\Longrightarrow \quad |u_m(x) - u_m(y)| < \epsilon \quad \forall m.$$

We suppose that there exists $\tilde{\epsilon} > 0$ such that for every $\delta > 0$ there exist natural m and $x, y \in [0, r_1]$, $|x - y| < \delta$ for which $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$. We choose $\tilde{\tilde{\epsilon}} > 0$ such that $\tilde{\tilde{\epsilon}} < M\tilde{\epsilon}$. We note that $R(u_m)(x)$ is uniformly continuous function of $x \in [0, r_1]$(For $u \in N$ the function $R(u)(r)$ is uniformly continuous function of $r \in [0, r_1]$ because $R(u)(r) \in \mathcal{C}^2([0, r_1])$ and as in point 2) of the proof of theorem 2.1 we have $\Big|\frac{\partial}{\partial r} R(u)(r)\Big| \leq \frac{2}{AB}$). Then there exists $\delta_1 = \delta_1(\tilde{\tilde{\epsilon}}) > 0$ such that for every $u \in N$

$$|R(u)(x) - R(u)(y)| < \tilde{\tilde{\epsilon}} \quad for \quad \forall \quad x, y \in [0, r_1] \quad : \quad |x - y| < \delta_1.$$

Then we may choose $0 < \delta < min\Big\{\delta_1, \frac{(M\tilde{\epsilon} - \tilde{\tilde{\epsilon}})AB}{2M_1}\Big\}$ such that there exist natural m and $x_1 \in [0, r_1]$, $x_2 \in [0, r_1]$ for which $|x_1 - x_2| < \delta$ and $|u_m(x_1) - u_m(x_2)| \geq \tilde{\epsilon}$. In particular

$$|R(u_m)(x_1) - R(u_m)(x_2)| < \tilde{\tilde{\epsilon}}. \tag{8}$$

Then from the midle point theorem we have

$$\begin{aligned}
&R(0) = 0, R(u_m)(x_1) = R'(\xi)(x_1) u_m(x_1), \quad R(u_m)(x_2) = R'(\xi)(x_2) u_m(x_2),\\
&|R(u_m)(x_1) - R(u_m)(x_2)| = |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_2)u_m(x_2)| =\\
&= |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_1)u_m(x_2) + R'(\xi)(x_1)u_m(x_2) - R'(\xi)(x_2)u_m(x_2)| \geq\\
&\geq |R'(\xi)(x_1)u_m(x_1) - R'(\xi)(x_1)u_m(x_2)| - |R'(\xi)(x_1) - R'(\xi)(x_2)||u_m(x_2)| =\\
&= |R'(\xi)(x_1)||u_m(x_1) - u_m(x_2)| - \Big|\tfrac{\partial}{\partial r}(R'(\xi))(\eta)\Big||x_1 - x_2||u_m(x_2)| \geq\\
&\geq M\tilde{\epsilon} - M_1 \delta \tfrac{2}{AB} \geq \tilde{\tilde{\epsilon}},
\end{aligned}$$

which is contradiction with (8).

Consequently

$$for \quad \forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) > 0 \quad : \quad from \quad |x - y| < \delta$$
$$\Longrightarrow \quad |u_m(x) - u_m(y)| < \epsilon \quad \forall m. \tag{9}$$

On the other hand, from the definition of the set N we have

$$|u_m| \leq \frac{2}{AB} \quad \forall m. \tag{10}$$

From (9) and (10) we conclude that the set $\{u_n\}$ is compact subset of $\mathcal{C}([0,r_1])$. Then there exists subsequence $\{u_{n_k}\}$ and finction $u \in \mathcal{C}([0,r_1])$ for which: for every $\epsilon > 0$ there exists $M = M(\epsilon) > 0$ such that for every $n_k > M$ we have $|u_{n_k}(x) - u(x)| < \epsilon$ for every $x \in [0,r_1]$; $u(x) = 0$ for $x > r_1$. From here and from $\lim_{k\longrightarrow\infty} ||u_{n_k} - \tilde{u}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = 0$ we have : for every $\epsilon > 0$ $\exists M = M(\epsilon) > 0$ such that for every $n_k > M$ we have

$$\max_{x\in[0,r_1]} |u_{n_k} - u| < \epsilon, \quad ||u_{n_k} - \tilde{u}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon.$$

Then for every $n_k > M$ we have

$$|u - \tilde{u}| \le |u - u_{n_k}| + |u_{n_k} - \tilde{u}| < \epsilon + |\tilde{u} - u_{n_k}|,$$

$$\int_0^{r_1} |u - \tilde{u}|dx < \epsilon r_1 + \int_0^{r_1} |\tilde{u} - u_{n_k}|dx,$$

(here we use the Hölder's inequality)

$$||u - \tilde{u}||_{L^1[0,r_1]} < \epsilon r_1 + r_1^{\frac{1}{q_1}} \Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{1}{p}}, \frac{1}{p} + \frac{1}{q_1} = 1,$$

for $h > 0$ we have

$$h^{-1-q\gamma}||u - \tilde{u}||_{L^1[0,r_1]} < h^{-1-q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}} h^{-1-q\gamma}\Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{1}{p}},$$

$$\int_1^2 h^{-1-q\gamma}dh||u - \tilde{u}||_{L^1[0,r_1]} < \int_1^2 h^{-1-q\gamma}dh\epsilon r_1$$
$$+r_1^{\frac{1}{q_1}} \int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{1}{p}} dh,$$
$$\frac{1 - 2^{-q\gamma}}{q\gamma}||u - \tilde{u}||_{L^1[0,r_1]} < \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1$$
$$+r_1^{\frac{1}{q_1}} \int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{1}{p}} dh \le$$

(here we use the Hölder's inequality)

$$\le \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}}\Big(\int_1^2 h^{(-1-q\gamma)q}\Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} \le$$

(since $h > 1$ we have $h^{(-1-q\gamma)q} \le h^{-1-q\gamma}$)

$$\le \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}}\Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\tilde{u} - u_{n_k}|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} =$$

(here we use that for $x > r_1$ we have $u_{n_k}(x) = \tilde{u}(x) = 0$)

$$= \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}}\Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(\tilde{u} - u_{n_k})|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} \le$$

$$\le \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}}\Big(\int_0^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(\tilde{u} - u_{n_k})|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} =$$
$$= \frac{1 - 2^{-q\gamma}}{q\gamma}\epsilon r_1 + r_1^{\frac{1}{q_1}}||\tilde{u} - u_{n_k}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} <$$
$$< \epsilon(\frac{1 - 2^{-q\gamma}}{q\gamma} r_1 + r_1^{\frac{1}{q_1}}),$$

i.e.

$$\frac{1-2^{-q\gamma}}{q\gamma}||u-\tilde{u}||_{L^1[0,r_1]} < \epsilon\Big(\frac{1-2^{-q\gamma}}{q\gamma}r_1 + r_1^{\frac{1}{q_1}}\Big)$$

for every $\epsilon > 0$. Consequently $u = \tilde{u}$ a.e.(almost everywhere) in $[0, r_1]$, $|u|^p = |\tilde{u}|^p$ a.e. in $[0, r_1]$. From here $|u_n - u| = |u_n - \tilde{u}|$ a.e., $|u_n - u|^p = |u_n - \tilde{u}|^p$ a.e. in $[0, r_1]$. Since $u_n(x) = u(x) = 0$ for $x > r_1$ we have $|\Delta_h(u_n - u)|^p = |\Delta_h(u_n - \tilde{u})|^p$, $|\Delta_h u|^p = |\Delta_h \tilde{u}|^p$ a.e. in $[0, r_1]$, for $h > 0$. Therefore $u \in \dot{B}^{\gamma}_{p,q}([0, r_1])$ and

$$\int_0^{r_1} |u_n - u|^p dx = \int_0^{r_1} |u_n - \tilde{u}|^p dx.$$

Now we will see that $\lim_{n\longrightarrow\infty} ||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = 0$. Really,

$$||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = \Big(\int_0^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n - u)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} =$$

$$= \Big(\int_0^2 h^{-1-q\gamma}\int_0^{r_1} |\Delta_h(u_n - \tilde{u})|^q dxdh\Big)^{\frac{1}{q}} =$$

$$||u_n - \tilde{u}||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \longrightarrow_{n\longrightarrow\infty} 0.$$

Consequently for every sequence $\{u_n\}$ with elements from N, which converges in $\dot{B}^{\gamma}_{p,q}([0, r_1])$ there exists function $u \in \mathcal{C}([0, r_1])$, $u \in \dot{B}^{\gamma}_{p,q}([0, r_1])$, for which $||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \longrightarrow_{n\longrightarrow\infty} 0$.

Below we suppose that the sequence $\{u_n\}$ is a sequence of elements of the set N which converges in $\dot{B}^{\gamma}_{p,q}([0, r_1])$. Then there exists $u \in \mathcal{C}([0, r_1])$, $u(x) = 0$ for $x > r_1$, $u \in \dot{B}^{\gamma}_{p,q}([0, r_1])$, $||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} \longrightarrow_{n\longrightarrow\infty} 0$.

Now we suppose that $u(r_1) \neq 0$. Since $u \in \mathcal{C}([0, r_1])$, $u_n \in \mathcal{C}([0, r_1])$, $u_n(r_1) = 0$ for every natural n, there exist $\epsilon_2 > 0$ and $\Delta_1 \subset [0, r_1]$, $r_1 \in \Delta_1$, such that

$$|u_n| < \frac{\epsilon_2}{2}, |u| > \epsilon_2$$

for every natural n and every $x \in \Delta_1$. Then for every natural n and for every $x \in \Delta_1$ we have

$$|u_n(x) - u(x)| > \frac{\epsilon_2}{2}.$$

Let $\epsilon_3 > 0$ is such that

$$\epsilon_3 < \frac{\epsilon_2}{2}\frac{1-2^{-q\gamma}}{q\gamma}\mu(\Delta_1)r_1^{-\frac{1}{q_1}}, \quad \frac{1}{p} + \frac{1}{q_1} = 1, \tag{11}$$

where $\mu(\Delta_1)$ is the measure of the set Δ_1. There exists $M > 0$ such that for every $n > M$ we have $||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_3$. Consequently for every $n > M$ and for every $x \in \Delta_1$ we have

$$|u_n(x) - u(x)| > \frac{\epsilon_2}{2}, \quad ||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_3.$$

Also we have

$$\frac{\epsilon_2}{2}\mu(\Delta_1) < \int_{\Delta_1} |u_n(x) - u(x)| dx \leq \int_0^{r_1} |u_n(x) - u(x)| dx \leq$$

(here we use the Hölder's inequality)

$$\leq \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}}.$$

For $h > 0$ we have

$$h^{-1-q\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) \le h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}},$$

$$\int_1^2 h^{-1-q\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1)dh \le \int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}} dh,$$

(here we use the Hölder's inequality)

$$\frac{1-2^{-q\gamma}}{q\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) \le \Big(\int_1^2 h^{(-1-q\gamma)q}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \le$$

(since $h > 1$ we have $h^{(-1-q\gamma)q} \le h^{-1-q\gamma}$)

$$\le \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^q dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \le$$

(here we use that $u_n = u = 0$ for $x > r_1$)

$$\le \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x) - u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \le$$

$$\le \Big(\int_0^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x) - u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} =$$

$$||u_n - u||_{\dot{B}^\gamma_{p,q}([0,r_1])} r_1^{\frac{1}{q_1}} <$$

$$< \epsilon_3 r_1^{\frac{1}{q_1}}.$$

Therefore

$$\frac{1-2^{-q\gamma}}{q\gamma}\frac{\epsilon_2}{2}\mu(\Delta_1) < \epsilon_3 r_1^{\frac{1}{q_1}},$$

which is a contradiction with (11). Consequently $u(r_1) = 0$. From here $u(t, r) = 0$ for $r \ge r_1$. Then $u_r(t, r) = u_{rr}(t, r) = 0$ for every $r \ge r_1$.

Now we suppose that the

$$|u(t, r)| \le \frac{2}{AB}$$

is not hold for every $r \in [0, r_1]$. Since $u \in \mathcal{C}([0, r_1])$ we may take $\epsilon_4 > 0$ and $\Delta_2 \subset [0, r_1]$ such that

$$|u| \ge \frac{2}{AB} + \epsilon_4 \quad for \quad r \in \Delta_2.$$

Then for every natural n and for every $r \in \Delta_2$ we have

$$|u_n - u| \ge |u| - |u_n| \ge \frac{2}{AB} + \epsilon_4 - \frac{2}{AB} = \epsilon_4.$$

Let $\epsilon_5 > 0$ is such that

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_4\mu(\Delta_2) > \epsilon_5 r_1^{\frac{1}{q_1}}. \tag{12}$$

There exist $M > 0$ such that for every $n > M$ we have $||u_n - u||_{\dot{B}^\gamma_{p,q}([0,r_1])} < \epsilon_5$. Consequently for every $n > M$ and for every $x \in \Delta_2$ we have

$$|u_n(x) - u(x)| \ge \epsilon_4, \quad ||u_n - u||_{\dot{B}^\gamma_{p,q}([0,r_1])} < \epsilon_5.$$

Also we have

$$\epsilon_4\mu(\Delta_2) < \int_{\Delta_2} |u_n(x)-u(x)|dx \leq$$

(here we use the Hölder's inequality)

$$\leq \Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}}.$$

For $h>0$ we have

$$h^{-1-q\gamma}\epsilon_4\mu(\Delta_2) \leq h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}},$$

$$\int_1^2 h^{-1-q\gamma}\epsilon_4\mu(\Delta_2)dh \leq \int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}} dh,$$

(here we use the Hölder's inequality)

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_4\mu(\Delta_2) \leq \Big(\int_1^2 h^{(-1-q\gamma)q}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(since $h>1$ we have $h^{(-1-q\gamma)q} \leq h^{-1-q\gamma}$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x)-u(x)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(here we use that $u_n = u = 0$ for $x > r_1$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x)-u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

$$\leq \Big(\int_0^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x)-u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} =$$

$$||u_n-u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} r_1^{\frac{1}{q_1}} <$$

$$< \epsilon_5 r_1^{\frac{1}{q_1}}.$$

Therefore

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_4\mu(\Delta_2) < \epsilon_5 r_1^{\frac{1}{q_1}},$$

which is a contradiction with (12). Therefore $|u| \leq \frac{2}{AB}$ for every $r\in[0,r_1]$.

Now we suppose that the inequality

$$|u(t,r)| \geq \frac{1}{A^2}$$

is not true for every $r \in \left[\frac{1}{\alpha},\frac{1}{\beta}\right]$. Since $u\in\mathcal{C}([0,r_1])$ we may take $\epsilon_6>0$ and $\Delta_3 \subset \left[\frac{1}{\alpha},\frac{1}{\beta}\right]$ such that

$$|u| \leq \frac{1}{A^2} - \epsilon_6 \quad for \quad r\in\Delta_3.$$

Then for every natural n and for every $r\in\Delta_3$ we have

$$|u_n-u| \geq |u_n| - |u| \geq \frac{1}{A^2} + \epsilon_6 - \frac{1}{A^2} = \epsilon_6.$$

Let $\epsilon_7 > 0$ is such that

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_6\mu(\Delta_3) > \epsilon_7 r_1^{\frac{1}{q_1}}. \tag{13}$$

There exist $M > 0$ such that for every $n > M$ we have $||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_7$. Consequently for every $n > M$ and for every $x \in \Delta_3$ we have

$$|u_n(x) - u(x)| > \epsilon_6, \quad ||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_7.$$

Also we have

$$\epsilon_6\mu(\Delta_3) < \int_{\Delta_3} |u_n(x) - u(x)|dx \leq$$

(here we use the Hölder's inequality)

$$\leq \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}}.$$

For $h > 0$ we have

$$h^{-1-q\gamma}\epsilon_6\mu(\Delta_3) \leq h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}},$$

$$\int_1^2 h^{-1-q\gamma}\epsilon_6\mu(\Delta_3)dh \leq \int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}} dh,$$

(here we use the Hölder's inequality)

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_6\mu(\Delta_3) \leq \Big(\int_1^2 h^{(-1-q\gamma)q}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(since $h > 1$ we have $h^{(-1-q\gamma)q} \leq h^{-1-q\gamma}$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(here we use that $u_n = u = 0$ for $x > r_1$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x) - u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

$$\leq \Big(\int_0^2 h^{-1-q\gamma}\Big(\int_0^{r_1} |\Delta_h(u_n(x) - u(x))|^p dx\Big)^{\frac{q}{p}} dh\Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} =$$

$$||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} r_1^{\frac{1}{q_1}} <$$

$$< \epsilon_7 r_1^{\frac{1}{q_1}}.$$

Therefore

$$\frac{1-2^{-q\gamma}}{q\gamma}\epsilon_6\mu(\Delta_2) < \epsilon_7 r_1^{\frac{1}{q_1}},$$

which is a contradiction with (13). Therefore $|u| \geq \frac{1}{A^2}$ for every $r \in \Big[\frac{1}{\alpha}, \frac{1}{\beta}\Big]$.

Now we suppose that the inequality

$$u(t, r) \geq 0$$

is not true for every $r \in \left[\frac{1}{\alpha}, r_1\right]$. Then from $u \in \mathcal{C}([0, r_1])$ and from $u_n \geq 0$ for every natural n and for every $r \in \left[\frac{1}{\alpha}, r_1\right]$, we may take $\epsilon_8 > 0$ and $\Delta_4 \subset \left[\frac{1}{\alpha}, r_1\right]$ such that for every natural n and for every $r \in \Delta_4$ we have

$$|u_n - u| \geq \epsilon_8.$$

Let $\epsilon_9 > 0$ is such that

$$\frac{1 - 2^{-q\gamma}}{q\gamma} \epsilon_8 \mu(\Delta_3) > \epsilon_9 r_1^{\frac{1}{q_1}}. \tag{14}$$

There exist $M > 0$ such that for every $n > M$ we have $||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_9$. Consequently for every $n > M$ and for every $x \in \Delta_4$ we have

$$|u_n(x) - u(x)| > \epsilon_8, \quad ||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} < \epsilon_9.$$

Also we have

$$\epsilon_8 \mu(\Delta_4) < \int_{\Delta_4} |u_n(x) - u(x)| dx \leq$$

(here we use the Hölder's inequality)

$$\leq \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx \Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}}.$$

For $h > 0$ we have

$$h^{-1-q\gamma} \epsilon_8 \mu(\Delta_4) \leq h^{-1-q\gamma} \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx \Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}},$$

$$\int_1^2 h^{-1-q\gamma} \epsilon_8 \mu(\Delta_4) dh \leq \int_1^2 h^{-1-q\gamma} \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx \Big)^{\frac{1}{p}} r_1^{\frac{1}{q_1}} dh,$$

(here we use the Hölder's inequality)

$$\frac{1 - 2^{-q\gamma}}{q\gamma} \epsilon_8 \mu(\Delta_4) \leq \Big(\int_1^2 h^{(-1-q\gamma)q} \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx \Big)^{\frac{q}{p}} dh \Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(since $h > 1$ we have $h^{(-1-q\gamma)q} \leq h^{-1-q\gamma}$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma} \Big(\int_0^{r_1} |u_n(x) - u(x)|^p dx \Big)^{\frac{q}{p}} dh \Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

(here we use that $u_n = u = 0$ for $x > r_1$)

$$\leq \Big(\int_1^2 h^{-1-q\gamma} \Big(\int_0^{r_1} |\Delta_h (u_n(x) - u(x))|^p dx \Big)^{\frac{q}{p}} dh \Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} \leq$$

$$\leq \Big(\int_0^2 h^{-1-q\gamma} \Big(\int_0^{r_1} |\Delta_h (u_n(x) - u(x))|^p dx \Big)^{\frac{q}{p}} dh \Big)^{\frac{1}{q}} r_1^{\frac{1}{q_1}} =$$

$$||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} r_1^{\frac{1}{q_1}} <$$

$$< \epsilon_9 r_1^{\frac{1}{q_1}}.$$

Therefore

$$\frac{1 - 2^{-q\gamma}}{q\gamma} \epsilon_8 \mu(\Delta_4) < \epsilon_9 r_1^{\frac{1}{q_1}},$$

which is a contradiction with (14). Therefore $|u| \geq 0$ for every $r \in \left[\frac{1}{\alpha}, r_1\right]$.

Consequently $u \in N$.

Then for every sequence $\{u_n\} \subset N$, which converges in $\dot{B}^{\gamma}_{p,q}([0,r_1])$ there exists $u \in N$ for which

$$\lim_{n \longrightarrow \infty} ||u_n - u||_{\dot{B}^{\gamma}_{p,q}([0,r_1])} = 0.\bullet$$

As in the proof of lemma 4.1 it is easy to verify that the set N_1 is closed subset of the space $\mathcal{C}((0,1]\dot{B}^{\gamma}_{p,q}(\mathcal{R}^+))$ for $\gamma \in (0,1)$, $p > 1$, $q \geq 1$.

References

[1] Shatah, J., M. Struwe. Geometric wave equation. Courant lecture notes in mathematics 2(1998).

[2] Taylor, M. Partial differential equations III. Springer-Verlag, 1996.

[3] Runst, T., W. Sickel. Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations, 1996, New York.

[4] Georgiev, S. Blow up of solutions for Klein-Gordon equations in the Reissner-Nordström metric, EJDE, Vol 2005(2005), No 67, pp. 1-22.

[5] Rauch, J. The u^5- Klein - Gordon equation, Brezis. and Lions. eds., Pilman Research Notes in Math. 53, pp. 335-364.

[6] Jörgens, K. Das Anfangswertproblem im Grossen für eine Klasse nicht-linear Wellengleichungen, Math. Z., 77, 1961, pp. 295-308.

[7] John, F. Blow up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28, 1979, 235-268.

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