

Invariant measures for the Nonlinear Schrödinger equation on the disc

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ABSTRACT. We study Gibbs measures invariant under the flow of the NLS on the unit disc of \mathbb{R}^2 . For that purpose, we construct the dynamics on a phase space of limited Sobolev regularity and a weighted Wiener measure invariant by the NLS flow. The density of the measure is integrable with respect to the Wiener measure for sub cubic nonlinear interactions. The existence of the dynamics is obtained in Bourgain spaces of low regularity. The key ingredient are bilinear Strichartz estimates for the free evolution. The bilinear effect in our analysis results from simple properties of the Bessel functions and estimates on series of Bessel functions.

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1. Introduction

This work fits in the line of research initiated in [3] aiming to study the possible extensions of the work of Bourgain on nonlinear Schrödinger equation (NLS) posed on the flat torus to other compact manifolds. We are concerned here with the long time behavior of solutions of the nonlinear Schrödinger equation, posed on the unit disc of \mathbb{R}^2 . Our aim is to construct the dynamics on a phase space of limited Sobolev regularity and a wighted Wiener measure invariant by the NLS flow. Consider the Nonlinear Schrödinger equation

$$(1.1) \quad iu_t + \Delta u + F(u) = 0,$$

where $u(t) : \Theta \rightarrow \mathbb{C}$ is a function defined on the unit disc

$$\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}.$$

The nonlinear interaction in (1.1) is induced by $F(z)$, $z \in \mathbb{C}$ which is a smooth (non linear) complex valued function. We also assume that $F(0) = 0$ and $F = \bar{\partial}V$ with a real valued V satisfying the gauge invariance assumption

$$V(e^{i\theta}z) = V(z), \quad \forall \theta \in \mathbb{R}, \forall z \in \mathbb{C}.$$

In addition, we suppose that for some $\alpha > 0$,

$$(1.2) \quad |\partial^{k_1} \bar{\partial}^{k_2} V(z)| \leq C_{k_1, k_2} \langle z \rangle^{2+\alpha-k_1-k_2}.$$

The real number α involved in (1.2) corresponds to the “degree” of the nonlinear interaction. A typical example for $F(u)$ is

$$F(u) = \pm \left(1 + |u|^2\right)^{\alpha/2} u$$

or $|u|^\alpha u$ when α is an even integer. In this paper, we assume that the nonlinearity is sub-cubic which means that

$$(1.3) \quad \alpha < 2.$$

Assumption (1.3) on α will be assumed from now on in the rest of this paper. Notice that we do not suppose the defocusing assumption which in the context of (1.1) would be of type $V \leq 0$. In the (easier) defocusing case, one can expect to cover a larger set of possible values of α (see the final remarks at the end of the paper).

It is important that the problem (1.1) may, at last formally, be seen as the Hamiltonian PDE

$$iu_t = \partial_{\bar{u}} H(u, \bar{u})$$

in an infinite dimensional phase space, with Hamiltonian

$$(1.4) \quad H(u, \bar{u}) = \int_{\Theta} |\nabla u|^2 - \int_{\Theta} V(u)$$

and canonical coordinates (u, \bar{u}) .

We are interested in the solutions of the initial boundary value problem associated to (1.1). This means that we study (1.1) subject to an initial condition

$$(1.5) \quad u(0, x_1, x_2) = u_0(x_1, x_2),$$

where u_0 is a given function. In this paper, we will only consider initial data of Sobolev regularity $< 1/2$ and thus we will not need to specify the boundary conditions on $\mathbb{R} \times \partial\Theta$, where

$$\partial\Theta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

is the border of Θ (see also Remark 7.2 below). We will however use the Dirichlet eigenfunctions of Δ as basis of $L^2(\Theta)$ and this will be convenient for our well-posedness analysis of (1.1)-(1.5).

We will only consider radial solutions of (1.1), i.e. solutions depending only on t and $x_1^2 + x_2^2$. Thus, we suppose that the data is radially symmetric, i.e.

$$(1.6) \quad u_0(x_1, x_2) = \tilde{u}_0(r),$$

where

$$x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad 0 \leq r < 1, \quad \varphi \in [0, 2\pi].$$

Let J_0 be the Bessel function of order zero (see e.g. [14]) and let z_1, z_2, \dots be the zeros of J_0 . We have that

$$0 < z_1 < z_2 < \dots < z_n < \dots$$

and the zeroes are simple. We also have that $J_0(z_n r)$ are eigenfunctions of the Dirichlet self adjoint realization of $-\Delta$, corresponding to eigenvalues z_n^2 . Moreover any $L^2(\Theta)$ radial function can be expanded with respect to $J_0(z_n r)$. Let us set

$$(1.7) \quad e_n \equiv e_n(r) = \|J_0(z_n \cdot)\|_{L^2(\Theta)}^{-1} J_0(z_n r)$$

and

$$e_{n,s} = z_n^{-s} e_n.$$

We can decompose the solutions of (1.1) with data of type (1.6) as

$$u(t) = \sum_{n \geq 1} c_n(t) e_{n,s}.$$

The initial data is thus given by

$$\tilde{u}_0 = \sum_{n \geq 1} c_n(0) e_{n,s},$$

i.e the initial data is uniquely determined from the sequence $(c_n(0))$, $n \in \mathbb{N}$. Thus the equation (1.1) can be written as

$$(1.8) \quad i z_n^{-s} \dot{c}_n(t) - z_n^2 z_n^{-s} c_n(t) + \Pi_n \left(F \left(\sum_{m \geq 1} c_m(t) e_{m,s} \right) \right) = 0,$$

$n \geq 1$, where Π_n is the projection on the mode e_n . For instance if $f \in L^1(\Theta)$ (which will always be the case in this paper), we have

$$\Pi_n(f) = \langle f, e_n \rangle = \int_{\Theta} f \overline{e_n}.$$

Of course one can define the action of Π_n on distributions but here we will not need it. Notice that if $f \in L^2(\Theta)$, $\Pi_n(f)$ is simply the $L^2(\Theta)$ scalar product of f and

e_n . Formally, equation (1.8) is in fact a Hamiltonian equation with with canonical coordinates (c, \bar{c}) and Hamiltonian

$$H(c, \bar{c}) = \sum_{n \geq 1} z_n^{2-2s} |c_n|^2 - \int_0^1 V \left(\sum_{m \geq 1} c_m e_{m,s}(r) \right) r dr,$$

where $c = (c_n)$, $n \in \mathbb{N}$. More precisely equation (1.8) can be written as

$$i c_t = J \frac{\delta H}{\delta \bar{c}}, \quad i \bar{c}_t = -J \frac{\delta H}{\delta c},$$

where δ denotes the variational derivative and $J = \text{diag}(z_n^{2s})_{n \geq 1}$ is the map inducing the symplectic form in the coordinates (c, \bar{c}) . The only important consequence, for our analysis, of this discussion is that $H(c, \bar{c})$ is, at least formally, conserved by the flow of (1.8).

Let us now describe the construction of Lebowitz-Rose-Speer (cf. [11]) of a weighted Wiener measure which is at least formally invariant under under flow of (1.1). The rigorous justification of the invariance of the measure will require, among other things, a new well-posedness result for the initial value problem (1.1)-(1.5). Let us fix a real number s such that

$$(1.9) \quad 0 < s < \frac{\alpha}{\alpha + 2}.$$

This number s will be fixed from now on in all the rest the paper. Notice that thanks to the restriction (1.3) on the degree of the nonlinearity α ,

$$\frac{\alpha}{\alpha + 2} < \frac{1}{2}.$$

For $\sigma \in [0, 1/2[$, let us denote by $H_{rad}^\sigma(\Theta)$ the Sobolev space of radial functions on Θ , i.e. $u \in H_{rad}^\sigma(\Theta)$ if and only if

$$u = \sum_{n \geq 1} c_n e_{n,s}, \quad c_n \in \mathbb{C}$$

with

$$\sum_{n \geq 1} z_n^{2(\sigma-s)} |c_n|^2 < \infty.$$

The Sobolev space $H_{rad}^s(\Theta)$ is naturally a complex Hilbert space with orthonormal basis $e_{n,s}$. Denote by (\cdot, \cdot) the scalar product in $H_{rad}^s(\Theta)$. Our goal will be to construct a well defined (at least local in time) dynamics on $H_{rad}^s(\Theta)$ and to construct a bounded Borel measure on it, invariant by the flow of (1.1).

The free Hamiltonian is given by

$$H_0(c, \bar{c}) = \sum_{n \geq 1} z_n^{2-2s} |c_n|^2.$$

It turns out that a renormalization of the formal measure

$$e^{-H_0(c, \bar{c})} d^2 c = \prod_{n \geq 1} e^{-z_n^{2-2s} |c_n|^2} d^2 c_n$$

is a Wiener measure. More precisely, we can give a sense of the formal measure

$$\frac{e^{-H_0(c,\bar{c})}d^2c}{\int e^{-H_0(c,\bar{c})}d^2c} = \prod_{n \geq 1} \frac{e^{-z_n^{2-2s}|c_n|^2}d^2c_n}{\int_{\mathbb{C}} e^{-z_n^{2-2s}|c_n|^2}d^2c_n}$$

as a measure on the Hilbert space $H_{rad}^s(\Theta)$ (corresponding to a Gaussian distribution for each mode).

A set $U \subset H_{rad}^s(\Theta)$ is called cylindrical if there exists $N \in \mathbb{N}$ and a Borel set $V \subset \mathbb{C}^N$ such that

$$(1.10) \quad U = \left\{ u \in H_{rad}^s(\Theta) : ((u, e_{1,s}), \dots, (u, e_{N,s})) \in V \right\}.$$

Let us denote by $\tilde{\mu}$ the measure, defined on the cylindrical sets U determined by (1.10) as

$$(1.11) \quad \begin{aligned} \tilde{\mu}(U) &= \frac{\int_V e^{-\sum_{1 \leq n \leq N} z_n^{2-2s}|c_n|^2} d^2c_1 \dots d^2c_N}{\int_{\mathbb{C}^N} e^{-\sum_{1 \leq n \leq N} z_n^{2-2s}|c_n|^2} d^2c_1 \dots d^2c_N} \\ &= \pi^{-N} \left(\prod_{1 \leq n \leq N} z_n^{2-2s} \right) \int_V e^{-\sum_{1 \leq n \leq N} z_n^{2-2s}|c_n|^2} d^2c_1 \dots d^2c_N. \end{aligned}$$

The cylindrical sets form an algebra in $H_{rad}^s(\Theta)$. Moreover the minimal sigma algebra containing all cylindrical sets is the Borel sigma algebra. Since (see (2.6) below) $z_n \sim n$, we deduce that the series

$$\sum_{n \geq 1} z_n^{2s-2}$$

converges. It implies that the linear map defined on $H_{rad}^s(\Theta)$ by

$$e_{n,s} \mapsto z_n^{2s-2} e_{n,s}$$

is in the trace class. Therefore (see e.g. [12, 6, 16]) the measure $\tilde{\mu}$ is countably additive on the cylindrical sets of $H_{rad}^s(\Theta)$. We then denote by μ the Borel probability measure on $H_{rad}^s(\Theta)$ which is the unique extension (Caratheodory theorem) of $\tilde{\mu}$ to the Borel sigma algebra of $H_{rad}^s(\Theta)$. For the sake of completeness, in Section 3 we present the proof of the countable additivity $\tilde{\mu}$ on the algebra of the cylindrical sets of $H_{rad}^s(\Theta)$. As we will show in Proposition 3.3, for $\sigma \in [s, 1/2[$, $\mu(H_{rad}^\sigma(\Theta)) = 1$ and thus, we may consider μ as a measure on the space

$$(1.12) \quad \mathcal{X} = \bigcap_{s \leq \sigma < \frac{1}{2}} H_{rad}^\sigma(\Theta).$$

Thus one should not take the particular choice of s that we made too seriously. Notice that since $\sigma < 1/2$ the boundary conditions are not of importance in the definition of \mathcal{X} . In addition, in (1.12) the intersection may be assumed countable.

One may hope that the expression $\exp(\int_{\Theta} V(u))d\mu(u)$ which is a normalised version of the formal Gibbs measure $\exp(-H(u, \bar{u}))d^2u$ is a well defined measure. The expression $\exp(-H(u, \bar{u}))d^2u$ is formally invariant by the flow thanks to the Hamiltonian conservation. If we were in finite dimensions the invariance would follow from the invariance of the Lebesgue measure by the flow (Liouville's theorem). There is however a problem with the integrability of the above density with respect

to μ . We will solve this problem by using the L^2 cut-off idea of Lebowitz-Rose-Speer [11].

This paper is devoted to the proof of the following statement.

THEOREM 1. *Fix $R > 0$. Let us denote by $\chi : \mathbb{R} \rightarrow \{0, 1\}$ the characteristic function of the interval $[0, R]$. For $u \in \mathcal{X}$, we define the functional $f(u)$ by*

$$f(u) = \chi(\|u\|_{L^2(\Theta)}) \exp\left(\int_{\Theta} V(u)\right).$$

Then for every $q \in [1, +\infty[$,

$$(1.13) \quad f(u) \in L^q(d\mu(u)).$$

Moreover, if we set $d\rho(u) = f(u)d\mu(u)$ then there exists a set Σ of full ρ measure such that for every $u_0 \in \Sigma$ the Cauchy problem (1.1)-(1.5) has a unique (in a suitable functional framework) global in time solution. Finally, if we denote by $\Phi(t)$, $t \in \mathbb{R}$ the flow of (1.1) acting on Σ then the measure ρ is invariant under the flow of (1.1), i.e. for every ρ measurable set $A \subset \Sigma$, every $t \in \mathbb{R}$, $\rho(A) = \rho(\Phi(t)(A))$.

REMARK 1.1. *The uniqueness statement of Theorem 1 can be precised as follows : for every $T > 0$ there exists a Banach space X_T continuously embedded in $C([-T, T]; H_{rad}^s(\Theta))$ such that the solution of (1.1) with data $u_0 \in \Sigma$ is unique in X_T .*

Notice that thanks to the growth assumption (1.2) and the Sobolev embedding, the functional $f(u)$ is well-defined for $u \in \mathcal{X}$.

As a consequence of Theorem 1, the Poincaré recurrence theorem implies that almost surely on the support of $d\rho$ the solutions of (1.1) are stable according to Poisson (see [16] and the references therein for more details).

Similar results to Theorem 1 in the case of the circle S^1 are known thanks to the works [1, 16]. Gibbs type invariant measures for a Wicked ordered cubic defocusing NLS, posed on the two dimensional rational torus are constructed in [2].

Invariant measures for defocusing NLS of type (1.1) posed on an arbitrary compact riemannian manifold are constructed in [10]. These measures are not of Gibbs type (but still related to the conservation laws), and are living on functions in the Sobolev space H^2 . Let us notice that Dirac measures concentrated on a stationary (independent of t) solutions of (1.1) are clearly invariant. The measures constructed in [10] are not of this trivial type since the defocusing nature of the problem excludes the existence of stationary solutions.

The proof of Theorem 1 follows the ideas developed by Zhidkov (see [16] and the references by the same author therein) and Bourgain [1]. The main difficulties we should overcome are to prove a new local well-posedness results for (1.1), posed on the unit disc as well as adapting some estimates on random Fourier series to the case of functions on the unit disc of \mathbb{R}^2 . In the local well-posedness analysis, we need some bilinear Strichartz estimates. Starting from the work of Bourgain, estimates in this spirit were already used by many authors in the context of dispersive PDE's. In the analysis here, the crucial bilinear effect results from simple properties of the

Bessel functions and estimates on some series of Bessel functions. Notice that the bilinear approach and the Bourgain spaces are needed to be employed here since the well-posedness analysis of [3] based only on linear Strichartz inequalities and Sobolev spaces requires the restriction $\sigma > 1/2$ (thus missing \mathcal{X}) coming from the Sobolev embedding $W^{\sigma,4} \subset L^\infty$, $\sigma > 1/2$ in two dimensions.

The rest of the paper is organized as follows. The next section is devoted to some properties of the Bessel functions needed for our analysis of NLS (1.1). In Section 3, we collect some properties of Wiener type measures on Sobolev spaces of radial functions on the disc. Section 4 is devoted to bilinear Strichartz type inequalities which are the basic analytical tool in this paper. In Section 5, we introduce the Bourgain spaces of radial functions on Θ . The main nonlinear estimate are established in Section 6. As a first consequence of these estimates, in Section 7 we prove some local well-posedness results for NLS and its finite dimensional approximation. Next, in section 8, we improve the result for the finite dimensional model. In Section 9, we transfer the result of Section 8 to the NLS. The proof of Theorem 1, we will be completed in Section 10. The final section is devoted to some straightforward extensions of Theorem 1 and open problems that seem of interest to the author of the present paper.

Notation. Let us now introduce several notations that will be used in the paper. For two positive real numbers N_1 and N_2 , we denote by $N_1 \wedge N_2 \equiv \min(N_1, N_2)$ the smaller one. For $x \in \mathbb{R}$, we set $\langle x \rangle \equiv 1 + |x|$. We use the notations \sim or \approx for the equivalence of two quantities, uniformly with respect to some parameters which will be clear in each appearance of these two symbols. Several positive constants uniform with respect to some parameters, which will be clear in each appearance, will be denoted by C or c . The parameter set will always be a set of numbers or a set of functions.

2. On the Bessel functions and their zeros

In this section, we collect several facts on the zero order Bessel function that will be used in the sequel. These facts are essentially in the literature (see e.g. [14, 15]) but, in order to keep the paper as self contained as possible, here we give the proofs. We will be interested on $J_0(x)$ for $x \geq 0$ and its zeros z_n since $J_0(z_n r)$, $0 \leq r < 1$ form a basis for the radial L^2 functions on the disc Θ . The Bessel function $J_0(x)$ is defined by

$$J_0(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j!)^2} \left(\frac{x}{2}\right)^{2j}.$$

The function $J_0(x)$ solves the ordinary differential equation

$$J_0''(x) + \frac{1}{x} J_0'(x) + J_0(x) = 0.$$

The function $J_0(x)$ may be seen as the zero Fourier coefficient of the function $\exp(ix \sin \theta)$, $\theta \in [-\pi, \pi]$ and thus

$$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \sin \theta} d\theta.$$

Moreover, by the Lebesgue differentiation theorem,

$$J'_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (i \sin \theta) e^{ix \sin \theta} d\theta.$$

Recall that $e_n : \Theta \rightarrow \mathbb{R}$, defined by (1.7) form an orthonormal basis of the L^2 radial functions on the disc Θ . Observe that $e_n(r)$ are real valued. The next lemma provides $L^p(\Theta)$ bounds for e_n in the regime $n \gg 1$.

LEMMA 2.1. *Let $p \in [2, \infty]$. There exists C such that for every $n \geq 1$,*

$$(2.1) \quad \|e_n\|_{L^p(\Theta)} \leq C\delta(n)\|e_n\|_{L^2(\Theta)} = C\delta(n), \quad \|e'_n\|_{L^p(\Theta)} \leq C\delta(n)\|e'_n\|_{L^2(\Theta)},$$

where

$$\delta(n) = \begin{cases} 1 & \text{when } 2 \leq p < 4, \\ (\log(1+n))^{\frac{1}{4}} & \text{when } p = 4, \\ n^{-\frac{2}{p} + \frac{1}{2}} & \text{when } p > 4. \end{cases}$$

In particular for every $\varepsilon > 0$ there exists C_ε such that for every $n_1, n_2 \geq 1$,

$$(2.2) \quad \|e_{n_1} e_{n_2}\|_{L^2(\Theta)} \leq C_\varepsilon (\min(n_1, n_2))^\varepsilon \|e_{n_1}\|_{L^2(\Theta)} \|e_{n_2}\|_{L^2(\Theta)} = C_\varepsilon (\min(n_1, n_2))^\varepsilon$$

and

$$(2.3) \quad \|e_{n_1} e'_{n_2}\|_{L^2(\Theta)} \leq C_\varepsilon (\min(n_1, n_2))^\varepsilon \|e_{n_1}\|_{L^2(\Theta)} \|e'_{n_2}\|_{L^2(\Theta)}.$$

Finally, there exist two positive constants C_1 and C_2 such that for every $n \in \mathbb{N}$,

$$(2.4) \quad C_1 n = C_1 n \|e_n\|_{L^2(\Theta)} \leq \|e'_n\|_{L^2(\Theta)} \leq C_2 n \|e_n\|_{L^2(\Theta)} = C_2 n.$$

PROOF. The proof is based on the asymptotics for $J_0(x)$ and $J'_0(x)$ for large values of x . These asymptotics may be found by applying the stationary phase formula to the integrals defining $J_0(x)$ and $J'_0(x)$. Indeed, in both cases the phase $\sin \theta$ has two non-degenerate critical points $\pm \frac{\pi}{2}$ on $[-\pi, \pi]$. Therefore, there exists $C > 0$ and a function $r_1(x)$ defined on $[1, +\infty[$ such that

$$J_0(x) = \sqrt{\frac{2}{\pi}} \frac{\cos(x - \frac{\pi}{4})}{\sqrt{x}} + r_1(x), \quad |r_1(x)| \leq Cx^{-\frac{3}{2}}$$

(the two critical points contribute with phases $\exp(i(\pm x \mp \pi/4))$). Similarly, we have

$$J'_0(x) = -\sqrt{\frac{2}{\pi}} \frac{\sin(x - \frac{\pi}{4})}{\sqrt{x}} + \tilde{r}_1(x), \quad |\tilde{r}_1(x)| \leq Cx^{-\frac{3}{2}}.$$

A first consequence of the above representations of $J_0(x)$ and $J'_0(x)$ is that the n 'th zero of $J_0(x)$ satisfy $z_n \sim n$. We can therefore write that for $n \gg 1$

$$\begin{aligned} \|J_0(z_n \cdot)\|_{L^2(\Theta)}^2 &= \int_0^1 |J_0(z_n r)|^2 r dr \\ &= z_n^{-2} \int_0^{z_n} |J_0(\rho)|^2 \rho d\rho \\ &\geq cn^{-2} \int_0^{cn} |J_0(\rho)|^2 \rho d\rho \\ &\geq Cn^{-2} \int_1^{cn} \left(\frac{1 + \cos(2\rho - \pi/2)}{2\rho} - \frac{C}{\rho^2} \right) \rho d\rho \\ &\geq cn^{-2}(cn - C \log(n)) \geq cn^{-1}. \end{aligned}$$

Therefore

$$(2.5) \quad \|J_0(z_n \cdot)\|_{L^2(\Theta)} \geq cn^{-1/2}.$$

Similarly, we can show that

$$\|J'_0(z_n \cdot)\|_{L^2(\Theta)} \geq cn^{-1/2}.$$

On the other hand, using that $|J_0(x)| \leq Cx^{-1/2}$, $x \geq 1$, and, $|J_0(x)| \leq C$, $x \leq 1$, we obtain that for $p \in [2, \infty[$,

$$\begin{aligned} \|J_0(z_n \cdot)\|_{L^p(\Theta)}^p &= \int_0^1 |J_0(z_n r)|^p r \, dr \\ &= z_n^{-2} \int_0^{z_n} |J_0(\rho)|^p \rho \, d\rho \\ &\leq Cn^{-2} \left(C + \int_1^{cn} \rho^{-p/2} \rho \, d\rho \right) \end{aligned}$$

which gives the bound (2.1) for e_n and $p < +\infty$ by distinguishing the three regimes for p involved in the definition of $\delta(n)$. The last estimate also implies that

$$\|J_0(z_n \cdot)\|_{L^2(\Theta)} \leq Cn^{-1/2}$$

and thus

$$\|J_0(z_n \cdot)\|_{L^2(\Theta)} \sim n^{-1/2}.$$

Estimate (2.1) for $p = \infty$ and e_n follows from the bound $|J_0(x)| \leq C$ for all $x \geq 0$ and the inequality (2.5). This completes the proof of (2.1) as far as e_n is concerned. The bound for e'_n in (2.1) can be established in a completely analogous way, once we have the stationary phase approximation of $J'_0(x)$. We also have

$$\|J'_0(z_n \cdot)\|_{L^2(\Theta)} \leq Cn^{-1/2}$$

and thus

$$\|J'_0(z_n \cdot)\|_{L^2(\Theta)} \sim n^{-1/2}.$$

Since

$$e'_n(r) = z_n \|J_0(z_n \cdot)\|_{L^2(\Theta)}^{-1} J'_0(z_n r)$$

we get estimate (2.4). Finally, the assertion of (2.2) results from (2.1) and Hölder inequality

$$\|e_{n_1} e_{n_2}\|_{L^2(\Theta)} \leq \|e_{n_1}\|_{L^p(\Theta)} \|e_{n_2}\|_{L^q(\Theta)}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2},$$

with p, q close to 4 and according to the order of n_1, n_2 , the bigger of p, q is attached to the smaller of n_1, n_2 . A similar argument yields (2.3). This completes the proof of Lemma 2.1. \square

The next lemma provides a more precise asymptotics for the zeros z_n , $n \gg 1$.

LEMMA 2.2. *For every $\kappa > 0$ there exists $C > 0$ such that the zeros of $J_0(x)$ can be written as*

$$(2.6) \quad z_n = \pi \left(n - \frac{1}{4} \right) + \frac{1}{8\pi \left(n - \frac{1}{4} \right)} + r(n), \quad |r(n)| \leq Cn^{-(2-\kappa)}.$$

REMARK 2.3. *In fact, much better bounds on $r(n)$ may be proved. However, estimate (2.6) will be sufficient for our applications.*

PROOF OF PROPOSITION 2.2. Using the stationary phase formula at order 2 in the integral representation of $J_0(x)$ gives the existence of a constant $C > 0$ and a function $r_2(x)$ defined on $[1, +\infty[$ such that

$$J_0(x) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(x - \frac{\pi}{4}\right)}{x^{1/2}} + \sqrt{\frac{2}{\pi}} \frac{\sin\left(x - \frac{\pi}{4}\right)}{8x^{3/2}} + r_2(x), \quad |r_2(x)| \leq Cx^{-\frac{5}{2}}.$$

Therefore, for $n \gg 1$, the zero z_n solves the equation $F(z_n) = 0$, where $F(x)$ (with $x - \pi/4$ near the positive odd integer multiples of $\pi/2$) is a continuous function of the form

$$F(x) = \frac{1}{\tan\left(x - \frac{\pi}{4}\right)} + \frac{1}{8x} + \mathcal{O}(n^{-2}).$$

Here $\mathcal{O}(n^{-2})$ denotes a quantity $\leq Cn^{-2}$ with C independent of n and x . For $\kappa > 0$, we set

$$z_n^\pm = \pi\left(n - \frac{1}{4}\right) + \frac{1}{8\pi\left(n - \frac{1}{4}\right)} \pm \frac{1}{n^{2-\kappa}}.$$

Further, we set

$$\varepsilon_n^\pm = \frac{1}{8\pi\left(n - \frac{1}{4}\right)} \pm \frac{1}{n^{2-\kappa}}.$$

Notice that $\cos(z_n^\pm - \pi/4) = (-1)^n \sin \varepsilon_n^\pm$ and $\sin(z_n^\pm - \pi/4) = (-1)^{n+1} \cos \varepsilon_n^\pm$. Therefore, by expanding, we get

$$F(z_n^\pm) = -\tan(\varepsilon_n^\pm) + \frac{1}{8\pi\left(n - \frac{1}{4}\right) + 8\varepsilon_n^\pm} + \mathcal{O}(n^{-2}) = \mp \frac{1}{n^{2-\kappa}} + \mathcal{O}(n^{-2}).$$

Therefore for $n \gg 1$ the zero z_n lies between z_n^- and z_n^+ . This completes the proof of Lemma 2.2. \square

3. The measures μ and ρ

In this section, we prove (1.13) and we collect some properties of the measures μ and ρ . Let us first observe that the minimal sigma algebra containing the algebra of cylindrical sets (1.10) contains the closed balls of $H_{rad}^\sigma(\Theta)$, $\sigma \in [s, 1/2[$. Indeed, if for $r > 0$ and $v \in H_{rad}^\sigma(\Theta)$, we set

$$B_\sigma(r, v) = \left\{ u \in H_{rad}^s(\Theta) : u \in H_{rad}^\sigma(\Theta) \quad \text{and} \quad \|u - v\|_{H^\sigma(\Theta)} \leq r \right\}$$

then

$$B_\sigma(r, v) = \bigcap_{N \geq 1} U_{\sigma, N}(r, v),$$

where the cylindrical sets $U_{\sigma, N}(r, v)$ are defined by

$$U_{\sigma, N}(r, v) = \left\{ u \in H_{rad}^s(\Theta) : \sum_{1 \leq j \leq N} z_n^{\sigma-s} |(u - v, e_{j,s})|^2 \leq r^2 \right\}.$$

Since $H_{rad}^s(\Theta)$ is separable, we obtain that the minimal sigma algebra containing all cylindrical sets is the Borel sigma algebra.

As mentioned in the introduction, for a sake of completeness, we give the proof of the countable additivity of the measure $\check{\mu}$.

PROPOSITION 3.1. *The measure $\tilde{\mu}$, defined on the algebra of cylindrical sets (1.10) by formula (1.11) is countably additive, i.e. for every sequence U_n , $n \in \mathbb{N}$ of cylindrical sets such that $U_{n+1} \subset U_n$ and*

$$(3.1) \quad \bigcap_{n \in \mathbb{N}} U_n = \emptyset,$$

one has

$$\lim_{n \rightarrow \infty} \tilde{\mu}(U_n) = 0.$$

Thus $\tilde{\mu}$ has a unique extension that we denote by μ to the Borel sigma algebra of $H_{rad}^s(\Theta)$ which is a Borel probability measure on $H_{rad}^s(\Theta)$.

PROOF. Let $\sigma > 0$ be such that $s + \sigma < 1/2$. For $R \geq 1$, we consider the set

$$K_R = \{u \in H_{rad}^s(\Theta) : \|u\|_{H^{s+\sigma}(\Theta)} \leq R\}.$$

Thanks to the compactness of the embedding $H_{rad}^{s+\sigma}(\Theta)$ into $H_{rad}^s(\Theta)$, we obtain that K_R is a compact set of $H_{rad}^s(\Theta)$. Since U_n , $n \in \mathbb{N}$ are cylindrical sets, there exists a function $r : \mathbb{N} \rightarrow \mathbb{N}$ such that for every n the set U_n can be seen as a subset of the finite dimensional space $E_{r(n)}$ defined by $E_{r(n)} = \text{span}(e_{j,s})_{1 \leq j \leq r(n)}$. More precisely, there exists a Borel set \tilde{U}_n of $E_{r(n)}$ such that

$$U_n = \left(u \in H_{rad}^s(\Theta) : (u, e_{1,s})e_{1,s} + \dots + (u, e_{r(n),s})e_{r(n),s} \in \tilde{U}_n \right).$$

Consider the cylindrical sets $F_{r(n)}$ defined as

$$F_{r(n)} \equiv \left(u \in H_{rad}^s(\Theta) : (u, e_{1,s})e_{1,s} + \dots + (u, e_{r(n),s})e_{r(n),s} \in K_R \right).$$

Then

$$(3.2) \quad \tilde{\mu}(F_{r(n)}) \geq 1 - CR^{-2},$$

where C is a constant independent of R and what is more important, independent of $n \in \mathbb{N}$. Set $m = r(n)$. In order to prove (3.2), we observe that

$$1 - \tilde{\mu}(F_{r(n)}) \leq I,$$

where I is given by the integral

$$I = \pi^{-m} \left(\prod_{j=1}^m z_j^{2-2s} \right) \int_V e^{-\sum_{1 \leq j \leq m} z_j^{2-2s} |c_j|^2} d^2 c_1 \dots d^2 c_m,$$

where V is given by

$$V = \{(c_1, \dots, c_m) \in \mathbb{C}^m : \sum_{j=1}^m z_j^{2\sigma} |c_j|^2 \geq R^2\}.$$

Set $\theta \equiv s + \sigma < 1/2$. By the change of the variable $c_j \rightarrow z_j^\sigma c_j$, we obtain that

$$I = \pi^{-m} \left(\prod_{j=1}^m z_j^{2-2\theta} \right) \int_W e^{-\sum_{1 \leq j \leq m} z_j^{2-2\theta} |c_j|^2} d^2 c_1 \dots d^2 c_m,$$

where W is given by

$$W = \{(c_1, \dots, c_m) \in \mathbb{C}^m : \sum_{j=1}^m |c_j|^2 \geq R^2\}.$$

By introducing polar coordinates in each c_j integration, we may estimate

$$\begin{aligned} R^2 I &\leq \pi^{-m} \left(\prod_{j=1}^m z_j^{2-2\theta} \right) \int_W \left(\sum_{j=1}^m |c_j|^2 \right) e^{-\sum_{1 \leq j \leq m} z_j^{2-2\theta} |c_j|^2} d^2 c_1 \dots d^2 c_m \\ &\leq \pi^{-m} \left(\prod_{j=1}^m z_j^{2-2\theta} \right) \int_{\mathbb{C}^m} \left(\sum_{j=1}^m |c_j|^2 \right) e^{-\sum_{1 \leq j \leq m} z_j^{2-2\theta} |c_j|^2} d^2 c_1 \dots d^2 c_m \\ &= \sum_{j=1}^m z_j^{2\theta-2} \leq C \end{aligned}$$

where C is a constant independent of $m = r(n)$. This proves (3.2).

Let us fix $\varepsilon > 0$. By the regularity of the Lebesgue measure, using that $U_{n+1} \subset U_n$ one can construct closed sets $\tilde{V}_n \subset E_{r(n)}$ such that

$$(3.3) \quad V_n = \left(u \in H_{rad}^s(\Theta) : (u, e_{1,s})e_{1,s} + \dots + (u, e_{r(n),s})e_{r(n),s} \in \tilde{V}_n \right)$$

satisfy

$$V_n \subset U_n, \quad V_{n+1} \subset V_n, \quad \tilde{\mu}(U_n \setminus V_n) < \varepsilon/2.$$

Indeed, one first constructs closed sets $\tilde{W}_n \subset E_{r(n)}$ such that

$$W_n = \left(u \in H_{rad}^s(\Theta) : (u, e_{1,s})e_{1,s} + \dots + (u, e_{r(n),s})e_{r(n),s} \in \tilde{W}_n \right)$$

satisfy

$$W_n \subset U_n, \quad \tilde{\mu}(U_n \setminus W_n) < \varepsilon/2^{n+3}.$$

Then, we set

$$V_n \equiv \bigcap_{j=1}^n W_j$$

and one easily verifies that V_n satisfies (3.3).

We have that $K_R \cap V_n$ is a compact set of $H_{rad}^s(\Theta)$ included in U_n . Therefore (3.1) yields

$$\bigcap_{n \in \mathbb{N}} (K_R \cap V_n) = \emptyset.$$

Hence, there exists $N \geq 1$ such that $K_R \cap V_n = \emptyset$ for $n \geq N$. Moreover, $F_{r(n)} \cap V_n = \emptyset$ for $n \geq N$. Indeed, if $u \in F_{r(n)} \cap V_n$ then by setting

$$u_n \equiv (u, e_{1,s})e_{1,s} + \dots + (u, e_{r(n),s})e_{r(n),s},$$

we observe that $u_n \in K_R$ and $u_n \in V_n$ which is a contradiction. Thus $F_{r(n)} \cap V_n = \emptyset$. Therefore, using (3.2), we infer that

$$1 \geq \tilde{\mu}(F_{r(n)} \cup V_n) = \tilde{\mu}(F_{r(n)}) + \tilde{\mu}(V_n) \geq 1 - CR^{-2} + \tilde{\mu}(V_n).$$

Hence $\tilde{\mu}(V_n) \leq CR^{-2}$ and thus for $R \gg 1$ (independently of n),

$$\tilde{\mu}(U_n) \leq \tilde{\mu}(V_n) + \tilde{\mu}(U_n \setminus V_n) < CR^{-2} + \varepsilon/2 < \varepsilon.$$

This completes the proof of Proposition 3.1. \square

REMARK 3.2. *One may show that for $s \geq 1/2$, the measure $\tilde{\mu}$ is not countably additive on the algebra of the cylindrical set on $H_{rad}^s(\Theta)$ (see e.g. [6]).*

If $u \in H_{rad}^s(\Theta)$ is given by

$$u = \sum_{n \geq 1} c_n e_{n,s}$$

then we can consider a Littlewood-Paley decomposition of u defined by

$$u = \sum_{N\text{-dyadic}} \Delta_N(u),$$

where N is running over the set of dyadic integers, i.e. the nonnegative powers of 2, and, the projector Δ_N is defined by

$$\Delta_N(u) = \sum_{n: N \leq \langle z_n \rangle < 2N} c_n e_{n,s}.$$

We next state a bound on the μ measure of functions containing only high frequencies in their Littlewood-Paley decompositions.

PROPOSITION 3.3. *Let $\sigma \in [s, 1/2[$. There exist $C > 0$ and $c > 0$ such that for every $N_0 \geq 1$, every $\lambda \geq 1$,*

$$\mu\left(u \in H_{rad}^s(\Theta) : \left\| \sum_{\substack{N \geq N_0 \\ N\text{-dyadic}}} \Delta_N(u) \right\|_{H^\sigma(\Theta)} \leq \lambda\right) \geq 1 - Ce^{-c\lambda^2 N_0^{2(1-\sigma)}}.$$

In particular

$$(3.4) \quad \mu\left(u \in H_{rad}^s(\Theta) : \|u\|_{H^\sigma(\Theta)} \leq \lambda\right) \geq 1 - Ce^{-c\lambda^2}$$

and

$$\mu(H_{rad}^\sigma(\Theta)) = 1.$$

Therefore one can consider μ as a measure on the space \mathcal{X} defined by (1.12).

PROOF OF PROPOSITION 3.3. In view of (1.11), we obtain that the measure μ is the distribution of the random series

$$(3.5) \quad \varphi_\omega(r) = \sum_{n \geq 1} \frac{g_n(\omega)}{z_n^{1-s}} e_{n,s}(r) = \sum_{n \geq 1} \frac{g_n(\omega)}{z_n} e_n(r),$$

where $g_n(\omega)$ is a sequence of normalised ($\mathcal{N}(0, 1/\sqrt{2})$) independent identically distributed (i.i.d.) complex Gaussian random variables, defined in a probability space (Ω, \mathcal{F}, p) . More precisely, for U a μ -measurable set, we have

$$\mu(U) = p(\omega : \varphi_\omega \in U).$$

Consider a Littlewood-Paley decomposition of (3.5)

$$(3.6) \quad \varphi_\omega(r) = \sum_{N\text{-dyadic}} \Delta_N(\varphi_\omega(r))$$

with

$$\Delta_N(\varphi_\omega(r)) = \sum_{n: N \leq \langle z_n \rangle < 2N} z_n^{-1} g_n(\omega) e_n(r).$$

We need therefore to establish the bound

$$p\left(\omega \in \Omega : \left\| \sum_{\substack{N \geq N_0 \\ N\text{-dyadic}}} \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} > \lambda\right) \leq Ce^{-c\lambda^2 N_0^{2(1-\sigma)}}.$$

Let us next prove an inequality for Gaussians.

LEMMA 3.4. *Let $g_n(\omega)$ be a sequence of normalized i.i.d. complex Gaussian random variables defined in a probability space (Ω, \mathcal{F}, p) . Then there exist positive numbers c_1, c_2 such that for every finite set of indexes $\Lambda \subset \mathbb{N}$, every $\lambda > 0$,*

$$p\left(\omega \in \Omega : \sum_{n \in \Lambda} |g_n(\omega)|^2 > \lambda\right) \leq e^{c_1|\Lambda| - c_2\lambda}.$$

PROOF. For every $\zeta > 0$,

$$(3.7) \quad p\left(\omega \in \Omega : \sum_{n \in \Lambda} |g_n(\omega)|^2 > \lambda\right) = p\left(\omega \in \Omega : \prod_{n \in \Lambda} e^{\zeta |g_n(\omega)|^2} > e^{\zeta\lambda}\right).$$

For $\zeta < 1$, using the Tchebishev inequality and the independence of $g_n(\omega)$, we deduce that (3.7) is bounded by

$$e^{-\zeta\lambda} \prod_{n \in \Lambda} \mathbb{E}(e^{\zeta |g_n(\omega)|^2}) = e^{-\zeta\lambda} z^{|\Lambda|},$$

where the positive number z is given by

$$z = \pi^{-1} \left(\int_{-\infty}^{\infty} e^{-(1-\zeta)x^2} dx \right)^2 = \frac{1}{1-\zeta} > 1.$$

This completes the proof of Lemma 3.4. \square

Let us now turn to the proof of Proposition 3.3. For $N_0 \geq 1$, we set

$$A_{N_0} = \left(\omega \in \Omega : \left\| \sum_{\substack{N \geq N_0 \\ N\text{-dyadic}}} \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} > \lambda \right).$$

Let θ be a real number such that

$$(3.8) \quad 0 < 2\theta < 1 - 2\sigma.$$

For $\kappa > 0$ and $N \geq N_0$, N being a dyadic integer, we set

$$A_{N,\kappa} = \left(\omega \in \Omega : \left\| \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} > \lambda\kappa(N^{-\theta} + (N^{-1}N_0)^{1-\sigma}) \right).$$

Now, we observe that there exists κ sufficiently small depending on σ but independent of N_0 and N such that

$$(3.9) \quad A_{N_0} \subset \bigcup_{\substack{N \geq N_0 \\ N\text{-dyadic}}} A_{N,\kappa}.$$

The restriction

$$\left\| \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} > \lambda\kappa(N^{-\theta} + (N^{-1}N_0)^{1-\sigma})$$

implies that

$$\sum_{n: N \leq \langle z_n \rangle < 2N} z_n^{2\sigma} z_n^{-2} |g_n(\omega)|^2 > \lambda^2 \kappa^2 (N^{-\theta} + (N^{-1}N_0)^{1-\sigma})^2$$

and therefore, in view of (2.6),

$$\sum_{n: N \leq \langle z_n \rangle < 2N} |g_n(\omega)|^2 > c\lambda^2 \kappa^2 N^{2-2\sigma} (N^{-2\theta} + (N^{-1}N_0)^{2-2\sigma}).$$

Once again invoking (2.6), we infer that

$$\#\{n : N \leq \langle z_n \rangle < 2N\} \leq CN$$

and therefore Lemma 3.4 yields the bound

$$p(A_{N,\kappa}) \leq e^{c_1 N - c_2 \lambda^2 \kappa^2 N_0^{2-2\sigma} - c_2 \lambda^2 \kappa^2 N^{2-2\sigma-2\theta}}.$$

The assumption (3.8) implies that $1 < 2 - 2\sigma - 2\theta$ and thus

$$p(A_{N,\kappa}) \leq C e^{-c\lambda^2 N_0^{2-2\sigma}} e^{-cN^{2-2\sigma-2\theta}}.$$

Using (3.9), a summation over N yields

$$p(A_{N_0}) \leq C e^{-c\lambda^2 N_0^{2-2\sigma}}$$

which completes the proof of Proposition 3.3. \square

REMARK 3.5. *One can use the method of proof of Proposition 3.3 to improve (3.2) to exponential bounds.*

Let us now turn to the proof of (1.13) of Theorem 1. It is a consequence of the following statement.

PROPOSITION 3.6. *Let $g_n(\omega)$ be a sequence of normalised i.i.d. complex Gaussian random variables, defined in a probability space (Ω, \mathcal{F}, p) . Let $\chi : \mathbb{R} \rightarrow \{0, 1\}$ be the characteristic function of the interval $[0, R]$, $R > 0$. Define the random series φ_ω by*

$$\varphi_\omega(r) = \sum_{n \geq 1} \frac{g_n(\omega)}{z_n} e_n(r).$$

Then for every $q > 0$,

$$\mathbb{E}\left(\chi(\|\phi_\omega\|_{L^2(\Theta)}) \exp\left(q \int_{\Theta} |V(\phi_\omega)|\right)\right) < \infty.$$

PROOF. Thanks to (1.2)

$$\int_{\Theta} |V(\phi_\omega)| \leq C + C \|\phi_\omega\|_{L^{\alpha+2}(\Theta)}^{\alpha+2}.$$

Therefore, we have to show that

$$\mathbb{E}\left(\chi(\|\phi_\omega\|_{L^2(\Theta)}) \exp(Cq \|\phi_\omega\|_{L^{\alpha+2}(\Theta)}^{\alpha+2})\right) < \infty.$$

Observe that it suffices to show that

$$\int_1^\infty f(\lambda) d\lambda < \infty,$$

where

$$f(\lambda) = p\left(\omega \in \Omega : \|\phi_\omega\|_{L^{\alpha+2}(\Theta)} \geq \left(\frac{\log \lambda}{Cq}\right)^{\frac{1}{\alpha+2}}, \|\phi_\omega\|_{L^2(\Theta)} \leq R\right).$$

Set

$$\gamma := \left(\frac{\log \lambda}{Cq}\right)^{\frac{1}{\alpha+2}}.$$

Let us now fix the real number σ according to the Sobolev embedding restriction

$$\sigma = 2\left(\frac{1}{2} - \frac{1}{\alpha+2}\right).$$

Notice that thanks to (1.3) and (1.9), $1/2 > \sigma \geq s$ (of course the important point here is that $\sigma < 1/2$). The Sobolev embedding $H^\sigma(\Theta) \subset L^{\alpha+2}(\Theta)$ thus yields the bound

$$(3.10) \quad \|\phi_\omega\|_{L^{\alpha+2}(\Theta)} \leq C_{sob} \|\phi_\omega\|_{H^\sigma(\Theta)}.$$

Therefore

$$f(\lambda) \leq p\left(\omega \in \Omega : \|\phi_\omega\|_{H^\sigma(\Theta)} \geq \gamma/C_{sob}, \quad \|\phi_\omega\|_{L^2(\Theta)} \leq R\right).$$

Consider again the Littlewood-Paley decomposition (3.6). In the spirit of the Brézis-Gallouet argument, we set

$$N_0 = \kappa \gamma^{\frac{1}{\sigma}},$$

where $\kappa > 0$ is a small number to be fixed later. Then

$$\left(\omega \in \Omega : \|\phi_\omega\|_{H^\sigma(\Theta)} \geq \gamma/C_{sob}, \quad \|\phi_\omega\|_{L^2(\Theta)} \leq R\right) \subset A_1 \cup A_2$$

with

$$A_1 = \left(\omega \in \Omega : \left\| \sum_{\substack{N \leq N_0 \\ N\text{-dyadic}}} \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} \geq \gamma/(4C_{sob}), \quad \|\phi_\omega\|_{L^2(\Theta)} \leq R\right)$$

and

$$A_2 = \left(\omega \in \Omega : \left\| \sum_{\substack{N > N_0 \\ N\text{-dyadic}}} \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} \geq \gamma/(4C_{sob})\right).$$

On the other hand

$$\left\| \sum_{\substack{N \leq N_0 \\ N\text{-dyadic}}} \Delta_N(\varphi_\omega) \right\|_{H^\sigma(\Theta)} \leq CN_0^\sigma \|\varphi_\omega\|_{L^2(\Theta)} \leq CR\kappa^\sigma \gamma.$$

Hence for $\kappa \ll 1$, the set A_1 is empty. This fixes the parameter κ . On the other hand, thanks to Proposition 3.3,

$$p(A_2) \leq Ce^{-c\gamma^2 N_0^{2(1-\sigma)}}.$$

Therefore

$$f(\lambda) \leq Ce^{-c[\log \lambda]^{\frac{2}{\alpha+2}} N_0^{2(1-\sigma)}}.$$

Coming back to the definitions of σ and N_0 , we get

$$[\log \lambda]^{\frac{2}{\alpha+2}} N_0^{2(1-\sigma)} = c[\log \lambda]^{\frac{2}{\alpha}}.$$

The assumption $\alpha < 2$ implies $2/\alpha > 1$ and therefore $f(\lambda)$ is integrable on $[1, +\infty[$. This completes the proof of Proposition 3.6. \square

We now state a corollary of Proposition 3.3 and Proposition 3.6.

PROPOSITION 3.7. *Let $\sigma \in [s, 1/2[$. Then there exist $C > 0$ and $c > 0$ such that*

$$\rho(u \in H_{rad}^s(\Theta) : \|u\|_{H^\sigma(\Theta)} > \lambda) \leq Ce^{-c\lambda^2}.$$

PROOF. Set

$$A_\lambda = (u \in H_{rad}^s(\Theta) : \|u\|_{H^\sigma(\Theta)} > \lambda).$$

Then using Proposition 3.3 and Proposition 3.6, we can write

$$\begin{aligned} \rho(A_\lambda) &= \int_{A_\lambda} d\rho = \int_{A_\lambda} f(u) d\mu(u) \\ &\leq \left(\int_{A_\lambda} f^2(u) d\mu(u) \right)^{1/2} \left(\int_{A_\lambda} d\mu(u) \right)^{1/2} \\ &\leq C(\mu(A_\lambda))^{\frac{1}{2}} \leq Ce^{-c\lambda^2}. \end{aligned}$$

This completes the proof of Proposition 3.7. \square

Next, we define the finite dimensional sup-spaces of $H_{rad}^s(\Theta)$,

$$E_N = \text{span}(e_{1,s}, \dots, e_{N,s}).$$

We equip E_N with the measures μ_N and ρ_N which are the natural restrictions to E_N of μ and ρ respectively. More precisely for a Borel set $V \subset \mathbb{C}^N$, we set

$$(3.11) \quad \tilde{V} = \{c_1 e_{1,s} + \dots + c_N e_{N,s}, \quad (c_1, \dots, c_N) \in V\}.$$

We define the measures μ_N and ρ_N on the sigma algebra of sets of type (3.11) by

$$\mu_N(\tilde{V}) = \pi^{-N} \left(\prod_{1 \leq n \leq N} z_n^{2-2s} \right) \int_V e^{-\sum_{1 \leq n \leq N} z_n^{2-2s} |c_n|^2} d^2 c_1 \dots d^2 c_N$$

and

$$\begin{aligned} \rho_N(\tilde{V}) &= \pi^{-N} \left(\prod_{1 \leq n \leq N} z_n^{2-2s} \right) \times \\ &\quad \times \int_V f(c_1 e_{1,s} + \dots + c_N e_{N,s}) e^{-\sum_{1 \leq n \leq N} z_n^{2-2s} |c_n|^2} d^2 c_1 \dots d^2 c_N. \end{aligned}$$

It is now clear that to every cylindrical set of $H_{rad}^s(\Theta)$ we may naturally associate a μ_N and ρ_N measurable set on E_N , provided N being sufficiently large. For $u \in H_{rad}^s(\Theta)$, we define the projector S_N ,

$$S_N : H_{rad}^s(\Theta) \longrightarrow E_N$$

via the formula

$$(3.12) \quad S_N(u) = \sum_{n=1}^N (u, e_{n,s}) e_{n,s}.$$

We have the following statement.

LEMMA 3.8. *Let U be an open set of $H_{rad}^\sigma(\Theta)$, $\sigma \in [s, 1/2[$. Then*

$$(3.13) \quad \rho(U) \leq \liminf_{N \rightarrow \infty} \rho_N(U \cap E_N).$$

Moreover, if F is a closed set of $H_{rad}^\sigma(\Theta)$, $\sigma \in [s, 1/2[$ then

$$(3.14) \quad \limsup_{N \rightarrow \infty} \rho_N(F \cap E_N) \leq \rho(F).$$

PROOF. Define the sets

$$U_N \equiv \{u \in H_{rad}^s(\Theta) : S_N(u) \in U\}.$$

Observe that $U \cap E_N$ is ρ_N measurable and

$$\rho_N(U \cap E_N) = \rho(U_N).$$

We have the inclusion

$$(3.15) \quad U \subset \liminf_N(U_N),$$

where

$$\liminf_N(U_N) \equiv \bigcup_{N \geq 1} \bigcap_{N_1 \geq N} U_{N_1}.$$

Indeed, we have that for every $u \in H_{rad}^s(\Theta)$,

$$(3.16) \quad \lim_{N \rightarrow \infty} \|u - S_N(u)\|_{H^s(\Theta)} = 0.$$

Therefore, using that U is an open set, we conclude that for every $u \in U$ there exists $N_0 \geq 1$ such that for $N \geq N_0$ one has $u \in U_N$. Hence we have (3.15). If A is a ρ -measurable set, we denote by χ_A the characteristic function of A . Define the non negative functions f_N by

$$f_N(u) \equiv \chi_{U_N}(u)f(u).$$

Notice that thanks to (3.15),

$$\liminf_{N \rightarrow \infty} \chi_{U_N} \geq \chi_U.$$

Next, we set

$$F(u) \equiv \liminf_{N \rightarrow \infty} f_N(u).$$

Thus

$$F(u) \geq \chi_U(u)f(u).$$

Since

$$\rho_N(U \cap E_N) = \int_{H_{rad}^s(\Theta)} f_N(u) d\mu(u),$$

using the Fatou lemma, we get

$$\liminf_{N \rightarrow \infty} \rho_N(U \cap E_N) \geq \int_{H_{rad}^s(\Theta)} F(u) d\mu(u) \geq \int_U f(u) d\mu(u) = \rho(U).$$

Next, we define the sets

$$F_N \equiv \{u \in H_{rad}^s(\Theta) : S_N(u) \in F\}.$$

Thus

$$\rho_N(F \cap E_N) = \rho(F_N).$$

We have that

$$(3.17) \quad \limsup_N(F_N) \subset F,$$

where

$$\limsup_N(F_N) \equiv \bigcap_{N \geq 1} \bigcup_{N_1 \geq N} F_{N_1}.$$

Indeed, suppose that $u \in \limsup_N(F_N)$. Thus there exists a sequence $(N_k)_{k \in \mathbb{N}}$ tending to infinity such that $u \in F_{N_k}$ which means that for every k one has $S_{N_k}(u) \in F$

F . Since F is closed, coming back to (3.16), we obtain that $u \in F$ and therefore we get (3.17). If we set

$$G(u) \equiv \limsup_{N \rightarrow \infty} \tilde{f}_N(u),$$

where

$$\tilde{f}_N(u) \equiv \chi_{F_N}(u) f(u).$$

then we have

$$G(u) \leq \chi_F(u) f(u)$$

and the Fatou lemma gives,

$$\begin{aligned} \limsup_{N \rightarrow \infty} \rho_N(F \cap E_N) &= \limsup_{N \rightarrow \infty} \int_{H_{rad}^s(\Theta)} \tilde{f}_N(u) d\mu(u) \\ &\leq \int_{H_{rad}^s(\Theta)} G(u) d\mu(u) \\ &\leq \int_F f(u) d\mu(u) \\ &= \rho(F). \end{aligned}$$

This completes the proof of Lemma 3.8. \square

The next lemma shows that every ρ measurable set can be approximated by subsets of compact sets in $H_{rad}^s(\Theta)$.

LEMMA 3.9. *Lets $\sigma \in]s, 1/2[$ and denote by $K_R, R > 0$ the ball*

$$K_R \equiv \{u \in H_{rad}^s(\Theta) : \|u\|_{H^\sigma(\Theta)} \leq R\}.$$

Then, for every ρ measurable set A ,

$$\rho(A) = \lim_{R \rightarrow \infty} \rho(A \cap K_R).$$

PROOF. Since $\mu(H_{rad}^s(\Theta)) = \mu(H_{rad}^\sigma(\Theta))$ and since $f(u) \in L^q(d\mu(u))$, $1 \leq q < \infty$, we obtain that $\rho(H_{rad}^s(\Theta)) = \rho(H_{rad}^\sigma(\Theta))$. Therefore, using Proposition 3.7, we can write

$$\begin{aligned} 0 \leq \rho(A) - \rho(A \cap K_R) &= \rho(A \cup K_R) - \rho(K_R) \\ &\leq \rho(H_{rad}^s(\Theta)) - \rho(K_R) \\ &= \rho(H_{rad}^\sigma(\Theta)) - \rho(K_R) \\ &\leq C e^{-CR^2} \end{aligned}$$

which completes the proof of Lemma 3.9. \square

4. Bilinear Strichartz estimates

We now state a localized Strichartz type bilinear estimate associated to the linear Schrödinger group on the unit disc.

PROPOSITION 4.1. *For every $\varepsilon > 0$, there exists $\beta < 1/2$, there exists $C > 0$ such that for every $N_1, N_2 \geq 1$, every $L_1, L_2 \geq 1$, every u_1, u_2 two functions on $\mathbb{R} \times \Theta$ of the form*

$$u_j(t, r) = \sum_{N_j \leq \langle z_n \rangle < 2N_j} c_j(n, t) e_n(r), \quad j = 1, 2$$

where the Fourier transform of $c_j(n, t)$ with respect to t satisfies

$$\text{supp } \widehat{c}_j(n, \tau) \subset \{\tau \in \mathbb{R} : L_j \leq \langle \tau + z_n^2 \rangle \leq 2L_j\}, \quad j = 1, 2$$

one has the bound

$$\|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C(N_1 \wedge N_2)^\varepsilon (L_1 L_2)^\beta \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

PROOF. Let us first notice that for $j = 1, 2$,

$$(4.1) \quad \|u_j\|_{L^2(\mathbb{R} \times \Theta)}^2 = c \sum_{N_j \leq \langle z_n \rangle < 2N_j} \int_{-\infty}^{\infty} |\widehat{c}_j(n, \tau)|^2 d\tau.$$

Denote $v(t, r) = u_1(t, r)u_2(t, r)$. Our purpose is thus to estimate $\|v\|_{L^2(\mathbb{R} \times \Theta)}$. Equivalently, we need to bound $\|\widehat{v}(\tau, \cdot)\|_{L^2(\mathbb{R}_r \times \Theta)}$. Write

$$\|\widehat{v}(\tau, r)\|_{L^2(\mathbb{R} \times \Theta)}^2 = c \int_0^1 \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) d\tau_1 \right|^2 r d\tau dr.$$

On the other hand $\widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r)$ is equal to

$$\sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} \widehat{c}_1(n_1, \tau_1) \widehat{c}_2(n_2, \tau - \tau_1) e_{n_1}(r) e_{n_2}(r).$$

Therefore, by invoking (2.2), we can write

$$\begin{aligned} & \left\| \int_{-\infty}^{\infty} \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) d\tau_1 \right\|_{L^2(\Theta)} \leq \int_{-\infty}^{\infty} \left\| \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) \right\|_{L^2(\Theta)} d\tau_1 \\ & \leq \int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| |\widehat{c}_2(n_2, \tau - \tau_1)| \|e_{n_1} e_{n_2}\|_{L^2(\Theta)} d\tau_1 \\ & \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon \int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| |\widehat{c}_2(n_2, \tau - \tau_1)| d\tau_1. \end{aligned}$$

Our aim is estimate the $L^2(\mathbb{R}_\tau)$ norm of the last expression. For this purpose, we will of course make use of the support properties of $\widehat{c}_j(n, \tau)$. Using the Cauchy-Schwarz inequality in (τ_1, n_1, n_2) gives the bound

$$\begin{aligned} & \int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| |\widehat{c}_2(n_2, \tau - \tau_1)| d\tau_1 \leq \\ & \leq \left(\int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)|^2 |\widehat{c}_2(n_2, \tau - \tau_1)|^2 d\tau_1 \right)^{\frac{1}{2}} \\ & \quad |\Lambda(\tau, L_1, L_2, N_1, N_2)|^{\frac{1}{2}}, \end{aligned}$$

where $\Lambda(\tau, L_1, L_2, N_1, N_2)$ is the following set of $\mathbb{R} \times \mathbb{N} \times \mathbb{N}$,

$$(4.2) \quad \Lambda(\tau, L_1, L_2, N_1, N_2) = \{(\tau_1, n_1, n_2) \in \mathbb{R} \times \mathbb{N} \times \mathbb{N} : L_1 \leq \langle \tau_1 + z_{n_1}^2 \rangle \leq 2L_1, \\ L_2 \leq \langle \tau - \tau_1 + z_{n_2}^2 \rangle \leq 2L_2, \langle z_{n_1} \rangle \in [N_1, 2N_1], \langle z_{n_2} \rangle \in [N_2, 2N_2]\}.$$

The next lemma contains the main combinatorial ingredient of our analysis.

LEMMA 4.2. *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for every $\tau \in \mathbb{R}$, every $N_1, N_2 \geq 1$, every $L_1, L_2 \geq 1$,*

$$|\Lambda(\tau, L_1, L_2, N_1, N_2)| \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon (L_1 L_2).$$

PROOF. Consider the set $\tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)$ of $\mathbb{N} \times \mathbb{N}$, defined by

$$\begin{aligned} \tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2) = \{ & (n_1, n_2) \in \mathbb{N} \times \mathbb{N} : \langle \tau + z_{n_1}^2 + z_{n_2}^2 \rangle \leq 2(L_1 + L_2), \\ & \langle z_{n_1} \rangle \in [N_1, 2N_1], \langle z_{n_2} \rangle \in [N_2, 2N_2]\}. \end{aligned}$$

Let $(\tau_1, n_1, n_2) \in \Lambda(\tau, L_1, L_2, N_1, N_2)$. Then the triangle inequality yields

$$\langle \tau_1 + z_{n_1}^2 + z_{n_2}^2 \rangle \leq \langle \tau_1 + z_{n_1}^2 \rangle + \langle \tau - \tau_1 + z_{n_2}^2 \rangle \leq 2(L_1 + L_2).$$

Therefore $(\tau_1, n_1, n_2) \in \Lambda(\tau, L_1, L_2, N_1, N_2)$ implies that $(n_1, n_2) \in \tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)$.

On the other hand for a fixed $(n_1, n_2) \in \tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)$ the Lebesgue measure of the possible τ_1 such that $(\tau_1, n_1, n_2) \in \Lambda(\tau, L_1, L_2, N_1, N_2)$ is bounded by $C(L_1 \wedge L_2)$. Therefore

$$(4.3) \quad |\Lambda(\tau, L_1, L_2, N_1, N_2)| \leq C(L_1 \wedge L_2) |\tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)|.$$

We next estimate $|\tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)|$. Observe that $z_{n_1}^2 + z_{n_2}^2$ ranges in an interval of size $\leq C(L_1 + L_2)$. Hence, thanks to (2.6) the expression $(4n_1 - 1)^2 + (4n_2 - 1)^2$ also ranges in an interval of size $C(L_1 + L_2)$, where the constant C is independent of τ, L_1, L_2, N_1, N_2 . Indeed, suppose that for some $A \in \mathbb{R}$,

$$(4.4) \quad A \leq z_{n_1}^2 + z_{n_2}^2 \leq A + C(L_1 + L_2).$$

In (4.4), A is the parameter we have no control on. Using (2.6), we obtain that (4.4) implies

$$(4.5) \quad \frac{16A}{\pi^2} \leq (4n_1 - 1)^2 + (4n_2 - 1)^2 + R(n_1, n_2) \leq \frac{16A + 16C(L_1 + L_2)}{\pi^2},$$

where, thanks to the estimate on the remainder in (2.6), the function $R(n_1, n_2)$ in (4.5) satisfies

$$|R(n_1, n_2)| \leq \tilde{C}.$$

Therefore, (4.5) implies

$$\frac{16A}{\pi^2} - \tilde{C} \leq (4n_1 - 1)^2 + (4n_2 - 1)^2 \leq \frac{16A + 16C(L_1 + L_2)}{\pi^2} + \tilde{C}.$$

Thus $(4n_1 - 1)^2 + (4n_2 - 1)^2$ ranges in an interval of size

$$\frac{16C(L_1 + L_2)}{\pi^2} + 2\tilde{C} \leq \left(\frac{16C}{\pi^2} + \tilde{C} \right) (L_1 + L_2)$$

exactly as we claimed. Denote the interval where $(4n_1 - 1)^2 + (4n_2 - 1)^2$ can range by Δ . Another appeal to (2.6) yields that the restrictions $\langle z_{n_1} \rangle \in [N_1, 2N_1]$ and $\langle z_{n_2} \rangle \in [N_2, 2N_2]$ imply the bounds

$$0 \leq n_1 \leq cN_1, \quad 0 \leq n_2 \leq cN_2.$$

Let l be an integer in the interval Δ . Then we have the bound

$$(4.6) \quad \#\left((n_1, n_2) \in \mathbb{N} \times \mathbb{N} : l = (4n_1 - 1)^2 + (4n_2 - 1)^2, \right. \\ \left. 0 \leq n_1 \leq cN_1, \quad 0 \leq n_2 \leq cN_2 \right) \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon$$

Indeed, if $l \leq 2c^2(N_1 \wedge N_2)^{2005}$ then the left hand-side of (4.6) is bounded by $C_\varepsilon (\min(N_1, N_2))^\varepsilon$ by the standard bound (see e.g. [8, 13]) on the number of divisors in the ring of Gaussian integers $\mathbb{Z}[i]$. Let us next suppose that

$$l \geq 2c^2(N_1 \wedge N_2)^{2005} + 1.$$

By symmetry, we can suppose that $N_2 \geq N_1$. We have that $4n_2 - 1 \in I$, where the interval I is defined by

$$I = [\sqrt{l - (4cN_1 - 1)^2}, \sqrt{l}].$$

But the size of I is bounded by

$$\frac{(4cN_1 - 1)^2}{\sqrt{l}} \leq \frac{CN_1^2}{\sqrt{c^2N_1^{2005} + 1}} \leq C.$$

Therefore the size of I is bounded by a constant uniform in N_1 , N_2 and l . Thus in the case $l \geq 2c_3^2(N_1 \wedge N_2)^{2005} + 1$, we can get even better than (4.6), namely we have a bound by a uniform constant. This completes the proof of (4.6). Using (4.6) we infer that

$$|\tilde{\Lambda}(\tau, L_1, L_2, N_1, N_2)| \leq C_\varepsilon |\Delta| (N_1 \wedge N_2)^\varepsilon = C_\varepsilon (L_1 + L_2) (N_1 \wedge N_2)^\varepsilon.$$

Observe that $(L_1 \wedge L_2)(L_1 + L_2) \leq 2(L_1 L_2)$. Therefore coming back to (4.3) completes the proof of Lemma 4.2. \square

Thanks to Lemma 4.2, we may write

$$\begin{aligned} \left\| \int_{-\infty}^{\infty} \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) d\tau_1 \right\|_{L^2(\Theta)} &\leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon (L_1 L_2)^{\frac{1}{2}} \\ &\left(\int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)|^2 |\widehat{c}_2(n_2, \tau - \tau_1)|^2 d\tau_1 \right)^{\frac{1}{2}}. \end{aligned}$$

Squaring the above inequality, integration over $\tau \in \mathbb{R}$ and using (4.1) gives the bound

$$(4.7) \quad \|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon (L_1 L_2)^{\frac{1}{2}} \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

We however claimed that the power of $L_1 L_2$ can be smaller than $1/2$. In order to obtain this better bound with respect to the L_1, L_2 localization, we will get an inequality which is better than (4.7) as far as $(L_1 L_2)$ is concerned but which is very weak with respect to the N_1, N_2 localization. Using the formula for the inverse of the Fourier transform, the support properties of the Fourier transform of $c_j(n, t)$, and the Cauchy-Schwarz inequality, we obtain the bound

$$|c_j(n, t)|^2 \leq CL_j \int_{-\infty}^{\infty} |\widehat{c}_j(n, \tau)|^2 d\tau, \quad j = 1, 2.$$

Hence, we infer that

$$\begin{aligned} \|u_j(t, \cdot)\|_{L^2(\Theta)}^2 &= \sum_{N_j \leq \langle z_n \rangle < 2N_j} |c_j(n, t)|^2 \\ &\leq CL_j \sum_{N_j \leq \langle z_n \rangle < 2N_j} \int_{-\infty}^{\infty} |\widehat{c}_j(n, \tau)|^2 d\tau \\ &= CL_j \|u_j\|_{L^2(\mathbb{R} \times \Theta)}^2. \end{aligned}$$

Therefore

$$(4.8) \quad \|u_j\|_{L^\infty(\mathbb{R}; L^2(\Theta))} \leq CL_j^{\frac{1}{2}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}.$$

Interpolation (it is in fact simply the Hölder inequality) with the equality

$$\|u_j\|_{L^2(\mathbb{R}; L^2(\Theta))} = \|u_j\|_{L^2(\mathbb{R} \times \Theta)}$$

gives the bound

$$(4.9) \quad \|u_j\|_{L^4(\mathbb{R}; L^2(\Theta))} \leq CL_j^{\frac{1}{4}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}, \quad j = 1, 2.$$

Recall that (2.6) implies that

$$\#\{n \in \mathbb{N} : N \leq \langle z_n \rangle < 2N\} \leq CN.$$

Therefore, using (2.1) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |u_j(t, r)| &\leq CN_j^{\frac{1}{2}} \sum_{N_j \leq \langle z_n \rangle < 2N_j} |c_j(n, t)| \\ &\leq CN_j^{\frac{1}{2}} N_j^{\frac{1}{2}} \left(\sum_{N_j \leq \langle z_n \rangle < 2N_j} |c_j(n, t)|^2 \right)^{\frac{1}{2}} \\ &\leq CN_j L_j^{\frac{1}{2}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

Thus

$$(4.10) \quad \|u_j\|_{L^\infty(\mathbb{R}; L^\infty(\Theta))} \leq CN_j L_j^{\frac{1}{2}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}.$$

Next, we can write

$$\begin{aligned} \|u_j(t, \cdot)\|_{L^\infty(\Theta)}^2 &\leq CN_j \left(\sum_{N_j \leq \langle z_n \rangle < 2N_j} |c_j(n, t)| \right)^2 \\ &\leq CN_j^2 \sum_{N_j \leq \langle z_n \rangle < 2N_j} |c_j(n, t)|^2 \\ &= CN_j^2 \|u_j(t, \cdot)\|_{L^2(\Theta)}^2. \end{aligned}$$

Integration of the last inequality over $t \in \mathbb{R}$ gives

$$(4.11) \quad \|u_j\|_{L^2(\mathbb{R}; L^\infty(\Theta))} \leq CN_j \|u_j\|_{L^2(\mathbb{R} \times \Theta)}.$$

Interpolation between (4.10) and (4.11) now gives

$$(4.12) \quad \|u_j\|_{L^4(\mathbb{R}; L^\infty(\Theta))} \leq CL_j^{\frac{1}{4}} N_j \|u_j\|_{L^2(\mathbb{R} \times \Theta)}.$$

Suppose that $N_1 \leq N_2$. Then using (4.9), (4.12) and the Hölder inequality, we obtain

$$\begin{aligned} \|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} &\leq \|u_1\|_{L^4(\mathbb{R}; L^\infty(\Theta))} \|u_2\|_{L^4(\mathbb{R}; L^2(\Theta))} \\ &\leq C(L_1 L_2)^{\frac{1}{4}} N_1 \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

Therefore, we arrive at

$$(4.13) \quad \|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C(L_1 L_2)^{\frac{1}{4}} (N_1 \wedge N_2) \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

Interpolation between (4.7) and (4.13) completes the proof of Proposition 4.1. \square

We will also need the following variant of Proposition 4.1.

PROPOSITION 4.3. *For every $\varepsilon > 0$, there exists $\beta < 1/2$, there exists $C > 0$ such that for every $N_1, N_2 \geq 1$, every $L_1, L_2 \geq 1$, every u_1, u_2 two functions on $\mathbb{R} \times \Theta$ of the form*

$$u_1(t, r) = \sum_{N_1 \leq \langle z_n \rangle < 2N_1} c_1(n, t) e_n(r)$$

and

$$u_2(t, r) = \sum_{N_2 \leq \langle z_n \rangle < 2N_2} c_2(n, t) e'_n(r)$$

where the Fourier transform of $c_j(n, t)$ with respect to t satisfies

$$\text{supp } \widehat{c}_j(n, \tau) \subset \{\tau \in \mathbb{R} : L_j \leq \langle \tau + z_n^2 \rangle \leq 2L_j\}, \quad j = 1, 2$$

one has the bound

$$\|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C(N_1 \wedge N_2)^\varepsilon (L_1 L_2)^\beta \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

PROOF. Recall that the function e_n satisfies the equation

$$r e_n''(r) + e_n'(r) = -z_n^2 r e_n(r).$$

Therefore, using that for $m \neq n$, e_m and e_n are orthogonal in $L^2(\Theta)$ and vanishing at $r = 1$, an integration by parts gives for $m \neq n$,

$$\begin{aligned} \int_0^1 e_m'(r) e_n'(r) r dr &= - \int_0^1 e_m(r) (e_n'(r) r)' dr + [e_m(r) e_n'(r) r]_0^1 \\ &= - \int_0^1 e_m(r) (e_n'(r) + r e_n''(r)) dr \\ &= z_n^2 \int_0^1 e_m(r) e_n(r) r dr = 0. \end{aligned}$$

Therefore, we obtain that

$$(4.14) \quad \|u_2\|_{L^2(\mathbb{R} \times \Theta)}^2 = c \sum_{N_2 \leq \langle z_n \rangle < 2N_2} \|e_n'\|_{L^2(\Theta)}^2 \int_{-\infty}^{\infty} |\widehat{c}_2(n, \tau)|^2 d\tau$$

and from now on the proof of Proposition 4.3 follows the lines of the proof of Proposition 4.1. Indeed, using (2.3), we can write

$$\begin{aligned} &\left\| \int_{-\infty}^{\infty} \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) d\tau_1 \right\|_{L^2(\Theta)} \leq \int_{-\infty}^{\infty} \|\widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r)\|_{L^2(\Theta)} d\tau_1 \\ &\leq \int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| |\widehat{c}_2(n_2, \tau - \tau_1)| \|e_{n_1} e'_{n_2}\|_{L^2(\Theta)} d\tau_1 \\ &\leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon \int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| \|e'_{n_2}\|_{L^2(\Theta)} |\widehat{c}_2(n_2, \tau - \tau_1)| d\tau_1. \end{aligned}$$

Again, our goal is to estimate the $L^2(\mathbb{R}_\tau)$ norm of the last expression. Using the Cauchy-Schwarz inequality in (τ_1, n_1, n_2) yields

$$\begin{aligned} &\int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)| \|e'_{n_2}\|_{L^2(\Theta)} |\widehat{c}_2(n_2, \tau - \tau_1)| d\tau_1 \leq \\ &\leq \left(\int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)|^2 \|e'_{n_2}\|_{L^2(\Theta)}^2 |\widehat{c}_2(n_2, \tau - \tau_1)|^2 d\tau_1 \right)^{\frac{1}{2}} \\ &\quad \times |\Lambda(\tau, L_1, L_2, N_1, N_2)|^{\frac{1}{2}}, \end{aligned}$$

where $\Lambda(\tau, L_1, L_2, N_1, N_2)$ is defined by (4.2). A use of Lemma 4.2 now gives

$$\begin{aligned} & \left\| \int_{-\infty}^{\infty} \widehat{u}_1(\tau_1, r) \widehat{u}_2(\tau - \tau_1, r) d\tau_1 \right\|_{L^2(\Theta)} \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon (L_1 L_2)^{\frac{1}{2}} \\ & \left(\int_{-\infty}^{\infty} \sum_{N_1 \leq \langle z_{n_1} \rangle < 2N_1} \sum_{N_2 \leq \langle z_{n_2} \rangle < 2N_2} |\widehat{c}_1(n_1, \tau_1)|^2 \|e'_{n_2}\|_{L^2(\Theta)}^2 |\widehat{c}_2(n_2, \tau - \tau_1)|^2 d\tau_1 \right)^{\frac{1}{2}}. \end{aligned}$$

and therefore

$$\|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C_\varepsilon (N_1 \wedge N_2)^\varepsilon (L_1 L_2)^{\frac{1}{2}} \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

Next, using the localisation of the Fourier transforms of $c_1(n, t)$, as in the proof of Proposition 4.1, we get the bound

$$\|u_1\|_{L^4(\mathbb{R}; L^2(\Theta))} \leq CL_1^{\frac{1}{4}} \|u_1\|_{L^2(\mathbb{R} \times \Theta)}.$$

Next, we estimate u_2 as follows

$$\begin{aligned} \|u_2(t, \cdot)\|_{L^2(\Theta)}^2 &= \sum_{N_2 \leq \langle z_n \rangle < 2N_2} |c_2(n, t)|^2 \|e'_n\|_{L^2(\Theta)}^2 \\ &\leq C \sum_{N_2 \leq \langle z_n \rangle < 2N_2} \left(L_2 \int_{-\infty}^{\infty} |\widehat{c}_2(n, \tau)|^2 d\tau \right) \|e'_n\|_{L^2(\Theta)}^2 \\ &= CL_2 \|u_2\|_{L^2(\mathbb{R} \times \Theta)}^2. \end{aligned}$$

Therefore

$$\|u_2\|_{L^\infty(\mathbb{R}; L^2(\Theta))} \leq CL_2^{\frac{1}{2}} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

Interpolating with the equality

$$\|u_2\|_{L^2(\mathbb{R}; L^2(\Theta))} = \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

gives

$$\|u_2\|_{L^4(\mathbb{R}; L^2(\Theta))} \leq CL_2^{\frac{1}{4}} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

Thus

$$\|u_j\|_{L^4(\mathbb{R}; L^2(\Theta))} \leq CL_j^{\frac{1}{4}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}, \quad j = 1, 2.$$

Next, using (2.1), we get the bound

$$(4.15) \quad \|u_j\|_{L^\infty(\mathbb{R}; L^\infty(\Theta))} \leq CN_j L_j^{\frac{1}{2}} \|u_j\|_{L^2(\mathbb{R} \times \Theta)}, \quad j = 1, 2.$$

Indeed, for $j = 1$ such an inequality is already proved in Proposition 4.1. For $j = 2$, we can write by using (2.1),

$$\begin{aligned} |u_2(t, r)| &\leq CN_2^{\frac{1}{2}} \sum_{N_2 \leq \langle z_n \rangle < 2N_2} |c_2(n, t)| \|e'_n\|_{L^2(\Theta)} \\ &\leq CN_2 \left(\sum_{N_2 \leq \langle z_n \rangle < 2N_2} |c_2(n, t)|^2 \|e'_n\|_{L^2(\Theta)}^2 \right) \\ &\leq CN_2 L_2^{\frac{1}{2}} \|u_2\|_{L^2(\mathbb{R} \times \Theta)} \end{aligned}$$

and thus (4.15) for $j = 2$. Moreover,

$$(4.16) \quad \|u_j\|_{L^2(\mathbb{R}; L^\infty(\Theta))} \leq CN_j \|u_j\|_{L^2(\mathbb{R} \times \Theta)}, \quad j = 1, 2.$$

Indeed, for $j = 1$ it is already proved in Proposition 4.1. For $j = 2$, by invoking once again (2.1), and the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \|u_2(t, \cdot)\|_{L^\infty(\Theta)}^2 &\leq CN_2 \left(\sum_{N_2 \leq \langle z_n \rangle < 2N_2} |c_2(n, t)| \|e'_n\|_{L^2(\Theta)} \right)^2 \\ &\leq CN_2^2 \sum_{N_2 \leq \langle z_n \rangle < 2N_2} |c_2(n, t)|^2 \|e'_n\|_{L^2(\Theta)}^2. \end{aligned}$$

Integration of the last inequality over t gives (4.16) for $j = 2$. An interpolation gives

$$\|u_j\|_{L^4(\mathbb{R}; L^\infty(\Theta))} \leq CL_j^{\frac{1}{4}} N_j \|u_j\|_{L^2(\mathbb{R} \times \Theta)}, \quad j = 1, 2.$$

Then the Hölder inequality gives

$$\|u_1 u_2\|_{L^2(\mathbb{R} \times \Theta)} \leq C(L_1 L_2)^{\frac{1}{4}} (N_1 \wedge N_2) \|u_1\|_{L^2(\mathbb{R} \times \Theta)} \|u_2\|_{L^2(\mathbb{R} \times \Theta)}.$$

A final interpolation completes the proof of Proposition 4.3. \square

5. Bourgain spaces

We denote by $L_{rad}^2(\Theta)$ the L^2 radial functions on the unit disc. We endow $L_{rad}^2(\Theta)$ with the natural Hilbert space structure. Similarly, we denote by $L_{rad}^2(\mathbb{R} \times \Theta)$ the L^2 functions on $\mathbb{R} \times \Theta$, radial with respect to the second argument. For $\sigma < 1/2$, the norm in $H_{rad}^\sigma(\Theta)$ of a radial function

$$v = \sum_{n \geq 1} c_n e_n$$

can be expressed as

$$\|v\|_{H_{rad}^\sigma(\Theta)}^2 = \sum_{n \geq 1} z_n^{2\sigma} |c_n|^2.$$

In this paper, we will only consider spaces of Sobolev regularity $< 1/2$ and thus there is no need to specify the boundary conditions on $\partial\Theta$ (in our context it simply means $r = 1$). More precisely the choice of the Dirichlet eigenfunctions e_n as basis of $L_{rad}^2(\Theta)$ is not of importance in the definition of $H_{rad}^\sigma(\Theta)$, $\sigma < 1/2$.

Next, we define the Bourgain spaces $X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)$ of functions on $\mathbb{R} \times \Theta$ which are radial with respect to the second argument. These spaces are equipped with the norm

$$\|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}^2 = \sum_{n \geq 1} z_n^{2\sigma} \|\langle \tau + z_n^2 \rangle^b \widehat{c}_n(\tau)\|_{L^2(\mathbb{R}_\tau)}^2,$$

where

$$u(t) = \sum_{n \geq 1} c_n(t) e_n.$$

Notice that

$$(5.1) \quad \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)} = \|\exp(-it\Delta)(u(t))\|_{H^b(\mathbb{R}; H_{rad}^\sigma(\Theta))}.$$

Indeed, using that

$$\exp(-it\Delta)(u(t)) = \sum_{n \geq 1} \exp(itz_n^2) c_n(t) e_n$$

and since

$$\exp(itz_n^2) c_n(t) = \widehat{c}_n(\tau - z_n^2),$$

we arrive at

$$\begin{aligned}
 \|\exp(-it\Delta)(u(t))\|_{H^b(\mathbb{R}; H_{rad}^\sigma(\Theta))}^2 &= \sum_{n \geq 1} z_n^{2\sigma} \|\exp(itz_n^2)c_n(t)\|_{H^b(\mathbb{R})}^2 \\
 &= \sum_{n \geq 1} z_n^{2\sigma} \|\langle \tau \rangle^b \widehat{c}_n(\tau - z_n^2)\|_{L^2(\mathbb{R})}^2 \\
 &= \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}^2.
 \end{aligned}$$

This proves (5.1). Clearly $X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)$ have a Hilbert space structure and for $0 \leq \sigma < 1/2$ we can see $X_{rad}^{-\sigma,-b}(\mathbb{R} \times \Theta)$ as its dual via the $L^2(\mathbb{R} \times \Theta)$ pairing. A one dimensional Sobolev embedding (for functions with values in $H_{rad}^\sigma(\Theta)$) yields the estimate

$$(5.2) \quad \|u\|_{L^\infty(\mathbb{R}; H_{rad}^\sigma(\Theta))} \leq C_b \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}, \quad b > \frac{1}{2}.$$

Next, for $T > 0$, we define the restriction spaces $X_{rad}^{\sigma,b}([-T, T] \times \Theta)$, equipped with the natural norm

$$\|u\|_{X_{rad}^{\sigma,b}([-T, T] \times \Theta)} = \inf\{\|w\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}, \quad w \in X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta) \quad \text{with} \quad w|_{[-T, T] \times \Theta} = u\}.$$

Therefore (5.2) yields

$$\|u\|_{L^\infty([-T, T]; H_{rad}^\sigma(\Theta))} \leq C_b \|u\|_{X_{rad}^{\sigma,b}([-T, T] \times \Theta)}, \quad b > \frac{1}{2}$$

which implies that for $b > 1/2$ the space $X_{rad}^{\sigma,b}([-T, T] \times \Theta)$ is continuously embedded in $C([-T, T]; H_{rad}^\sigma(\Theta))$. Similarly, for $I \subset \mathbb{R}$ an interval, we can define the restriction spaces $X_{rad}^{\sigma,b}(I \times \Theta)$, equipped with the natural norm.

Following [5], our next purpose is to express the norm in $X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)$ in terms of some basic localisation operators. Recall that for $u = \sum_{n \geq 1} c_n e_n$, the projector Δ_N is defined by

$$\Delta_N(u) = \sum_{n: N \leq \langle z_n \rangle < 2N} c_n e_n.$$

For $N \geq 2$ a dyadic integer, we define the projector \tilde{S}_N by

$$\tilde{S}_N = \sum_{\substack{N_1 \leq N/2 \\ N_1 \text{-dyadic}}} \Delta_{N_1}.$$

For a notational convenience, we assume that \tilde{S}_1 is zero. Notice that \tilde{S}_N is essentially equivalent to S_N , where the projector S_N is defined in (3.12). For N, L positive integers, we define $\Delta_{N,L}$ by

$$(5.3) \quad \Delta_{N,L}(u) = \frac{1}{2\pi} \sum_{n: N \leq \langle z_n \rangle < 2N} \left(\int_{L \leq \langle \tau + z_n^2 \rangle \leq 2L} \widehat{c}_n(\tau) e^{it\tau} d\tau \right) e_n,$$

where

$$u(t) = \sum_{n \geq 1} c_n(t) e_n.$$

Then for $u \in X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)$ (with the natural interpretation of the τ integration in (5.3) if $b < 0$), we can write the identity

$$u = \sum_{L,N\text{-dyadic}} \Delta_{N,L}(u)$$

in $X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)$. Next, we have that there exists a constant $C_{\sigma,b} > 1$ which depends continuously on σ and b such that

$$\begin{aligned} C_{\sigma,b}^{-1} L^b N^\sigma \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R} \times \Theta)} &\leq \|\Delta_{N,L}(u)\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)} \\ &\leq C_{\sigma,b} L^b N^\sigma \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R} \times \Theta)} \end{aligned}$$

and

$$\begin{aligned} C_{\sigma,b}^{-1} \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}^2 &\leq \sum_{L,N\text{-dyadic}} L^{2b} N^{2\sigma} \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R} \times \Theta)}^2 \\ (5.4) \qquad \qquad \qquad &\leq C_{\sigma,b} \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}^2. \end{aligned}$$

Moreover there exists $C_b > 1$, a continuous function of b such that

$$\begin{aligned} C_b^{-1} \|\Delta_N(u)\|_{X_{rad}^{0,b}(\mathbb{R} \times \Theta)}^2 &\leq \sum_{L\text{-dyadic}} L^{2b} \|\Delta_{N,L}(u)\|_{L^2(\mathbb{R} \times \Theta)}^2 \\ (5.5) \qquad \qquad \qquad &\leq C_b \|\Delta_N(u)\|_{X_{rad}^{0,b}(\mathbb{R} \times \Theta)}^2 \end{aligned}$$

and there exists $C_\sigma > 1$, a continuous function of σ such that

$$\begin{aligned} C_\sigma^{-1} \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}^2 &\leq \sum_{N\text{-dyadic}} N^{2\sigma} \|\Delta_N(u)\|_{X_{rad}^{0,b}(\mathbb{R} \times \Theta)}^2 \\ (5.6) \qquad \qquad \qquad &\leq C_\sigma \|u\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)}^2. \end{aligned}$$

Proposition 4.1 now has a natural formulation in terms of the basic localization projectors.

PROPOSITION 5.1. *For every $\varepsilon > 0$, there exist $\beta < 1/2$ and $C > 0$ such that for every $N_1, N_2, L_1, L_2 \geq 1$, every $u_1, u_2 \in L_{rad}^2(\mathbb{R} \times \Theta)$,*

$$\begin{aligned} \|\Delta_{N_1, L_1}(u_1) \Delta_{N_2, L_2}(u_2)\|_{L^2(\mathbb{R} \times \Theta)} &\leq \\ &\leq C (L_1 L_2)^\beta \min(N_1, N_2)^\varepsilon \|\Delta_{N_1, L_1}(u_1)\|_{L^2(\mathbb{R} \times \Theta)} \|\Delta_{N_2, L_2}(u_2)\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

PROOF. It suffices to observe that $\Delta_{N_j, L_j}(u)$, $j = 1, 2$ satisfy the localisation properties needed to apply Proposition 4.1. \square

The following global linear Strichartz estimates is a direct consequence of Proposition 5.1.

PROPOSITION 5.2. *Let us fix $\varepsilon \in]0, 1/2[$ and $b > 1/2$. Then there exists $C > 0$ such that for every $u \in X_{rad}^{\varepsilon,b}(\mathbb{R} \times \Theta)$,*

$$\|u\|_{L^4(\mathbb{R} \times \Theta)} \leq C \|u\|_{X_{rad}^{\varepsilon,b}(\mathbb{R} \times \Theta)}.$$

PROOF. It suffices to apply Proposition 5.1 with $L_1 = L_2$, $N_1 = N_2$ and sum up over the dyadic values of these parameters by applying the Cauchy-Schwarz inequality and (5.4). \square

6. Nonlinear estimates

The next statement contains the main analytical ingredient in the proof of Theorem 1.

PROPOSITION 6.1. *Let $0 < \sigma_1 \leq \sigma < 1/2$. Then there exist two positive numbers b, b' such that $b + b' < 1$, $b' < 1/2 < b$, there exists $C > 0$ such that for every $u, v \in X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)$,*

$$(6.1) \quad \|F(u)\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)} \leq C \left(1 + \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \right) \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}$$

and

$$(6.2) \quad \|F(u) - F(v)\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)} \leq C \left(1 + \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}^2 + \|v\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}^2 \right) \|u - v\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}.$$

PROOF. Using the gauge invariance of the nonlinearity $F(u)$, we observe that $F(u) - (\partial F)(0)u$ is vanishing at order 3 at $u = 0$. It therefore suffices to prove that

$$(6.3) \quad \|F(u)\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)} \leq C \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)},$$

under the additional assumption that $F(u)$ is vanishing at order 3 at $u = 0$. Indeed, by writing

$$\|F(u) - (\partial F)(0)u\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)} \geq \|F(u)\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)} - C \|u\|_{X_{rad}^{\sigma, -b'}(\mathbb{R} \times \Theta)},$$

we deduce that the claimed estimate (6.1) follows from (6.3) applied to $F(u) - (\partial F)(0)u$. By duality, in order to prove (6.3), it suffices to establish the bound

$$(6.4) \quad \left| \int_{\mathbb{R} \times \Theta} F(u) \bar{v} \right| \leq C \|v\|_{X_{rad}^{-\sigma, b'}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}.$$

Next, we have the decomposition

$$v = \sum_{N_0\text{-dyadic}} \Delta_{N_0}(v).$$

Moreover, using Proposition 5.2 and the sub-cubic nature of the nonlinearity, we infer that for $u \in X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)$, $\sigma \in]0, 1/2[$, $b > 1/2$, one has $F(u) \in X_{rad}^{-\sigma, -b}(\mathbb{R} \times \Theta)$. In addition, we can write

$$F(u) = \sum_{N_1\text{-dyadic}} \left(F(\tilde{S}_{2N_1}(u)) - F(\tilde{S}_{N_1}(u)) \right)$$

in $X_{rad}^{-\sigma, -b}(\mathbb{R} \times \Theta)$, with the convention that $\tilde{S}_1(u) = 0$. Since $\Delta_N = \tilde{S}_{2N} - \tilde{S}_N$, we can therefore write

$$F(u) = \sum_{N_1\text{-dyadic}} \Delta_{N_1}(u) G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) + \sum_{N_1\text{-dyadic}} \overline{\Delta_{N_1}(u)} G_2(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \equiv F_1(u) + F_2(u),$$

where $G_1(z_1, z_2)$ and $G_2(z_1, z_2)$ are smooth functions with a control on their growth at infinity coming from (1.2). We are going only to show that

$$\left| \int_{\mathbb{R} \times \Theta} F_1(u) \bar{v} \right| \leq C \|v\|_{X_{rad}^{-\sigma, b'}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}$$

since the argument for

$$\left| \int_{\mathbb{R} \times \Theta} F_2(u) \bar{v} \right|$$

is completely analogous. We can write

$$F_1(u) = \sum_{N_1\text{-dyadic}} \Delta_{N_1}(u) G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)).$$

Next, we set

$$I = \left| \int_{\mathbb{R} \times \Theta} F_1(u) \bar{v} \right|$$

and

$$I(N_0, N_1) = \left| \int_{\mathbb{R} \times \Theta} \Delta_{N_1}(u) \overline{\Delta_{N_0}(v)} G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \right|.$$

Then $I \leq I_1 + I_2$, where

$$I_1 = \sum_{\substack{N_0 \leq N_1 \\ N_0, N_1\text{-dyadic}}} I(N_0, N_1), \quad I_2 = \sum_{\substack{N_0 \geq N_1 \\ N_0, N_1\text{-dyadic}}} I(N_0, N_1).$$

We estimate first I_1 . Similarly to the above expansion for F , using the vanishing property at the origin of F , we now decompose $G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u))$ as follows,

$$\sum_{N_2\text{-dyadic}} \left(G_1(\tilde{S}_{2N_2} \Delta_{N_1}(u), \tilde{S}_{2N_2} \tilde{S}_{N_1}(u)) - G_1(\tilde{S}_{N_2} \Delta_{N_1}(u), \tilde{S}_{N_2} \tilde{S}_{N_1}(u)) \right).$$

Therefore, using that $\Delta_{N_1} \Delta_{N_2} = \Delta_{N_1}$, if $N_1 = N_2$ and zero elsewhere, we obtain that

$$G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) = \sum_{\substack{N_2 \leq N_1 \\ N_2\text{-dyadic}}} \Delta_{N_2}(u) G_{11}^{N_2}(\Delta_{N_2}(u), \tilde{S}_{N_2}(u)) + \sum_{\substack{N_2 \leq N_1 \\ N_2\text{-dyadic}}} \overline{\Delta_{N_2}(u)} G_{12}^{N_2}(\Delta_{N_2}(u), \tilde{S}_{N_2}(u)).$$

Finally, we expand for $j = 1, 2$,

$$G_{1j}^{N_2}(\Delta_{N_2}(u), \tilde{S}_{N_2}(u)) = \sum_{\substack{N_3 \leq N_2 \\ N_3\text{-dyadic}}} \Delta_{N_3}(u) G_{1j1}^{N_3}(\Delta_{N_3}(u), \tilde{S}_{N_3}(u)) + \sum_{\substack{N_3 \leq N_2 \\ N_3\text{-dyadic}}} \overline{\Delta_{N_3}(u)} G_{1j2}^{N_3}(\Delta_{N_3}(u), \tilde{S}_{N_3}(u)),$$

where, thanks to the growth assumption on the nonlinearity $F(u)$, we obtain that the functions $G_{1j_1 j_2}^{N_3}(z_1, z_2)$, $j_1, j_2 = 1, 2$ satisfy

$$|G_{1j_1 j_2}^{N_3}(z_1, z_2)| \leq C.$$

We therefore have the bound

$$I_1 \leq C \sum_{\substack{N_0 \leq N_1 \\ N_0, N_1\text{-dyadic}}} \sum_{\substack{N_1 \geq N_2 \geq N_3 \\ N_2, N_3\text{-dyadic}}} \int_{\mathbb{R} \times \Theta} |\Delta_{N_0}(v) \Delta_{N_1}(u) \Delta_{N_2}(u) \Delta_{N_3}(u)|$$

and moreover using the equality

$$\Delta_N = \sum_{L\text{-dyadic}} \Delta_{N,L},$$

we arrive at

$$I_1 \leq C \sum_{L_0, L_1, L_2, L_3\text{-dyadic}} \sum_{\substack{N_1 \geq N_2 \geq N_3, N_1 \geq N_0 \\ N_0, N_1, N_2, N_3\text{-dyadic}}} \int_{\mathbb{R} \times \Theta} |\Delta_{N_0, L_0}(v) \Delta_{N_1, L_1}(u) \Delta_{N_2, L_2}(u) \Delta_{N_3, L_3}(u)|.$$

Using Proposition 5.1 and the Cauchy-Schwarz inequality, we have that for every $\varepsilon > 0$ there exist $\beta < 1/2$ and C_ε such that

$$\begin{aligned} \int_{\mathbb{R} \times \Theta} |\Delta_{N_0, L_0}(v) \Delta_{N_1, L_1}(u) \Delta_{N_2, L_2}(u) \Delta_{N_3, L_3}(u)| &\leq \\ &\leq \|\Delta_{N_0, L_0}(v) \Delta_{N_2, L_2}(u)\|_{L^2(\mathbb{R} \times \Theta)} \|\Delta_{N_1, L_1}(u) \Delta_{N_3, L_3}(u)\|_{L^2(\mathbb{R} \times \Theta)} \leq \\ &\leq C_\varepsilon (N_2 N_3)^\varepsilon (L_0 L_1 L_2 L_3)^\beta \|\Delta_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \prod_{j=1}^3 \|\Delta_{N_j, L_j}(u)\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

Therefore, if we set

$$(6.5) \quad Q \equiv Q(N_0, N_1, N_2, N_3, L_0, L_1, L_2, L_3) = C N_0^{-\sigma} N_1^\sigma (N_2 N_3)^{\sigma_1} L_0^{b'} (L_1 L_2 L_3)^b \\ \times \|\Delta_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \prod_{j=1}^3 \|\Delta_{N_j, L_j}(u)\|_{L^2(\mathbb{R} \times \Theta)},$$

we can write

$$I_1 \leq \sum_{L_0, L_1, L_2, L_3\text{-dyadic}} \sum_{\substack{N_1 \geq N_2 \geq N_3, N_1 \geq N_0 \\ N_0, N_1, N_2, N_3\text{-dyadic}}} L_0^{\beta-b'} (L_1 L_2 L_3)^{\beta-b} \left(\frac{N_0}{N_1}\right)^\sigma (N_2 N_3)^{\varepsilon-\sigma_1} Q$$

Let us take $\varepsilon > 0$ such that $\varepsilon - \sigma_1 < 0$. This fixes β . Then we choose b' such that $\beta < b' < 1/2$. We finally choose $b > 1/2$ such that $b + b' < 1$. With this choice of the parameters, using (5.5) and after summing geometric series in $L_0, L_1, L_2, L_3, N_2, N_3$, we can write that

$$I_1 \leq C \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \sum_{\substack{N_0 \leq N_1 \\ N_0, N_1\text{-dyadic}}} \left(\frac{N_0}{N_1}\right)^\sigma c(N_0) d(N_1),$$

where

$$(6.6) \quad c(N_0) = N_0^{-\sigma} \|\Delta_{N_0}(v)\|_{X_{rad}^{0, b'}(\mathbb{R} \times \Theta)}, \quad d(N_1) = N_1^\sigma \|\Delta_{N_1}(u)\|_{X_{rad}^{0, b}(\mathbb{R} \times \Theta)}.$$

We now make appeal to the following lemma which is a discrete variant of the Schur test.

LEMMA 6.2. *For every $\sigma > 0$ there exists $C > 0$ such that for every couple of functions $c_j(N)$, $j = 1, 2$, defined on the set of the dyadic integers such that*

$$\|c_j\| \equiv \sum_{N\text{-dyadic}} |c_j(N)|^2 < \infty, \quad j = 1, 2$$

one has

$$(6.7) \quad \left| \sum_{\substack{N_0 \leq N_1 \\ N_0, N_1 \text{-dyadic}}} \left(\frac{N_0}{N_1} \right)^\sigma c_1(N_0) c_2(N_1) \right| \leq C \|c_1\| \|c_2\|.$$

PROOF. Write $N_1 = 2^j N_0$ with $j \geq 0$, $j \in \mathbb{Z}$. Thus the left hand-side of (6.7) can be rewritten as

$$\left| \sum_{j=0}^{\infty} \sum_{N_0 \text{-dyadic}} 2^{-j\sigma} c_1(N_0) c_2(2^j N_0) \right|$$

which by the Cauchy-Schwartz inequality in N_0 is bounded by $C \|c_1\| \|c_2\|$ with

$$C = \sum_{j=0}^{\infty} 2^{-j\sigma}.$$

This completes the proof of Lemma 6.2. \square

Next using (5.6) and Lemma 6.2, we deduce that

$$I_1 \leq C \|v\|_{X_{rad}^{-\sigma, b'}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}.$$

This ends the analysis for I_1 . We next turn to the estimate for I_2 . The basic idea is that after an integration by parts, the structure of I_2 becomes very close to the structure of I_1 , by simply exchanging the roles of N_0 and N_1 . In this context the Proposition 4.3 gives the relevant bound. We start by some preliminary observations. For $u \in L_{rad}^2(\mathbb{R} \times \Theta)$ we can write

$$\Delta_{N, L}(u) = \sum_{N \leq \langle z_n \rangle < 2N} c(n, t) e_n(r),$$

where

$$\text{supp } \widehat{c}(n, \tau) \subset \{\tau \in \mathbb{R} : L \leq \langle \tau + z_n^2 \rangle \leq 2L\}$$

and

$$\|\Delta_{N, L}(u)\|_{L^2(\mathbb{R} \times \Theta)}^2 = c \sum_{N \leq \langle z_n \rangle < 2N} \int_{-\infty}^{\infty} |\widehat{c}(n, \tau)|^2 d\tau.$$

Moreover

$$\partial_r (\Delta_{N, L}(u)) = \sum_{N \leq \langle z_n \rangle < 2N} c(n, t) e'_n(r).$$

Recall that for $m \neq n$, e'_m and e'_n are orthogonal in $L^2(\Theta)$. Moreover, thanks to (2.4),

$$\|e'_n\|_{L^2(\Theta)} \approx n \|e_n\|_{L^2(\Theta)}$$

and thus using that

$$\|\partial_r (\Delta_{N, L}(u))\|_{L^2(\mathbb{R} \times \Theta)}^2 = c \sum_{N \leq \langle z_n \rangle < 2N} \|e'_n\|_{L^2(\Theta)}^2 \int_{-\infty}^{\infty} |\widehat{c}(n, \tau)|^2 d\tau$$

we arrive at the crucial relation

$$(6.8) \quad \|\partial_r (\Delta_{N, L}(u))\|_{L^2(\mathbb{R} \times \Theta)} \approx N \|\Delta_{N, L}(u)\|_{L^2(\mathbb{R} \times \Theta)}.$$

Let us observe that

$$e_n(r) = -\frac{1}{z_n^2} \frac{1}{r} \partial_r (r \partial_r e_n(r)).$$

Since $\Delta_{N_1}(u)G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u))$ is vanishing on the boundary of Θ , an integration by parts yields

$$\begin{aligned} \int_0^1 e_n(r)\Delta_{N_1}(u)G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u))rdr &= \\ &= \frac{1}{z_n^2} \int_0^1 e'_n(r)\partial_r \left(\Delta_{N_1}(u)G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \right) r dr. \end{aligned}$$

Write

$$(6.9) \quad \Delta_{N_0, L_0}(v) = \sum_{N_0 \leq \langle z_{n_0} \rangle < 2N_0} c(n_0, t) e_{n_0}(r),$$

where

$$\text{supp } \hat{c}(n_0, \tau) \subset \{ \tau \in \mathbb{R} : L_0 \leq \langle \tau + z_{n_0}^2 \rangle \leq 2L_0 \}.$$

Then, for $n \in \mathbb{N}$ such that $\langle z_n \rangle \in [N_0, 2N_0[$, we set

$$\tilde{c}(n, t) = \frac{c(n, t)}{z_n^2},$$

where $c(n, t)$ are the coefficients involved in (6.9). Define $\tilde{\Delta}_{N_0, L_0}$ as

$$\tilde{\Delta}_{N_0, L_0}(v) = \sum_{N_0 \leq \langle z_{n_0} \rangle \leq 2N_0} \tilde{c}(n_0, t) e'_{n_0}(r).$$

Clearly $\tilde{\Delta}_{N_0, L_0}(v)$ is an object which fits in the scope of applicability of Proposition 4.3 and

$$(6.10) \quad \|\tilde{\Delta}_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \approx N_0^{-1} \|\Delta_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)}.$$

Recall that $e_n(r)$ are real valued. In view of the above discussion, we need to control the expression

$$E = \sum_{\substack{N_0 \geq N_1 \\ L_0, N_0, N_1 \text{-dyadic}}} \left| \int_{\mathbb{R} \times \Theta} \overline{\tilde{\Delta}_{N_0, L_0}(v)} \partial_r \left(\Delta_{N_1}(u)G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \right) \right|.$$

Now, we can write $E \leq E_1 + E_2$, where

$$E_1 = \sum_{\substack{N_0 \geq N_1 \\ L_0, N_0, N_1 \text{-dyadic}}} \left| \int_{\mathbb{R} \times \Theta} \overline{\tilde{\Delta}_{N_0, L_0}(v)} \partial_r \left(\Delta_{N_1}(u) \right) G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \right|$$

and

$$E_2 = \sum_{\substack{N_0 \geq N_1 \\ L_0, N_0, N_1 \text{-dyadic}}} \left| \int_{\mathbb{R} \times \Theta} \overline{\tilde{\Delta}_{N_0, L_0}(v)} \Delta_{N_1}(u) \partial_r \left(G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) \right) \right|.$$

By expanding $G_1(z_1, z_2)$ and using the growth and vanishing assumptions on the nonlinear interaction F , we can write

$$\begin{aligned} E_1 \leq C & \sum_{L_0, L_1, L_2, L_3 \text{-dyadic}} \\ & \sum_{\substack{N_0 \geq N_1 \geq N_2 \geq N_3 \\ N_0, N_1, N_2, N_3 \text{-dyadic}}} \int_{\mathbb{R} \times \Theta} |\tilde{\Delta}_{N_0, L_0}(v) \partial_r \left(\Delta_{N_1, L_1}(u) \right) \Delta_{N_2, L_2}(u) \Delta_{N_3, L_3}(u)|. \end{aligned}$$

Using Proposition 5.1, Proposition 4.3, the Cauchy-Schwarz inequality, (6.8) and (6.10), we have that for every $\varepsilon > 0$ there exist $\beta < 1/2$ and C_ε such that

$$\begin{aligned} & \int_{\mathbb{R} \times \Theta} |\tilde{\Delta}_{N_0, L_0}(v) \partial_r(\Delta_{N_1, L_1}(u)) \Delta_{N_2, L_2}(u) \Delta_{N_3, L_3}(u)| \leq \\ & \leq \|\tilde{\Delta}_{N_0, L_0}(v) \Delta_{N_2, L_2}(u)\|_{L^2(\mathbb{R} \times \Theta)} \|\partial_r(\Delta_{N_1, L_1}(u)) \Delta_{N_3, L_3}(u)\|_{L^2(\mathbb{R} \times \Theta)} \leq \\ & \leq C_\varepsilon (N_2 N_3)^\varepsilon (L_0 L_1 L_2 L_3)^\beta \|\tilde{\Delta}_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \\ & \quad \times \|\partial_r(\Delta_{N_1, L_1}(u))\|_{L^2(\mathbb{R} \times \Theta)} \prod_{j=2}^3 \|\Delta_{N_j, L_j}(u)\|_{L^2(\mathbb{R} \times \Theta)} \leq \\ & \leq C_\varepsilon (N_2 N_3)^\varepsilon (L_0 L_1 L_2 L_3)^\beta \frac{N_1}{N_0} \|\Delta_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \prod_{j=1}^3 \|\Delta_{N_j, L_j}(u)\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

Therefore, with Q defined by (6.5), we can write

$$(6.11) \quad E_1 \leq \sum_{L_0, L_1, L_2, L_3 \text{-dyadic}} \sum_{\substack{N_0 \geq N_1 \geq N_2 \geq N_3 \\ N_0, N_1, N_2, N_3 \text{-dyadic}}} L_0^{\beta-b'} (L_1 L_2 L_3)^{\beta-b} \left(\frac{N_0}{N_1}\right)^{\sigma-1} (N_2 N_3)^{\varepsilon-\sigma_1} Q$$

Let us take $\varepsilon > 0$ such that $\varepsilon - \sigma_1 < 0$. Then as we did for the bound for I_1 , we choose b' such that $\beta < b' < 1/2$. We finally choose $b > 1/2$ such that $b + b' < 1$. Using (5.5) and after summing geometric series in $L_0, L_1, L_2, L_3, N_2, N_3$, we can write that

$$E_1 \leq C \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \sum_{\substack{N_0 \geq N_1 \\ N_0, N_1 \text{-dyadic}}} \left(\frac{N_1}{N_0}\right)^{1-\sigma} c(N_0) d(N_1),$$

where $c(N_0)$ and $d(N_1)$ are defined by (6.6). Therefore, using (5.6) and Lemma 6.2, we arrive at the bound

$$E_1 \leq C \|v\|_{X_{rad}^{-\sigma, b'}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}.$$

Let us now turn to the bound for E_2 . Using the formula

$$\partial_r(f(z(r))) = (\partial_r z) \partial f + (\partial_r \bar{z}) \bar{\partial} f,$$

we can write

$$\begin{aligned} \partial_r(G_1(\Delta_{N_1}(u), \tilde{S}_{N_1}(u))) &= \sum_{\substack{N_2 \leq N_1 \\ N_2 \text{-dyadic}}} \partial_r(\Delta_{N_2}(u)) \tilde{G}_{11}^{N_2}(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)) + \\ & \quad \sum_{\substack{N_2 \leq N_1 \\ N_2 \text{-dyadic}}} \overline{\partial_r(\Delta_{N_2}(u))} \tilde{G}_{12}^{N_2}(\Delta_{N_1}(u), \tilde{S}_{N_1}(u)), \end{aligned}$$

where thanks to the growth assumption on the nonlinearity, $\tilde{G}_{1j}^{N_2}(z_1, z_2)$, $j = 1, 2$ satisfy

$$(6.12) \quad \sum_{k=1}^2 \left(|\partial_{z_k} \tilde{G}_{1j}^{N_2}(z_1, z_2)| + |\bar{\partial}_{z_k} \tilde{G}_{1j}^{N_2}(z_1, z_2)| \right) \leq C.$$

By expanding $\tilde{G}_{1j}^{N_2}(z_1, z_2)$, $j = 1, 2$ in a telescopic series and using (6.12), we get the bound

$$E_2 \leq C \sum_{L_0, L_1, L_2, L_3\text{-dyadic}} \sum_{\substack{N_0 \geq N_1 \geq N_2, N_1 \geq N_3 \\ N_0, N_1, N_2, N_3\text{-dyadic}}} \int_{\mathbb{R} \times \Theta} |\tilde{\Delta}_{N_0, L_0}(v) \Delta_{N_1, L_1}(u) \partial_r (\Delta_{N_2, L_2}(u)) \Delta_{N_3, L_3}(u)|.$$

Using Proposition 5.1, Proposition 4.3, the Cauchy-Schwarz inequality, (6.8) and (6.10), we have that for every $\varepsilon > 0$ there exist $\beta < 1/2$ and C_ε such that

$$\begin{aligned} & \int_{\mathbb{R} \times \Theta} |\tilde{\Delta}_{N_0, L_0}(v) \partial_r (\Delta_{N_1, L_1}(u)) \Delta_{N_2, L_2}(u) \Delta_{N_3, L_3}(u)| \leq \\ & \leq \|\tilde{\Delta}_{N_0, L_0}(v) \Delta_{N_3, L_3}(u)\|_{L^2(\mathbb{R} \times \Theta)} \|\partial_r (\Delta_{N_2, L_2}(u)) \Delta_{N_1, L_1}(u)\|_{L^2(\mathbb{R} \times \Theta)} \leq \\ & \leq C_\varepsilon (N_2 N_3)^\varepsilon (L_0 L_1 L_2 L_3)^\beta \|\tilde{\Delta}_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \\ & \times \|\partial_r (\Delta_{N_2, L_2}(u))\|_{L^2(\mathbb{R} \times \Theta)} \|\Delta_{N_1, L_1}(u)\|_{L^2(\mathbb{R} \times \Theta)} \|\Delta_{N_3, L_3}(u)\|_{L^2(\mathbb{R} \times \Theta)} \leq \\ & \leq C_\varepsilon (N_2 N_3)^\varepsilon (L_0 L_1 L_2 L_3)^\beta \frac{N_2}{N_0} \|\Delta_{N_0, L_0}(v)\|_{L^2(\mathbb{R} \times \Theta)} \prod_{j=1}^3 \|\Delta_{N_j, L_j}(u)\|_{L^2(\mathbb{R} \times \Theta)}. \end{aligned}$$

Next, for $N_2 \leq N_1$, we can write,

$$\left(\frac{N_0}{N_1}\right)^\sigma \frac{N_2}{N_0} \leq \left(\frac{N_0}{N_1}\right)^{\sigma-1}$$

and therefore, with Q defined by (6.5), we can write

$$E_2 \leq \sum_{L_0, L_1, L_2, L_3\text{-dyadic}} \sum_{\substack{N_0 \geq N_1 \geq N_2, N_1 \geq N_3 \\ N_0, N_1, N_2, N_3\text{-dyadic}}} L_0^{\beta-b'} (L_1 L_2 L_3)^{\beta-b} \left(\frac{N_0}{N_1}\right)^{\sigma-1} (N_2 N_3)^{\varepsilon-\sigma_1} Q$$

But the right hand-side of the above inequality is exactly the same as the right hand-side of (6.11). Therefore

$$E_2 \leq C \|v\|_{X_{rad}^{-\sigma, b'}(\mathbb{R} \times \Theta)} \|u\|_{X_{rad}^{\sigma_1, b}(\mathbb{R} \times \Theta)}^2 \|u\|_{X_{rad}^{\sigma, b}(\mathbb{R} \times \Theta)}.$$

This completes the proof of (6.1). In the proof of (6.1), we analysed the expression $\|F(u)\|_{X_{rad}^{\sigma, -b'}}$. The argument is based on successive expansions of $F(u)$ in telescopic series and thus it works equally well if we replace $F(u)$ by $u G(v, w)$ where $G(z_1, z_2)$ satisfies the growth assumption

$$(6.13) \quad \left| \partial_{z_1}^{k_1} \bar{\partial}_{z_1}^{k_2} \partial_{z_2}^{l_1} \bar{\partial}_{z_2}^{l_2} G(z_1, z_2) \right| \leq C_{k_1, k_2, l_1, l_2} (1 + |z_1| + |z_2|)^{\max(0, \alpha - k_1 - k_2 - l_1 - l_2)}.$$

But this is exactly the situation that occurs in the analysis of (6.2). Indeed, one can write

$$F(u) - F(v) = (u - v)G_1(u, v) + (\bar{u} - \bar{v})G_2(u, v)$$

with $G_j(z_1, z_2)$, $j = 1, 2$ satisfying (6.13). Since the analysis is very similar to the proof of (6.1), we shall only outline the estimate for $(u - v)G_1(u, v)$. Again, we can suppose that $F(u)$ is vanishing at order 3 at $u = 0$. Let us set

$$w_1 = u - v, \quad w_2 = u, \quad w_3 = v.$$

One needs to bound

$$\left| \int_{\mathbb{R} \times \Theta} w_1 G_1(w_2, w_3) \overline{w_4} \right|$$

by

$$C(1 + \|w_2\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)} + \|w_3\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)})^2 \|w_1\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)} \|w_4\|_{X_{rad}^{-\sigma,b'}(\mathbb{R} \times \Theta)}.$$

Next, we expand

$$w_1 = \sum_{N_1\text{-dyadic}} \Delta_{N_1}(w_1), \quad w_4 = \sum_{N_0\text{-dyadic}} \Delta_{N_0}(w_4)$$

and

$$G_1(w_2, w_3) = \sum_{N_2\text{-dyadic}} \left(G_1(\tilde{S}_{2N_2}(w_2), \tilde{S}_{2N_2}(w_3)) - G_1(\tilde{S}_{N_2}(w_2), \tilde{S}_{N_2}(w_3)) \right).$$

Thus, modulo complex conjugations irrelevant in this discussion, one has to evaluate quantities of type

$$(6.14) \quad \sum_{N_0, N_1, N_2\text{-dyadic}} \left| \int_{\mathbb{R} \times \Theta} \overline{\Delta_{N_0}(w_4)} \Delta_{N_1}(w_1) \Delta_{N_2}(w_j) H_j^{N_2}(\Delta_{N_2}(w_2), \tilde{S}_{N_2}(w_2), \Delta_{N_2}(w_3), \tilde{S}_{N_2}(w_3)) \right|, \quad j = 2, 3,$$

where $H_j^{N_2}(z_1, z_2, z_3, z_4)$ are smooth functions satisfying growth restrictions at infinity coming from (1.2). In the analysis of (6.14), we distinguish two cases for N_0, N_1, N_2 in the sum defining (6.14). The first case is when $N_0 \leq \max(N_1, N_2)$, in this case, we expand once more $H_j^{N_2}$ which introduces a sum over N_3 -dyadic, $N_3 \leq N_2$ of terms $\Delta_{N_3}(w_j)$ (or complex conjugate) times a bounded function (thanks to the sub cubic nature of the nonlinearity). The analysis is then exactly the same as for that of I_1 in the proof of (6.1). If $N_0 \geq \max(N_1, N_2)$, then we integrate by parts by the aid of $\Delta_{N_0}(w_4)$ and analysis is the same as in the bound for I_2 in the proof of (6.1). This completes the proof of Proposition 6.1. \square

Let us now consider the integral equation corresponding to the problem (1.1)-(1.5)

$$(6.15) \quad u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau.$$

With Proposition 6.1 in hand, we can deduce the following estimates for the terms in the right hand-side of (6.15).

PROPOSITION 6.3. *Let $0 < \sigma_1 \leq \sigma < 1/2$. Then there exist two positive numbers b, b' such that $b + b' < 1$, $b' < 1/2 < b$, there exists $C > 0$ such that for every $T \in]0, 1]$, every $u, v \in X_{rad}^{\sigma,b}([-T, T] \times \Theta)$, every $u_0 \in H_{rad}^{\sigma}(\Theta)$,*

$$(6.16) \quad \|e^{it\Delta} u_0\|_{X_{rad}^{\sigma,b}([-T, T] \times \Theta)} \leq C \|u_0\|_{H_{rad}^{\sigma}(\Theta)},$$

$$(6.17) \quad \left\| \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau \right\|_{X_{rad}^{\sigma,b}([-T, T] \times \Theta)} \leq \\ \leq CT^{1-b-b'} \left(1 + \|u\|_{X_{rad}^{\sigma_1,b}([-T, T] \times \Theta)}^2 \right) \|u\|_{X_{rad}^{\sigma,b}([-T, T] \times \Theta)}$$

and

$$(6.18) \quad \left\| \int_0^t e^{i(t-\tau)\Delta} (F(u(\tau)) - F(v(\tau))) d\tau \right\|_{X_{rad}^{\sigma,b}([-T,T] \times \Theta)} \leq \\ \leq CT^{1-b-b'} \left(1 + \|u\|_{X_{rad}^{\sigma,b}([-T,T] \times \Theta)}^2 + \|v\|_{X_{rad}^{\sigma,b}([-T,T] \times \Theta)}^2 \right) \|u - v\|_{X_{rad}^{\sigma,b}([-T,T] \times \Theta)}.$$

PROOF. Let $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $[-1, 1]$. Then, using (5.1), we can write

$$\|e^{it\Delta} u_0\|_{X_{rad}^{\sigma,b}([-T,T] \times \Theta)} \leq \|\psi(t)e^{it\Delta} u_0\|_{X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)} = \|\psi\|_{H^b(\mathbb{R})} \|u_0\|_{H_{rad}^\sigma(\Theta)}$$

which proves (6.16). Let us remark that if $\tilde{u} \in X_{rad}^{\sigma,b}(\mathbb{R} \times \Theta)$ realises the $X_{rad}^{\sigma,b}([-T, T] \times \Theta)$ norm of u then the same \tilde{u} realises all $X_{rad}^{\sigma',b}([-T, T] \times \Theta)$, $\sigma' < \sigma$ norms of u . With this remark in hand, now the proofs of (6.17) and (6.18) follow from (6.1) and (6.2) respectively, (5.1) and the inequality

$$(6.19) \quad \|\psi(t/T) \int_0^t f(\tau) d\tau\|_{H^b(\mathbb{R})} \leq CT^{1-b-b'} \|f\|_{H^{-b'}(\mathbb{R})}.$$

We refer to [7] for the proof of (6.19). This completes the proof of Proposition 6.3. \square

7. Basic local well-posedness results for NLS and the truncated NLS

Recall that we are interested in constructing solutions of the initial value problem

$$(7.1) \quad iu_t + \Delta u + F(u) = 0, \quad u|_{t=0} = u_0.$$

We will approximate the solutions of (7.1) by the solutions of the ODE

$$(7.2) \quad iu_t + \Delta u + S_N(F(u)) = 0, \quad u|_{t=0} = u_0 \in E_N,$$

for $N \gg 1$ (for the definition of the projector S_N , see Section 3, (3.12) above). Equation (7.2) can be seen as a Hamiltonian ODE for $u = S_N(u)$. More precisely, if

$$u = S_N(u) = \sum_{n=1}^N c_n e_{n,s},$$

then the Hamiltonian of the ODE (7.2) is given by

$$H(u, \bar{u}) = \sum_{n=1}^N z_n^{-2s} |c_n|^2 - \int_0^1 V\left(\sum_{m=1}^N c_m e_{m,s}(r)\right) r dr.$$

Multiplying (7.2) by \bar{u} and integrating over Θ yields that the L^2 norm is still a conserved quantity for (7.2). Therefore, the Cauchy-Lipschitz theorem for ODE's implies the existence of global dynamics for (7.2) for every $u_0 \in L^2(\Theta)$. The L^2 conservation provides the bound

$$\sum_{n=1}^N z_n^{-2s} |c_n(t)|^2 \leq C$$

uniformly in t . However, bounds on the quantities

$$\sum_{n=1}^N n^\sigma n^{-2s} |c_n(t)|^2, \quad \sigma > 0$$

for long times are non trivial and go beyond the scope of the basic Cauchy-Lipschitz theorem. We next state the basic local well-posedness result for (7.1), which unfortunately misses the L^2 theory.

PROPOSITION 7.1. *Let us fix $0 < \sigma_1 \leq \sigma < 1/2$. Then there exist $b > 1/2$, $\beta > 0$, $C > 0$, $\tilde{C} > 0$, $c \in]0, 1]$ such that for every $A > 0$ if we set $T = c(1 + A)^{-\beta}$ then for every $u_0 \in H_{rad}^{\sigma_1}(\Theta)$ satisfying $\|u_0\|_{H^{\sigma_1}} \leq A$ there exists a unique solution of (7.1) in $X_{rad}^{\sigma_1, b}([-T, T] \times \Theta)$. Moreover*

$$\|u\|_{L^\infty([-T, T]; H^{\sigma_1}(\Theta))} \leq C\|u\|_{X_{rad}^{\sigma_1, b}([-T, T] \times \Theta)} \leq \tilde{C}\|u_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0 \in H_{rad}^\sigma(\Theta)$ then

$$\|u\|_{L^\infty([-T, T]; H^\sigma(\Theta))} \leq C\|u\|_{X_{rad}^{\sigma, b}([-T, T] \times \Theta)} \leq \tilde{C}\|u_0\|_{H^\sigma(\Theta)}.$$

Finally if u and v are two solutions with data u_0, v_0 respectively, satisfying

$$\|u_0\|_{H^{\sigma_1}} \leq A, \quad \|v_0\|_{H^{\sigma_1}} \leq A$$

then

$$\|u - v\|_{L^\infty([-T, T]; H^{\sigma_1}(\Theta))} \leq C\|u_0 - v_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0, v_0 \in H_{rad}^\sigma(\Theta)$ then

$$\|u - v\|_{L^\infty([-T, T]; H^\sigma(\Theta))} \leq C\|u_0 - v_0\|_{H^\sigma(\Theta)}.$$

PROOF. It is a direct application of Proposition 6.3 and the contraction mapping principle to the map $\Phi_{u_0}(u)$ defined by the right hand-side of (6.15). Indeed, for shortness, let us denote by X_T^σ the Bourgain space $X_{rad}^{\sigma, b}([-T, T] \times \Theta)$, where b is fixed in Proposition 6.3. Then there exists $\theta > 0$ ($\theta \equiv 1 - b - b'$ with b' fixed by Proposition 6.3) such that

$$(7.3) \quad \|\Phi_{u_0}(u)\|_{X_T^{\sigma_1}} \leq C\|u_0\|_{H_{rad}^{\sigma_1}(\Theta)} + CT^\theta(1 + \|u\|_{X_T^{\sigma_1}}^2)\|u\|_{X_T^{\sigma_1}}$$

and

$$(7.4) \quad \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_T^{\sigma_1}} \leq CT^\theta\|u - v\|_{X_T^{\sigma_1}}(1 + \|u\|_{X_T^{\sigma_1}}^2 + \|v\|_{X_T^{\sigma_1}}^2).$$

Using (7.3), we obtain that for every u_0 such that $\|u_0\|_{H^{\sigma_1}} \leq A$ if we take

$$T \sim (1 + A)^{-\theta/2}$$

then the map Φ_{u_0} sends the ball of radius $2C\|u_0\|_{H^{\sigma_1}}$ of $X_T^{\sigma_1}$ into the same ball. Thanks to (7.4), with the same restriction on T the map Φ_{u_0} is also a contraction on the ball of radius $2C\|u_0\|_{H^{\sigma_1}}$ of $X_T^{\sigma_1}$. The fixed point of this contraction is the needed local solution of (7.1). Proposition 6.3 also yields the bound

$$\|\Phi_{u_0}(u)\|_{X_T^\sigma} \leq C\|u_0\|_{H_{rad}^\sigma(\Theta)} + CT^\theta(1 + \|u\|_{X_T^{\sigma_1}}^2)\|u\|_{X_T^\sigma}$$

We obtain thus the propagation of higher regularity with the same restrictions on T . Using Proposition 6.3, we get the bound

$$(7.5) \quad \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X_T^\sigma} \leq CT^\theta\|u - v\|_{X_T^\sigma}(1 + \|u\|_{X_T^\sigma}^2 + \|v\|_{X_T^\sigma}^2).$$

Applying Proposition 6.3, (7.4) and (7.5) to the difference of two solutions yields the quoted Lipschitz bound. This completes the proof of Proposition 7.1. \square

REMARK 7.2. *If we are interested to prove propagation of higher Sobolev regularity, with our methods we only can treat the domains of the powers of the Dirichlet Laplacian. For example we may expect to get that H_0^1 regularity is propagated by the flow. Similar results for the Neumann Laplacian do not follow from our analysis. As mention in the introduction, we do not pursue this here since the measure ρ "lives" on functions of Sobolev regularity $H^{1/2-}$.*

We state the analog of Proposition 7.1 for (7.2).

PROPOSITION 7.3. *Let us fix $0 < \sigma_1 \leq \sigma < 1/2$. Then there exist $b > 1/2$, $\beta > 0$, $C > 0$, $\tilde{C} > 0$, $c \in]0, 1]$ such that for every $A > 0$ if we set $T = c(1 + A)^{-\beta}$ then for every $N \geq 1$, every $u_0 \in H_{rad}^{\sigma_1}(\Theta)$ satisfying $\|u_0\|_{H^{\sigma_1}} \leq A$ there exists a unique solution $u = S_N(u)$ of (7.2) in $X_{rad}^{\sigma_1, b}([-T, T] \times \Theta)$. Moreover*

$$\|u\|_{L^\infty([-T, T]; H^{\sigma_1}(\Theta))} \leq C \|u\|_{X_{rad}^{\sigma_1, b}([-T, T] \times \Theta)} \leq \tilde{C} \|u_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0 \in H_{rad}^\sigma(\Theta)$ then

$$\|u\|_{L^\infty([-T, T]; H^\sigma(\Theta))} \leq C \|u\|_{X_{rad}^{\sigma, b}([-T, T] \times \Theta)} \leq \tilde{C} \|u_0\|_{H^\sigma(\Theta)}.$$

Finally if u and v are two solutions with data u_0, v_0 respectively, satisfying

$$\|u_0\|_{H^{\sigma_1}} \leq A, \quad \|v_0\|_{H^{\sigma_1}} \leq A$$

then

$$\|u - v\|_{L^\infty([-T, T]; H^{\sigma_1}(\Theta))} \leq C \|u_0 - v_0\|_{H^{\sigma_1}(\Theta)}.$$

If in addition $u_0, v_0 \in H_{rad}^\sigma(\Theta)$ then

$$\|u - v\|_{L^\infty([-T, T]; H^\sigma(\Theta))} \leq C \|u_0 - v_0\|_{H^\sigma(\Theta)}.$$

PROOF. The only new point compared to Proposition 7.1 is to observe that S_N is bounded, uniformly in N on the Bourgain spaces $X_{rad}^{\sigma, b}([-T, T] \times \Theta)$, namely for every $u \in X_{rad}^{\sigma, b}([-T, T] \times \Theta)$,

$$\|S_N(u)\|_{X_{rad}^{\sigma, b}([-T, T] \times \Theta)} \leq \|u\|_{X_{rad}^{\sigma, b}([-T, T] \times \Theta)},$$

a bound which is direct consequence of the definitions of $X_{rad}^{\sigma, b}([-T, T] \times \Theta)$ and the projector S_N . \square

REMARK 7.4. *The main point in Proposition 7.3 is the uniformness of the bounds with respect to N .*

8. Improved bounds for the truncated NLS

In this sections, we improve the result of Proposition 7.3. More precisely, we show bounds on the H^σ norm of the solutions of (7.2), uniform in N for initial data of "large ρ_N measure". Let us denote by $\Phi_N(t)$ the smooth flow map of (7.2) which is defined globally thanks to the L^2 conservation for (7.2). The next statement results from an application of Liouville's theorem to (7.2).

PROPOSITION 8.1. *The measure ρ_N defined in Section 3 is invariant under the flow of the (7.2).*

PROOF. Set $c(t) \equiv (c_n(t))_{1 \leq n \leq N}$, where

$$u(t) = \sum_{n=1}^N c_n(t) e_{n,s}.$$

In the coordinates c_n , the equation (7.2) can be written as

$$(8.1) \quad iz_n^{-s} \dot{c}_n(t) - z_n^2 z_n^{-s} c_n(t) + \int_{\Theta} S_N(F(u(t))) \bar{e}_n = 0, \quad 1 \leq n \leq N.$$

Next, equation (8.1) can be written in a Hamiltonian format as follows

$$\partial_t c_n = -iz_n^{2s} \frac{\partial H}{\partial \bar{c}_n}, \quad \partial_t \bar{c}_n = iz_n^{2s} \frac{\partial H}{\partial c_n}, \quad 1 \leq n \leq N,$$

with

$$H(c, \bar{c}) = \sum_{n=1}^N z_n^{2-2s} |c_n|^2 - \int_0^1 V\left(\sum_{m=1}^N c_m e_{m,s}(r)\right) r dr.$$

Since

$$\sum_{n=1}^N \left(\frac{\partial}{\partial c_n} \left(-iz_n^{2s} \frac{\partial H}{\partial \bar{c}_n} \right) + \frac{\partial}{\partial \bar{c}_n} \left(iz_n^{2s} \frac{\partial H}{\partial c_n} \right) \right) = 0,$$

we can apply the Liouville theorem for divergence free vector fields to conclude that the measure $dc d\bar{c}$ is invariant under the flow of (7.2). Recall that we denote by $\Phi_N(t)$, the flow of (7.2) and that the quantities $H(c, \bar{c})$ and

$$\|c\|^2 \equiv \sum_{n=1}^N |c_n|^2$$

are conserved under $\Phi_N(t)$. Let A be a Borel set of E_N . Recall that we denote by χ the characteristic function of the interval $[0, R]$, $R > 0$. Then

$$\rho_N(A) = \kappa_N \int_A e^{-H(c, \bar{c})} \chi(\|c\|) dc d\bar{c},$$

where

$$\kappa_N = \pi^{-N} \left(\prod_{1 \leq n \leq N} z_n^{2-2s} \right).$$

In addition

$$(8.2) \quad \rho_N\left(\Phi_N(t)(A)\right) = \kappa_N \int_{\Phi_N(t)(A)} e^{-H(c, \bar{c})} \chi(\|c\|) dc d\bar{c}.$$

We can write

$$\Phi_N(t)(A) = \{(c, \bar{c}) : (c, \bar{c}) = \Phi_N(t)(b, \bar{b}), \quad (b, \bar{b}) \in A\}.$$

Let us perform the change of variables $(c, \bar{c}) = \Phi_N(t)(b, \bar{b})$ in the right hand-side of (8.2). Since $dc d\bar{c}$ is invariant under $\Phi_N(t)$ the Jacobian of this variable change is one. Next by the conservation laws

$$H(\Phi_N(t)(b, \bar{b})) = H(b, \bar{b}), \quad \|\Phi_N(t)(b)\| = \|b\|.$$

Therefore

$$\rho_N\left(\Phi_N(t)(A)\right) = \kappa_N \int_A e^{-H(b, \bar{b})} \chi(\|b\|) db d\bar{b} = \rho_N(A).$$

This completes the proof of Proposition 8.1. \square

Next, we state a bound for the solutions of (7.2) which gives a control, independent of N on norms which are stronger than L^2 but weaker than H^1 .

PROPOSITION 8.2. *For every integer $i \geq 1$, $\sigma \in [s, 1/2[$ there exists a set*

$$\Sigma_{N,\sigma}^i \subset E_N$$

such that

$$(8.3) \quad \rho_N(E_N \setminus \Sigma_{N,\sigma}^i) \leq 2^{-i},$$

and for $u_0 \in \Sigma_{N,\sigma}^i$ one has the bound

$$(8.4) \quad \|\Phi_N(t)(u_0)\|_{H^\sigma} \leq C_\sigma(i + \log(1 + |t|))^{\frac{1}{2}}.$$

Moreover, for $N_1 \leq N_2$ we have the inclusion $\Sigma_{N_1,\sigma}^i \subset \Sigma_{N_2,\sigma}^i$.

PROOF. We will consider only the positive values of t , the analysis for $t < 0$ being the same. For $\sigma \in [s, 1/2[$, i, j integers ≥ 1 , we set

$$B_{N,\sigma}^{i,j}(D_\sigma) = \left\{ u \in E_N : \|u\|_{H^\sigma(\Theta)} \leq D_\sigma(i+j)^{\frac{1}{2}}, \quad \|u\|_{L^2(\Theta)} \leq R \right\},$$

where the number $D_\sigma \gg 1$ (independent of i, j, N) will be fixed later. Thanks to Proposition 7.3, there exists $\tau \in]0, 1]$, $\tau \sim D_\sigma^{-\beta}(i+j)^{-\beta/2}$ for some $\beta > 0$ and such that for $t \in [0, \tau]$,

$$(8.5) \quad \Phi_N(t)(B_{N,\sigma}^{i,j}(D_\sigma)) \subset B_{N,\sigma}^{i,j}(CD_\sigma),$$

where $B_{N,\sigma}^{i,j}(CD_\sigma)$ is defined similarly to $B_{N,\sigma}^{i,j}(D_\sigma)$ simply replacing D_σ by CD_σ in the H^σ bound for u . Next, we set

$$\Sigma_{N,\sigma}^{i,j}(D_\sigma) = \bigcap_{k=0}^{[2^j/\tau]} \Phi_N(-k\tau)(B_{N,\sigma}^{i,j}(D_\sigma)),$$

where $[2^j/\tau]$ stays for the integer part of $2^j/\tau$. Using Proposition 8.1, we can write

$$\begin{aligned} \rho_N(E_N \setminus \Sigma_{N,\sigma}^{i,j}(D_\sigma)) &= \rho_N\left(\bigcup_{k=0}^{[2^j/\tau]} (E_N \setminus \Phi_N(-k\tau)(B_{N,\sigma}^{i,j}(D_\sigma)))\right) \\ &\leq ([2^j/\tau] + 1)\rho_N(E_N \setminus B_{N,\sigma}^{i,j}(D_\sigma)) \\ &\leq C2^j D_\sigma^\beta (i+j)^{\beta/2} \rho_N(E_N \setminus B_{N,\sigma}^{i,j}(D_\sigma)). \end{aligned}$$

Let us now observe that

$$\begin{aligned} \rho_N(E_N \setminus B_{N,\sigma}^{i,j}(D_\sigma)) &= \rho\left(u \in H_{rad}^s(\Theta) : \|S_N(u)\|_{H^\sigma(\Theta)} > D_\sigma(i+j)^{\frac{1}{2}}\right) \\ &\leq \rho\left(u \in H_{rad}^s(\Theta) : \|u\|_{H^\sigma(\Theta)} > D_\sigma(i+j)^{\frac{1}{2}}\right). \end{aligned}$$

Therefore, using Proposition 3.7, we can write

$$(8.6) \quad \rho_N(E_N \setminus \Sigma_{N,\sigma}^{i,j}(D_\sigma)) \leq C_\sigma 2^j D_\sigma^\beta (i+j)^{\beta/2} e^{-cD_\sigma^2(i+j)} \leq 2^{-(i+j)},$$

provided $D_\sigma \gg 1$, depending on σ but independent of i, j, N . Thanks to (8.5), we obtain that for $u_0 \in \Sigma_{N,\sigma}^{i,j}(D_\sigma)$, the solution u of (7.2) with data u_0 satisfies

$$(8.7) \quad \|u(t)\|_{H^\sigma(\Theta)} \leq CD_\sigma(i+j)^{\frac{1}{2}}, \quad 0 \leq t \leq 2^j.$$

Next, we set

$$\Sigma_{N,\sigma}^i = \bigcap_{j \geq 1} \Sigma_{N,\sigma}^{i,j}(D_\sigma).$$

Thanks (8.6),

$$(8.8) \quad \rho_N(E_N \setminus \Sigma_{N,\sigma}^i) \leq 2^{-i}.$$

Next, using (8.7), we get (8.4). Observe that for $N_1 \leq N_2$, we have the inclusion $B_{N_1,\sigma}^{i,j}(D_\sigma) \subset B_{N_2,\sigma}^{i,j}(D_\sigma)$ which implies that $\Sigma_{N_1,\sigma}^{i,j}(D_\sigma) \subset \Sigma_{N_2,\sigma}^{i,j}(D_\sigma)$. This in turn implies that for $N_1 \leq N_2$, $\Sigma_{N_1,\sigma}^i \subset \Sigma_{N_2,\sigma}^i$. This completes the proof of Proposition 8.2. \square

Next, we prove an invariance property of the sets $\Sigma_{N,\sigma}^i$ constructed in Proposition 8.2.

PROPOSITION 8.3. *For every $\sigma \in]s, 1/2[$ every $\sigma_1 \in [s, \sigma[$ every $t \in \mathbb{R}$ every integer $i \geq 1$ there exists i_1 such that for every $N \geq 1$, if $u_0 \in \Sigma_{N,\sigma}^i$ then one has*

$$\Phi_N(t)(u_0) \in \Sigma_{N,\sigma_1}^{i+i_1}.$$

PROOF. Again, we can suppose that $t > 0$. Set $u(t) \equiv \Phi_N(t)(u_0)$. If $u_0 \in \Sigma_{N,\sigma}^i$ then for every integer $j \geq 1$, we have the bound

$$\|\Phi_N(t_1)(u_0)\|_{H^\sigma} \leq C_\sigma(i+j)^{\frac{1}{2}}, \quad 0 \leq t_1 \leq 2^j.$$

Let $j_0 \in \mathbb{N}$ (depending on t) be such that for every $j \geq 1$, $2^j + t \leq 2^{j+j_0}$. Therefore, we have that

$$\|\Phi_N(t_1)(u(t))\|_{H^\sigma} = \|\Phi_N(t+t_1)(u_0)\|_{H^\sigma} \leq C_\sigma(i+j+j_0)^{\frac{1}{2}}, \quad 0 \leq t_1 \leq 2^j.$$

The crucial observation is that thanks to the L^2 conservation law, interpolating between the last bound and the L^2 conservation provides the existence of $\theta \in]0, 1[$ (depending on σ and σ_1) such that

$$\|\Phi_N(t_1)(u(t))\|_{H^{\sigma_1}} \leq c \left[C_\sigma(i+j+j_0) \right]^{\frac{\theta}{2}}, \quad 0 \leq t_1 \leq 2^j.$$

Next, we observe that since $\theta < 1$, for $j_0 \gg 1$,

$$c \left[C_\sigma(i+j+j_0) \right]^{\frac{\theta}{2}} \leq D_{\sigma_1}(i+j+j_0)^{\frac{1}{2}}.$$

Thus

$$\|\Phi_N(t_1)(u(t))\|_{H^{\sigma_1}} \leq D_{\sigma_1}(i+j+j_0)^{\frac{1}{2}}, \quad 0 \leq t_1 \leq 2^j.$$

We can now conclude that $u(t) \in \Sigma_{N,\sigma_1}^{i+j_0,j}(D_\sigma)$ for every $j \geq 1$. Therefore

$$u(t) \in \Sigma_{N,\sigma_1}^{i+j_0}.$$

This completes the proof of Proposition 8.3. \square

REMARK 8.4. *The number i_1 is the same for every i , i.e. it depends only on t, σ, σ_1 . This fact is however not of importance for the sequel.*

9. Global existence for NLS on a set of full ρ measure

The goal of this section is to compare the flows of (7.1) and (7.2) on a set of full ρ measure. For an integer $i \geq 1$ and $\sigma \in [s, 1/2[$, we set

$$\Sigma_\sigma^i \equiv \bigcup_{N \geq 1} \Sigma_{N,\sigma}^i.$$

where $\Sigma_{N,\sigma}^i$ are defined in Proposition 8.2. Let us denote by $\overline{\Sigma_\sigma^i}$ the closure of Σ_σ^i in $H_{rad}^\sigma(\Theta)$. Thus Σ_σ^i is a closed set of $H_{rad}^\sigma(\Theta)$. Then thanks to Lemma 3.8 and Proposition 8.2, we can write

$$(9.1) \quad \rho(\overline{\Sigma_\sigma^i}) \geq \limsup_{N \rightarrow \infty} \rho_N(\Sigma_{N,\sigma}^i) \geq \limsup_{N \rightarrow \infty} (\rho_N(E_N) - 2^{-i}) = \rho(H_{rad}^s(\Theta)) - 2^{-i}.$$

Next, we set

$$\Sigma_\sigma \equiv \bigcup_{i \geq 1} \overline{\Sigma_\sigma^i}.$$

In view of (9.1), we obtain that Σ_σ is of full ρ measure.

Let $l = (l_j)_{j \in \mathbb{N}}$ be a increasing sequence of real numbers such that $l_0 = s$, $l_j < 1/2$ and

$$\lim_{j \rightarrow \infty} l_j = 1/2.$$

Then, we set

$$(9.2) \quad \Sigma = \bigcap_{\sigma \in l} \Sigma_\sigma$$

The set Σ is of full ρ measure since every Σ_σ is of full ρ measure and the intersection in (9.2) is countable. The set Σ is the statistical ensemble for the problem (7.1) and the solutions of (7.1) with data in Σ are globally defined. We have the following statement.

PROPOSITION 9.1. *For every $u_0 \in \Sigma$, the local solution of (7.1) given by Proposition 7.1 is globally defined. Moreover for every $t \in \mathbb{R}$, if we denote by $\Phi(t)$ the flow map of (7.1) acting on Σ then $\Phi(t)(\Sigma) = \Sigma$.*

PROOF. Let us fix $u_0 \in \overline{\Sigma_\sigma^i}$, $\sigma \in l$, $\sigma_1 \in]0, \sigma[$ and $T > 0$. Thus there exists a sequence $u_{0,k} \in \Sigma_{N_k,\sigma}^i$, where N_k is tending to infinity, such that $u_{0,k}$ converges to u_0 in $H^\sigma(\Theta)$. Thanks to Proposition 8.2

$$(9.3) \quad \|\Phi_{N_k}(t)(u_{0,k})\|_{H^\sigma} \leq C_\sigma(i + \log(1 + |t|))^{\frac{1}{2}}.$$

Set

$$u_{N_k}(t) \equiv \Phi_{N_k}(t)(u_{0,k}).$$

Thanks to (9.3), there exists $\Lambda > 1$, independent of N_k , such that

$$(9.4) \quad \|u_{N_k}(t)\|_{H^\sigma} \leq \Lambda, \quad |t| \leq T.$$

Let us observe that (9.4), applied for $t = 0$ implies that $\|u_0\|_{H^\sigma} \leq \Lambda$ (after passing to the limit $N_k \rightarrow \infty$). Let $\tau > 0$ be the local existence time for (7.1), provided by Proposition 7.1 for $A = \Lambda + 1$. Recall that we can assume $\tau \sim \Lambda^{-\beta}$ for some $\beta > 0$. Denote by $u(t)$ the solution of (7.1) with data u_0 on the time interval $[-\tau, \tau]$. Set

$$v \equiv u - u_{N_k}.$$

Then v solves the equation

$$(9.5) \quad iv_t + \Delta v + F(u) - S_{N_k}(F(u_{N_k})) = 0, \quad v|_{t=0} = u_0 - u_{0,k}.$$

Next we write

$$F(u) - S_{N_k}(F(u_{N_k})) = S_{N_k}(F(u) - F(u_{N_k})) + (1 - S_{N_k})F(u).$$

Observe that the map $1 - S_N$ sends $H_{rad}^\sigma(\Theta)$ to $H_{rad}^{\sigma_1}(\Theta)$ with norm $\leq CN^{\sigma_1 - \sigma}$. Similarly, for $I \subset \mathbb{R}$ an interval, the map $1 - S_N$ sends $X_{rad}^{\sigma,b}(I \times \Theta)$ to $X_{rad}^{\sigma_1,b}(I \times \Theta)$ with norm $\leq CN^{\sigma_1 - \sigma}$. Moreover S_N acts as a bounded operator (with norm ≤ 1) on the Bourgain spaces $X_{rad}^{\sigma,b}$. Therefore, using Proposition 6.3, we can write the Duhamel formula associated to (9.5) and we obtain that there exists $b > 1/2$ and $\theta > 0$ (depending only on σ, σ_1) such that one has the bound

$$\begin{aligned} \|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)} &\leq C\|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} \\ &\quad + C\tau^\theta \|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)} (1 + \|u\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)}^2 + \\ &\quad + \|u_{N_k}\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)}^2) \\ &\quad + C\tau^\theta N_k^{\sigma_1 - \sigma} \|u\|_{X_{rad}^{\sigma,b}([- \tau, \tau] \times \Theta)} (1 + \|u\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)}^2). \end{aligned}$$

Using Proposition 7.1 and Proposition 7.3, we get

$$\begin{aligned} \|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)} &\leq C\|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} \\ &\quad + C\tau^\theta \|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)} (1 + C\|u_0\|_{H^{\sigma_1}(\Theta)}^2 + C\|u_{0,k}\|_{H^{\sigma_1}(\Theta)}^2) \\ &\quad + C\tau^\theta N_k^{\sigma_1 - \sigma} \|u_0\|_{H^\sigma(\Theta)} (1 + C\|u_0\|_{H^{\sigma_1}(\Theta)}^2) \\ &\leq C\|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} + C\tau^\theta \Lambda^2 N_k^{\sigma_1 - \sigma} \|u_0\|_{H^\sigma(\Theta)} \\ &\quad + C\tau^\theta \Lambda^2 \|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)}. \end{aligned}$$

Therefore, assuming in addition that $\tau \sim \Lambda^{-\theta/2}$, we obtain

$$\|v\|_{X_{rad}^{\sigma_1,b}([- \tau, \tau] \times \Theta)} \leq C\|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} + CN_k^{\sigma_1 - \sigma} \|u_0\|_{H^\sigma(\Theta)}, \quad \tau \sim \Lambda^{-\beta},$$

for some fixed positive real number β and where the constant C is independent of N_k . Since $b > 1/2$, the last inequality implies

$$(9.6) \quad \|v(t)\|_{H^{\sigma_1}(\Theta)} \leq C\|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} + CN_k^{\sigma_1 - \sigma} \|u_0\|_{H^\sigma(\Theta)},$$

where $|t| \leq \tau \sim \Lambda^{-\beta}$, $\beta > 0$. By taking $N_k \gg 1$ and using the triangle inequality, we get

$$(9.7) \quad \|u(t)\|_{H^{\sigma_1}(\Theta)} \leq \Lambda + 1, \quad |t| \leq \tau.$$

The key quantity in this discussion is

$$\|v(t)\|_{H^{\sigma_1}(\Theta)} + N_k^{\sigma_1 - \sigma} \|u(t)\|_{H^\sigma(\Theta)}.$$

We can iterate the argument for obtaining (9.6) on $[\tau, 2\tau]$ thanks to the definition of τ and the bounds (9.4) and (9.7). We obtain

$$\|v(t)\|_{H^{\sigma_1}(\Theta)} \leq C\|v(\tau)\|_{H^{\sigma_1}(\Theta)} + CN_k^{\sigma_1 - \sigma} \|u(\tau)\|_{H^\sigma(\Theta)},$$

where $t \in [\tau, 2\tau]$ and $\tau \sim \Lambda^{-\beta}$. Moreover, by taking $N_k \gg 1$,

$$\|u(t)\|_{H^{\sigma_1}(\Theta)} \leq \Lambda + 1, \quad \tau \leq t \leq 2\tau.$$

Then, we can continue by covering the interval $[-T, T]$ with intervals of size τ , which yields the existence of $u(t)$ on $[-T, T]$. Moreover v satisfies the bound

$$\|v(t)\|_{H^{\sigma_1}(\Theta)} \leq C^{1+T} \left(N_k^{\sigma_1 - \sigma} \|u_0\|_{H^\sigma(\Theta)} + \|u_0 - u_{0,k}\|_{H^{\sigma_1}(\Theta)} \right), \quad |t| \leq T.$$

Therefore by taking $N_k \gg 1$ (depending in particular on T), we obtain that for every $\varepsilon > 0$ there exists N_0 such that for $N_k \geq N_0$ one has the inequality

$$\sup_{|t| \leq T} \|u(t) - \Phi_{N_k}(t)(u_{0,k})\|_{H^{\sigma_1}(\Theta)} < \varepsilon.$$

Hence for every $t \in [-T, T]$,

$$(9.8) \quad \lim_{k \rightarrow \infty} \|u(t) - \Phi_{N_k}(t)(u_{0,k})\|_{H^{\sigma_1}(\Theta)} = 0.$$

Since $T > 0$ was chosen arbitrary, we obtain that for every $u_0 \in \overline{\Sigma_\sigma^i}$ the local solution of (7.1) is globally defined. Since i and $\sigma \in l$ are also arbitrary, we obtain that for every $u_0 \in \Sigma$, the the local solution of (7.1) is globally defined. Let us denote by $\Phi(t)$ the flow of (7.1) acting on Σ . Let us show the inclusion

$$(9.9) \quad \Phi(t)(\Sigma) \subset \Sigma.$$

Fix $u_0 \in \Sigma$. It suffices to show that for every $\sigma_1 \in l$, we have

$$\Phi(t)(u_0) \in \Sigma_{\sigma_1}.$$

Let us take $\sigma \in]\sigma_1, 1/2[$, $\sigma \in l$. Since $u_0 \in \Sigma$, we have that $u_0 \in \Sigma_\sigma$. Therefore there exists i such that $u_0 \in \overline{\Sigma_\sigma^i}$. Let again $u_{0,k} \in \Sigma_{N_k, \sigma}^i$ be a sequence which tends to u_0 in $H^\sigma(\Theta)$. Thanks to Proposition 8.3 there exists i_1 such that

$$\Phi_{N_k}(t)(u_{0,k}) \in \Sigma_{N_k, \sigma_1}^{i+i_1}.$$

Therefore using (9.8), we obtain that

$$\Phi(t)(u_0) \in \overline{\Sigma_{\sigma_1}^{i+i_1}}.$$

Thus $\Phi(t)(u_0) \in \Sigma_{\sigma_1}$ which proves (9.9). Moreover the flow $\Phi(t)$ is reversible which implies that $\Phi(t)(\Sigma) = \Sigma$. Indeed, if $u \in \Sigma$ and $t \in \mathbb{R}$, we set $u_0 \equiv \Phi(-t)u \in \Sigma$ (which is well-defined thanks to the previous analysis) and thus $u = \Phi(t)u_0$, i.e. $\Sigma \subset \Phi(t)(\Sigma)$. This completes the proof of Proposition 9.1. \square

We complete this section by getting a continuity property of $\Phi(t)$.

PROPOSITION 9.2. *Let $u \in \Sigma$ and $u_n \in \Sigma$ be a sequence such that $u_n \rightarrow u$ in $H^s(\Theta)$. Then for every $t \in \mathbb{R}$, $\Phi(t)(u_n) \rightarrow \Phi(t)(u)$ in $H^s(\Theta)$. In particular, for every A , a closed set in $H_{rad}^s(\Theta)$ one has*

$$\Phi(t)(A \cap \Sigma) = \overline{\Phi(t)(A \cap \Sigma)} \cap \Sigma,$$

where $\overline{\Phi(t)(A \cap \Sigma)}$ denotes the closure in $H_{rad}^s(\Theta)$ of $\Phi(t)(A \cap \Sigma)$.

PROOF. Since $u \in \Sigma$ there exists $\Lambda \geq 1$ such that

$$\sup_{|\tau| \leq |t|} \|\Phi(\tau)(u)\|_{H^s(\Theta)} \leq \Lambda.$$

Let us denote by τ_0 the local existence time in Proposition 7.1, associated to $A = 2\Lambda$. Then, by the continuity of the flow $\Phi(\tau_0)(u_n) \rightarrow \Phi(\tau_0)(u)$ in $H^s(\Theta)$. Next, we cover the interval $[0, t]$ by intervals of size τ_0 and we apply the continuity of

the flow established in Proposition 7.1 at each step. Therefore, we obtain that $\Phi(t)(u_n) \rightarrow \Phi(t)(u)$ in $H^s(\Theta)$. Since $\Phi(t)(\Sigma) \subset \Sigma$, it is clear that

$$(9.10) \quad \Phi(t)(A \cap \Sigma) \subset \overline{\Phi(t)(A \cap \Sigma)} \cap \Sigma.$$

Next, let us fix $u \in \overline{\Phi(t)(A \cap \Sigma)} \cap \Sigma$. Thus there exists $v_n \in A \cap \Sigma$ such that $u_n \equiv \Phi(t)(v_n)$ converges to u in $H^s(\Theta)$. Since $v_n \in \Sigma$ and $u \in \Sigma$, we obtain that $u_n \in \Sigma$ and $\Phi(-t)(u) \in \Sigma$. Therefore, using the continuity property we have just established, we obtain that $v_n = \Phi(-t)(u_n)$ converges to $\Phi(-t)(u)$ in $H^s(\Theta)$. Since the set A is assumed closed, we obtain that $\Phi(-t)(u) \in A$. Thus $u \in \Phi(t)(A \cap \Sigma)$ which gives the opposite to (9.10) inclusion. This completes the proof of Proposition 9.2. \square

10. Invariance of the measure ρ

In this section, we complete the proof of Theorem 1. Recall that we denote by $\Phi(t)$, $t \in \mathbb{R}$ the flow of (7.1) acting on Σ , defined in (9.2). Thanks to the continuity properties of $\Phi(t)$ displayed by Proposition 9.2, we have that if $A \subset \Sigma$ is a ρ measurable set then so is $\Phi(t)(A)$. Let us observe that thanks to the reversibility of the flow $\Phi(t)$, it suffices to prove that for every $t \in \mathbb{R}$ and every ρ measurable set $A \subset \Sigma$ one has the inequality

$$(10.1) \quad \rho(\Phi(t)(A)) \geq \rho(A).$$

Let us show that it suffices to prove (10.1) only for closed sets of $H_{rad}^s(\Theta)$. Indeed, by the regularity of the bounded Borel measures for every ρ measurable set $A \subset \Sigma$, we can find a sequence of closed sets $F_n \subset A$ such that

$$\rho(A) = \lim_{n \rightarrow \infty} \rho(F_n).$$

Hence if we can prove (10.1) for the sets F_n then we can write

$$\rho(A) = \lim_{n \rightarrow \infty} \rho(F_n) \leq \limsup_{n \rightarrow \infty} \rho(\Phi(t)(F_n)) \leq \rho(\Phi(t)(A)).$$

Therefore, it suffices to prove (10.1) for closed sets of $H_{rad}^s(\Theta)$ which are included in Σ .

Fix $\sigma \in]s, 1/2[$, $\sigma \in l$. Let us next show that it suffices to prove (10.1) for subsets of Σ which are compacts of $H_{rad}^s(\Theta)$ which are bounded in $H_{rad}^\sigma(\Theta)$. Indeed, using Lemma 3.9, we can write that for every closed in $H_{rad}^s(\Theta)$ set $A \subset \Sigma$, one has

$$\rho(A) = \lim_{R \rightarrow \infty} \rho(A \cap K_R),$$

where K_R is the closed ball of radius R in $H_{rad}^\sigma(\Theta)$, $\sigma \in]s, 1/2[$. Thus $A \cap K_R$ is a compact in $H_{rad}^s(\Theta)$ and if we can prove (10.1) for compacts which are bounded in $H_{rad}^\sigma(\Theta)$ then

$$\rho(A) \leq \limsup_{R \rightarrow \infty} \rho(\Phi(t)(A \cap K_R)) \leq \rho(\Phi(t)(A)).$$

Thus, it suffices to prove (10.1) for subsets of Σ which are compacts in $H_{rad}^s(\Theta)$ and bounded in $H_{rad}^\sigma(\Theta)$.

Let us now fix $t \in \mathbb{R}$ and $K \subset \Sigma$, a bounded set of $H_{rad}^\sigma(\Theta)$ which is a compact in $H_{rad}^s(\Theta)$. Then we have the following lemma.

LEMMA 10.1. *There exists a ball \mathcal{B} , centered at the origin of $H_{rad}^s(\Theta)$ containing all $\Phi(\tau)(K)$, $|\tau| \leq |t|$.*

PROOF. The sets $\Phi(\tau)(K)$ are contained in a ball of $H_{rad}^s(\Theta)$ for $|\tau|$ small enough, given by Proposition 7.1. We then argue by contradiction by supposing that there exists T and a sequence $u_n \in K$ such that

$$(10.2) \quad \lim_{n \rightarrow \infty} \|\Phi(T)(u_n)\|_{H^s(\Theta)} = \infty.$$

Since K is a compact, there exists a subsequence still denoted by u_n and $u \in K$ such that $u_n \rightarrow u$ in $H_{rad}^s(\Theta)$. Since $u \in \Sigma$, we can apply Proposition 9.2 and we obtain that $\Phi(T)(u_n) \rightarrow \Phi(T)(u)$ in $H^s(\Theta)$ which contradicts (10.2). This completes the proof of Lemma 10.1. \square

Let us denote by R_1 the radius of \mathcal{B} . Set

$$\tau_1 \equiv c(1 + R_1)^{-M},$$

where $0 < c \ll 1$ and $M \gg 1$ are two parameters to be fixed later. A first restriction on c and M is to chose them so that τ_1 is smaller than the time existence provided by Propositions 7.1,7.3 associated to $A = R_1$ (and $\sigma_1 = s$). It is then sufficient to prove that

$$(10.3) \quad \rho(K) \leq \rho(\Phi(\tau)(K)), \quad |\tau| \leq \tau_1.$$

Indeed, once (10.3) is established, it suffices to cover $[0, t]$ by intervals of size $\sim \tau_1$ and to apply (10.3) at each step. Such an iteration is possible since at each step the image under $\Phi(\tau)$ of the corresponding set remains in \mathcal{B} and is included in Σ .

Let us now prove (10.3). Fix $\varepsilon > 0$. Denote by B_ε the open ball centered at the origin and of radius ε of $H_{rad}^s(\Theta)$. Recall that we denote by $\Phi_N(t)$, $t \in \mathbb{R}$ the flow of (7.2). Then using Proposition 7.3, we infer that there exists $c > 0$ such that

$$(10.4) \quad \Phi_N(\tau)\left((K + B_\varepsilon) \cap E_N\right) \subset \Phi_N(\tau)(S_N(K)) + B_{c\varepsilon}, \quad N \gg 1.$$

We now make appeal to the following lemma.

LEMMA 10.2. *For $N \gg 1$ one has the inclusion*

$$\Phi_N(\tau)(S_N(K)) + B_{c\varepsilon} \subset \Phi(\tau)(K) + \overline{B_{2c\varepsilon}}.$$

PROOF. The argument is similar to the proof of Proposition 9.1. For $u_0 \in K$, we denote by u the solution of (7.1) with data u_0 and by u_N the solution of (7.2) with data $S_N(u_0)$. Next, we set $v \equiv u - u_N$. Then v is a solution of

$$(10.5) \quad iv_t + \Delta v + F(u) - S_N(F(u_N)) = 0, \quad v|_{t=0} = (1 - S_N)u_0.$$

By writing

$$F(u) - S_N(F(u_N)) = S_N(F(u) - F(u_N)) + (1 - S_N)F(u)$$

and using Proposition 6.3, we obtain that there exists $b > 1/2$ and $\theta > 0$ such that one has

$$\begin{aligned} \|v\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} &\leq CN^{s-\sigma} \|u_0\|_{H^\sigma(\Theta)} \\ &\quad + C\tau^\theta \|v\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} (1 + \|u\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)}^2 \\ &\quad + \|u_N\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)}^2) \\ &\quad + C\tau^\theta N^{s-\sigma} \|u\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} (1 + \|u\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)}^2). \end{aligned}$$

Using Proposition 7.1 and Proposition 7.3, we get

$$\begin{aligned} \|v\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} &\leq CN^{s-\sigma} \|u_0\|_{H^\sigma(\Theta)} \\ &\quad + C\tau^\theta \|v\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} (1 + C\|u_0\|_{H^s(\Theta)}^2) \\ &\quad + C\tau^\theta N^{s-\sigma} \|u_0\|_{H^\sigma(\Theta)} (1 + C\|u_0\|_{H^s(\Theta)}^2). \end{aligned}$$

Coming back to the definition of τ_1 , by taking $c \ll 1$ and $M \gg 1$, we infer that

$$\|v\|_{X_{rad}^{s,b}([- \tau, \tau] \times \Theta)} \leq CN^{s-\sigma} \|u_0\|_{H^\sigma(\Theta)}.$$

Using that u_0 is in a bounded set of $H_{rad}^\sigma(\Theta)$ and since $b > 1/2$, the last inequality implies

$$\|v(t)\|_{H^s(\Theta)} \leq CN^{s-\sigma} \|u_0\|_{H^\sigma(\Theta)} \leq \tilde{C}N^{s-\sigma}, \quad |t| \leq \tau.$$

This completes the proof of Lemma 10.2. \square

Using (10.4), Lemma 10.2, Lemma 3.8 and Proposition 8.1, we can write

$$\begin{aligned} \rho\left(\Phi(\tau)(K) + \overline{B_{2c\varepsilon}}\right) &\geq \limsup_{N \rightarrow \infty} \rho_N\left(\left(\Phi(\tau)(K) + \overline{B_{2c\varepsilon}}\right) \cap E_N\right) \\ &\geq \liminf_{N \rightarrow \infty} \rho_N\left(\Phi_N(\tau)\left((K + B_\varepsilon) \cap E_N\right)\right) \\ &= \liminf_{N \rightarrow \infty} \rho_N\left((K + B_\varepsilon) \cap E_N\right) \\ &\geq \rho(K + B_\varepsilon) \geq \rho(K). \end{aligned}$$

By letting $\varepsilon \rightarrow 0$, we obtain that $\rho(\Phi(\tau)(K)) \geq \rho(K)$. This completes the proof of (10.3) which in turn completes the proof of (10.1).

This completes the proof of Theorem 1. \square

REMARK 10.3. *Let us notice that in the proof of Theorem 1, we did not make appeal to the conservation laws of (7.1). We only used the conservation laws of (7.2) and thus the propagation of higher Sobolev regularity for (7.1) was not needed.*

11. Final remarks

The result of Theorem 1 is obtained under the assumption $\alpha < 2$. Let us recall that if $\alpha = 2$ with $F(u) = |u|^2 u$ then one can construct initial data for (7.1) such that the local solutions constructed in Proposition 7.1 develop singularities in finite time (see [9, 4]). Observe that the data giving blow-up solutions in [9] has to be sufficiently smooth (at least H^1) in order to give sense of the quantities involved in the well-known viriel identity. But one can show that for φ_ω defined by (3.5) we have that $\|\varphi_\omega\|_{H^1(\Theta)}$ is infinity almost surely. It would be interesting to decide whether the obstruction to make work the proof of Theorem 1 is related to a blow

up phenomenon, i.e. can one prove a blow up of the solutions of (7.1) with data on a set A such that $\rho(A) > 0$? A related and probably simpler question is whether one can construct a blow up solution of NLS with data which is in H^s , $s < 1$ but not in H^1 ?

If we suppose the defocusing assumption $V(z) \leq 0$ then there is no problem with the integrability of $f(u)$ and the L^2 cut-off is not needed.

Let us notice that the restriction $\alpha < 2$ is too strong for the well-posedness analysis of (1.1) with data in \mathcal{X} . Indeed this analysis seems to hold true for $\alpha < 4$. Here is a rough explanation. Essentially speaking, in order to make work the nonlinear estimates with data of Sobolev regularity $< 1/2$, after $k \in \mathbb{N}$ expansions of the nonlinearity, for

$$N_2 \geq N_3 \geq \dots \geq N_k$$

one should control the expression

$$(11.1) \quad (N_2 N_3)^\varepsilon N_4 \dots N_k$$

by

$$C(N_2 N_3 \dots N_k)^\sigma$$

for some $\sigma < 1/2$. This leads to the restriction $k - 3 < \frac{1}{2}(k - 1)$, i.e. $k < 5$ which corresponds to $\alpha < 4$. In (11.1) the factor $N_4 \dots N_k$ appear from Sobolev embeddings which in $2d$ costs $\frac{d}{2} = 1$ derivatives (see [5] for a similar discussion). However for $\alpha \geq 2$, the Sobolev inequality is no longer available to give sense of $\int_\Theta V(u)$ for $u \in \mathcal{X}$. On the other hand one only needs to show that $\int_\Theta V(u)$ is finite μ almost surely. This seems to be tractable by some Gaussian estimates and the bounds of Lemma 2.1. We plan to pursue this issue elsewhere.

The measure μ constructed in Theorem 1 is obtained for functions on the disc of radius $r = 1$. Similar measures can be constructed for any finite radius r and the limiting behaviour of these measures as $r \rightarrow \infty$ seems to be an interesting problem.

One can also ask the question about ergodicity properties of the measure ρ , i.e. the existence of “non trivial” ρ measurable sets invariant under the flow.

Let us finally mention an extension of Theorem 1. One can construct invariant measures leaving on functions invariant by the rotations of the disc (see [4]). In this case, in the polar coordinates (r, φ) on Θ , the measure “lives” on the set of functions

$$(11.2) \quad e^{in\varphi} \sum_{k \geq 1} \frac{g_k(\omega)}{z_{nk}} \frac{J_{nk}(z_{nk}r)}{\|J_{nk}(z_{nk}\cdot)\|_{L^2(\Theta)}},$$

where J_n , $n \geq 0$, $n \in \mathbb{Z}$ is the Bessel function of order n and z_{nk} are its zeros (Theorem 1 corresponds to $n = 0$). In (11.2), $g_k(\omega)$ is again a sequence of normalized i.i.d. complex random variables.

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