

Multiple-Spike Ground State Solutions of the Gierer-Meinhardt Equations for Biological Activator-Inhibitor Systems

Yuncheng You

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ABSTRACT. In many biological pattern formation processes and in some chemical or biochemical reactions, an activator-inhibitor system of two reaction-diffusion equations serves as a mathematical model, typically, the Gierer-Meinhardt equations. This type of model equations features two largely different diffusion coefficients and an essentially nonlocal nonlinearity. In this paper, the one-dimensional Gierer-Meinhardt equations are considered,

$$\begin{aligned} A_t &= d\Delta A - A + \frac{A^2}{H} = 0 \quad \text{in } \mathbb{R}, \\ H_t &= D\Delta H - H + A^2 = 0 \quad \text{in } \mathbb{R}, \\ A, H &> 0 \text{ and } A, H \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{aligned}$$

where $\sigma^2 = d/D \ll 1$. By the Lyapunov-Schmidt method, a sharp order-estimate of the number k of multiple spikes of the ground state solutions is made. The k -spike solutions are constructed by adding small perturbation to the function which has k appropriately distributed spikes resembling the solution of the problem

$$\begin{aligned} \Delta u - u + u^2 &= 0 \quad \text{in } \mathbb{R}, \\ 0 < u &\rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned}$$

The main result is that, for sufficiently small $\sigma > 0$, there exists such a ground state solution with $k = \text{const } \sigma^{-\beta}$, where $0 < \beta < 1/2$ and β can be arbitrarily close to $1/2$. In the proof of this conjecture, *a priori* estimates of linear and nonlinear parts are conducted by means of cut-off decomposition, sharp calculations of multiple spike interactions at all levels, and finally a fine-tuned adjustment of spike centers.

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1. Introduction

In 1952, A. M. Turing [21] showed that a reaction-diffusion system modeling two chemical substances, called morphogens, could generate a spatially nonuniform pattern from a spatially almost uniform steady-state, which is stable in the absence of diffusion, by using a symmetry-breaking instability analysis driven by the distinguished diffusion coefficients. Then it has been found experimentally and numerically [2, 7, 8, 12, 14] that, for many reaction-diffusion systems, the ground states as well as the evolving dynamics can exhibit a common phenomenon called *point condensation* [20, 26, 28], which consists of spike-type patterns with bumps around certain spatial points. Although the microscopic and/or biochemical reasons causing such phenomena of pattern formation are still under investigation in many cases, mathematically a coherent theory for the existence and stability of the multi-spike solutions (or called multi-bump solutions, multi-spot solutions) for several typical reaction-diffusion systems has been established based on the methods in nonlinear elliptic partial differential equations, linear stability analysis for parabolic equations, and singular perturbation analysis. Many of these results are listed in the References.

The Gierer-Meinhardt model [7] proposed in 1972 and the Gray-Scott model [8] proposed in 1983, each having certain variations in terms of the nonlinearity, are two typical models of reaction-diffusion systems which have been intensively studied by a number of authors in the last two decades, cf. [1, 3–5, 11, 23, 25, 27, 28, 30–32, 35, 36] for the Gierer-Meinhardt system and [6, 15, 22, 26, 29, 33, 34] for the Gray-Scott system.

The Gierer-Meinhardt model describes the activator-inhibitor coupled behavior for many systems in cell biology and physiology [7, 12, 14], whereas the Gray-Scott model characterizes self-replicating patterns of some autocatalytic and feedback systems in reaction kinetics and biochemistry [2, 6, 8]. It is interesting to note that both model systems admit specific symmetric and asymmetric multi-spike (multi-bump) patterns in one-dimensional and two-dimensional spatial domains, as seen in the listed references and additional papers cited therein.

Motivated by and based on the pioneering work during 1986–1996 [13, 16–20] on the profiles, especially the location and the shape of peaks, of the variational solutions to Neumann problems of semilinear elliptic PDEs, researches have been conducted for Neumann problems of single nonlinear elliptic PDEs [9, 10] in the

form

$$\epsilon^2 \Delta u - u + f(u) = 0,$$

and for Gierer-Meinhardt system and Gray-Scott system on the existence and construction of K -spike solutions, where K is any given positive integer, by allowing the coefficient ϵ^2 in the above equation and the ratio σ^2 of two largely different diffusion coefficients in Gierer-Meinhardt or Gray-Scott system to be sufficiently small, while K is fixed and can be large. A common and notable feature in these works, cf. [3, 4, 11, 23–25, 28–36], is the leverage of using the *Lyapunov-Schmidt method* of finite dimensional reduction to find the multi-spike profiles of ground state solutions as a small perturbation of a basis function whose multiple spikes are well shaped and distributed.

Consider a prescaled Gierer-Meinhardt system of two coupled nonlinear parabolic equations,

$$\begin{aligned} (GM) \quad & A_t = d\Delta A - A + \frac{A^2}{H}, \quad x \in \Omega, t > 0, \\ & H_t = D\Delta H - H + A^2, \quad x \in \Omega, t > 0, \\ & \frac{\partial A}{\partial n} = \frac{\partial H}{\partial n} = 0, \quad x \in \partial\Omega, t \geq 0, \end{aligned}$$

where Ω is a 1D or 2D bounded, Lipschitzian domain, $A = A(x, t)$ and $H = H(x, t)$ are the concentrations of an activator substance and an inhibitor substance, respectively. As we know, an important thing is to study the positive, non-constant, steady states called ground states of this GM system. They are the solutions of the elliptic system:

$$\begin{aligned} d\Delta A - A + \frac{A^2}{H} &= 0, \quad x \in \Omega, \\ D\Delta H - H + A^2 &= 0, \quad x \in \Omega, \\ \frac{\partial A}{\partial n} = \frac{\partial H}{\partial n} &= 0, \quad x \in \partial\Omega. \end{aligned}$$

In the original model in [7] and in many other biological settings, it is suggested or has been justified that one can make the assumption of a slowly diffusing activator and a rapidly diffusing inhibitor, which implies that $d \ll D$ and $\sigma^2 = d/D \ll 1$.

We can rescale the independent variables and two unknowns A and H as follows; set

$$u(x) = \sigma^2 A(d^{1/2}x), \quad v(x) = \sigma^2 H(d^{1/2}x).$$

Then the above elliptic system reduces to the following equivalent system,

$$\begin{aligned} \Delta u - u + \frac{u^2}{v} &= 0, \quad x \in \Omega_d, \\ \Delta v - \sigma^2 v + u^2 &= 0, \quad x \in \Omega_d, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0, \quad x \in \partial\Omega_d. \end{aligned}$$

Here, $\Omega_d = d^{-1/2}\Omega$. When $d \rightarrow 0$ but σ remains quantitatively stabilized, for the one-dimensional domain we come up with the limiting system

$$(1.1) \quad \begin{aligned} u'' - u + \frac{u^2}{v} &= 0, & x \in \mathbb{R}, \\ v'' - \sigma^2 v + u^2 &= 0, & x \in \mathbb{R}, \\ u, v > 0 \text{ and } u, v &\rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned}$$

The system (1.1) and its two-dimensional counterpart serve as a natural approximation to the Gierer-Meinhardt ground state equations when Ω_d is very large and a pattern formation occurs in the interior of the domain away from the boundary.

A particularly notable question concerning the pattern formation of the ground states of the Gierer-Meinhardt system (1.1) and its two-dimensional counterpart stands as follows:

Is it true that there exist solutions with an arbitrarily large number of spikes as the diffusion ratio parameter σ gets smaller and smaller? If so, then what is the estimate of the maximal multiple spike number k in terms of a sufficiently small given σ ?

The first half of this question has been affirmatively answered by [3] for the 1D case and by [4] for the 2D case. There are other related results in the listed references too. What they have proved is that given an arbitrarily large integer K , there is a small constant $\sigma_K > 0$ such that if $0 < \sigma \leq \sigma_K$, then there exists a ground state solution of (1.1), which exhibits exactly K spikes in the activator component. There are a couple of different mechanisms to mathematically construct these multi-spike ground state solutions.

In this work, we shall answer the second half of the aforementioned question. Specifically, the stand point is: given an arbitrarily small $\sigma > 0$, how many (namely up to what order k relative to σ) spikes can one expect to occur in a ground state solution of the limiting Gierer-Meinhardt system? As far as we are aware, this is an open problem.

We shall take the approach of using the Lyapunov-Schmidt method to tackle this problem. The entire work consists of three stages: the linear part, the nonlinear part, and the final solution to the finite dimensional reduction. In dealing with the variable number k of multiple spikes in an unknown order of power σ as conjectured, we have to conduct sophisticated *a priori* estimates in order to get the sharpest estimates in each step throughout the process of analysis.

In treating the linear part, which is the principal approximation of the Fréchet derivative of the nonlinear, nonlocal operator of the reduced single equation, a cut-off decomposition plays a key role to achieve the bounded invertibility of this linear operator. In the nonlinear part, we have to integrate all the aspects of multiple spike interactions at several levels and to assemble all the estimates together for a sharpest attempt.

In the rest of this section, we shall set up some basic facts and concepts involved in this paper. Most importantly, we shall establish several Tool Lemmas which will be very instrumental and frequently used in the subsequent sections.

First of all, what we call spike (or bump) means a function which resembles the unique analytic solution $U(x)$ to the ODE problem:

$$(1.2) \quad \begin{aligned} u'' - u + u^2 &= 0, \quad x \in \mathbb{R}, \\ 0 < u(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

We can solve (1.2) to get the explicit form of $U(x)$ (see Appendix A):

$$(1.3) \quad U(x) = 6 \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{6e^x}{(1+e^x)^2} = \frac{6e^{-x}}{(1+e^{-x})^2}, \quad x \in \mathbb{R}.$$

This $U(x)$ is an even, positive function. The following properties of U will be used frequently.

LEMMA 1.1. *For the spike function $U(x)$ given by (1.3), the following properties hold.*

$$(1) \quad U(x) = \frac{6e^{-|x|}}{(1+e^{-|x|})^2}, \quad x \in \mathbb{R}. \quad \text{We have}$$

$$(1.4) \quad \begin{aligned} U(x) &= |U(x)| \leq 6e^{-|x|}, \quad x \in \mathbb{R}, \\ U(x) &= 6e^{-|x|} \left(1 + O\left(e^{-|x|}\right)\right), \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

$$(2) \quad U'(x) = \frac{6e^x(1-e^x)}{(1+e^x)^3} = \frac{-6e^{-x}(1-e^{-x})}{(1+e^{-x})^3}, \quad x \in \mathbb{R}, \quad \text{and } U' \text{ is an odd function,}$$

$$(1.5) \quad \begin{aligned} |U'(x)| &\leq 6e^{-|x|}, \quad x \in \mathbb{R}, \\ |U'(x)| &= 6e^{-|x|} \left(1 + O\left(e^{-|x|}\right)\right), \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

$$(3) \quad U''(x) = \frac{6e^{-|x|}}{(1+e^{-|x|})^4} (1 - 4e^{-|x|} + e^{-2|x|}), \quad x \in \mathbb{R}.$$

$$(4) \quad \text{We have}$$

$$(1.6) \quad \begin{aligned} \int_{\mathbb{R}} U(x) dx &= \int_{\mathbb{R}} U^2(x) dx = 6, \quad \int_{\mathbb{R}} U^3(x) dx = \frac{36}{5}, \\ \int_{\mathbb{R}} |U'(x)| dx &= 3, \quad \int_{\mathbb{R}} |U'(x)|^2 dx = \frac{6}{5}, \\ \int_{\mathbb{R}} |x|U(x) dx &= 12 \log 2, \quad \int_{\mathbb{R}} |x|U^2(x) dx = 12 \log 2 - 3, \\ \int_{\mathbb{R}} e^{-|x|}U^2(x) dx &= 3, \quad \int_{\mathbb{R}} e^{-x}U^2(x) dx = 12. \end{aligned}$$

PROOF. These integrals can be calculated directly. See Appendix B. \square

The following two lemmas are most instrumental throughout this paper, which allows us to make the sharpest estimates.

LEMMA 1.2. *Let r be a given real number and $p_r(x)$ be the function*

$$p_r(x) = e^{-|x|}e^{-|x+r|}, \quad x \in \mathbb{R}.$$

Then the following properties hold:

$$(1.7) \quad \begin{aligned} |p_r(x)| &= p_r(x) \leq e^{-|r|}, \quad x \in \mathbb{R}, \\ \int_{\mathbb{R}} p_r(x) dx &= \int_{\mathbb{R}} e^{-|x|} e^{-|x+r|} dx = (1 + |r|) e^{-|r|}, \end{aligned}$$

and

$$(1.8) \quad \int_{\mathbb{R}} |x| p_r(x) dx = \int_{\mathbb{R}} |x| e^{-|x|} e^{-|x+r|} dx = \frac{1}{2} e^{-|r|} (1 + |r| + r^2),$$

$$(1.9) \quad \begin{aligned} \int_{\mathbb{R}} |x|^2 p_r(x) dx &= \int_{\mathbb{R}} |x|^2 e^{-|x|} e^{-|x+r|} dx = e^{-|r|} \left[\frac{1}{2} (|r| + r^2) + \frac{1}{3} |r|^3 \right], \\ \int_{\mathbb{R}} |x|^2 p_r(x)^2 dx &= \int_{\mathbb{R}} |x|^2 e^{-2|x|} e^{-2|x+r|} dx = e^{-2|r|} \left(\frac{1}{16} + \frac{|r|}{8} + \frac{r^2}{4} + \frac{|r|^3}{3} \right) \\ &\leq \frac{1}{3} e^{-2|r|} (1 + |r|)^3, \end{aligned}$$

$$(1.10) \quad \int_{\mathbb{R}} |x| e^{-|x|} e^{-2|x+r|} dx \leq 2e^{-|r|} (|r| + e^{-|r|}).$$

PROOF. The function $p_r(x)$ can be explicitly written as follows:

$$p_r(x) = \begin{cases} e^{-(2x+r)}, & x \in [0, \infty), \\ e^{-r}, & x \in [-r, 0], \\ e^{2x+r}, & x \in (-\infty, -r], \end{cases} \quad \text{if } r > 0;$$

$$p_r(x) = \begin{cases} e^{-(2x+r)}, & x \in [-r, \infty), \\ e^r, & x \in [0, -r], \\ e^{2x+r}, & x \in (-\infty, 0], \end{cases} \quad \text{if } r < 0.$$

Then (1.7)–(1.10) can be validated by straightforward integration, using substitutions or integration by parts. See Appendix C for the details. \square

REMARK 1. Since these relevant integrals we deal with in the subsequent sections are calculated exactly in Lemma 1.2, without any possible improvement, the corresponding estimates will be deemed as sharp estimates.

Let β be a constant, $0 < \beta < 1$. For sufficiently small $\sigma > 0$, let

$$(1.11) \quad r_\sigma = (1 - \beta) |\log \sigma|.$$

We have, for any positive integer m ,

$$(1.12) \quad e^{-mr_\sigma} = e^{-m(1-\beta)|\log \sigma|} = \sigma^{m(1-\beta)}.$$

The following estimates on the sums of finite geometric and mixed series of uncertain length are very useful in the sequel.

LEMMA 1.3. Let $k \geq 1$ be an integer and let $0 \leq b_j \leq 2$ for $j = 1, \dots, k$. For $r = r_\sigma$ given by (1.11), we have

$$(1.13) \quad \sum_{j=1}^k b_j e^{-jr} \leq \frac{2\sigma^{1-\beta}}{1 - \sigma^{1-\beta}},$$

and

$$(1.14) \quad \sum_{j=1}^k b_j (j |\log \sigma|) e^{-jr} \leq \frac{2 |\log \sigma| \sigma^{1-\beta}}{(1 - \sigma^{1-\beta})^2},$$

for sufficiently small σ .

PROOF. (1.13) is obviously true. We show (1.14) by using Abel's transformation

$$\sum_{i=1}^m a_i d_i = a_m D_m - \sum_{i=1}^{m-1} (a_{i+1} - a_i) D_i, \quad \text{with } D_i = \sum_{j=1}^i d_j.$$

This is the discrete version of integration by parts. Thus we can regard jr as a_j and e^{-jr} as d_j to get

$$\begin{aligned} \sum_{j=1}^k b_j(j|\log \sigma|)e^{-jr} &= \sum_{j=1}^k \frac{b_j}{1-\beta} (jr)e^{-jr} \leq \frac{2}{1-\beta} \sum_{j=1}^k jre^{-jr} \\ &= a \left\{ \frac{kre^{-r}(1-e^{-kr})}{1-e^{-r}} - \sum_{j=1}^{k-1} \frac{re^{-r}(1-e^{-jr})}{1-e^{-r}} \right\} \quad (\text{here } a = \frac{2}{1-\beta}) \\ &\leq a \left\{ \frac{kre^{-r}}{1-e^{-r}} - \frac{re^{-r}}{1-e^{-r}} \left(k-1 - \sum_{j=1}^{k-1} e^{-jr} \right) \right\} \\ &= a \left\{ \frac{re^{-r}}{1-e^{-r}} + \frac{re^{-r}}{1-e^{-r}} \cdot \frac{e^{-r}(1-e^{-(k-1)r})}{1-e^{-r}} \right\} \\ &\leq \frac{are^{-r}}{1-e^{-r}} \left(1 + \frac{e^{-r}}{1-e^{-r}} \right) = \frac{are^{-r}}{(1-e^{-r})^2} = \frac{2|\log \sigma|\sigma^{1-\beta}}{(1-\sigma^{1-\beta})^2}. \end{aligned}$$

□

As a corollary of (1.14), for sufficiently small σ , there is a uniform constant $C_0 > 0$, such that

$$(1.15) \quad \sum_{j=1}^k b_j[j(1+|\log \sigma|)]e^{-jr} \leq \frac{C_0|\log \sigma|\sigma^{1-\beta}}{(1-\sigma^{1-\beta})^2}.$$

Let integer $k \geq 2$ and $0 < \beta < 1$ be given. Define set in \mathbb{R}^k as follows,

$$(1.16) \quad \Lambda = \Lambda(\sigma, \beta) = \left\{ \xi = (\xi_1, \dots, \xi_k) : \begin{array}{l} \xi_j < \xi_{j+1}, j = 1, \dots, k-1, \sum_{j=1}^k \xi_j = 0, \\ \text{and } (1-\beta)|\log \sigma| \leq |\xi_j - \xi_{j+1}| \leq 1 + |\log \sigma|, \\ j = 1, \dots, k-1. \end{array} \right\}.$$

With the observation of (1.15), in the subsequent sections, we can replace the inequality $|\xi_j - \xi_{j+1}| \leq 1 + |\log \sigma|$ by writing $|\xi_j - \xi_{j+1}| \leq |\log \sigma|$, just for notational simplicity, that will not affect any estimate. We shall not repeat this remark.

For any given $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$, define a corresponding multi-spike function

$$(1.17) \quad W(x) = \sum_{j=1}^k U(x - \xi_j), \quad x \in \mathbb{R}.$$

The partial derivatives of W will be denoted by

$$(1.18) \quad W_j(x) = \frac{\partial W}{\partial \xi_j} = -U'(x - \xi_j), \quad j = 1, \dots, k.$$

By (1.6), we have

$$(1.19) \quad \int_{\mathbb{R}} |W_j(x)| dx = 3 \text{ and } \int_{\mathbb{R}} |W_j(x)|^2 dx = \frac{6}{5}, \quad j = 1, \dots, k.$$

The following is a known result [4].

LEMMA 1.4. *The unique variational solution in $H^1(\mathbb{R})$ to the problem*

$$\begin{aligned} -\varphi'' + (1 - 2U(x))\varphi &= 0 \quad \text{in } \mathbb{R}, \\ \varphi(x) &\rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{aligned}$$

is $\varphi(x) = U'(x)$. Let L_1 be the operator $L_1(\varphi) = -\varphi'' + (1 - 2W(x))\varphi$. Then the kernel of L_1 is

$$\ker L_1 = \text{Span} \{W_1, \dots, W_k\}.$$

The following lemma shows that the integral in (1.20) has the same order of k which affects some estimates we will encounter.

LEMMA 1.5. *For any given $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$, given in (1.16), if $\mu > 0$ is a constant, then*

$$(1.20) \quad \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx \leq \frac{2}{\mu} \left[k - (k-1)\sigma^{\mu/2} \right],$$

for $0 < \sigma < 1$.

PROOF. We can directly compute this integral to get

$$\begin{aligned} \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx &= \int_{-\infty}^{\xi_1} e^{-\mu(\xi_1 - x)} + \int_{\xi_1}^{\frac{1}{2}(\xi_1 + \xi_2)} e^{-\mu(x - \xi_1)} \\ &\quad + \int_{\frac{1}{2}(\xi_1 + \xi_2)}^{\xi_2} e^{-\mu(\xi_2 - x)} + \dots + \int_{\xi_k}^{\infty} e^{-\mu(x - \xi_k)} \\ &= \frac{2k}{\mu} - \frac{2(k-1)}{\mu} e^{-\frac{\mu}{2}(1 + |\log \sigma|)} = \frac{2}{\mu} \left[k - (k-1)\sigma^{\mu/2} \right]. \end{aligned}$$

□

Let us also introduce a cut-off decomposition for functions. Define $\theta(x)$ to be a truncating function:

$$(1.21) \quad \theta(x) = \theta_{\sigma}(x) = \begin{cases} 1, & \text{for } |x| \leq r_{\sigma}/4, \\ \text{smooth and decreasing,} & \text{for } |x| \in [r_{\sigma}/4, r_{\sigma}/2], \\ 0, & \text{for } |x| \geq r_{\sigma}/2, \end{cases}$$

where r_{σ} is given in (1.11), such that $\theta \in C^2(\mathbb{R})$. For small σ , we can make $|\theta'(x)| \leq 1$. Then, for any given $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$, a function $\varphi(x)$ defined on \mathbb{R} can be written as

$$(1.22) \quad \varphi(x) = \sum_{j=0}^k \varphi_j(x) = \varphi_0(x) + \sum_{j=1}^k \varphi(x)\theta(x - \xi_j),$$

namely, $\varphi_j(x) = \varphi(x)\theta(x - \xi_j)$, $j = 1, \dots, k$. This will be referred to as the cut-off decomposition with respect to ξ . By the separation of the points ξ_1, \dots, ξ_k we have

$$\text{supp } \varphi_i \cap \text{supp } \varphi_j = \emptyset, \quad \text{for } i \neq j \text{ in } \{1, \dots, k\}.$$

Let

$$(1.23) \quad \Omega_0 = \mathbb{R} \setminus \bigcup_{j=1}^k \Omega_j, \quad \Omega_j = \left[\xi_j - \frac{r_\sigma}{4}, \xi_j + \frac{r_\sigma}{4} \right], \quad j = 1, \dots, k.$$

Then $\text{supp } \varphi_0 \subset \Omega_0$ and

$$(1.24) \quad \Omega = \bigcup_{j=0}^k \Omega_j.$$

2. Main Result and Approach of Proof

We use Δ to denote the second-order distributional derivative of a function defined on \mathbb{R} , namely, $\Delta\varphi = \varphi''$. We shall use I to denote the identity operator. Consider the linear differential operator

$$(2.1) \quad \begin{aligned} L_o(\varphi) &= (-\Delta + \sigma^2 I)(\varphi), \quad \varphi \in H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}), \\ 0 < \varphi(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

The Green's function associated with L_o is found to be

$$(2.2) \quad G(x, y) = \frac{1}{2\sigma} e^{-\sigma|x-y|}, \quad \text{for } x, y \in \mathbb{R}.$$

Its verification is straightforward and omitted.

First of all, we can rescale the problem (1.1). Let $\tau > 0$ be a parameter, which will be specified later. We can replace u and v in (1.1) by the rescaled τu and τv , then (1.1) becomes the problem

$$(2.3) \quad \begin{aligned} u'' - u + u^2/v &= 0, \\ v'' - \sigma^2 v + \tau u^2 &= 0, \end{aligned} \quad \text{in } \mathbb{R}$$

$u, v > 0$ and $u, v \rightarrow 0$ as $|x| \rightarrow \infty$.

Since $L_o : H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is invertible and its inverse operator is bounded, the second ODE in the system (2.3) can be solved for v , and we have

$$(2.4) \quad v = T[u^2] \triangleq L_o^{-1}(\tau u^2) = (-\Delta + \sigma^2 I)^{-1}(\tau u^2).$$

By substituting (2.4) into the first ODE in (2.3), the rescaled problem (2.3) is reduced to a single nonlocal nonlinear ODE problem:

$$(2.5) \quad \begin{aligned} S[u] \triangleq u'' - u + \frac{u^2}{T[u^2]} &= 0 \quad \text{in } \mathbb{R}, \\ u > 0 \text{ and } u &\rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned}$$

The operator S is Fréchet differentiable. In order to distinguish the solution to the original problem (1.1) and the solution to the rescaled problems (2.3) and (2.5), we shall retain the notation (u, v) for (2.3) and (2.5) and denote the solution to (1.1) by (u^*, v^*) .

We take a normalization parameter τ to be

$$(2.6) \quad \tau = \tau_\sigma = \frac{2\sigma}{\omega}, \quad \text{where } \omega = \int_{\mathbb{R}} W^2(x) dx,$$

where $W(x)$ is the multi-spike function given in (1.17).

In this work, we shall prove the following main result, Theorem 2.1.

THEOREM 2.1. *There exists a positive number σ_0 such that for any given σ , $0 < \sigma \leq \sigma_0$, there exists a solution (u, v) to the one-dimensional Gierer-Meinhardt system (2.3) with the following properties.*

(1) *The u component has k spikes, with*

$$(2.7) \quad k = \text{const } \sigma^{-\beta},$$

where $0 < \beta < 1/2$ can be arbitrarily close to $1/2$,

and such that

$$(2.8) \quad \lim_{\sigma \rightarrow 0} \left| u(x) - \sum_{j=1}^k U(x - \xi_j) \right| = 0, \quad \text{uniformly in } x \in \mathbb{R}.$$

(2) *The v component satisfies*

$$(2.9) \quad \lim_{\sigma \rightarrow 0} |v(\xi_j + x) - 1| = 0, \quad \text{uniformly on any compact set of } x,$$

for $j = 1, \dots, k$.

Here $\xi = (\xi_1, \dots, \xi_k)$ is some vector point in the set Λ defined by (1.16). Moreover, the corresponding multi-spike ground state solution to the original Gierer-Meinhardt system (1.1) exists and is given by $(u^, v^*) = (\tau u, \tau v)$.*

The rest of the paper is devoted to proving this main result. Our goal is to reach a sharp estimate on the order of the multiple spike number k in terms of σ . Here let us lay out the approach of the proof.

Since the problem (2.3) has been reduced to a single ODE problem (2.5), our strategy is to seek for a solution of (2.5) in the form

$$(2.10) \quad u(x) = W(x) + \varphi(x), \quad x \in \mathbb{R},$$

where W is the function of multiple standard spikes introduced in (1.17), and φ ought to be a small perturbation function, which tends to zero as $\sigma \rightarrow 0$ with respect to an appropriate norm.

Let $V(x)$ be the unique solution of the ODE problem:

$$(2.11) \quad \begin{aligned} V'' - \sigma^2 V + \tau W^2 &= 0, \quad x \in \mathbb{R}, \\ 0 < V(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

According to (2.2) and (2.4), $V = T[W^2]$ is expressed as

$$(2.12) \quad V(x) = \frac{\tau}{2\sigma} \int_{\mathbb{R}} e^{-\sigma|x-y|} W^2(y) dy = \frac{1}{\omega} \int_{\mathbb{R}} e^{-\sigma|x-y|} W^2(y) dy,$$

where $\omega = \int_{\mathbb{R}} W^2(y) dy$ is shown in (2.6). When we substitute $u = W + \varphi$ into (2.5), it becomes

$$(2.13) \quad (W'' - W) + (\varphi'' - \varphi) + \frac{W^2}{V} + \frac{2W\varphi}{V} - \frac{2W^2}{V^2} T[W\varphi] + (\text{r.t.}) = 0,$$

where (r.t.) stands for all the remainder terms of the order $O(\varphi^2)$ as φ small. Thus φ satisfies the following equation

$$(2.14) \quad -\varphi'' + \varphi - \frac{2W\varphi}{V} + \frac{2W^2}{V^2} T[W\varphi] = S[W] + N(W, \varphi),$$

where the left-hand side is a linear nonlocal operator on φ , and the right-hand side consists of nonlinear terms

$$(2.15) \quad \begin{aligned} S[W] &= W'' - W + \frac{W^2}{V}, \quad \text{and} \\ N(W, \varphi) &= (r.t.) = \frac{(W + \varphi)^2}{T[(W + \varphi)^2]} - \left(\frac{W^2}{V} + \frac{2W\varphi}{V} - \frac{2W^2}{V^2} T[W\varphi] \right). \end{aligned}$$

We shall take the approach of the Lyapunov-Schmidt method to show the existence of a solution φ of this equation (2.14), which has the properties we demand. As for the specific procedure, we divide the entire proof into the following stages.

STAGE 1. First we shall study the linear part in equation (2.14). But instead of its full form, we will consider the following linear operator, which we call the principal approximation of the linear part in (2.14),

$$(2.16) \quad \begin{aligned} L(\varphi) &\triangleq -\varphi'' + \varphi - 2W\varphi + \frac{2W^2}{\omega} \langle W, \varphi \rangle \\ &= -\varphi'' + (1 - 2W)\varphi + \frac{2W^2}{\omega} \langle W, \varphi \rangle, \quad \varphi \in H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}), \end{aligned}$$

where $\omega = \int_{\mathbb{R}} W^2(x) dx$ and $\langle \cdot, \cdot \rangle$ is the L^2 inner product. Comparing L with the linear operator on the left side of (2.14), we have made two simplifications in (2.16) but preserved all the essentials in the linear part. These two simplifications are: in (2.16),

$$(2.17) \quad \text{we replace } V(x) \text{ by } 1 \text{ and } T[W\varphi] \text{ by } \frac{1}{\omega} \langle W, \varphi \rangle.$$

In Section 3, we shall consider the following problem: Given a function h whose weighted L^∞ norm is finite, and given a set of points $(\xi_1, \dots, \xi_k) \in \Lambda$, find a solution φ and $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$ such that

$$(2.18) \quad \begin{aligned} L(\varphi) &= h + \gamma \cdot \nabla_\xi W = h + \sum_{j=1}^k \gamma_j W_j \quad \text{in } \mathbb{R}, \\ \varphi(x) &\rightarrow 0 \text{ as } |x| \rightarrow \infty, \text{ and} \\ \langle \varphi, W_j \rangle &= 0, \quad \text{for } j = 1, \dots, k. \end{aligned}$$

Under the assumption we made in Theorem 2.1, we shall prove the solution operator is a bounded linear operator with respect to a weighted L^∞ space by using the *a priori* estimates, the cut-off technique, and the Fredholm alternative principle.

STAGE 2. In Section 4, we shall estimate the cost of the replacements made in (2.17), which essentially proves the statement in the main theorem (Theorem 2.1) on v component. Moreover, we shall establish a key result on the estimate of integrals

$$\int_{\mathbb{R}} S[W](x) U'(x - \xi_j) dx, \quad j = 1, \dots, k,$$

which constitutes the main part of the nonlinear side in (2.14) under investigation. We also want to estimate the weighted L^∞ norm of $S[W]$.

STAGE 3. By the Lyapunov-Schmidt method, the problem of seeking for a solution φ of the original equation (2.14) is imbedded into an augmented problem of solving the problem:

$$(2.19) \quad \begin{aligned} S[W + \varphi_{\sigma, \xi}](x) &= \sum_{j=0}^k \gamma_j U'(x - \xi_j), \quad x \in \mathbb{R}, \\ \langle \varphi_{\sigma, \xi}, U'(\cdot - \xi_j) \rangle &= 0, \quad j = 1, \dots, k, \text{ and } \varphi_{\sigma, \xi}(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \end{aligned}$$

for both $\varphi_{\sigma, \xi}$ and $\gamma = (\gamma_1, \dots, \gamma_k)$. This augmented problem is equivalent to solving

$$(2.20) \quad \begin{aligned} L(\varphi) &= S[W] + P(W, \varphi) + \sum_{j=1}^k \gamma_j W_j \\ \langle \varphi, U'(\cdot - \xi_j) \rangle &= 0, \quad j = 1, \dots, k, \\ \varphi(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty, \end{aligned}$$

where L is the linear operator of the principal approximation studied in Section 3, and the nonlinear part $P(W, \varphi)$ is given by

$$(2.21) \quad P(W, \varphi) \triangleq \frac{(W + \varphi)^2}{T[(W + \varphi)^2]} - \left(\frac{W^2}{V} + 2W\varphi - \frac{2W^2}{\omega} \langle W, \varphi \rangle \right).$$

In Section 5, we shall make estimates of the nonlinear terms in $P(W, \varphi)$. Then we shall use the contraction mapping argument and the fixed point theory to show that there exists a solution $(\varphi_{\sigma, \xi}, \gamma)$ of the problem (2.20) under the assumptions we make. Furthermore, an estimate on the integrals

$$\int_{\mathbb{R}} (S[W + \varphi_{\sigma, \xi}](x) - S[W](x)) U'(x - \xi_j) dx, \quad j = 1, \dots, k,$$

will be made as a preparation for the final stage.

STAGE 4. Now the problem is reduced to solving the finite-dimensional problem

$$(2.22) \quad \begin{aligned} \sum_{j=1}^k \gamma_j \int_{\mathbb{R}} U'(x - \xi_j) U'(x - \xi_\ell) dx &= \int_{\mathbb{R}} S[W] U'(x - \xi_\ell) dx \\ &+ \int_{\mathbb{R}} (S[W + \varphi_{\sigma, \xi}] - S[W]) U'(x - \xi_\ell) dx, \\ &\ell = 1, \dots, k, \end{aligned}$$

in the sense that there exists a choice $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$ such that the right-hand side of (2.22) turns out to be zero, so that the component $\gamma_{\sigma, \xi} = (\gamma_1, \dots, \gamma_k)$ of the solution of the augmented problem (2.19) or, equivalently, (2.20), is a zero vector corresponding to this ξ . Hence the linear coefficient matrix of γ is nonsingular for small σ . This proof will be fulfilled by using the Brouwer fixed point theorem. Finally, since there exists a $\xi \in \Lambda$ which makes $\gamma_{\sigma, \xi} = 0$, then the solution $\varphi_{\sigma, \xi}$ to this augmented problem (2.19) is exactly the solution of the problem

$$(2.23) \quad S[W + \varphi_{\sigma, \xi}] = 0.$$

Hence there exists a solution $u = W + \varphi_{\sigma, \xi}$ to the original problem (2.5), which has the property stated in Theorem 2.1.

3. The Bounded Invertibility of the Linear Operator

Here we study the linear operator of the principal approximation L defined by

$$(3.1) \quad \begin{aligned} L(\varphi) &\triangleq -\varphi'' + (1 - 2W)\varphi + \frac{2W^2}{\omega} \langle W, \varphi \rangle, \quad \varphi \in H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}), \\ \text{where } \omega &= \int_{\mathbb{R}} W^2(x) dx, \end{aligned}$$

as we mentioned in (2.16). The formal adjoint operator of L with respect to the L^2 inner product is given by

$$(3.2) \quad L^*(\varphi) \triangleq -\varphi'' + (1 - 2W)\varphi + \frac{2W}{\omega} \langle W^2, \varphi \rangle, \quad \varphi \in H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}).$$

In the sequel, we shall always refer the set Λ to (1.16) and make the assumption (2.7), which will not be repeated in each lemma and theorem. We may only mention the assumption in Theorem 2.1 for this. We shall denote a positive constant without need of specification by a generic C .

First we estimate

$$(3.3) \quad \begin{aligned} L(W_j) &= L(-U'(\cdot - \xi_j)) = 2(W(x) - U(x - \xi_j))U'(x - \xi_j) \\ &\quad - \frac{2W^2}{\omega} \langle W, U'(\cdot - \xi_j) \rangle \\ &= \sum_{i(\neq j)=1}^k 2U(x - \xi_i)U'(x - \xi_j) - \frac{2W^2(x)}{\omega} \langle W, U'(\cdot - \xi_j) \rangle, \end{aligned}$$

where

$$(3.4) \quad \begin{aligned} \left| \sum_{i(\neq j)=1}^k 2U(x - \xi_i)U'(x - \xi_j) \right| &\leq 2 \sum_{i \neq j} U(y) |U'(y + \xi_i - \xi_j)| \quad (\text{by Lemma 1.1}) \\ &\leq 72 \sum_{i \neq j} e^{-|y|} e^{-|y + \xi_i - \xi_j|} \leq 72 \sum_{i \neq j} e^{-|\xi_i - \xi_j|} \quad (\text{by Lemma 1.2}) \\ &\leq 72 \sum_{i \neq j} \sigma^{|i-j|(1-\beta)} \quad (\text{by (1.12)}) \\ &\leq 144 \frac{\sigma^{1-\beta}}{1 - \sigma^{1-\beta}} \leq 288 \sigma^{1-\beta}, \end{aligned}$$

where, and hereafter, we assume that σ is so small that

$$(3.5) \quad 1 - \sigma^{1-\beta} > 1/2.$$

Moreover, we have

$$\left| -\frac{2W^2(x)}{\omega} \langle W, U'(\cdot - \xi_j) \rangle \right|$$

$$= \frac{2}{\omega} \left(\sum_{j=1}^k U^2(x - \xi_j) + \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) \right) \\ \times \sum_{i(\neq j)=1}^k \left| \int_{\mathbb{R}} U(y - \xi_i) U'(y - \xi_j) dy \right|$$

in which,

$$\omega = \int_{\mathbb{R}} W^2(x) dx \geq \sum_{i=1}^k \int_{\mathbb{R}} U^2(x - \xi_i) dx = 6k, \\ (3.6) \quad \sum_{j=1}^k U^2(x - \xi_j) = U^2(x - \xi_{j_0}) + \sum_{j \neq j_0} U^2(x - \xi_j) \\ \leq 36 + \frac{2\sigma^{2(1-\beta)}}{1 - \sigma^{2(1-\beta)}} \leq 36 + 4\sigma^{2(1-\beta)}$$

where $\min_{1 \leq j \leq k} |x - \xi_j| = |x - \xi_{j_0}|$, and

$$\sum_{\substack{i,j=1 \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) \leq 36 \sum_{\substack{i,j \\ i \neq j}} \sigma^{|i-j|(1-\beta)} \leq 36k (4\sigma^{1-\beta}) \leq C\sigma^{1-2\beta}.$$

Besides we have

$$\sum_{i(\neq j)=1}^k \left| \int_{\mathbb{R}} U(y - \xi_i) U'(y - \xi_j) dy \right| \leq \sum_{i \neq j} (1 + |\xi_i - \xi_j|) e^{-|\xi_i - \xi_j|} \\ \text{(by Lemma 1.2)} \\ \leq \sum_{i \neq j} \left(1 + \frac{1}{1-\beta} |i-j|(1-\beta) |\log \sigma| \right) \sigma^{|i-j|(1-\beta)} \quad \text{(by Lemma 1.3)} \\ \leq 4\sigma^{1-\beta} + 8|\log \sigma| \sigma^{1-\beta}.$$

Then we get

$$(3.7) \quad \left| -\frac{2W^2(x)}{\omega} \langle W, U'(\cdot - \xi_j) \rangle \right| \leq \frac{1}{3k} (36 + C\sigma^{1-2\beta}) (4 + 8|\log \sigma|) \sigma^{1-\beta} \\ \leq C(1 + |\log \sigma|) \sigma, \quad \text{for some constant } C > 0,$$

because $k = C\sigma^{-\beta}$. It follows from (3.3), (3.4) and (3.5) that

$$(3.8) \quad |L(W_j)| \leq 288\sigma^{1-\beta} + C(1 + |\log \sigma|) \sigma.$$

Similarly, we have

$$\begin{aligned}
(3.9) \quad & \left| -\frac{2W(x)}{\omega} \langle W^2, U'(\cdot - \xi_j) \rangle \right| \leq \frac{1}{3k} (6 + 4\sigma^{1-\beta}) \left| \int_{\mathbb{R}} W^2(x) U'(x - \xi_j) dx \right| \\
& \leq \frac{1}{3k} (6 + 4\sigma^{1-\beta}) \int_{\mathbb{R}} |W(x)| \sum_{i(\neq j)=1}^k U(x - \xi_i) |U'(x - \xi_j)| dx \\
& \leq \frac{1}{3k} (6 + 4\sigma^{1-\beta})^2 (4\sigma^{1-\beta} + 8|\log \sigma|) \sigma^{1-\beta} \\
& \leq \tilde{C}(1 + |\log \sigma|)\sigma, \quad \text{for some constant } \tilde{C} > 0.
\end{aligned}$$

Then from (3.2), (3.4) and (3.9) it follows that

$$(3.10) \quad |L^*(W_j)| \leq 288\sigma^{1-\beta} + \tilde{C}(1 + |\log \sigma|)\sigma.$$

Thus we have proved the following lemma.

LEMMA 3.1. *Under the assumption of Theorem 2.1, it holds that*

$$(3.11) \quad \begin{aligned} & |L(W_j)| \leq C\sigma^{1-\beta} \quad \text{or} \quad L(W_j) = O(\sigma^{1-\beta}), \\ & |L^*(W_j)| \leq \tilde{C}\sigma^{1-\beta} \quad \text{or} \quad L^*(W_j) = O(\sigma^{1-\beta}), \end{aligned} \quad j = 1, \dots, k.$$

PROOF. (3.11) follows from (3.8) and (3.10), since $0 < \beta < 1$, the term $(1 + |\log \sigma|)\sigma$ is dominated by the $\sigma^{1-\beta}$ term when σ is sufficiently small. \square

For a fixed $0 < \mu \leq \frac{1}{3}$, define $L_\mu^\infty = L_\mu^\infty(\mathbb{R})$ to be the weighted space:

$$\begin{aligned}
(3.12) \quad & L_\mu^\infty = \left\{ \psi \in H_0^1(\mathbb{R}) : \limsup_{|x| \rightarrow \infty} e^{\mu|x|} |\psi(x)| < \infty \right\}, \\
& \text{with the norm } \|\psi\|_\mu = \sup \left\{ e^{\mu \min_{1 \leq i \leq k} |x - \xi_i|} |\psi(x)| \right\}.
\end{aligned}$$

Note that once k and (ξ_1, \dots, ξ_k) are fixed, L_μ^∞ is a Banach space. Moreover, for a given k and different $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$, the norms $\|\cdot\|_\mu$ are all equivalent.

Now consider the following problem: Given a function $h \in L_\mu^\infty (\subset L^2(\mathbb{R}))$ and a $\xi \in \Lambda$, find a solution $(\varphi, \gamma) \in L_\mu^\infty \times \mathbb{R}^k$, $\gamma = (\gamma_1, \dots, \gamma_k)$, of the following problem,

$$\begin{aligned}
(3.13) \quad & L(\varphi) = h + \gamma \cdot \nabla_\xi W = h + \sum_{j=1}^k \gamma_j W_j \quad \text{in } \mathbb{R}, \\
& \langle \varphi, W_j \rangle = 0, \quad j = 1, \dots, k, \quad \text{and} \\
& \varphi(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.
\end{aligned}$$

We address the property of such a vector $\gamma = (\gamma_1, \dots, \gamma_k)$ which satisfies equation (3.13) in the following lemma.

LEMMA 3.2. *Suppose $\varphi \in L_\mu^\infty$ and $\gamma = (\gamma_1, \dots, \gamma_k)$ satisfy equation (3.13), where $h \in L_\mu^\infty$ is given. Then it holds that, for $j = 1, \dots, k$,*

$$(3.14) \quad |\gamma_j| \leq \|\gamma\|_{\mathbb{R}^k} \leq C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu,$$

where C_1 and $C_2(\sigma)$ are positive constants, and $C_2(\sigma) = O(\sigma^{1-\frac{3\beta}{2}})$.

PROOF. Taking the L^2 inner product of (3.13) with each W_j , we can get

$$(3.15) \quad \langle \varphi, L^*(W_j) \rangle = \langle h, W_j \rangle + \gamma_j \int_{\mathbb{R}} |W_j(x)|^2 dx + \sum_{i(\neq j)=1}^k \gamma_i \langle W_i, W_j \rangle, \quad j = 1, \dots, k.$$

From Lemma 1.1, we know $\int_{\mathbb{R}} |W_j|^2 dx = 6/5$. By Lemma 3.1, we have

$$|\langle \varphi, L^*(W_j) \rangle| \leq \int_{\mathbb{R}} \tilde{C} \sigma^{1-\beta} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx \|\varphi\|_{\mu} \leq C \sigma^{1-\beta} \|\varphi\|_{\mu}.$$

We also have

$$\begin{aligned} |\langle h, W_j \rangle| &\leq 6 \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} e^{-|x - \xi_j|} dx \|h\|_{\mu} \\ &\leq 6 \|h\|_{\mu} \left\{ \int_{|x - \xi_j| \leq r_{\sigma}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx + \left(\int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx \right) \sigma^{1-\beta} \right\} \end{aligned}$$

For $i \neq j$, we find that by Lemma 1.2,

$$\begin{aligned} |\langle W_i, W_j \rangle| &\leq \int_{\mathbb{R}} 36 e^{-|x|} e^{-|x + \xi_i - \xi_j|} dx \leq 36 (1 + |\xi_i - \xi_j|) e^{-|\xi_i - \xi_j|} \\ &\leq 36 (1 + |i - j| |\log \sigma|) \sigma^{(1-\beta)|i-j|}. \end{aligned}$$

Since (3.15) can be written as a linear system with respect to γ_j 's,

$$(3.16) \quad \left(\frac{6}{5} I + A \right) \gamma = \begin{pmatrix} \langle \varphi, L^*(W_1) \rangle - \langle h, W_1 \rangle \\ \vdots \\ \langle \varphi, L^*(W_k) \rangle - \langle h, W_k \rangle \end{pmatrix},$$

and the matrix A is given by

$$A = \begin{pmatrix} 0 & \langle W_2, W_1 \rangle & \cdots & \langle W_k, W_1 \rangle \\ \langle W_1, W_2 \rangle & 0 & \cdots & \langle W_k, W_2 \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle W_1, W_k \rangle & \langle W_2, W_k \rangle & \cdots & \langle W_k, W_k \rangle \end{pmatrix},$$

whose operator norm relative to \mathbb{R}^k satisfies

$$(3.17) \quad \begin{aligned} \|A\| &\leq \sqrt{k} \max_{i \neq j} |\langle W_i, W_j \rangle| \leq 36 \sqrt{k} (1 + |\log \sigma|) \sigma^{1-\beta} \\ &\leq C (1 + |\log \sigma|) \sigma^{1 - \frac{3\beta}{2}}. \end{aligned}$$

We see that for σ small, the matrix $(6/5)I + A$ is boundedly invertible. Let

$$C_A = \left\| \left(\frac{6}{5} I + A \right)^{-1} \right\|.$$

Then solving (3.16), we obtain

$$\begin{aligned}
\|\gamma\|_{\mathbb{R}^k} &\leq C_A \|\text{col}(\langle \varphi, L^*(W_j) \rangle - \langle h, W_j \rangle)\|_{\mathbb{R}^k} \\
&\leq C_A \left[\sum_{j=1}^k (|\langle \varphi, L^*(W_j) \rangle| + |\langle h, W_j \rangle|)^2 \right]^{1/2} \\
(3.18) \quad &\leq C_A \left\{ \sqrt{k} C \sigma^{1-\beta} \|\varphi\|_{\mu} + 36 \left(\int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx \right) \|h\|_{\mu} \right. \\
&\quad \left. + 12\sqrt{k} \left(\int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx \right) \sigma^{1-\beta} \|h\|_{\mu} \right\} \\
&\leq C_1 \|h\|_{\mu} + C_2(\sigma) \|\varphi\|_{\mu}
\end{aligned}$$

where

$$\begin{aligned}
(3.19) \quad C_1 &= C_A \left(36 + 12 C \sigma^{1-\frac{3\beta}{2}} \right) \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dx, \\
C_2(\sigma) &= C_A C \sigma^{1-\frac{3\beta}{2}}.
\end{aligned}$$

Thus, the lemma is proved. \square

As a corollary, if (3.13) admits a unique solution (φ, γ) such that

$$(3.20) \quad \|\varphi\|_{\mu} \leq \text{const} \|h\|_{\mu},$$

then $\gamma = (\gamma_1, \dots, \gamma_k)$ is uniquely determined by (3.16) and satisfies

$$(3.21) \quad \|\gamma\|_{\mathbb{R}^k} \leq \text{const} \|h\|_{\mu},$$

provided that $\beta \leq 2/3$.

Define a self-adjoint linear operator

$$(3.22) \quad L_1 = -\Delta + (1 - 2W) I : H^2(\mathbb{R}) \cap H_0^1(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

which is a part of the linear operator L in (3.1). Set

$$(3.23) \quad Z(x) = \sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) + 2W(x).$$

We have

$$\begin{aligned}
(3.24) \quad L_1(Z) &= (-\Delta + (1 - 2W)) \left(\sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) + 2W(x) \right) \\
&= - \left(\sum_j U'(x - \xi_j) + \sum_j (x - \xi_j) U''(x - \xi_j) \right)' \\
&\quad - 2W'' + (1 - 2W) \left(\sum_j (x - \xi_j) U'(x - \xi_j) + 2W(x) \right) \\
&= -4W'' + (1 - 2W)2W + \sum_j (x - \xi_j) [-U'''(x - \xi_j) + U'(x - \xi_j) \\
&\quad - 2WU'(x - \xi_j)] \\
&= -2W - 4 \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) - 2 \sum_{\substack{i,j \\ i \neq j}} (x - \xi_j) U(x - \xi_i) U'(x - \xi_j),
\end{aligned}$$

where

$$\begin{aligned}
(3.25) \quad & \left\| -4 \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) \right\|_{L^2} \leq 144 \sum_{\substack{i,j \\ i \neq j}} \left(\int_{\mathbb{R}} e^{-2|x|} e^{-2|x+\xi_i-\xi_j|} dx \right)^{1/2} \\
&= 144 \sum_{i \neq j} \left(\left[\frac{1}{2} + |\xi_i - \xi_j| \right] e^{-2|\xi_i - \xi_j|} \right)^{1/2} \\
&\leq 144 \sum_{i \neq j} \left(1 + |\xi_i - \xi_j|^{1/2} \right) e^{-|\xi_i - \xi_j|} \\
&\leq 144 \sum_{i \neq j} \left(1 + \sqrt{|i - j| |\log \sigma|} \right) \sigma^{|i-j|(1-\beta)} \\
&\leq 576 k \sigma^{1-\beta} + 288 \left(\sum_{i \neq j} |i - j| |\log \sigma| \sigma^{2|i-j|(1-\beta)} \right)^{1/2} \quad (\text{by Lemma 1.3}) \\
&\leq C \sigma^{1-2\beta} + C \sqrt{k} \sigma^{1-\beta} |\log \sigma|^{1/2} \leq C \sigma^{1-2\beta} + C \sigma^{1-\frac{3\beta}{2}} |\log \sigma|^{1/2} \\
&= O(\sigma^{1-2\beta}) + O\left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma|^{1/2}\right),
\end{aligned}$$

and

$$\begin{aligned}
(3.26) \quad & \left\| -2 \sum_{\substack{i,j \\ i \neq j}} (x - \xi_j) U(x - \xi_i) U'(x - \xi_j) \right\|_{L^2} \\
& \leq 72 \sum_{i \neq j} \left\{ \int_{\mathbb{R}} |x|^2 e^{-2|x|} e^{-2|x+\xi_j-\xi_i|} dx \right\}^{1/2} \quad (\text{by using (1.9) in Lemma 1.2}) \\
& \leq \frac{72}{\sqrt{3}} \sum_{i \neq j} (1 + |i - j| |\log \sigma|) \sigma^{|i-j|(1-\beta)} \quad (\text{by using Lemma 1.3}) \\
& \leq C (1 + |\log \sigma|) \sigma^{1-2\beta} = O(\sigma^{1-2\beta} |\log \sigma|).
\end{aligned}$$

In (3.13) let us write φ in the form

$$(3.27) \quad \varphi = aW + \psi, \quad \text{where } \langle W, \psi \rangle = 0, \quad \text{and } a \in \mathbb{R}.$$

Since $L(\psi) = L_1(\psi)$, (3.13) yields the following equality

$$(3.28) \quad a \langle L(W), Z \rangle + \langle L_1(\psi), Z \rangle = \langle h, Z \rangle + \sum_{j=1}^k \gamma_j \langle W_j, Z \rangle.$$

From (3.24)–(3.26) it follows that

$$\begin{aligned}
(3.29) \quad & |\langle L_1(\psi), Z \rangle| = |\langle \psi, L_1(Z) \rangle| \\
& \leq \left\{ O(\sigma^{1-2\beta}) + O\left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma|^{1/2}\right) + O(\sigma^{1-2\beta} |\log \sigma|) \right\} \|\psi\|_{L^2} \\
& \leq O(\sigma^{1-2\beta} |\log \sigma|) \|\varphi\|_{L^2},
\end{aligned}$$

since $\|\psi\|_{L^2} \leq \|\varphi\|_{L^2}$. Then for $\varphi \in L_\mu^\infty$, by Lemma 1.5, we obtain

$$\begin{aligned}
(3.30) \quad & |\langle L_1(\psi), Z \rangle| \leq O(\sigma^{1-2\beta} |\log \theta|) \left(\int_{\mathbb{R}} e^{-2 \min_{1 \leq i \leq k} |x - \xi_i|} dx \right)^{1/2} \|\varphi\|_\mu \\
& \leq O(\sigma^{1-2\beta} |\log \theta|) \left(\frac{1}{\mu} (k - (k-1)\sigma^\mu) \right)^{1/2} \|\varphi\|_\mu \\
& \leq O\left(\sigma^{1-\frac{5\beta}{2}} \|\log \sigma|\right) \|\varphi\|_\mu.
\end{aligned}$$

Next we estimate $\langle L(W), Z \rangle$ in (3.28). In fact, by using (1.7) we get

$$\begin{aligned}
(3.31) \quad \langle L(W), Z \rangle &= \left\langle -W'' + (1 - 2W)W + \frac{2W^2}{\omega} \langle W, W \rangle, Z \right\rangle \\
&= \left\langle -W^2 + \frac{12k}{\omega} W^2, Z \right\rangle - \sum_{\substack{i,j \\ i \neq j}} \langle U(x - \xi_i) U(x - \xi_j), Z \rangle \\
&\quad + \left\langle \frac{2W^2}{\omega} \sum_{\substack{i,j \\ i \neq j}} \langle U(x - \xi_i), U'(x - \xi_j) \rangle, Z \right\rangle \\
&= \left(\frac{12k}{\omega} - 1 \right) \left\langle W^2, \sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) + 2W \right\rangle + \sum_{i \neq j} \sigma^{|i-j|(1-\beta)} \|Z\|_{L^1} \\
&\quad + O(\sigma^{1-\beta} |\log \sigma|) \|W^2\|_{L^2} \|Z\|_{L^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left(\frac{12k}{\omega} - 1 \right) \left\langle W^2, \sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) + 2W \right\rangle \\
&= \left(\frac{12k}{\omega} - 1 \right) \left\{ \frac{5}{3} \int_{\mathbb{R}} W^3(x) dx \right. \\
&\quad \left. + \sum_{j=1}^k \sum_{\substack{i,\ell \\ \text{not both } j}} \int_{\mathbb{R}} (x - \xi_j) U(x - \xi_i) U(x - \xi_\ell) U'(x - \xi_j) dx \right\}
\end{aligned}$$

where the triple sum is estimated as follows,

$$\begin{aligned}
&\left| \sum_{j=1}^k \sum_{\substack{i,\ell \\ \text{not both } j}} \int_{\mathbb{R}} (x - \xi_j) U'(x - \xi_j) U(x - \xi_i) U(x - \xi_\ell) dx \right| \\
&\leq \left| \sum_{j=1}^k \sum_{\substack{i=j \\ \text{or } \ell=j}} \dots \right| + \left| \sum_{j=1}^k \sum_{i=\ell \neq j} \dots \right| + \left| \sum_{j=1}^k \sum_{\substack{i \neq j \\ \ell \neq j \\ i \neq \ell}} \dots \right| = \Pi_1 + \Pi_2 + \Pi_3,
\end{aligned}$$

and

$$\begin{aligned}
\Pi_1 &\leq 2 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} \left| \int_{\mathbb{R}} (x - \xi_j) U' (x - \xi_j) U (x - \xi_j) U (x - \xi_\ell) dx \right| \\
&\leq 432 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} \int_{\mathbb{R}} |x| e^{-2|x|} e^{-|x+\xi_j-\xi_\ell|} dx \\
&\leq 432 e^{-1} \sum_{j=1}^k \sum_{|\ell-j| \geq 1} (1 + |\ell - j| |\log \sigma|) \sigma^{|\ell-j|(1-\beta)} \\
&\leq Ck |\log \sigma| \sigma^{1-\beta} \leq C\sigma^{1-2\beta} |\log \sigma|, \\
\Pi_2 &\leq 2 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} \left| \int_{\mathbb{R}} (x - \xi_j) U' (x - \xi_j) U^2 (x - \xi_\ell) dx \right| \\
&\leq 432 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} \int_{\mathbb{R}} |x| e^{-|x|} e^{-2|x+\xi_j-\xi_\ell|} dx \quad (\text{by (1.10)}) \\
&\leq 864 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} e^{-|\xi_j-\xi_\ell|} (|\xi_j - \xi_\ell| + e^{-|\xi_j-\xi_\ell|}) \\
&\leq 864 \sum_{j=1}^k \sum_{|\ell-j| \geq 1} \sigma^{|i-\ell|(1-\beta)} (|i - \ell| |\log \sigma| + \sigma^{|i-\ell|(1-\beta)}) \\
&\leq Ck \left[\sigma^{1-\beta} |\log \sigma| + \sigma^{2(1-\beta)} \right] = O(\sigma^{1-2\beta} |\log \sigma|), \\
\Pi_3 &\leq \sum_{j=1}^k \sum_{\substack{\text{either } |i-\ell| \geq 2 \\ \text{or } |i-j| \geq 2 \\ \text{or } |\ell-j| \geq 2}} \left| \int_{\mathbb{R}} (x - \xi_j) U' (x - \xi_j) U (x - \xi_i) U (x - \xi_\ell) dx \right| \\
&\leq 216 \sum_{j=1}^k \sum_{i, \ell \text{ as above}} \int_{\mathbb{R}} |x| e^{-|x|} e^{-|x+\xi_j-\xi_i|} e^{-|x+\xi_j-\xi_\ell|} dx \\
&\leq 216 \sum_{j=1}^k \sum_{i, \ell \text{ as above}} \begin{cases} \sigma^{|i-\ell|(1-\beta)} \int_{\mathbb{R}} |x| e^{-|x|} dx & (\text{if } |i - \ell| \geq 2), \text{ or} \\ \sigma^{|i-j|(1-\beta)} \int_{\mathbb{R}} |x| e^{-|x+\xi_j-\xi_\ell|} dx & (\text{if } |i - j| \geq 2), \text{ or} \\ \sigma^{|\ell-j|(1-\beta)} \int_{\mathbb{R}} |x| e^{-|x+\xi_j-\xi_i|} dx & (\text{if } |\ell - j| \geq 2) \end{cases} \\
&\leq 216 k^2 C |\log \sigma| \sigma^{2(1-\beta)} \leq C\sigma^{2(1-2\beta)} |\log \sigma|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
(3.32) \quad &\left(\frac{12k}{\omega} - 1 \right) \left\langle W^2, \sum_{j=1}^k (x - \xi_j) U' (x - \xi_j) + 2W \right\rangle \\
&\geq \frac{1}{2} \left\{ \frac{5}{3} \int_{\mathbb{R}} W^3(x) dx + O(\sigma^{1-2\beta} |\log \sigma|) \right\} \geq 6k + O(\sigma^{1-2\beta} |\log \sigma|),
\end{aligned}$$

because

$$\frac{12k}{\omega} - 1 = \frac{12k}{6k + C\sigma^{1-2\beta} |\log \sigma|} - 1 \geq \left(2 - \frac{1}{2} \right) - 1 = \frac{1}{2} \quad (\text{for } \sigma \text{ small})$$

and by Lemma 1.1,

$$\int_{\mathbb{R}} W^3(x) dx > k \int_{\mathbb{R}} U^3(x) dx = \frac{36}{5} k.$$

Moreover, we have

$$\begin{aligned} \|Z\|_{L^1} &\leq \left\| \sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) \right\|_{L^1} + 2\|W\|_{L^1} \\ &= \|-W\|_{L^1} + 2\|W\|_{L^1} = 3\|W\|_{L^1} = 18k, \end{aligned}$$

$$\begin{aligned} \|Z\|_{L^2} &\leq \left\| \sum_{j=1}^k (x - \xi_j) U'(x - \xi_j) \right\|_{L^2} + 2\|W\|_{L^2} \\ &\leq k \left(\int_{\mathbb{R}} |x|^2 |U'(x)|^2 dx \right)^{1/2} + 2k\sqrt{6} \\ &\leq k \left[6 \left(\int_{\mathbb{R}} |x|^2 e^{-2|x|} \right)^{1/2} + 2\sqrt{6} \right] = k \left(\frac{3}{2} + 2\sqrt{6} \right), \end{aligned}$$

and

$$\|W^2\|_{L^2} \leq \|W\|_{L^\infty} \|W\|_{L^2} \leq \frac{12}{1 - \sigma^{1-\beta}} \|W\|_{L^2} \leq 24\sqrt{6} k, \quad \text{for } \sigma \text{ small.}$$

Substituting these and (3.32) into (3.31), we obtain

$$\begin{aligned} (3.33) \quad \langle L(W), Z \rangle &\geq 6k + O(\sigma^{1-2\beta} |\log \sigma|) + O(\sigma^{1-2\beta}) k + O(\sigma^{1-2\beta} |\log \sigma|) k \\ &= k [6 + O(\sigma^{1-2\beta} |\log \sigma|)]. \end{aligned}$$

On the other hand, we have

$$(3.34) \quad |\langle h, Z \rangle| \leq \|h\|_{\mu} \|Z\|_{L^1} \leq 18k \|h\|_{\mu},$$

and

$$\left| \sum_{j=1}^k \gamma_j \langle W_j, Z \rangle \right| \leq \|\gamma\| \sum_{j=1}^k \left| \int_{\mathbb{R}} W_j(x) Z(x) dx \right|,$$

where

$$\int_{\mathbb{R}} W_j(x) Z(x) dx = - \sum_{i(\neq j)=1}^k \int_{\mathbb{R}} U'(x - \xi_j) [(x - \xi_i) U'(x - \xi_i) + 2U(x - \xi_i)] dx$$

because, when $i = j$, we have

$$\begin{aligned} & - \int_{\mathbb{R}} U'(x - \xi_j) [(x - \xi_j) U'(x - \xi_j) + 2U(x - \xi_j)] dx \\ &= - \int_{\mathbb{R}} (x - \xi_j) U'(x - \xi_j)^2 dx = - \int_{\mathbb{R}} x |U'(x)|^2 dx = 0. \end{aligned}$$

(odd function)

Thus we get

$$\begin{aligned}
(3.35) \quad \left| \sum_{j=1}^k \gamma_j \langle W_j, Z \rangle \right| &\leq \|\gamma\| \sum_{j=1}^k \sum_{i(\neq j)=1}^k \left[\int_{\mathbb{R}} 36|x|e^{-|x|}e^{|x+\xi_i-\xi_j|} dx \right. \\
&\quad \left. + \int_{\mathbb{R}} 72e^{-|x|}e^{-|x+\xi_i-\xi_j|} dx \right] \\
&\leq \|\gamma\| \sum_{\substack{i,j \\ i \neq j}} \left[36 \left(\int_{\mathbb{R}} |x|e^{-\epsilon_0|x|} dx \right) e^{-(1-\epsilon_0)|\xi_i-\xi_j|} \right. \\
&\quad \left. + 72(1 + |i-j| |\log \sigma|) \sigma^{|i-j|(1-\beta)} \right] \\
&\leq Ck \|\gamma\| \left[\sigma^{(1-\epsilon_0)(1-\beta)} + \sigma^{(1-\beta)} |\log \sigma| \right] \\
&\leq k (C_1 \|h\|_{\mu} + C_2(\sigma) \|\varphi\|_{\mu}) O \left(\sigma^{(1-\epsilon_0)(1-\beta)} \right), \quad \text{for any small } \epsilon_0 > 0,
\end{aligned}$$

where we used (3.14) in Lemma 3.2.

LEMMA 3.3. *Under the assumption that*

$$(3.36) \quad 0 < \beta < \frac{1}{2},$$

it holds that

$$(3.37) \quad \left| \frac{2}{\omega} \langle W, \varphi \rangle \right| \leq C_3 \|h\|_{\mu} + C_4(\sigma) \|\varphi\|_{\mu},$$

for a solution (φ, γ) to the problem (3.13), where $C_3 > 0$ and $C_4(\sigma) > 0$ are constants, with the property that $C_4(\sigma) = O \left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma| \right)$.

PROOF. From (3.28), (3.30), (3.33), (3.34) and (3.35), we obtain

$$\begin{aligned}
(3.38) \quad |a| &\leq \frac{1}{k [6 + O(\sigma^{1-2\beta} |\log \sigma|)]} \left\{ O \left(\sigma^{1-\frac{5\beta}{2}} |\log \sigma| \right) \|\varphi\|_{\mu} + 18k \|h\|_{\mu} \right. \\
&\quad \left. + k (C_1 \|h\|_{\mu} + C_2(\sigma) \|\varphi\|_{\mu}) O \left(\sigma^{(1-\epsilon_0)(1-\beta)} \right) \right\} \\
&\leq \frac{1}{6 + O(\sigma^{1-2\beta} |\log \sigma|)} \left\{ (18 + C_1 O \left(\sigma^{(1-\epsilon_0)(1-\beta)} \right)) \|h\|_{\mu} \right. \\
&\quad \left. + \left(O \left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma| \right) + C_2(\sigma) O \left(\sigma^{(1-\epsilon_0)(1-\beta)} \right) \right) \|\varphi\|_{\mu} \right\} \\
&\leq \hat{C}_3 \|h\|_{\mu} + \hat{C}_4(\sigma) \|\varphi\|_{\mu},
\end{aligned}$$

for σ sufficiently small, where $\hat{C}_3 > 0$ is a uniform constant, and $C_2(\sigma) = O \left(\sigma^{1-\frac{3\beta}{2}} \right)$ implies that

$$\begin{aligned}
(3.39) \quad \hat{C}_4(\sigma) &= O \left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma| \right) + O \left(\sigma^{1-\frac{3\beta}{2}+(1-\epsilon_0)(1-\beta)} \right) \\
&= O \left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma| \right).
\end{aligned}$$

Since (3.27) implies that

$$(3.40) \quad \langle W, \varphi \rangle = a \|W\|_{L^2}^2 + \langle W, \psi \rangle = a \|W\|_{L^2}^2,$$

and we see that

$$\begin{aligned}
\|W\|_{L^2}^2 &= 6k + \sum_{\substack{i,j \\ i \neq j}} \int_{\mathbb{R}} U(x - \xi_i) U(x - \xi_j) dx \\
&\leq 6k + \sum_{i \neq j} 36 \int_{\mathbb{R}} e^{-|x|} e^{-|x + \xi_i - \xi_j|} dx \quad (\text{by (1.7)}) \\
&\leq 6k + \sum_{i \neq j} 36(1 + |i - j| |\log \sigma|) \sigma^{|i-j|(1-\beta)} \quad (\text{by Lemma 1.3}) \\
&\leq 6k + Ck\sigma^{1-\beta} |\log \sigma| = k(6 + O(\sigma^{1-\beta} |\log \sigma|)).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
(3.41) \quad \left| \frac{2}{\omega} \langle W, \varphi \rangle \right| &\leq \frac{2}{6k} |\langle W, \varphi \rangle| \leq \frac{1}{3k} |a| \|W\|_{L^2}^2 \\
&\leq \frac{1}{3k} \left(\hat{C}_3 \|h\|_{\mu} + \hat{C}_4(\sigma) \|\varphi\|_{\mu} \right) k(6 + O(\sigma^{1-\beta} |\log \sigma|)) \\
&= C_3 \|h\| + C_4(\sigma) \|\varphi\|_{\mu},
\end{aligned}$$

where

$$\begin{aligned}
(3.42) \quad C_3 &\left(\geq \hat{C}_3 \left(2 + \frac{1}{3} O(\sigma^{1-\beta} |\log \sigma|) \right) \right) \text{ is a constant, and} \\
C_4(\sigma) &= \hat{C}_4(\sigma) \left(2 + \frac{1}{3} O(\sigma^{1-\beta} |\log \sigma|) \right) = O\left(\sigma^{1-\frac{3\beta}{2}} |\log \sigma|\right).
\end{aligned}$$

Thus the lemma is proved. \square

With all these preparations, we are going to conduct an *a priori* estimate of the solution φ of the problem (3.13), if it exists. According to (3.1), (2.1) and (2.2), a solution φ to Eq. (3.13) satisfies the integral equation

$$(3.43) \quad \varphi(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left(2W\varphi - \frac{2W^2}{\omega} \langle W, \varphi \rangle + h + \gamma \cdot \nabla_{\xi} W \right) dy.$$

In order to get a sharp estimate of the integral term involving $2W\varphi$ on the right-hand side of (3.43), we use the cut-off decomposition of φ introduced in (1.22). Recall that by (1.23),

$$\begin{aligned}
\Omega &= \Omega_0 \cup \Omega_1 \cup \dots \cup \Omega_k, \quad (\text{mutually disjoint}) \\
\Omega_0 &= \mathbb{R} \setminus \bigcup_{j=1}^k \Omega_j \quad \text{and} \quad \Omega_j = [\xi_j - x_{\sigma}, \xi_j + x_{\sigma}], \quad j = 1, \dots, k,
\end{aligned}$$

$$\begin{aligned}
&\text{where } x_{\sigma} = r_{\sigma}/4 = \frac{1}{4}(1-\beta)|\log \sigma|, \text{ so that} \\
&e^{-x_{\sigma}} = e^{-r_{\sigma}/4} \leq \sigma^{\frac{1}{4}(1-\beta)}.
\end{aligned}$$

For any $j = 0, 1, \dots, k$, we shall consider the integral terms

$$\begin{aligned}
(3.44) \quad \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} &\left[2W(y) \sum_{\ell=0}^k \varphi_{\ell}(y) - \frac{2W^2}{\omega} \langle W, \varphi \rangle + h + \gamma \cdot \nabla_{\xi} W \right] dy \\
&= J_j^1 + J_j^2 + J_j^3 + \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} h(y) dy.
\end{aligned}$$

Since we choose $0 < \mu \leq 1/3$, we have $\frac{1}{1-\mu} \leq \frac{3}{2}$. For the simplicity of notation, we set

$$(3.45) \quad \rho(x) = \rho(x, \xi) \triangleq \min_{1 \leq i \leq k} |x - \xi_i|, \quad x \in \mathbb{R}.$$

First of all, it is easy to see that

$$(3.46) \quad \begin{aligned} \left| \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} h(y) dy \right| &\leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} e^{-\mu\rho(y)} \|h\|_{\mu} dy \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{-(1-\mu)|x-y|} e^{-\mu\left(|x-y| + \min_{1 \leq i \leq k} |y - \xi_i|\right)} dy \|h\|_{\mu} \\ &\leq \frac{1}{2} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left(\int_{\mathbb{R}} e^{-(1-\mu)|x-y|} dy \right) \|h\|_{\mu} \\ &= \frac{1}{1-\mu} e^{-\mu\rho(x)} \|h\|_{\mu}. \end{aligned}$$

In (3.44),

$$\begin{aligned} J_j^1 &= \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} 2W(y) \sum_{\ell=0}^k \varphi_{\ell}(y) dy, \\ J_j^2 &= \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} \left(-\frac{2W^2(y)}{\omega} \langle W, \varphi \rangle \right) dy, \text{ and} \\ J_j^3 &= \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} \gamma \cdot \nabla_{\xi} W(y) dy. \end{aligned}$$

We start with the estimate of J_j^1 . Then $J^1 = \sum_{j=0}^k J_j^1$. Note that for any $y \in \Omega_0$,

$$(3.47a) \quad \min_{1 \leq i \leq k} |y - \xi_i| \geq x_{\sigma}.$$

We find that

$$\begin{aligned} |J_0^1| &= \left| \int_{\Omega_0} e^{-|x-y|} W(y) \varphi(y) dy \right| \leq \int_{\Omega_0} e^{-|x-y|} |W(y)| |\varphi(y)| dy \\ &\leq \int_{\Omega_0} e^{-|x-y|} \left(6 \sum_{j=1}^k e^{-|y-\xi_j|} \right) \left(e^{-\mu \min_{1 \leq i \leq k} |y-\xi_i|} \|\varphi\|_{\mu} \right) dy \\ &\leq 6 \int_{\Omega_0} e^{-|x-y|} \left(\sum_{j=1}^k e^{-(1-\mu)|y-\xi_j|} \right) \left(e^{-2\mu \min_{1 \leq i \leq k} |y-\xi_i|} \right) dy \|\varphi\|_{\mu} \\ &\leq 6e^{-2\mu\rho(x)} \int_{\Omega_0} e^{-(1-2\mu)|x-y|} \left(\sum_{j=1}^k e^{-(1-\mu)|y-\xi_j|} \right) dy \|\varphi\|_{\mu}, \end{aligned}$$

because

$$e^{-2\mu|x-y| - 2\mu \min_{1 \leq i \leq k} |y-\xi_i|} \leq e^{-2\mu \min_{1 \leq i \leq k} |x-\xi_i|} = e^{-2\mu\rho(x)},$$

$$\text{where } |x-y| + \min_{1 \leq i \leq k} |y-\xi_i| = \min_{1 \leq i \leq k} \{|x-y| + |y-\xi_i|\} \geq \min_{1 \leq i \leq k} |x-\xi_i|.$$

Moreover,

$$\begin{aligned}
(3.47b) \quad & \sum_{j=1}^k e^{-(1-\mu)|y-\xi_j|} = e^{-(1-\mu)|y-\xi_{j_0}|} + \sum_{j \neq j_0} e^{-(1-\mu)|y-\xi_j|} \\
& \quad \text{(where } |y - \xi_{j_0}| = \min_{1 \leq j \leq k} |y - \xi_j| \text{)} \\
& \leq e^{-(1-\mu)x_\sigma} + 2e^{-(1-\mu)(x_\sigma+r_\sigma)} + 2e^{-(1-\mu)(x_\sigma+2r_\sigma)} + \dots \\
& \leq \frac{2e^{-(1-\mu)x_\sigma}}{1 - e^{-(1-\mu)r_\sigma}} = \frac{2\sigma^{(1-\mu)(1-\beta)/4}}{1 - \sigma^{(1-\mu)(1-\beta)}} \leq 4\sigma^{(1-\mu)(1-\beta)/4},
\end{aligned}$$

for σ small. Therefore, we get

$$\begin{aligned}
(3.48) \quad & |J_0^1| \leq 6e^{-2\mu\rho(x)} \left(\int_{\Omega_0} e^{-(1-2\mu)|x-y|} dy \right) 4\sigma^{(1-\mu)(1-\beta)/4} \|\varphi\|_\mu \\
& = \frac{48}{1-2\mu} \sigma^{(1-\mu)(1-\beta)/4} e^{-2\mu\rho(x)} \|\varphi\|_\mu \\
& \leq 144\sigma^{(1-\mu)(1-\beta)/4} e^{-2\mu\rho(x)} \|\varphi\|_\mu.
\end{aligned}$$

For $j = 1, \dots, k$, due to $\text{supp } \theta(\cdot - \xi_\ell) \cap \Omega_j = \emptyset$ for $\ell \neq j$ and $|\theta(y - \xi_j)| \leq 1$ for $y \in \mathbb{R}$, we have

$$\begin{aligned}
(3.49) \quad & |J_j^1| = \left| \int_{\Omega_j} e^{-|x-y|} W(y) \sum_{\ell=0}^k \varphi_\ell(y) dy \right| \\
& = \left| \int_{\Omega_j} e^{-|x-y|} W(y) \theta(y - \xi_j) \varphi(y) dy \right| \\
& \leq \sum_{m=1}^k \int_{\Omega_j} e^{-|x-y|} U(y - \xi_m) |\varphi(y)| dy \\
& = \sum_{m=1}^k \int_{\xi_j - x_\sigma}^{\xi_j + x_\sigma} e^{-|x-y|} U(y - \xi_m) |\varphi(y)| dy \\
& = \sum_{m=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-|x-y-\xi_j|} U(y + \xi_j - \xi_m) |\varphi(y + \xi_j)| dy \\
& \leq 6 \sum_{m=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-|x-y-\xi_j|} e^{-|y+\xi_j-\xi_m|} e^{-\mu \min_{1 \leq i \leq k} |y+\xi_j-\xi_i|} dy \|\varphi\|_\mu \\
& \leq 6 \sum_{m=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-|x-y-\xi_j|} e^{-(1-\mu)|y+\xi_j-\xi_m|} e^{-2\mu \min_{1 \leq i \leq k} |y+\xi_j-\xi_i|} dy \|\varphi\|_\mu \\
& \leq 6 \sum_{m=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-(1-\mu)|y+\xi_j-\xi_m|} e^{-2\mu \min_{1 \leq i \leq k} |x-\xi_i|} dy \|\varphi\|_\mu \\
& = 6e^{-2\mu\rho(x)} \left(\int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-(1-\mu)|y|} dy \right) \|\varphi\|_\mu \\
& + 6e^{-2\mu\rho(x)} \left(\sum_{m \neq j} \int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-(1-\mu)|y+\xi_j-\xi_m|} dy \right) \|\varphi\|_\mu,
\end{aligned}$$

where, for $m \neq j$, we have

$$(3.50) \quad \sum_{m \neq j} e^{-(1-\mu)|y+\xi_j-\xi_m|} \leq \frac{2\sigma^{\frac{3}{4}(1-\mu)(1-\beta)}}{1-\sigma^{1-\beta}} \leq \frac{2\sqrt{2}}{\sqrt{2}-1} \sigma^{\frac{3}{4}(1-\mu)(1-\beta)}.$$

With the key observation that for any $x \in \mathbb{R}$, $y \in [-x_\sigma, x_\sigma]$, there is an index $j_0 = j_0(x, y) \in \{1, \dots, k\}$, such that

$$|x - y - \xi_{j_0}| = \min_{1 \leq j \leq k} |x - y - \xi_j|,$$

we can calculate

$$(3.51) \quad \begin{aligned} \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} &= e^{-(1-2\mu)|x-y-\xi_{j_0}|} + \sum_{j \neq j_0} e^{-(1-2\mu)|x-y-\xi_j|} \\ &\leq 1 + e^{-(1-2\mu)|x-y-\xi_{j_0-1}|} + e^{-(1-2\mu)|x-y-\xi_{j_0+1}|} + \dots \\ &\leq 1 + 2 \left(e^{-(1-2\mu)r_\sigma/2} + e^{-(1-2\mu)3r_\sigma/2} + e^{-(1-2\mu)5r_\sigma/2} + \dots \right) \\ &= 1 + \frac{2e^{-(1-2\mu)r_\sigma/2}}{1 - e^{-(1-2\mu)r_\sigma}} = 1 + \frac{2\sigma^{\frac{1}{2}(1-2\mu)(1-\beta)}}{1 - \sigma^{(1-2\mu)(1-\beta)}} < C_5, \end{aligned}$$

where $C_5 > 0$ is a constant independent of small σ . By summing up $|J_j^1|$ for all $j = 1, \dots, k$, and using (3.50) and (3.51), we get

$$(3.52) \quad \begin{aligned} |J^1| &= \left| \sum_{j=0}^k J_j^1 \right| \leq \sum_{j=0}^k |J_j^1| \leq e^{-2\mu\rho(x)} \|\varphi\|_\mu \left\{ 144 \sigma^{(1-\mu)(1-\beta)/4} \right. \\ &\quad \left. + \int_{-x_\sigma}^{x_\sigma} 6 \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} e^{-(1-\mu)|y|} dy \right. \\ &\quad \left. + \sum_{m \neq j} \int_{-x_\sigma}^{x_\sigma} 6 \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} \sigma^{\frac{3}{4}(1-\mu)(1-\beta)} dy \right\} \\ &\leq e^{-2\mu\rho(x)} \|\varphi\|_\mu \left\{ 144 \sigma^{1/12} + 6 C_5 \frac{2}{1-\mu} + 6 C_5 \int_{-x_\sigma}^{x_\sigma} \sum_{m(\neq j)=1}^k e^{-(1-\mu)|y+\xi_j-\xi_m|} dy \right\} \\ &\leq e^{-2\mu\rho(x)} \|\varphi\|_\mu \left\{ 144 \sigma^{1/12} + 18 C_5 + 6 C_5 \frac{2\sqrt{2}}{\sqrt{2}-1} \sigma^{\frac{3}{4}(1-\mu)(1-\beta)} 2x_\sigma \right\} \\ &\leq e^{-2\mu\rho(x)} \|\varphi\|_\mu \left\{ 18 C_5 + 144 \sigma^{1/12} + 15\sqrt{2} C_5 \sigma^{\frac{3}{4}(1-\mu)(1-\beta)} (C_0 + |\log \sigma|) \right\}, \end{aligned}$$

where we computed

$$\frac{2\sqrt{2}}{\sqrt{2}-1} < \frac{2\sqrt{2}}{0.4} = 5\sqrt{2}, \quad \text{and } 2x_\sigma = \frac{r_\sigma}{2} \leq \frac{1}{2} (1-\beta) |\log \sigma|.$$

Secondly we estimate J_j^2 and $J^2 = \sum_{j=0}^k J_j^2$. With similar observations as in the processing of J^1 , by using Lemma 3.3 and (3.47) and (3.51), we can get the following

estimates. First,

$$\begin{aligned}
(3.53) \quad |J_0^2| &= \left| \frac{1}{2} \int_{\Omega_0} e^{-|x-y|} \left(-\frac{2W^2(y)}{\omega} \langle W, \varphi \rangle \right) dy \right| \\
&\leq \frac{1}{2} (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \int_{\Omega_0} e^{-|x-y|} W^2(y) dy \\
&\leq 18 (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \int_{\Omega_0} e^{-|x-y|} \left(\sum_{j=1}^k e^{-|y-\xi_j|} \right)^2 dy \\
&\leq 18 (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \int_{\Omega_0} e^{-|x-y|} \left(\sum_{j=1}^k e^{-(1-\mu)|y-\xi_j|} \right)^2 e^{-2\mu \min_{1 \leq i \leq k} |y-\xi_i|} dy \\
&\leq 18 e^{-2\mu\rho(x)} (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \int_{\Omega_0} e^{-(1-2\mu)|x-y|} 16 \sigma^{(1-\mu)(1-\beta)/2} dy \\
&= 576 e^{-2\mu\rho(x)} \frac{\sigma^{(1-\mu)(1-\beta)/2}}{1-2\mu} (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \\
&= e^{-2\mu\rho(x)} (C_6(\sigma) \|h\|_\mu + C_7(\sigma) \|\varphi\|_\mu),
\end{aligned}$$

where

$$\begin{aligned}
(3.54) \quad C_6(\sigma) &= C_3 \frac{576}{1-2\mu} \sigma^{(1-\mu)(1-\beta)/2} = O\left(\sigma^{(1-\mu)(1-\beta)/2}\right), \\
C_7(\sigma) &= C_4(\sigma) O\left(\sigma^{(1-\mu)(1-\beta)/2}\right) = O\left(\sigma^{1-\frac{3\beta}{2}+(1-\mu)(1-\beta)/2} |\log \sigma|\right) \\
&= O\left(\sigma^{1-\frac{3\beta}{2}}\right),
\end{aligned}$$

where $(1-\mu)(1-\beta)/2 \geq \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/6$. Next, for $j = 1, \dots, k$, we have

$$\begin{aligned}
(3.55) \quad |J_j^2| &= \left| \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} \left(-\frac{2W^2(y)}{\omega} \langle W, \varphi \rangle \right) dy \right| \\
&\leq \frac{1}{2} (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \left\{ \int_{\Omega_j} e^{-|x-y|} \sum_{\ell=1}^k U^2(y-\xi_\ell) dy \right. \\
&\quad \left. + \int_{\Omega_j} e^{-|x-y|} \sum_{\substack{m,\ell \\ m \neq \ell}} U(y-\xi_m) U(y-\xi_\ell) dy \right\} \\
&\leq 18 (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \left\{ \sum_{\ell=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-|x-y-\xi_j|} e^{-2|y+\xi_j-\xi_\ell|} dy \right. \\
&\quad \left. + \sum_{\substack{m,\ell \\ m \neq \ell}} \int_{-x_\sigma}^{x_\sigma} e^{-|x-y-\xi_j|} e^{-|y+\xi_j-\xi_m|} e^{-|y+\xi_j-\xi_\ell|} dy \right\}
\end{aligned}$$

$$\begin{aligned} &\leq 18 (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) e^{-2\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left\{ \int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-2(1-\mu)|y|} dy \right. \\ &\quad + \sum_{\ell(\neq j)=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-2(1-\mu)|y+\xi_j-\xi_\ell|} dy \\ &\quad \left. + \sum_{\substack{m,\ell \\ m \neq \ell}} \int_{-x_\sigma}^{x_\sigma} e^{-(1-2\mu)|x-y-\xi_j|} e^{-(1-\mu)|y+\xi_j-\xi_m|} e^{-(1-\mu)|y+\xi_j-\xi_\ell|} dy \right\}. \end{aligned}$$

Summing up (3.53) and (3.55) for $j = 1, \dots, k$, we obtain

$$\begin{aligned} |J^2| &\leq \sum_{j=0}^k |J_j^2| \leq e^{-2\mu\rho(x)} \left\{ C_6(\sigma) \|h\|_\mu + C_7(\sigma) \|\varphi\|_\mu \right. \\ &\quad + 18 (C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) \left[\int_{-x_\sigma}^{x_\sigma} \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} e^{-2(1-\mu)|y|} dy \right. \\ &\quad + \int_{-x_\sigma}^{x_\sigma} \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} \left(\sum_{\ell(\neq j)=1}^k e^{-2(1-\mu)|y+\xi_j-\xi_\ell|} \right) dy \\ &\quad \left. \left. + \int_{-x_\sigma}^{x_\sigma} \sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} \left(\sum_{\substack{m,\ell \\ m \neq \ell}} e^{-(1-\mu)|y+\xi_j-\xi_m|} e^{-(1-\mu)|y+\xi_j-\xi_\ell|} \right) dy \right] \right\}, \end{aligned}$$

where

$$\begin{aligned} &\sum_{j=1}^k e^{-(1-2\mu)|x-y-\xi_j|} \leq C_5, \quad \text{by (3.51),} \\ &\sum_{\ell(\neq j)=1} e^{-2(1-\mu)|y+\xi_j-\xi_\ell|} \leq e^{-2(1-\mu)\frac{3}{4}r_\sigma} \left[1 + 2e^{-2(1-\mu)r_\sigma} + \dots \right] \\ &\leq \frac{2\sigma^{\frac{3}{2}(1-\mu)(1-\beta)}}{1 - \sigma^{2(1-\mu)(1-\beta)}} \leq C \sigma^{\frac{3}{2}(1-\mu)(1-\beta)}, \quad \text{for all } y \in [-x_\sigma, x_\sigma], \end{aligned}$$

and

$$\begin{aligned} &\sum_{\substack{m,\ell \\ m \neq \ell}} e^{-(1-\mu)|y+\xi_j-\xi_m|} e^{-(1-\mu)|y+\xi_j-\xi_\ell|} \leq \sum_{\substack{m,\ell \\ m \neq \ell}} e^{-(1-\mu)|\xi_m-\xi_\ell|} \\ &\leq \sum_{m=1}^k \sum_{\ell \neq m} \sigma^{|m-\ell|(1-\mu)(1-\beta)} \\ &\leq k \frac{2\sigma^{(1-\mu)(1-\beta)}}{1 - \sigma^{(1-\mu)(1-\beta)}} \leq C \sigma^{(1-\mu)(1-\beta)-\beta}, \quad \forall y \in [-x_\sigma, x_\sigma]. \end{aligned}$$

Therefore, we come up with

$$\begin{aligned}
|J^2| &\leq e^{-2\mu\rho(x)} \{C_6(\sigma) \|h\|_\mu + C_7(\sigma) \|\varphi\|_\mu \\
&\quad + 18(C_3 \|h\|_\mu + C_4(\sigma) \|\varphi\|_\mu) C_5 \left[\frac{1}{1-\mu} \sigma^{\frac{1}{2}(1-\mu)(1-\beta)} \right. \\
(3.56) \quad &\quad \left. + C\sigma^{\frac{3}{2}(1-\mu)(1-\beta)} (C_0 + |\log \sigma|) + C\sigma^{(1-\mu)(1-\beta)-\beta} (C_0 + |\log \sigma|) \right\} \\
&\leq e^{-2\mu\rho(x)} [C_8(\sigma) \|h\|_\mu + C_9(\sigma) \|\varphi\|_\mu],
\end{aligned}$$

where

$$\begin{aligned}
(3.57) \quad C_8(\sigma) &= C_6(\sigma) + 18C_3C_5[\dots] = O\left(\sigma^{\frac{1}{2}(1-\mu)(1-\beta)}\right) + O\left(\sigma^{(1-\mu)(1-\beta)-\beta} |\log \sigma|\right), \\
C_9(\sigma) &= C_7(\sigma) + 18C_4(\sigma)C_5[\dots] = O\left(\sigma^{1-\frac{3\beta}{2}}\right) + O\left(\sigma^{1-\frac{3\beta}{2}} \sigma^{(1-\mu)(1-\beta)-\beta} |\log \sigma|^2\right) \\
&= O\left(\sigma^{1-\frac{3\beta}{2}}\right),
\end{aligned}$$

provided that

$$(3.58) \quad (1-\mu)(1-\beta) - \beta = 1-\mu - (2-\mu)\beta > 0, \quad \text{or } \beta < \frac{1-\mu}{2-\mu}.$$

When we choose $\mu = 1/3$, then (3.58) implies $\beta < \frac{2}{5}$. For μ close to 0, β is close to $1/2$.

Thirdly we estimate J_0^3 and $J^3 = \sum_{j=0}^k J_j^3$, with the aid of Lemma 3.2. In fact,

$$\begin{aligned}
|J_0^3| &= \left| \frac{1}{2} \int_{\Omega_0} e^{-|x-y|} \gamma \cdot \nabla_\xi W(y) dy \right| = \left| \frac{1}{2} \int_{\Omega_0} e^{-|x-y|} \sum_{j=1}^k \gamma_j W_j(y) dy \right| \\
&\leq \frac{1}{2} \|\gamma\| \sum_{j=1}^k \int_{\Omega_0} e^{-|x-y|} 6 e^{-|y-\xi_j|} dy \\
&\leq 3\|\gamma\| \sum_{j=1}^k \int_{\Omega_0} e^{-(1-\mu)(|x-y|+|y-\xi_j|)} dy e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \\
&\leq 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \sum_{j=1}^k \int_{\Omega_0} e^{-(1-\mu)(|x-y|+|y-\xi_j|)} dy,
\end{aligned}$$

Where, by (3.47a) and (3.47b), we have

$$\sum_{j=1}^k e^{-(1-\mu)|y-\xi_j|} \leq 4\sigma^{(1-\mu)(1-\beta)/4},$$

so that

$$\begin{aligned}
(3.59) \quad |J_0^3| &\leq 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \left(4\sigma^{(1-\mu)(1-\beta)/4}\right) \int_{\Omega_0} e^{-(1-\mu)|x-y|} dy \\
&\leq \frac{24}{1-\mu} e^{-\mu\rho(x)} (C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) \sigma^{\frac{1}{4}(1-\mu)(1-\beta)}.
\end{aligned}$$

For $j = 1, \dots, k$, we have

$$\begin{aligned}
(3.60) \quad |J_j^3| &= \left| \frac{1}{2} \int_{\Omega_j} e^{-|x-y|} \sum_{i=1}^k \gamma_i W_i(y) dy \right| \leq \frac{1}{2} \|\gamma\| \sum_{i=1}^k \int_{\Omega_j} e^{-|x-y|} |U'(y - \xi_i)| dy \\
&\leq 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \sum_{i=1}^k \int_{\Omega_j} e^{-(1-\mu)(|x-y|+|y-\xi_i|)} dy \\
&= 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \sum_{i=1}^k \int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)(|x-y-\xi_j|+|y+\xi_j-\xi_i|)} dy \\
&\leq 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \left[\int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)|x-y-\xi_j|} e^{-(1-\mu)|y|} dy \right. \\
&\quad \left. + \int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)|x-y-\xi_j|} \sum_{i(\neq j)=1}^k e^{-(1-\mu)|y+\xi_j-\xi_i|} dy \right] \\
&\leq 3(C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) e^{-\mu\rho(x)} \left[\int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)|x-y-\xi_j|} e^{-(1-\mu)|y|} dy \right. \\
&\quad \left. + \int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)|x-y-\xi_j|} \cdot \frac{2\sigma^{\frac{3}{4}(1-\mu)(1-\beta)}}{1-\sigma^{(1-\mu)(1-\beta)}} dy \right].
\end{aligned}$$

Summing up (3.59) and (3.60) for $j = 1, \dots, k$, by (3.51), we obtain

$$\begin{aligned}
(3.61) \quad |J^3| &\leq e^{-\mu\rho(x)} (C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) \left\{ \frac{24}{1-\mu} \sigma^{\frac{1}{4}(1-\mu)(1-\beta)} \right. \\
&\quad \left. + 3C_5 \left[\int_{-x_\sigma}^{x_\sigma} e^{-(1-\mu)|y|} dy + \int_{-x_\sigma}^{x_\sigma} 4\sigma^{\frac{3}{4}(1-\mu)(1-\beta)} dy \right] \right\} \quad (\text{for } \sigma \text{ small}) \\
&\leq e^{-\mu\rho(x)} (C_1 \|h\|_\mu + C_2(\sigma) \|\varphi\|_\mu) \left\{ \frac{24}{1-\mu} \sigma^{\frac{1}{4}(1-\mu)(1-\beta)} \right. \\
&\quad \left. + 3C_5 \left[\frac{2}{1-\mu} \sigma^{\frac{1}{4}(1-\mu)(1-\beta)} + 2\sigma^{\frac{3}{4}(1-\mu)(1-\beta)} (C_0 + |\log \sigma|) \right] \right\} \\
&= e^{-\mu\rho(x)} (C_{10}(\sigma) \|h\|_\mu + C_{11}(\sigma) \|\varphi\|_\mu),
\end{aligned}$$

where

$$\begin{aligned}
(3.62) \quad C_{10}(\sigma) &= O\left(\sigma^{\frac{1}{4}(1-\mu)(1-\beta)}\right), \quad \text{and} \\
C_{11}(\sigma) &= C_2(\sigma) O\left(\sigma^{\frac{1}{4}(1-\mu)(1-\beta)}\right) = O\left(\sigma^{1-\frac{3\beta}{2}+\frac{1}{4}(1-\mu)(1-\beta)}\right).
\end{aligned}$$

We summarize these estimates in the following lemma.

LEMMA 3.4. *Under the assumption*

$$(3.63) \quad 0 < \beta < \min \left\{ \frac{1}{2}, \frac{1-\mu}{2-\mu} \right\} = \frac{1-\mu}{2-\mu},$$

for any solution (φ, γ) to the problem (3.13), it holds that for $x \in \mathbb{R}$,

$$\begin{aligned}
(3.64) \quad |\varphi(x)| &\leq e^{-2\mu \min_{1 \leq i \leq k} |x-\xi_i|} (K_1 \|\varphi\|_\mu + C_8(\sigma) \|h\|_\mu) \\
&\quad + e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} (C_{11}(\sigma) \|\varphi\|_\mu + K_2 \|h\|_\mu),
\end{aligned}$$

where $K_1 \geq 18$, $K_2 > 0$ are constants, and

$$(3.65) \quad C_8(\sigma) = O\left(\sigma^{(1-\mu)(1-\beta)-\beta} |\log \sigma|\right),$$

$$(3.66) \quad C_{11}(\sigma) = O\left(\sigma^{1-\frac{3\beta}{2}+\frac{1}{4}(1-\mu)(1-\beta)}\right).$$

PROOF. First of all, note that the assumption made in (3.63) is necessary due to the assumption (3.36) required in Lemma 3.3 and earlier proofs, as well as the assumption (3.58) required in (3.56) and (3.57).

We assemble (3.43), (3.44), (3.46), (3.52), (3.56) and (3.61) altogether to reach the following inequality,

$$(3.67) \quad \begin{aligned} |\varphi(x)| &\leq |J^1| + |J^2| + |J^3| + \left| \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} h(y) dy \right| \\ &\leq e^{-2\mu\rho(x)} [C_{12}\|\varphi\|_{\mu} + C_8(\sigma)\|h\|_{\mu} + C_9(\sigma)\|\varphi\|_{\mu}] \\ &+ e^{-\mu\rho(x)} \left[C_{10}(\sigma)\|h\|_{\mu} + C_{11}(\sigma)\|\varphi\|_{\mu} + \frac{1}{1-\mu}\|h\|_{\mu} \right] \\ &= e^{-2\mu \min_{1 \leq i \leq k} |x-\xi_i|} [C_8(\sigma)\|h\|_{\mu} + (C_{12} + C_9(\sigma))\|\varphi\|_{\mu}] \\ &+ e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \left[\left(\frac{1}{1-\mu} + C_{10}(\sigma) \right) \|h\|_{\mu} + C_{11}(\sigma)\|\varphi\|_{\mu} \right], \quad \forall x \in \mathbb{R}, \end{aligned}$$

where the constant C_{12} comes from (3.52) for $|J^1|$, with

$$C_{12} \geq 18C_5 + 144\sigma^{1/12} + 15\sqrt{2}C_5\sigma^{\frac{3}{4}(1-\mu)(1-\beta)}(C_0 + |\log \sigma|),$$

and the other relevant constants are introduced in the estimates of $|J^2|$ and $|J^3|$. Finally, let

$$(3.68) \quad \begin{aligned} K_1 (\geq C_{12} + C_9(\sigma)) &\text{ be a constant, } K_1 \geq 18C_5 \geq 18, \text{ and} \\ K_2 \left(\geq \frac{1}{1-\mu} + C_{10}(\sigma) \right) &\text{ be a constant,} \end{aligned}$$

then we reach the conclusion of this lemma. \square

REMARK 2 (The order of $k = O(\sigma^{-\beta})$ and the effect of μ). According to the proof procedure we have accomplished so far, the assumption (3.63) on the order of $k = O(\sigma^{-\beta})$,

$$0 < \beta < \frac{1-\mu}{2-\mu} \quad (< \frac{1}{2})$$

is sharp. As we choose $\mu = \frac{1}{3}$, then accordingly $0 < \beta < \frac{2}{5} \in \left(\frac{1}{3}, \frac{1}{2}\right)$. But we can reduce the parameter $\mu > 0$ of the weighted norm to increase the number $\frac{1-\mu}{2-\mu}$ and to make it very close to $1/2$. We shall make another remark on the order β after we finish the estimates in the nonlinear part later in Section 6.

LEMMA 3.5. Under the assumption (3.63), if sequences $\{\sigma_n\} \subset \mathbb{R}^+$ and $\{h_n\} \subset L_{\mu}^{\infty}$ satisfy $0 < \sigma_n \rightarrow 0$ and $\|h_n\|_{\mu} \rightarrow 0$, as $n \rightarrow \infty$, and if $(\varphi_n, \gamma^n) \in L_{\mu}^{\infty} \times \mathbb{R}^k$ is a solution to the problem (3.13), then it holds that

$$(3.69) \quad \|\varphi_n\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

PROOF. Suppose (3.69) does not hold. Without loss of generality, we can assume that

$$\|\varphi_n\|_\mu = 1, \quad \text{for all } n.$$

Substituting $\varphi = \varphi_n$, $h = h_n$, and $\sigma = \sigma_n$ into (3.64), in Lemma 3.4, and then taking the limit as $n \rightarrow \infty$, we get

$$(3.70) \quad |\varphi_n(x)| \leq K_1 e^{-2\mu \min_{1 \leq i \leq k} |x - \xi_i|}, \quad x \in \mathbb{R}$$

or equivalently,

$$(3.71) \quad e^{\mu \min_{1 \leq i \leq k} |x - \xi_i|} |\varphi_n(x)| \leq K_1 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|}, \quad x \in \mathbb{R}.$$

We claim that there exists at least one index $m \in \{1, \dots, k\}$ such that

$$(3.72) \quad \sup \left\{ |\varphi_n(x)| : x \in I_m = \left[\xi_m - \frac{\log K_1}{\mu}, \xi_m + \frac{\log K_1}{\mu} \right] \right\} \geq \frac{1}{K_1}.$$

Otherwise, for any

$$x \in \bigcup_{j=1}^k \left[\xi_j - \frac{\log K_1}{\mu}, \xi_j + \frac{\log K_1}{\mu} \right],$$

we would have

$$e^{\mu \min_{1 \leq i \leq k} |x - \xi_i|} |\varphi_n(x)| < \frac{K_1}{K_1} = 1.$$

Moreover, for any

$$x \in \mathbb{R} \setminus \bigcup_{j=1}^k \left[\xi_j - \frac{\log K_1}{\mu}, \xi_j + \frac{\log K_1}{\mu} \right],$$

by (3.71) we have

$$e^{\mu \min_{1 \leq i \leq k} |x - \xi_i|} |\varphi_n(x)| \leq K_1 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} < \frac{K_1}{K_1} = 1.$$

Then it follows that

$$e^{\mu \min_{1 \leq i \leq k} |x - \xi_i|} |\varphi_n(x)| < 1, \quad \text{for every } x \in \mathbb{R},$$

which contradicts the assumption $\|\varphi_n\|_\mu = 1$. (Note that the norm $\|\varphi_n\|_\mu$ must be attained on a subset of \mathbb{R} with a positive Lebesgue measure.) Therefore, the claim (3.72) holds. However, the index m may depend on n .

Define the translated function $\tilde{\varphi}_n(x) = \varphi_n(x + \xi_m)$, $x \in \mathbb{R}$, where ξ_m is specified in (3.72). Then for all $n \geq 1$, one has

$$(3.73) \quad \sup_{|x| \leq \frac{1}{\mu} \log K_1} |\tilde{\varphi}_n(x)| \geq \frac{1}{K_1}.$$

Note that

$$\frac{1}{\mu} \log K_1 > 0 \quad \text{due to } K_1 \geq 18.$$

On the other hand, (3.70) implies that

$$0 \leq |\tilde{\varphi}_n(x)| \leq K_1, \quad x \in \mathbb{R}.$$

By the Heine-Borel Theorem on compactness and by a diagonal sequence selection, we can confirm that there exists a subsequence of $\{\tilde{\varphi}_n\}$ which converges uniformly over compact intervals to a limit function denoted by $\tilde{\varphi}(x)$, $x \in \mathbb{R}$. We relabel the

convergent subsequence as the same as $\{\tilde{\varphi}_n\}$. Since $\tilde{\varphi}(x)$ satisfies Eq. (3.13), we have

$$L(\tilde{\varphi}_n) = \widetilde{h}_n + \sum_{j=1}^h \gamma_j^n \tilde{W}_j, \quad x \in \mathbb{R},$$

here $\widetilde{h}_n(x) = h_n(x + \xi_m)$ and $\tilde{W}_j = W_j(x + \xi_m)$. Thus $\tilde{\varphi}_n$ satisfies the integral equation

$$(3.74) \quad \tilde{\varphi}_n(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left(2\tilde{W}\tilde{\varphi}_n - \frac{2\tilde{W}^2}{\omega} \langle \tilde{W}, \tilde{\varphi}_n \rangle + \widetilde{h}_n + \gamma^n \cdot \nabla_{\xi} \tilde{W} \right) dy,$$

where $\tilde{W}(x) = W(x + \xi_m)$. Now take limit as $n \rightarrow \infty$ and use the Dominated Convergence Theorem to see that the limit function $\tilde{\varphi}$ satisfies the equation

$$(3.75) \quad \tilde{\varphi}(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} \left(2\tilde{W}\tilde{\varphi} \right) (y) dy = \int_{\mathbb{R}} e^{-|x-y|} \tilde{W}(y) \tilde{\varphi}(y) dy, \quad x \in \mathbb{R},$$

because the remaining terms in (3.74) converge to zero, namely,

$$\int_{\mathbb{R}} e^{-|x-y|} \left(-\frac{2\tilde{W}^2}{\omega} \langle \tilde{W}, \tilde{\varphi}_n \rangle \right) dy \rightarrow 0 \quad \text{because of Lemma 3.3,}$$

$$\int_{\mathbb{R}} e^{-|x-y|} \widetilde{h}_n(y) dy \rightarrow 0 \quad \text{by the assumption } \|h_n\|_{\mu} \rightarrow 0, \text{ and}$$

$$\int_{\mathbb{R}} e^{-|x-y|} \gamma^n \cdot \nabla_{\xi} \tilde{W} dy \rightarrow 0 \quad \text{because of Lemma 3.2.}$$

It follows that $\tilde{\varphi}$ is a solution to the problem:

$$(3.76) \quad \begin{aligned} L_1(\tilde{\varphi}) &= -\tilde{\varphi}'' + (1 - 2W)\tilde{\varphi} = 0, \quad x \in \mathbb{R}, \\ \langle \tilde{\varphi}, \tilde{W}_j \rangle &= 0, \quad j = 1, \dots, k, \text{ and} \\ \tilde{\varphi}(x) &\rightarrow 0 \text{ exponentially, as } |x| \rightarrow \infty. \end{aligned}$$

By Lemma 1.4, we know that

$$\text{Ker } L_1 = \text{Span} \{W_1, \dots, W_k\} = \text{Span} \{ \tilde{W}_1, \dots, \tilde{W}_k \}.$$

Hence, (3.76) implies that

$$(3.77) \quad \tilde{\varphi} \in \text{Ker } L_1 \cap (\text{Ker } L_1)^{\perp} = \{0\}, \quad \text{i.e., } \tilde{\varphi}(x) \equiv 0.$$

However, $\tilde{\varphi}_n$ also satisfies the equation

$$\tilde{\varphi}_n = (-\Delta + I)^{-1} \left[2\tilde{W}\tilde{\varphi}_n - \frac{2\tilde{W}^2}{\omega} \langle \tilde{W}, \tilde{\varphi}_n \rangle + \widetilde{h}_n + \gamma^n \cdot \nabla_{\xi} \tilde{W} \right],$$

by the regularity of the solutions to harmonic equations, this implies $\tilde{\varphi}_n \in H^2(\mathbb{R}) \cap H_0^1(\mathbb{R})$ and

$$\|\tilde{\varphi}_n\|_{H^2(\mathbb{R})} \leq \text{const}, \quad \text{for } n \geq 1.$$

The Sobolev imbedding property $H^2(\mathbb{R}) \rightarrow C_{\text{loc}}^{0,1}(\mathbb{R})$ infers that $\{\tilde{\varphi}_n\}$ is uniformly Lipschitz continuous on the compact interval $I_0 = [-\mu^{-1} \log K_1, \mu^{-1} \log K_1]$. This together with (3.73) implies that there is a positive number $\nu_0 > 0$ such that

$$\text{meas} \left\{ x \in I_0 : |\tilde{\varphi}_n(x)| \geq \frac{1}{2K_1} \right\} \geq \nu_0.$$

Therefore, there exists a measurable subset $\tilde{I} \subset I_0 \subset \mathbb{R}$, such that $\text{meas}(\tilde{I}) > 0$ and

$$(3.78) \quad \tilde{\varphi}(x) > 0 \text{ for } x \in \tilde{I}.$$

Finally, (3.77) and (3.78) constitute a contradiction. This contradiction shows that (3.69) holds. The lemma is proved. \square

Now we can prove the key result concerning the solution of the augmented linear problem (3.13), which is the principal approximation of the linear part of the original equation (2.14).

THEOREM 3.6. *Under the assumption*

$$(3.79) \quad k = \text{const } \sigma^{-\beta}, \text{ with } 0 < \beta < \frac{1-\mu}{2-\mu},$$

there exists a positive number $\hat{\sigma} \leq \frac{1}{2}$ such that for any given $0 < \sigma \leq \hat{\sigma}$ and for any $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$ in (1.16) and a given $h \in L_\mu^\infty$, the problem (3.13) has a unique solution (φ, γ) , which is denoted by $\varphi = \mathcal{J}(h)$ and $\gamma = \Pi(h)$. Moreover, there exist two positive constants Γ_0 and Γ_1 such that

$$(3.80) \quad \begin{aligned} \|\varphi\|_\mu &= \|\mathcal{J}(h)\|_\mu \leq \Gamma_0 \|h\|_\mu, \\ \|\gamma\|_{\mathbb{R}^k} &= \|\Pi(h)\|_{\mathbb{R}^k} \leq \Gamma_1 \|h\|_\mu. \end{aligned}$$

PROOF. Define a Hilbert space H by

$$H = \left\{ \varphi \in H_0^1(\mathbb{R}) : \langle \varphi, W_j \rangle = 0, j = 1, \dots, k, \text{ and } \varphi(x) \xrightarrow[\mu]{\text{exp}} 0 \text{ as } |x| \rightarrow \infty \right\},$$

where $\varphi(x) \xrightarrow[\mu]{\text{exp}} 0$, as $|x| \rightarrow \infty$, means that

$$\limsup_{|x| \rightarrow \infty} e^{\mu|x|} |\varphi(x)| < \infty,$$

and H is endowed with the norm of Sobolev space $H_0^1(\mathbb{R})$. A solution to the problem (3.13) ought to be such a pair (φ, γ) that $\varphi \in H$ satisfying the equation

$$(3.81) \quad \int_{\mathbb{R}} (\varphi' \psi' + \varphi \psi) dy - \langle 2W\varphi, \psi \rangle + \frac{2}{\omega} \langle W, \varphi \rangle \langle W^2, \psi \rangle = \langle h, \psi \rangle, \quad \forall \psi \in H,$$

and vice-versa, and $\gamma \in \mathbb{R}^k$ satisfying the equation (3.16). Note that this variational equation (3.81) is equivalent to the following version,

$$(3.82) \quad \varphi - (1 - \Delta)^{-1} R(\varphi) = (1 - \Delta)^{-1} h,$$

where $R(\varphi) = 2W\varphi - \frac{2W^2}{\omega} \langle W, \varphi \rangle$. Since $(I - \Delta)^{-1}$ is a compact linear operator and R is a bounded linear operator, so that $(I - \Delta)^{-1} R$ is a compact operator on H . And $(I - \Delta)^{-1} h \in H$. Thus by the Fredholm alternative principle, Eq. (3.81) is uniquely solvable if and only if it admits only the trivial zero solution for $h = 0$.

Suppose the existence and uniqueness part of this theorem does not hold, then there exists a positive sequence $\{\sigma_n\}$, $\sigma_n \rightarrow 0$, as $n \rightarrow \infty$, and a corresponding solution sequence $\{\varphi_n\} \subset H$ of the problem (3.13) with $h_n \equiv 0$, such that $\|\varphi_n\|_\mu = 1$ for $n \geq 1$, since we can always normalize a nontrivial solution φ_n with $\|\varphi_n\|_\mu \neq 0$. However, this directly contradicts Lemma 3.5 as we have shown. Therefore, applying the Fredholm alternative principle and Lemma 3.2, we can assert that

problem (3.13) has a unique solution for any $0 < \sigma \leq \hat{\sigma}$, where $\hat{\sigma}$ is some constant sufficiently small.

To show the boundedness of the solution mapping $\varphi = \mathcal{J}(h)$ and $\gamma \in \Pi(h)$ in (3.80), we can use (3.82). By the inverse operator theorem in Banach spaces, the unique solvability we proved in the first part of this theorem implies that

$$[I - (I - \Delta)^{-1}R]^{-1} \text{ is a bounded linear operator,}$$

so that the linear operator \mathcal{J} for the φ -component,

$$(3.83) \quad \mathcal{J} = [I - (I - \Delta)^{-1}R]^{-1} (I - \Delta)^{-1} \in \mathcal{L}(L_\mu^\infty, L_\mu^\infty),$$

is bounded. Then by Lemma 3.2, the linear operator Π for the γ -component is also bounded. The proof is completed. \square

4. Nonlinear Estimates

In this section, we have two objectives. First we shall take the full consideration of the σ effect in $T[W\varphi]$, which is replaced by $\frac{1}{\omega} \langle W, \varphi \rangle$ in Section 3, and estimate the difference between $V(x)$ and constant 1, while we set $V(x) = 1$ in Section 3. Second is a key result on the estimate of the integrals

$$\int_{\mathbb{R}} S[W](x)U'(x - \xi_j) dx, \quad j = 1, \dots, k,$$

which constitutes the main part of the nonlinear side in (2.14). Here, $W(x)$ defined by (1.17), Λ defined by (1.16) and L_μ^∞ defined by (3.12) remain the same.

I) For each given $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$ and $W(x)$ given in (1.17) accordingly, the function $V = T[W^2]$ is the unique solution of the ODE problem (2.11), which has the explicit expression (2.12). We first want to know the behavior of this function V around the points ξ_1, \dots, ξ_k . For each $\ell = 1, \dots, k$, we have

(4.1)

$$\begin{aligned} V(\xi_\ell) &= \frac{1}{\omega} \int_{\mathbb{R}} e^{-\sigma|\xi_\ell - x|} \left[\sum_{j=1}^k U^2(x - \xi_j) + \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i)U(x - \xi_j) \right] dx \\ &= 1 + \frac{1}{\omega} \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) \left(\sum_{j=1}^k U^2(x - \xi_j) + \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i)U(x - \xi_j) \right) dx \end{aligned}$$

where

$$\omega = \int_{\mathbb{R}} W^2(x) dx, \text{ as we introduced in (2.16).}$$

By the instrumental inequality

$$(4.2) \quad 0 \leq 1 - e^{-\sigma y} \leq \sigma y, \quad \text{which is valid for all } y \geq 0,$$

we can make the following estimates, in which C may represent different positive constants which need not be further specified.

A) For $j \neq \ell$ in the first sum in (4.1), we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) U^2(x - \xi_j) dx \right| \leq \int_{\mathbb{R}} \left| 1 - e^{-\sigma|\xi_\ell - \xi_j - x|} \right| U^2(x) dx \\
(4.3) \quad & \leq \int_{\mathbb{R}} \sigma |\xi_\ell - \xi_j - x| U^2(x) dx \leq \sigma \left(\int_{\mathbb{R}} |x| U^2(x) dx + \int_{\mathbb{R}} |\xi_\ell - \xi_j| U^2(x) dx \right) \\
& = \sigma (12 \log 2 - 3 + 6 |\xi_\ell - \xi_j|) \quad (\text{by Lemma 1.1}) \\
& \leq \sigma (12 \log 2 - 3 + 6 |\ell - j| |\log \sigma|) \leq \sigma (12 \log 2 - 3 + 6(k-1) |\log \sigma|) \\
& \leq C \sigma^{1-\beta} |\log \sigma|, \quad \text{for some constant } C > 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
(4.4) \quad & \left| \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) \sum_{j=1}^k U^2(x - \xi_j) dx \right| \leq \left| \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) U^2(x - \xi_\ell) dx \right| \\
& + \sum_{j(\neq \ell)=1}^k \left| \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) U^2(x - \xi_j) dx \right| \\
& \leq \sigma \int_{\mathbb{R}} |x| U^2(x) dx + (k-1) C \sigma^{1-\beta} |\log \sigma| \\
& \leq (12 \log 2 - 3) \sigma + (k-1) C \sigma^{1-\beta} |\log \sigma|.
\end{aligned}$$

B) For those cross-product integral terms with either $i = \ell \neq j$ or $j = \ell \neq i$, we have

$$\begin{aligned}
(4.5) \quad & \sum_{i=\ell \neq j \text{ or } j=\ell \neq i} \left| \int_{\mathbb{R}} \left(e^{-\sigma|\xi_\ell - x|} - 1 \right) U(x - \xi_i) U(x - \xi_j) dx \right| \\
& \leq 2 \sum_{j(\neq \ell)=1}^k \int_{\mathbb{R}} \left| e^{-\sigma|y|} - 1 \right| U(y) U(y + \xi_\ell - \xi_j) dy \\
& \leq 72 \sigma \sum_{j(\neq \ell)=1}^k \int_{\mathbb{R}} |y| e^{-|y|} e^{-|y + \xi_\ell - \xi_j|} dy \quad (\text{by using (1.8)}) \\
& = 36 \sigma \sum_{j(\neq \ell)=1}^k e^{-|\xi_\ell - \xi_j|} \left(1 + |\xi_\ell - \xi_j| + |\xi_\ell - \xi_j|^2 \right) \\
& \leq 36 \sigma \sum_{j(\neq \ell)=1}^k \sigma^{|\ell - j|(1-\beta)} \left(1 + |\ell - j| |\log \sigma| + |\ell - j|^2 |\log \sigma|^2 \right) \quad (\text{by Lemma 1.3}) \\
& \leq 36 \sigma \left[C \sigma^{1-\beta} + C \sigma^{1-\beta} |\log \sigma| + k |\log \sigma| C \sigma^{1-\beta} |\log \sigma| \right] \\
& \leq C \sigma^{2-\beta} (1 + |\log \sigma|) + C \sigma^{2(1-\beta)} |\log \sigma|^2 \\
& = O \left(\sigma^{2(1-\beta)} |\log \sigma|^2 \right).
\end{aligned}$$

C) For those cross-product integral terms with (i, j, ℓ) mutually unequal, we have

$$\begin{aligned}
& \sum_{\substack{i,j(\neq\ell) \\ i\neq j}} \left| \int_{\mathbb{R}} \left(e^{|\xi_\ell - x|} - 1 \right) U(x - \xi_i) U(x - \xi_j) dx \right| \\
& \leq 36\sigma \sum_{\substack{i,j(\neq\ell) \\ i\neq j}} \int_{\mathbb{R}} |\xi_\ell - \xi_i - x| e^{-|x|} e^{-|x+\xi_i-\xi_j|} dx \\
& \leq 36\sigma \sum_{\substack{i,j(\neq\ell) \\ i\neq j}} \left\{ |\xi_\ell - \xi_i| \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_i-\xi_j|} dx + \int_{\mathbb{R}} |x| e^{-|x|} e^{-|x+\xi_i-\xi_j|} dx \right\} \\
& \quad \text{(by (1.7) and (1.8))} \\
& \leq 36\sigma \sum_{i,j(\neq\ell)} \left\{ |\ell - i| |\log \sigma| (1 + |\xi_i - \xi_j|) e^{-|\xi_i - \xi_j|} \right. \\
(4.6) \quad & \quad \left. + \frac{1}{2} e^{-|\xi_i - \xi_j|} (1 + |\xi_i - \xi_j| + |\xi_i - \xi_j|^2) \right\} \\
& \leq 36\sigma \sum_{\substack{i,j \\ i\neq j}} e^{-|\xi_i - \xi_j|} \left[\frac{1}{2} (1 + |\xi_i - \xi_j|) + k |\log \sigma| (1 + 2|\xi_i - \xi_j|) \right] \\
& \leq 36\sigma \sum_{\substack{i,j \\ i\neq j}} \sigma^{|i-j|(1-\beta)} \left[\frac{1}{2} (1 + |i-j| |\log \sigma|) + k |\log \sigma| (1 + 2|i-j| |\log \sigma|) \right] \\
& \quad \text{(by Lemma 1.3)} \\
& \leq 36\sigma k [C \sigma^{1-\beta} (1 + |\log \sigma|) + k |\log \sigma| C \sigma^{1-\beta} |\log \sigma|] \\
& \leq C \sigma^{2(1-\beta)} (1 + |\log \sigma|) + C \sigma^{2-3\beta} |\log \sigma|^2 \\
& = O(\sigma^{2-3\beta} |\log \sigma|^2).
\end{aligned}$$

Again, note that

$$(4.7) \quad \omega = \int_{\mathbb{R}} W^2(x) dx \geq 6k, \text{ and } k = O(\sigma^{-\beta}).$$

From (4.1), (4.4), (4.5), (4.6) and (4.7) we obtain

$$\begin{aligned}
V(\xi_\ell) - 1 & \leq \frac{1}{6k} \left\{ (12 \log 2 - 3)\sigma + (k-1)C\sigma^{1-\beta} |\log \sigma| \right. \\
& \quad \left. + O(\sigma^{2(1-\beta)} |\log \sigma|^2) + O(\sigma^{2-3\beta} |\log \sigma|^2) \right\} \\
& = O(\sigma^{1+\beta}) + O(\sigma^{1-\beta} |\log \sigma|) + O(\sigma^{2-\beta} |\log \sigma|^2) + O(\sigma^{2(1-\beta)} |\log \sigma|^2) \\
& = O(\sigma^{1-\beta} |\log \sigma|).
\end{aligned}$$

Therefore, we have the following lemma.

LEMMA 4.1. *It holds that*

$$(4.8) \quad V(\xi_\ell) = 1 + O(\sigma^{1-\beta} |\log \sigma|), \quad \ell = 1, \dots, k.$$

II) In order to know the information about $V(\xi_\ell + y)$, for $|y| \leq \frac{1}{2}|\log \sigma|$, $\ell = 1, \dots, k$, which covers the interval $[\xi_1 - \frac{1}{2}|\log \sigma|, \xi_k + \frac{1}{2}|\log \sigma|]$ of the real line, we now estimate the increment $V(\xi_\ell + y) - V(\xi_\ell)$. By (4.1), we have

$$(4.9) \quad \begin{aligned} V(\xi_\ell + y) - V(\xi_\ell) &= \frac{1}{\omega} \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell + y - x|} - e^{-\sigma|\xi_\ell - x|} \right] \\ &\times \left[\sum_{j=1}^k U^2(x - \xi_j) + \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) \right] dx. \end{aligned}$$

The estimates will be made part by part, as follows.

a) For $j < \ell$ in the first sum, by using the Taylor expansion

$$(4.10) \quad e^{-x} = 1 - x + O(x^2), \quad \text{for } x \text{ (small)} > 0,$$

we have

$$(4.11) \quad \begin{aligned} &\int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell + y - x|} - e^{-\sigma|\xi_\ell - x|} \right] U^2(x - \xi_j) dx \\ &= \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell - \xi_j + y - x|} - e^{-\sigma|\xi_\ell - \xi_j - x|} \right] U^2(x) dx \\ &= \int_{\mathbb{R}} \left[e^{-\sigma(\xi_\ell - \xi_j + y - x)} - e^{-\sigma(\xi_\ell - \xi_j - x)} \right] U^2(x) dx + R(y) \\ &= e^{-\sigma(\xi_\ell - \xi_j)} \int_{\mathbb{R}} \left[e^{-\sigma(y - x)} - e^{\sigma x} \right] U^2(x) dx + R(y) \\ &= e^{-\sigma|\xi_\ell - \xi_j|} (e^{-\sigma y} - 1) \int_{\mathbb{R}} e^{\sigma x} U^2(x) dx + R(y) \\ &= e^{-\sigma|\xi_\ell - \xi_j|} [-\sigma y + O(\sigma^2 y^2)] \hat{C}(\sigma) + R(y), \end{aligned}$$

where the constant $\hat{C}(\sigma)$ is given by

$$(4.12) \quad \hat{C}(\sigma) \triangleq \int_{\mathbb{R}} e^{\sigma x} U^2(x) dx \leq 36 \left(\frac{1}{2 - \sigma} + \frac{1}{2 + \sigma} \right) = \frac{144}{4 - \sigma^2} < 40,$$

and the remainder $R(y)$ can be expressed as

$$(4.13) \quad \begin{aligned} R(y) &= \int_{y + \xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x - y + \xi_j - \xi_\ell)} - e^{-\sigma(\xi_\ell - \xi_j + y - x)} \right] U^2(x) dx \\ &\quad - \int_{\xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x + \xi_j - \xi_\ell)} - e^{-\sigma(\xi_\ell - \xi_j - x)} \right] U^2(x) dx, \end{aligned}$$

where the two integrals in (4.13) will be denoted by $R_1(y)$ and $R_2(y)$, respectively.

Then,

$$(4.14) \quad \begin{aligned} |R_1(y)| &= \left| \int_{y + \xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x - y + \xi_j - \xi_\ell)} - e^{\sigma(x - y + \xi_j - \xi_\ell)} \right] U^2(x) dx \right| \\ &\leq \int_{y + \xi_\ell - \xi_j}^{\infty} |-2\sigma(x - y + \xi_j - \xi_\ell)| U^2(x) dx + (\text{h.o.t.}) \\ &\leq 72 \sigma \left\{ \int_{y + \xi_\ell - \xi_j}^{\infty} (|x| + |y|) e^{-2x} dx + |j - \ell| |\log \sigma| \int_{y + \xi_\ell - \xi_j}^{\infty} e^{-2x} dx \right\} + (\text{h.o.t.}) \end{aligned}$$

in which (h.o.t.) stands for higher-order terms. Since we have, for $j < \ell$, that

$$\begin{aligned} \int_{y+\xi_\ell-\xi_j}^{\infty} (|x| + |y|) e^{-2x} dx &= \int_{y+\xi_\ell-\xi_j}^{\infty} x e^{-2x} dx + |y| \int_{y+\xi_\ell-\xi_j}^{\infty} e^{-2x} dx \\ &= \left(-\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} \right) \Big|_{y+\xi_\ell-\xi_j}^{\infty} + |y| \left(-\frac{1}{2} e^{-2x} \right) \Big|_{y+\xi_\ell-\xi_j}^{\infty} \\ &\leq \frac{1}{2} (2|y| + |\ell - j| |\log \sigma| + 1) e^{-2(y+\xi_\ell-\xi_j)} \\ &\leq \frac{1}{2} (1 + (\ell - j + 1) |\log \sigma|) \sigma^{2(\ell-j)(1-\beta)-1}, \end{aligned}$$

because $|y| \leq \frac{1}{2} |\log \sigma|$, and

$$\int_{y+\xi_\ell-\xi_j}^{\infty} e^{-2x} dx \leq \frac{1}{2} \sigma^{2(\ell-j)(1-\beta)-1}.$$

Therefore,

$$\begin{aligned} |R_1(y)| &\leq 36 \sigma^{2(\ell-j)(1-\beta)} [1 + (2\ell - 2j + 1) |\log \sigma|] + (\text{h.o.t.}) \\ (4.15) \quad &\leq C \sigma^{2(\ell-j)(1-\beta)} (\ell - j + 1) |\log \sigma|. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} |R_2(y)| &= \left| \int_{\xi_\ell-\xi_j}^{\infty} \left[e^{-\sigma(x+\xi_j-\xi_\ell)} - e^{\sigma(x+\xi_j-\xi_\ell)} \right] U^2(x) dx \right| \\ (4.16) \quad &\leq 72 \sigma \left\{ \int_{\xi_\ell-\xi_j}^{\infty} x e^{-2x} dx + (\ell - j) |\log \sigma| \int_{\xi_\ell-\xi_j}^{\infty} e^{-2x} dx \right\} + (\text{h.o.t.}) \\ &\leq C \sigma^{2(\ell-j)(1-\beta)+1} [1 + (\ell - j) |\log \sigma|]. \end{aligned}$$

Substituting (4.15) and (4.16) into (4.13), we obtain that, for $j < \ell$, $|y| \leq \frac{1}{2} |\log \sigma|$,

$$\begin{aligned} (4.17) \quad &\int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] U^2(x - \xi_j) dx \\ &= \hat{C}(\sigma) e^{-\sigma|\xi_\ell-\xi_j|} [-\sigma y + O(\sigma^2 |\log \sigma|^2)] + O\left(\sigma^{2(\ell-j)(1-\beta)} (\ell - j + 1) |\log \sigma|\right). \end{aligned}$$

b) For $j > \ell$, we can proceed similarly to obtain

$$\begin{aligned} (4.18) \quad &\int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] U^2(x - \xi_j) dx \\ &= \int_{\mathbb{R}} \left[e^{-\sigma|\xi_j-\xi_\ell+x-y|} - e^{-\sigma|\xi_j-\xi_\ell+x|} \right] U^2(x) dx \\ &= \int_{\mathbb{R}} \left[e^{-\sigma(\xi_j-\xi_\ell+x-y)} - e^{-\sigma(\xi_j-\xi_\ell+x)} \right] U^2(x) dx + \tilde{R}(y) \\ &= \hat{C}(\sigma) e^{-\sigma|\xi_j-\xi_\ell|} [\sigma y + O(\sigma^2 |\log \sigma|^2)] + O\left(\sigma^{2(j-\ell)(1-\beta)} (j - \ell + 1) |\log \sigma|\right), \end{aligned}$$

for $|y| \leq \frac{1}{2} |\log \sigma|$, where $\hat{C}(\sigma)$ is the same constant in (4.12), and $\tilde{R}(y)$ represents the corresponding remainder which is treated similar to (4.13) through (4.16).

c) For $j = \ell$ in the first sum in (4.9), we have

$$\begin{aligned}
 (4.19) \quad & \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] U^2(x - \xi_\ell) dx \\
 &= \int_{\mathbb{R}} \left[e^{-\sigma|y-x|} - e^{-\sigma|x|} \right] U^2(x) dx \\
 &= -\sigma \int_{\mathbb{R}} [|y-x| - |x|] U^2(x) dx + O(\sigma^2 |\log \sigma|^2).
 \end{aligned}$$

Let

$$(4.20) \quad \Psi(y) = \int_{\mathbb{R}} [|y-x| - |x|] U^2(x) dx, \quad y \in \mathbb{R}.$$

Since Ψ satisfies

$$\begin{aligned}
 (4.21) \quad & \Psi''(y) = 2U^2(y), \quad y \in \mathbb{R}, \\
 & \Psi(0) = 0, \quad \Psi'(0) = 0,
 \end{aligned}$$

and $U^2(y)$ is an even function, Ψ is also an even function as the solution of (4.21). Therefore, $\Psi U^2 U'$ is an odd function, so that

$$(4.22) \quad \int_{\mathbb{R}} \Psi(y) U^2(y) U'(y) dy = 0.$$

This property will be used later.

d) Next let us estimate the cross-product integral terms in (4.9),

$$\begin{aligned}
 (4.23) \quad & \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) dx \\
 &= \sum_{j=1}^k \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] \sum_{i(\neq j)=1}^k U(x - \xi_i) U(x - \xi_j) dx.
 \end{aligned}$$

For $j < \ell$, similarly we can get

$$\begin{aligned}
 (4.24) \quad & \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] \sum_{i(\neq j)=1}^k U(x - \xi_i) U(x - \xi_j) dx \\
 &= \tilde{C}(\sigma) e^{-\sigma|\xi_\ell-\xi_j|} [-\sigma y + O(\sigma^2 y^2)] + \Phi^j(y)
 \end{aligned}$$

where the coefficient $\tilde{C}(\sigma)$ and the remainder $\Phi^j(y)$ have the following properties

$$\text{i) } \tilde{C}(\sigma) = \sum_{i(\neq j)=1}^k \tilde{C}_i(\sigma), \text{ with}$$

$$\begin{aligned} \tilde{C}_i(\sigma) &= \left| \tilde{C}_i(\sigma) \right| = \int_{\mathbb{R}} e^{\sigma x} U(x) U(x + \xi_j - \xi_i) dx \\ &\leq 36 \int_{\mathbb{R}} e^{\sigma x} e^{-|x|} e^{-|x + \xi_j - \xi_i|} dx \quad (\text{by direct integration}) \\ &\leq 36 \left(\frac{1}{2 - \sigma} + \frac{1}{2 + \sigma} + |\xi_j - \xi_i| \right) e^{-|\xi_j - \xi_i|} \\ &\leq 36 \left(\frac{4}{4 - \sigma^2} + |j - i| |\log \sigma| \right) \sigma^{|j - i|(1 - \beta)} \\ &< (48 + 36 |j - i| |\log \sigma|) \sigma^{|j - i|(1 - \beta)}, \end{aligned}$$

where we used the explicit expression of $p_r(x)$ shown in the proof of Lemma 1.2, so that

$$\begin{aligned} (4.25) \quad \tilde{C}(\sigma) &= \left| \tilde{C}(\sigma) \right| \leq \sum_{i(\neq j)=1}^k (48 + 36 |j - i| |\log \sigma|) \sigma^{|j - i|(1 - \beta)} \\ &\leq C (1 + |\log \sigma|) \sigma^{1 - \beta}, \quad \text{for a constant } C. \end{aligned}$$

ii) Similar to (4.13) and (4.14), the remainder Φ^j can be decomposed into

$$\begin{aligned} (4.26) \quad \Phi^j(y) &= \int_{y + \xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x - y + \xi_j - \xi_\ell)} - e^{\sigma(x - y + \xi_j - \xi_\ell)} \right] \sum_{i(\neq j)=1}^k U(x) U(x + \xi_j - \xi_i) dx \\ &\quad - \int_{\xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x + \xi_j - \xi_\ell)} - e^{\sigma(x + \xi_j - \xi_\ell)} \right] \sum_{i(\neq j)=1}^k U(x) U(x + \xi_j - \xi_i) dx \end{aligned}$$

where the two integrals in (4.26) will be denoted by $\Phi_1^j(y)$ and $\Phi_2^j(y)$, respectively. Then we have

$$\begin{aligned} (4.27) \quad \left| \Phi_1^j(y) \right| &= \left| \int_{y + \xi_\ell - \xi_j}^{\infty} \left[e^{-\sigma(x - y + \xi_j - \xi_\ell)} - e^{\sigma(x - y + \xi_j - \xi_\ell)} \right] \sum_{i(\neq j)=1}^k U(x) U(x + \xi_j - \xi_i) dx \right| \\ &\leq 72 \sigma \int_{y + \xi_\ell - \xi_j}^{\infty} (|x| + |y| + |\xi_j - \xi_\ell|) \sum_{i(\neq j)=1}^k e^{-|x|} e^{-|x + \xi_j - \xi_i|} dx + (\text{h.o.t.}), \end{aligned}$$

where (h.o.t.) represents the higher-order terms, and by using Lemmas 1.2 and 1.3 we find that

$$\begin{aligned}
& \sum_{i(\neq j)=1}^k \int_{y+\xi_\ell-\xi_j}^{\infty} |x| e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx \quad (\text{by (1.8)}) \\
& \leq \sum_{i(\neq j)=1}^k \frac{1}{2} e^{-|\xi_j-\xi_i|} (1 + |\xi_j - \xi_i|)^2 \\
& \leq \sum_{i(\neq j)=1}^k \frac{1}{2} \sigma^{|i-j|(1-\beta)} (1 + |i-j| |\log \sigma|)^2 \\
& \leq C \sigma^{1-\beta} |\log \sigma|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{i(\neq j)=1}^k \int_{y+\xi_\ell-\xi_j}^{\infty} (|y| + |\xi_\ell - \xi_j|) e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx \quad (\text{by (1.7)}) \\
& \leq \sum_{i(\neq j)=1}^k \left(\frac{1}{2} |\log \sigma| + |j - \ell| |\log \sigma| \right) \int_{y+\xi_\ell-\xi_j}^{\infty} e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx \\
& \leq \sum_{i(\neq j)=1}^k \left(\frac{1}{2} |\log \sigma| + |j - \ell| |\log \sigma| \right) (1 + |i-j| |\log \sigma|) \sigma^{|i-j|(1-\beta)} \\
& \leq C (1 + |j - \ell|) |\log \sigma|^2 \sigma^{1-\beta}.
\end{aligned}$$

Therefore, we obtain

$$(4.28) \quad \left| \Phi_1^j(y) \right| \leq C \sigma^{2-\beta} (1 + |j - \ell|) |\log \sigma|^2, \quad \text{for } j < \ell \text{ and } j > \ell.$$

Moreover, we have

$$\begin{aligned}
(4.29) \quad & \left| \Phi_2^j(y) \right| = \left| \int_{\xi_\ell-\xi_i}^{\infty} \left[e^{-\sigma(x+\xi_j-\xi_\ell)} - e^{\sigma(x+\xi_j-\xi_\ell)} \right] \sum_{i(\neq j)=1}^k U(x) U(x + \xi_j - \xi_i) dx \right| \\
& \leq 72 \sigma \int_{\xi_\ell-\xi_i}^{\infty} (|x| + |\xi_j - \xi_\ell|) \sum_{i(\neq j)=1}^k e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx + (\text{h.o.t.}) \\
& \leq 72 \sigma \sum_{i(\neq j)=1}^k \left\{ \int_{\xi_\ell-\xi_i}^{\infty} |x| e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx + |\xi_j - \xi_\ell| \int_{\xi_\ell-\xi_i}^{\infty} e^{-|x|} e^{-|x+\xi_j-\xi_i|} dx \right\} \\
& \leq C \sigma^{2-\beta} (1 + |j - \ell|) |\log \sigma|^2, \quad \text{for } j < \ell \text{ and } j > \ell.
\end{aligned}$$

Thus, we have

(4.30)

$$\begin{aligned}
T_\ell^c(y) &\triangleq \sum_{i(\neq j)=1}^k \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] \sum_{i(\neq j)=1}^k U(x-\xi_i) U(x-\xi_j) dx \\
&= \sum_{i(\neq j)=1}^k \tilde{C}(\sigma) e^{-\sigma|\xi_\ell-\xi_j|} \left[\sigma y \operatorname{sgn}(\xi_j - \xi_\ell) + O(\sigma^2 |\log \sigma|^2) \right] \\
&\quad + \sum_{i(\neq j)=1}^k C \sigma^{2-\beta} (1 + |j - \ell|) |\log \sigma|^2 \\
&\leq \sum_{i(\neq j)=1}^k C (1 + |\log \sigma|) \sigma^{1-\beta} e^{-\sigma|\xi_\ell-\xi_j|} \left[\sigma y \operatorname{sgn}(\xi_j - \xi_\ell) + O(\sigma^2 |\log \sigma|^2) \right] \\
&\quad + C k (k-1) \sigma^{2-\beta} |\log \sigma|^2.
\end{aligned}$$

Note that the first part $\sum_{i(\neq j)=1}^k \dots$ in (4.30) can be absorbed by the corresponding sum of the first part in (4.17) and in (4.18) for $j \neq \ell$.

For $j = \ell$ in (4.23), we have

$$\begin{aligned}
T_\ell(y) &\triangleq \int_{\mathbb{R}} \left[e^{-\sigma|\xi_\ell+y-x|} - e^{-\sigma|\xi_\ell-x|} \right] \sum_{i(\neq j)=1}^k U(x-\xi_i) U(x-\xi_\ell) dx \\
&= \int_{\mathbb{R}} \left[e^{-\sigma|y-x|} - e^{-\sigma|x|} \right] \sum_{i(\neq j)=1}^k U(x) U(x+\xi_\ell-\xi_i) dx \\
&= -\sigma \int_{\mathbb{R}} [|y-x| - |x|] \sum_{i(\neq j)=1}^k U(x) U(x-\xi_i+\xi_\ell) dx + (\text{h.o.t.})
\end{aligned}$$

so that

$$\begin{aligned}
|T_\ell(y)| &\leq C \sigma |y| \sum_{i(\neq j)=1}^k \int_{\mathbb{R}} e^{-|x|} e^{-|x-\xi_i+\xi_\ell|} dx \\
(4.31) \quad &\leq C \sigma |\log \sigma| \sum_{i(\neq j)=1}^k (1 + |i - \ell| |\log \sigma|) \sigma^{|i-\ell|(1-\beta)} \\
&\leq C \sigma^{2-\beta} |\log \sigma|^2 = O(\sigma^{2-\beta} |\log \sigma|^2), \text{ for } |y| \leq \frac{1}{2} |\log \sigma|.
\end{aligned}$$

Thus we have proved the following result.

LEMMA 4.2. For $\ell = 1, \dots, k$, and $|y| \leq \frac{1}{2} |\log \sigma|$, it holds that

$$\begin{aligned}
(4.32) \quad &V(\xi_\ell + y) - V(\xi_\ell) = \frac{1}{\omega} \left\{ -\sigma \Psi(y) + O(\sigma^{2-\beta} |\log \sigma|^2) \right. \\
&+ \sum_{i(\neq j)=1}^k \hat{C}(\sigma) e^{-\sigma|\xi_\ell-\xi_j|} \left[\sigma y \operatorname{sgn}(\xi_j - \xi_\ell) + O(\sigma^2 |\log \sigma|^2) \right] \\
&\quad \left. + O(\sigma^{2(1-\beta)} |\log \sigma|) + O(k^2 \sigma^{2-\beta} |\log \sigma|^2) \right\},
\end{aligned}$$

where $\hat{C}(\sigma) < 40$ is shown in (4.12) and $\omega \geq 6k$.

PROOF. Substituting (4.17), (4.18), (4.19), (4.20), (4.30) and (4.31) into (4.9), we obtain (4.32) with the observations as follows,

$$\sum_{j(\neq \ell)=1}^k O\left(\sigma^{2|j-\ell|(1-\beta)} (|j-\ell|+1) |\log \sigma|\right) = O\left(\sigma^{2(1-\beta)} |\log \sigma|\right)$$

by using Lemma 1.3, and

$$\begin{aligned} & Ck(k-1)\sigma^{2-\beta} |\log \sigma|^2 \quad (\text{in (4.30)}) \\ & + C\sigma^{2-\beta} |\log \sigma|^2 \quad (\text{in (4.31)}) = O\left(k^2\sigma^{2-\beta} |\log \sigma|^2\right). \end{aligned}$$

We shall note that

$$\frac{1}{\omega} O\left(k^2\sigma^{2-\beta} |\log \sigma|^2\right) = O\left(\sigma^{2(1-\beta)} |\log \sigma|^2\right)$$

□

III) As we said in the beginning of this section, the next is to make an estimate of the inner products

$$\langle S[W], U'(\cdot - \xi_\ell) \rangle, \quad \ell = 1, \dots, k,$$

where $S[W]$ is defined by (2.15). Indeed,

$$\begin{aligned} (4.33) \quad \langle S[W], U'(\cdot - \xi_\ell) \rangle &= \int_{\mathbb{R}} S[W](x) U'(x - \xi_\ell) dx \\ &= \int_{\mathbb{R}} \frac{1-V(x)}{V(x)} W^2(x) U'(x - \xi_\ell) dx + \sum_{\substack{i,j \\ i \neq j}} \int_{\mathbb{R}} U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) dx. \end{aligned}$$

The two parts on the right-hand side of (4.33) are denoted by E_1 and E_2 , respectively. First we have

$$\begin{aligned} (4.34) \quad E_1 &= \int_{\mathbb{R}} \frac{1-V(x)}{V(x)} U^2(x - \xi_\ell) U'(x - \xi_\ell) dx \\ &\quad + \sum_{\substack{i \neq j \text{ or} \\ i=j \neq \ell}} \int_{\mathbb{R}} \frac{1-V(x)}{V(x)} U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) dx. \end{aligned}$$

REMARK 3. In the following estimates, we shall use Lemmas 4.1 and 4.2, the latter is valid on the compact interval

$$(4.35) \quad I_\xi = \left[\xi_1 - \frac{1}{2} |\log \sigma|, \xi_k + \frac{1}{2} |\log \sigma| \right].$$

However, when we deal with the integrals over \mathbb{R} , we extend the utilization of Lemma 4.2 beyond the interval I_ξ , while the details of justification that the compensation is a small amount of higher order in σ are omitted.

a) The first term in (4.34) is

$$\begin{aligned} (4.36) \quad & \int_{\mathbb{R}} \frac{1-V}{V} U^2(x - \xi_\ell) U'(x - \xi_\ell) dx = \int_{\mathbb{R}} \frac{1-V(\xi_\ell + y)}{V(\xi_\ell + y)} U^2(y) U'(y) dy \\ & = \int_{\mathbb{R}} \frac{1-V(\xi_\ell)}{V(\xi_\ell + y)} U^2(y) U'(y) dy + \int_{\mathbb{R}} \frac{V(\xi_\ell) - V(\xi_\ell + y)}{V(\xi_\ell + y)} U^2(y) U'(y) dy. \end{aligned}$$

where

$$\begin{aligned}
(4.37) \quad & \int_{\mathbb{R}} \frac{1-V(\xi_\ell)}{V(\xi_\ell+y)} U^2(y)U'(y) dy = \int_{\mathbb{R}} \frac{1}{3} \frac{V(\xi_\ell)-1}{V^2(\xi_\ell+y)} V'(\xi_\ell+y) U^3(y) dy \\
& \leq \frac{1}{3} \int_{\mathbb{R}} \frac{O(\sigma^{1-\beta}|\log \sigma|)\sigma}{[V(\xi_\ell)+V(\xi_\ell+y)-V(\xi_\ell)]^2} U^3(y) dy \quad (\text{since } V'(x) \leq \sigma \text{ can be checked}) \\
& = \int_{\mathbb{R}} \frac{O(\sigma^{2-\beta}|\log \sigma|)U^3(y)}{[1+O(\sigma^{1-\beta}|\log \sigma|)+O(\sigma|\log \sigma|)+O(\sigma^{2(1-\beta)}|\log \sigma|^2)]^2} dy \quad (\text{from (4.32)}) \\
& = O(\sigma^{2-\beta}|\log \sigma|) \int_{\mathbb{R}} U^3(y) dy = O(\sigma^{2-\beta}|\log \sigma|),
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{V(\xi_\ell)-V(\xi_\ell+y)}{V(\xi_\ell+y)} U^2(y)U'(y) dy \\
& = \int_{\mathbb{R}} \frac{1}{1+O(\sigma^{1-\beta}|\log \sigma|)} \{\Pi_1(y)+\Pi_2(y)+\Pi_3(y)\} U^2(y)U'(y) dy,
\end{aligned}$$

in which, according to (4.32),

$$\begin{aligned}
(4.38) \quad & \Pi_1(y) = -\frac{\sigma}{\omega} \int_{\mathbb{R}} [|y-x|-|x|] = -\frac{\sigma}{\omega} \Psi(y), \\
& \Pi_2(y) = \sum_{j(\neq \ell)=1}^k \hat{C}(\sigma) e^{-\sigma|\xi_\ell-\xi_j|} \left[\frac{\sigma}{\omega} y \operatorname{sgn}(\xi_j-\xi_\ell) + O(\sigma^{2+\beta}|\log \sigma|^2) \right] \\
& \Pi_3(y) = \frac{1}{\omega} \left[O(\sigma^{2(1-\beta)}|\log \sigma|) + O(k^2\sigma^{2-\beta}|\log \sigma|^2) \right] \\
& = O(\sigma^{2(1-\beta)}|\log \sigma|^2).
\end{aligned}$$

By the property of (4.22) of the function Ψ , we have

$$\int_{\mathbb{R}} \Pi_1(y)U^2(y)U'(y) dy = 0,$$

so that

$$\begin{aligned}
& \int_{\mathbb{R}} \frac{1}{1+O(\sigma^{1-\beta}|\log \sigma|)} \Pi_1(y)U^2(y)U'(y) dy \\
& = \int_{\mathbb{R}} \left(\frac{1}{1+O(\sigma^{1-\beta}|\log \sigma|)} - 1 \right) \Pi_1(y)U^2(y)U'(y) dy \\
& = O(\sigma^{1-\beta}|\log \sigma|) \int_{\mathbb{R}} |\Pi_1(y)| U^2(y) |U'(y)| dy,
\end{aligned}$$

where

$$\int_{\mathbb{R}} |\Pi_1(y)| U^2(y) |U'(y)| dy \leq \int_{\mathbb{R}} \sigma^{1+\beta}|y| \cdot 6^3 e^{-3|y|} dy = 6^3 \left(\frac{2}{9}\right) \sigma^{1+\beta} = 48 \sigma^{1+\beta}.$$

Hence we have

$$(4.39) \quad \int_{\mathbb{R}} \frac{1}{1+O(\sigma^{1-\beta}|\log \sigma|)} \Pi_1(y)U^2(y)U'(y) dy = O(\sigma^2|\log \sigma|).$$

Next we get

$$\begin{aligned}
(4.40) \quad & \int_{\mathbb{R}} \frac{1}{1 + O(\sigma^{1-\beta} |\log \sigma|)} \Pi_2(y) U^2(y) U'(y) dy \\
&= \sum_{j(\neq \ell)=1}^k \hat{C}(\sigma) e^{-\sigma |\xi_\ell - \xi_j|} \frac{\sigma}{\omega} \int_{\mathbb{R}} y \operatorname{sgn}(\xi_j - \xi_\ell) U^2(y) U'(y) dy + O(\sigma^{2+\beta} |\log \sigma|^2) k \\
&= \sum_{j(\neq \ell)=1}^k d(\sigma) e^{-\sigma |\xi_\ell - \xi_j|} \sigma^{1+\beta} \operatorname{sgn} \frac{\xi_j - \xi_\ell}{|\xi_j - \xi_\ell|} + O(\sigma^2 |\log \sigma|^2),
\end{aligned}$$

where the constant $d(\sigma)$, given by

$$\begin{aligned}
(4.41) \quad d(\sigma) &= \hat{C}(\sigma) \frac{\sigma^{-\beta}}{\omega} \left(\frac{1}{3} \int_{\mathbb{R}} U^3(y) dy \right) = \hat{C}(\sigma) \frac{\sigma^{-\beta}}{\omega} \left(\frac{1}{3} \cdot \frac{36}{5} \right) \\
&= \frac{12}{5} \hat{C}(\sigma) \left(\frac{\sigma^{-\beta}}{\omega} \right),
\end{aligned}$$

is uniformly bounded because $\hat{C}(\sigma) < 40$ and $(\sigma^{-\beta}/\omega) \leq \sigma^{-\beta}/(6k) = O(1)$.

Finally we have

$$(4.42) \quad \int_{\mathbb{R}} \frac{1}{1 + O(\sigma^{1-\beta} |\log \sigma|)} \Pi_3(y) U^2(y) U'(y) dy = O(\sigma^{2(1-\beta)} |\log \sigma|^2).$$

From (4.36), (4.37), (4.39), (4.40) and (4.42) it follows that

$$\begin{aligned}
(4.43) \quad & \int_{\mathbb{R}} \frac{1 - V(x)}{V(x)} U^2(x - \xi_\ell) U'(x - \xi_\ell) dx \\
&= \sum_{j(\neq \ell)=1}^k d(\sigma) e^{-\sigma |\xi_\ell - \xi_j|} \sigma^{1+\beta} \operatorname{sgn}(\xi_j - \xi_\ell) + O(\sigma^{2(1-\beta)} |\log \sigma|^2).
\end{aligned}$$

b) Now we estimate the E_2 part in (4.33) and the second part of E_1 in (4.34), both involve the triplet product $U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell)$. Note that

$$(4.44) \quad \int_{\mathbb{R}} U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) dx = O(\sigma^{2(1-\beta)}),$$

if $\max\{|i - j|, |j - \ell|, |\ell - i|\} \geq 2$. Suppose $i = \ell$ (similarly if $j = \ell$). Then for $j < \ell$ we get

$$\begin{aligned}
(4.45) \quad & \int_{\mathbb{R}} U(x - \xi_\ell) U(x - \xi_j) U'(x - \xi_\ell) dx = -\frac{1}{2} \int_{\mathbb{R}} U^2(x - \xi_\ell) U'(x - \xi_j) dx \\
&= -\frac{1}{2} \int_{\mathbb{R}} U^2(x) U'(x - \xi_j + \xi_\ell) dx = 3 \int_{\mathbb{R}} U^2(x) e^{-|x - \xi_j + \xi_\ell|} dx + Q_1 \\
&= 3 \int_{\mathbb{R}} e^{-x} U^2(x) dx e^{-|\xi_j - \xi_\ell|} + Q_1 + Q_2.
\end{aligned}$$

where

$$(4.46) \quad \int_{\mathbb{R}} e^{-x} U^2(x) dx = 12, \text{ by Lemma 1.1,}$$

$$\begin{aligned}
|Q_1| &= \left| -\frac{1}{2} \int_{\mathbb{R}} U^2(x) U'(x - \xi_j + \xi_\ell) dx - 3 \int_{\mathbb{R}} U^2(x) e^{-|x - \xi_j + \xi_\ell|} dx \right| \\
&\leq \left| 3 \int_{\xi_j - \xi_\ell}^{\infty} U^2(x) e^{-(x - \xi_j + \xi_\ell)} \left[\frac{1 - e^{-(x - \xi_j + \xi_\ell)}}{(1 + e^{-(x - \xi_j + \xi_\ell)})^3} - 1 \right] dx \right| \\
(4.47) \quad &+ \left| 3 \int_{-\infty}^{\xi_j - \xi_\ell} U^2(x) e^{-(\xi_j - \xi_\ell - x)} \left[\frac{1 - e^{-(\xi_j - \xi_\ell - x)}}{(1 + e^{(\xi_j - \xi_\ell - x)})^3} - 1 \right] dx \right| \\
&\leq 108 \int_{\mathbb{R}} e^{-2|x|} e^{-|x - \xi_j + \xi_\ell|} \left[4 e^{-|x - \xi_j + \xi_\ell|} + O\left(e^{-2|x - \xi_j + \xi_\ell|}\right) \right] dx \\
&\leq C \int_{\mathbb{R}} e^{-2|x|} e^{-2|x - \xi_j + \xi_\ell|} dx \quad (\text{by (1.7)}) \\
&\leq C e^{-2|\xi_j - \xi_\ell|} (1 + |\xi_j - \xi_\ell|) \leq C \sigma^{2|j - \ell|(1 - \beta)} (1 + |j - \ell| |\log \sigma|),
\end{aligned}$$

and

$$\begin{aligned}
|Q_2| &= \left| 3 \int_{-\infty}^{\xi_j - \xi_\ell} U^2(x) \left[e^{-|x - \xi_j + \xi_\ell|} - e^{-(x - \xi_j + \xi_\ell)} \right] dx \right| \\
(4.48) \quad &\leq 108 \int_{-\infty}^{\xi_j - \xi_\ell} e^{2x} |2(x - \xi_j + \xi_\ell)| dx + (\text{h.o.t.}) \\
&= 54 e^{-2|\xi_j - \xi_\ell|} + (\text{h.o.t.}) \\
&\leq C \sigma^{2|j - \ell|(1 - \beta)}.
\end{aligned}$$

Thus we find that

$$\begin{aligned}
(4.49) \quad &\int_{\mathbb{R}} U(x - \xi_\ell) U(x - \xi_j) U'(x - \xi_\ell) dx \\
&= \begin{cases} 36 e^{-|\xi_j - \xi_\ell|} + O\left(\sigma^{2|j - \ell|(1 - \beta)} (1 + |j - \ell| |\log \sigma|)\right), & \text{if } j < \ell, \\ -36 e^{-|\xi_j - \xi_\ell|} + O\left(\sigma^{2|j - \ell|(1 - \beta)} (1 + |j - \ell| |\log \sigma|)\right), & \text{if } j > \ell, \end{cases}
\end{aligned}$$

We specify the sign difference between the case $j < \ell$, which we treated above, and the case $j > \ell$, for which the details are omitted. The difference is originated from different versions of $U'(x)$ we use in two cases.

For $j < \ell$, the version $U'(x) = -\frac{6e^{-x}(1 - e^{-x})}{(1 + e^{-x})^3}$ is used. Then we count on the smallness of the integrals in $|Q_2|$ over $(-\infty, \xi_j - \xi_\ell] \subset \mathbb{R}^-$.

For $j > \ell$, another version $U'(x) = \frac{6e^x(1 - e^x)}{(1 + e^x)^3}$ is used. Then we can count on the smallness of the corresponding integral over $[\xi_j - \xi_\ell, \infty) \subset \mathbb{R}^+$.

The only other possibility for the triplet product terms will be

$$\max\{|i - j|, |j - \ell|, |i - \ell|\} \geq 2,$$

then (4.44) applies and the sum of all these terms will be of the order

$$(4.50) \quad k O\left(\sigma^{2(1 - \beta)} \sum_{\nu=1}^{\infty} \sigma^{\nu(1 - \beta)}\right) = O(\sigma^{-\beta}) O\left(\sigma^{3(1 - \beta)}\right) = O(\sigma^{3 - 4\beta}).$$

Therefore, we obtain that, by Lemma 1.3, and since $\beta < 1/2$ implies $2(1 - \beta) < 3 - 4\beta$,

$$\begin{aligned}
 E_2 &= \sum_{\substack{i,j \\ i \neq j}} \int_{\mathbb{R}} U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) dx \\
 (4.51) \quad &= 36 \sum_{j(\neq \ell)=1}^k e^{-|\xi_j - \xi_\ell|} \operatorname{sgn} \frac{\xi_\ell - \xi_j}{|\xi_\ell - \xi_j|} + O\left(\sigma^{2(1-\beta)} |\log \sigma|\right) + O\left(\sigma^{3-4\beta}\right) \\
 &= 36 \sum_{j(\neq \ell)=1}^k e^{-|\xi_j - \xi_\ell|} \operatorname{sgn} \frac{\xi_\ell - \xi_j}{|\xi_\ell - \xi_j|} + O\left(\sigma^{2(1-\beta)} |\log \sigma|\right).
 \end{aligned}$$

Moreover, by Lemmas 4.1 and 4.2, similar to (4.51), we have the estimate on the second part of E_1 in (4.34) as follows,

$$\begin{aligned}
 &\sum_{\substack{i \neq j \text{ or} \\ i=j \neq \ell}} \int_{\mathbb{R}} \frac{1-V}{V} U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) dx \\
 (4.52) \quad &= O\left(\sigma^{1-\beta} |\log \sigma|\right) \left[6^3 \sum_{j(\neq \ell)=1}^k e^{-|\xi_j - \xi_\ell|} + O\left(\sigma^{3-4\beta}\right) \right] \\
 &= O\left(\sigma^{1-\beta} |\log \sigma|\right) \left[O\left(\sigma^{1-\beta}\right) + O\left(\sigma^{3-4\beta}\right) \right] \\
 &= O\left(\sigma^{2(1-\beta)} |\log \sigma|\right).
 \end{aligned}$$

With these preparations, we are now ready to prove a key result in the following theorem.

THEOREM 4.3. *Under the same assumptions as in Theorem 3.6, it holds that*

$$\begin{aligned}
 (4.53) \quad &\int_{\mathbb{R}} S[W](x) U'(x - \xi_\ell) dx = \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O\left(\sigma^{2(1-\beta)} |\log \sigma|^2\right), \\
 &\ell = 1, \dots, k,
 \end{aligned}$$

in which

$$(4.54) \quad F(r) = -d(\sigma) \sigma^{1+\beta} |r| - 36 e^{-|r|},$$

where $d(\sigma)$ is a uniformly bounded, positive constant given by (4.41). Moreover, for any given $0 < \mu < 1$, there is a constant $C_\mu > 0$ such that $S[W] \in L_\mu^\infty$ and

$$(4.55) \quad \|S[W]\|_\mu \leq C_\mu \sigma^{1-\beta-\frac{\mu}{2}}.$$

PROOF. By substituting (4.43), (4.51) and (4.52) into (4.33), we can confirm that

$$\begin{aligned}
 (4.56) \quad &\int_{\mathbb{R}} S[W](x) U'(x - \xi_\ell) dx = \sum_{j \neq \ell} d(\sigma) e^{-\sigma|\xi_j - \xi_\ell|} \sigma^{1+\beta} \operatorname{sgn}(\xi_j - \xi_\ell) \\
 &+ O\left(\sigma^{2(1-\beta)} |\log \sigma|^2\right) + \sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_\ell - \xi_j) \\
 &+ O\left(\sigma^{2(1-\beta)} |\log \sigma|\right).
 \end{aligned}$$

Moreover, we see that

(4.57)

$$\begin{aligned} & \left| \sum_{j \neq \ell} d(\sigma) \left[e^{-\sigma |\xi_j - \xi_\ell|} - 1 \right] \sigma^{1+\beta} \operatorname{sgn}(\xi_j - \xi_\ell) \right| \\ & \leq \sum_{j \neq \ell} d(\sigma) \left[\sigma |\xi_j - \xi_\ell| + O\left(\sigma^2 |\xi_j - \xi_\ell|^2\right) \right] \sigma^{1+\beta} = O\left(\sigma^{2+\beta} k(k-1) |\log \sigma|\right) \\ & = O\left(\sigma^{2-\beta} |\log \sigma|\right), \end{aligned}$$

which can be absorbed by the last term in (4.56), i.e., $O\left(\sigma^{2(1-\beta)} |\log \sigma|\right)$. Therefore, (4.56) yields

$$\begin{aligned} \int_{\mathbb{R}} S[W](u) U'(x - \xi_\ell) dx &= \sum_{j \neq \ell} \left[-d(\sigma) \sigma^{1+\beta} \operatorname{sgn}(\xi_\ell - \xi_j) + 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_\ell - \xi_j) \right] \\ & \quad + O\left(\sigma^{2(1-\beta)} |\log \sigma|^2\right) \\ &= \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O\left(\sigma^{2(1-\beta)} |\log \sigma|^2\right), \end{aligned}$$

which is (4.53).

Now we prove the second statement of this theorem and (4.55). For each $x \in \mathbb{R}$, there is a point ξ_{i_0} among ξ_1, \dots, ξ_k such that

$$(4.58) \quad |x - \xi_{i_0}| = \min_{1 \leq i \leq k} |x - \xi_i|.$$

Without loss of generality, we suppose that $x \in [\xi_{i_0}, \xi_{i_0+1})$. It is possible that $|x - \xi_{i_0}| = |x - \xi_{i_0+1}|$. Then we have

$$(4.59) \quad \begin{aligned} & \left| \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) \right| \leq \sum_{j \neq i_0} 36 e^{-|x - \xi_{i_0}|} e^{-|x - \xi_j|} \\ & + \sum_{j \neq i_0+1} 36 e^{-|x - \xi_{j_0+1}|} e^{-|x - \xi_j|} + \sum_{i \neq i_0, i_0+1} \sum_{j \neq i} 36 e^{-|x - \xi_i|} e^{-|x - \xi_j|}. \end{aligned}$$

In the first sum, the largest term is

$$\begin{aligned} & 36 e^{-|x - \xi_{i_0}|} e^{-|x - \xi_{i_0+1}|} = 36 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[e^{-(1-\mu)|x - \xi_{i_0}|} e^{-|x - \xi_{i_0+1}|} \right] \\ & = 36 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[e^{-(1-\mu)|x - \xi_{i_0}|} e^{-|\xi_{i_0+1} - \xi_{i_0}| + |x - \xi_{i_0}|} \right] \\ & \leq 36 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[\sigma^{1-\beta} e^{\mu |x - \xi_{i_0}|} \right] \\ & \leq 36 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[\sigma^{1-\beta} e^{\frac{\mu}{2} |\log \sigma|} \right] \\ & = 36 e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[\sigma^{1-\beta-\mu/2} \right], \end{aligned}$$

which implies that

$$(4.60) \quad \begin{aligned} \sum_{j \neq i_0} 36 e^{-|x-\xi_{i_0}|} e^{-|x-\xi_j|} &\leq 36 e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \left[\sigma^{1-\beta-\mu/2} \right. \\ &\quad \left. + 2\sigma^{2(1-\beta)-\mu/2} + 2\sigma^{3(1-\beta)-\mu/2} + \dots \right] \\ &\leq C e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{1-\beta-\mu/2}. \end{aligned}$$

It is easy to see that the second sum $\sum_{j \neq i_0+1} \dots$ satisfies the same estimate (4.60).

Moreover,

$$(4.61) \quad \begin{aligned} &36 \sum_{i \neq i_0, i_0+1} \left[e^{-|x-\xi_i|} \sum_{j \neq i} e^{-|x-\xi_j|} \right] \\ &\leq 36 \left[2\sigma^{1-\beta} + \sigma^{2(1-\beta)} + \dots \right] \left[2e^{-\min_{1 \leq i \leq k} |x-\xi_i|} \left(1 + \sigma^{1-\beta} + \sigma^{2(1-\beta)} + \dots \right) \right] \\ &\leq C e^{-\min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{1-\beta}. \end{aligned}$$

Therefore, we have the following estimate for $x \in I_\xi$, where I_ξ is the interval defined in (4.35).

1) If $x \in I_\xi$, then by (4.59), (4.60) and (4.61), with the observation

$$\frac{1-V(x)}{V(x)} = O(\sigma^{1-\beta} |\log \sigma|),$$

we have

$$(4.62) \quad \begin{aligned} S[W](x) &= \frac{1-V(x)}{V(x)} W^2(x) + \sum_{\substack{i,j \\ i \neq j}} U(x-\xi_i) U(x-\xi_j) \\ &= O(\sigma^{1-\beta} |\log \sigma|) \left[e^{-\min_{1 \leq i \leq k} |x-\xi_i|} \left(1 + \sigma^{1-\beta} + \sigma^{2(1-\beta)} + \dots \right) \right]^2 \\ &\quad + e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} O(\sigma^{1-\beta-\mu/2}) \\ &= e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \left[O(\sigma^{1-\beta} |\log \sigma|) + O(\sigma^{1-\beta-\mu/2}) \right] \\ &= e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} O(\sigma^{1-\beta-\mu/2}). \end{aligned}$$

2) If $x \in \mathbb{R} \setminus I_\xi = (-\infty, \xi_1 - \frac{1}{2} |\log \sigma|) \cup (\xi_k + \frac{1}{2} |\log \sigma|, \infty)$, then we can make an estimate of $S[W](x)$ as follows. Since (2.12) implies $0 < V(x) \leq 1$, we have

$$|1-V(x)| \leq 1.$$

By the Lebesgue Dominated Convergence Theorem,

$$\lim_{\sigma \rightarrow 0^+} V(x) = \frac{1}{\omega} \lim_{\sigma \rightarrow 0^+} \int_{\mathbb{R}} e^{-\sigma|z|} W^2(z+x) dz = 1,$$

which means for $\sigma > 0$ sufficiently small, $V(x) \geq 1 - \epsilon_0$ for some constant $\epsilon_0 < 1$. Thus

$$\left| \frac{1-V(x)}{V(x)} \right| \leq \frac{1}{1-\epsilon_0}.$$

On the other hand, we have

$$\begin{aligned}
 \sum_{j=1}^k U^2(x - \xi_j) &\leq 36 \sum_{j=1}^k e^{-2|x-\xi_j|} \quad (\text{since } x \in \mathbb{R} \setminus I_\xi) \\
 (4.63) \quad &\leq 36e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} e^{-(2-\mu)\frac{1}{2}|\log \sigma|} \left(1 + \sigma^{2(1-\beta)} + \sigma^{4(1-\beta)} + \dots\right) \\
 &\leq Ce^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{1-\mu/2},
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{\substack{i,j \\ i \neq j}} U(x - \xi_i) U(x - \xi_j) &\leq Ce^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{\frac{1}{2}(1-\mu)} \sigma^{\frac{1}{2}+(1-\beta)} \\
 (4.64) \quad &\leq Ce^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{2-\beta-\mu/2}.
 \end{aligned}$$

Hence we obtain the estimate

$$\begin{aligned}
 S[W](x) &\leq \frac{C}{1 - \epsilon_0} e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{1-\mu/2} + Ce^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} \sigma^{2-\beta-\mu/2} \\
 (4.65) \quad &= e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} O\left(\sigma^{1-\mu/2}\right).
 \end{aligned}$$

In any case, (4.62) and (4.64) imply that (4.55) holds. □

REMARK 4 (The second effect of μ). In the previous remark in Section 3, we noted the effect of μ on the order of $k = O(\sigma^{-\beta})$ in (3.63). Here is the second effect of the choice of the norm parameter μ . From (4.55), we see that for $0 < \mu \leq 1/3$ being chosen very close to zero, the upper bound of the μ -norm of $S[W]$ will be very close to the order $\sigma^{1-\beta}$, i.e.,

$$(4.66) \quad C_\mu \sigma^{1-\beta-\mu/2} \approx C_\mu \sigma^{1-\beta}.$$

5. Solution to the Augmented Nonlocal Problem

In this section, we shall prove that the augmented nonlocal problem (2.19) admits a solution $u = W + \varphi_{\sigma,\xi}$ and $\varphi_{\sigma,\xi}$ turns out to be a small perturbation. Recall that problem (2.19) is equivalent to the problem (2.20), which we write again as follows

$$\begin{aligned}
 L(\varphi) &= S[W] + P(W, \varphi) + \sum_{j=1}^k \gamma_j W_j \\
 (5.1) \quad &\langle \varphi, W_j \rangle = 0, \quad j = 1, \dots, k, \\
 &\varphi(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty,
 \end{aligned}$$

where L is the linear operator defined in (3.1), $S[W]$ is given in (2.15), and $P(W, \varphi)$ is given in (2.21).

By Theorem 4.3, for any $0 < \mu \leq 1/3$, $S[W] \in L_\mu^\infty$ and

$$(4.55) \quad \|S(W)\|_\mu \leq C_\mu \sigma^{1-\beta-\mu/2}.$$

We now consider the nonlinear term $P(W, \varphi)$. According to (2.21),

$$(5.2) \quad P(W, \varphi) = P_1(W, \varphi) + P_2(W, \varphi),$$

where

(5.3)

$$P_1(W, \varphi) = (W + \varphi)^2 \left[\frac{1}{T[(W + \varphi)^2]} - \frac{1}{V} + \frac{2T[W\varphi]}{V^2} \right] - (2W + \varphi) \varphi \frac{2T[W\varphi]}{V^2}$$

and

$$(5.4) \quad P_2(W, \varphi) = -2W\varphi \left(1 - \frac{1}{V} \right) + 2W^2 \left[\frac{\langle W, \varphi \rangle}{\omega} - \frac{T[W\varphi]}{V^2} \right] + \frac{\varphi^2}{V}.$$

We have

$$(5.5) \quad \begin{aligned} T[(W + \varphi)^2] &= V + 2T[W\varphi] + T[\varphi^2], \\ T[W\varphi] &= \frac{1}{\omega} \int_{\mathbb{R}} W\varphi dx + \frac{1}{\omega} \int_{\mathbb{R}} \left(e^{-\sigma|x-y|} - 1 \right) W(y)\varphi(y) dy, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\omega} \left| \int_{\mathbb{R}} \left(e^{-\sigma|x-y|} - 1 \right) W(y)\varphi(y) dy \right| &\leq \frac{1}{6k} \int_{\mathbb{R}} \sigma^{|x-y|} W(y) |\varphi(y)| dy \\ &\leq \frac{\sigma \|\varphi\|_{\mu}}{6k} \sum_{j=1}^k \int_{\mathbb{R}} |x-y| U(y - \xi_j) dy \\ &\leq \frac{\sigma \|\varphi\|_{\mu}}{k} \sum_{j=1}^k \int_{\mathbb{R}} |x-y| e^{-|y-\xi_j|} dy \\ &= \frac{\sigma \|\varphi\|_{\mu}}{k} \sum_{j=1}^k \int_{\mathbb{R}} |z| e^{-|x-\xi_j-z|} dz \\ &= \frac{\sigma \|\varphi\|_{\mu}}{k} \sum_{j=1}^k \left(2e^{-|x-\xi_j|} + 2|x - \xi_j| \right) \\ &\leq \frac{\sigma \|\varphi\|_{\mu}}{k} [O(1) + k(k+1)|\log \sigma|] \quad (\text{for any } x \in I_{\xi}) \\ &= \sigma^{1+\beta} \|\varphi\|_{\mu} [O(1) + O(\sigma^{-2\beta} |\log \sigma|)] = O(\sigma^{1-\beta} |\log \sigma|) \|\varphi\|_{\mu}. \end{aligned}$$

Hence, we obtain

$$(5.6) \quad \begin{aligned} T[W\varphi] &= O(\sigma^{\beta}) \|\varphi\|_{\mu} \int_{\mathbb{R}} W(x) dx + O(\sigma^{1-\beta} |\log \sigma|) \|\varphi\|_{\mu} \\ &= O(\sigma^{\beta}) \|\varphi\|_{\mu}, \end{aligned}$$

and likewise, by using Lemma 1.5, we can get

$$(5.7) \quad T[\varphi^2] = O(1) \|\varphi^2\|_{\mu}.$$

Then we obtain

$$(5.8) \quad \begin{aligned} |P_1(W, \varphi)| &\leq C(W^2 + \varphi^2) T[\varphi^2] + C|2W\varphi + \varphi^2| |T[W\varphi]| \\ &\leq C e^{-2\mu \min_{1 \leq j \leq k} |x-\xi_j|} \|\varphi\|_{\mu}^2. \end{aligned}$$

provided that $\|\varphi\|_{\mu} = O(1)$. Here we admit $x \in I_{\xi}$.

On the other hand, we have

$$\begin{aligned}
(5.9) \quad |P_2(W, \varphi)| &\leq 2W|\varphi|O(\sigma^{1-\beta}|\log \sigma|) + 2W^2O(\sigma^{1-\beta}|\log \sigma|)O(\sigma^\beta)\|\varphi\|_\mu + C|\varphi|^2 \\
&\leq e^{-2\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left[\frac{12}{1 - \sigma^{1-\beta}} O(\sigma^{1-\beta}|\log \sigma|)\|\varphi\|_\mu \right. \\
&\quad \left. + \frac{72}{(1 - \sigma^{1-\beta})^2} O(\sigma|\log \sigma|)\|\varphi\|_\mu + C\|\varphi\|_\mu^2 \right] \\
&\leq Ce^{-2\mu \min_{1 \leq i \leq k} |x - \xi_i|} [\|\varphi\|_\mu^2 + O(\sigma^{1-\beta}|\log \sigma|)\|\varphi\|_\mu].
\end{aligned}$$

Thus we have shown that in the range of $x \in I_\xi = [\xi_1 - \frac{1}{2}|\log \sigma|, \xi_k + \frac{1}{2}|\log \sigma|]$, there exists a constant $C > 0$ such that

$$(5.10) \quad \|P(W, \varphi)\|_\mu \leq C [\|\varphi\|_\mu + O(\sigma^{1-\beta}|\log \sigma|)] \|\varphi\|_\mu,$$

provided that $\|\varphi\|_\mu = O(1)$.

Next we consider the range $x \in \mathbb{R} \setminus I_\xi$. In that case,

$$\begin{aligned}
(5.11) \quad |P(W, \varphi)| &= \left| \frac{(W + \varphi)^2}{T[(W + \varphi)^2]} - \frac{W^2}{V} - 2W\varphi + \frac{2W^2}{\omega} \langle W, \varphi \rangle \right| \\
&\leq \left| \frac{2W\varphi + \varphi^2}{V} - 2W\varphi + \frac{2W^2}{\omega} \langle W, \varphi \rangle \right| + C(W + \varphi)^2 |2T[W\varphi] + T[\varphi^2]| \\
&= |\hat{P}_1(W, \varphi)| + |\hat{P}_2(W, \varphi)|,
\end{aligned}$$

where

$$\begin{aligned}
(5.12) \quad |\hat{P}_1(W, \varphi)| &\leq C \left(\sum_{j=1}^k e^{-|x - \xi_j|} \right) e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \|\varphi\|_\mu + Ce^{-2\mu \min_{1 \leq i \leq k} |x - \xi_i|} \|\varphi\|_\mu^2 \\
&\quad + \frac{72}{6k} \left(\sum_{j=1}^k e^{-|x - \xi_j|} \right)^2 \int_{\mathbb{R}} \sum_{j=1}^k e^{-|y - \xi_j|} e^{-\mu \min_{1 \leq i \leq k} |y - \xi_i|} dy \|\varphi\|_\mu \\
&\leq Ce^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} (\sigma^{1/2} \|\varphi\|_\mu + \|\varphi\|_\mu^2) \\
&\quad + Ce^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \sigma^{1 - \frac{\mu}{2}} \|\varphi\|_\mu \\
&\leq Ce^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} (\sigma^{1/2} \|\varphi\|_\mu + \|\varphi\|_\mu^2),
\end{aligned}$$

and

$$\begin{aligned}
|\hat{P}_2(W, \varphi)| &= C(W + \varphi)^2 |2T[W\varphi] + T[\varphi^2]| \\
&\leq C \left[\sum_{j=1}^k e^{-|x - \xi_j|} + e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \|\varphi\|_\mu \right]^2 |2T[W\varphi] + T[\varphi^2]| \\
&\leq Ce^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} (\sigma^{\frac{1}{2}(1-\mu)} + \|\varphi\|_\mu) (\sigma^{\frac{1}{2}} + \|\varphi\|_\mu) \\
&\quad \cdot \frac{C}{k} e^{-\sigma \min_{1 \leq i \leq k} |x - \xi_i|} \left[\int_{\mathbb{R}} W(y) dy \|\varphi\|_\mu + \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} dy \|\varphi\|_\mu^2 \right]
\end{aligned}$$

in which, by Lemma 1.5, we have

$$\frac{1}{k} \int_{\mathbb{R}} W(y) dy = 6 \quad \text{and} \quad \frac{1}{k} \int_{\mathbb{R}} e^{-\mu \min_{1 \leq i \leq k} |y - \xi_i|} dy = O(1),$$

so that

$$(5.13) \quad \left| \hat{P}_2(W, \varphi) \right| \leq C e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left(\sigma^{1 - \frac{\mu}{2}} \|\varphi\|_{\mu} + \|\varphi\|_{\mu}^2 \right),$$

provided that $\|\varphi\|_{\mu} = O(1)$. From (5.11) through (5.13) we get

$$(5.14) \quad |P(W, \varphi)| \leq C e^{-\mu \min_{1 \leq i \leq k} |x - \xi_i|} \left(\sigma^{\frac{1}{2}} \|\varphi\|_{\mu} + \|\varphi\|_{\mu}^2 \right),$$

for $x \in \mathbb{R} \setminus I_{\xi}$.

LEMMA 5.1. *Assume that for $0 < \mu \leq 1/3$, $\|\varphi\|_{\mu} = O(1)$ as σ being sufficiently small. Then $P(W, \varphi)$ given by (2.21) satisfies the estimate*

$$(5.15) \quad \|P(W, \varphi)\|_{\mu} \leq C \left[\|\varphi\|_{\mu} + O(\sigma^{1-\beta} |\log \sigma|) + O\left(\sigma^{\frac{1}{2}}\right) \right] \|\varphi\|_{\mu}.$$

PROOF. Simply combine (5.10) for $x \in I_{\xi}$ and (5.14) for $x \in \mathbb{R} \setminus I_{\xi}$ to reach the conclusion. \square

The following is a key result on the solution to the augmented nonlocal problem (5.1). Its proof is based on the investigation of the principal approximation linear operator L , which we presented in Theorem 3.6, and on the prepared nonlinear estimates shown in Theorem 4.3 and Lemma 5.1.

THEOREM 5.2. *There exists a constant $\check{\sigma} > 0$ such that for $0 < \sigma \leq \check{\sigma}$, under the assumption*

$$(3.79) \quad k = C\sigma^{-\beta} \quad \text{and} \quad 0 < \beta < \frac{1-\mu}{2-\mu} \quad \left(\text{with } 0 < \mu \leq \frac{1}{3}\right),$$

for any given $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$ in (1.16), there exists a unique solution

$$\varphi_{\sigma, \xi} \in L_{\mu}^{\infty}, \quad \gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$$

of the augmented nonlocal problem:

$$(5.16) \quad \begin{aligned} S[W + \varphi](x) &= \sum_{j=1}^k \gamma_j U'(x - \xi_j), \quad x \in \mathbb{R}, \\ \int_{\mathbb{R}} \varphi(x) U'(x - \xi_j) dx &= 0, \quad j = 1, \dots, k, \\ \varphi(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Moreover,

$$(5.17) \quad \|\varphi_{\sigma, \xi}\|_{\mu} \leq 2\Gamma_0 C_{\mu} \sigma^{1-\beta-\mu/2},$$

where Γ_0 is the constant in (3.80) and C_{μ} is the constant in (4.55).

PROOF. By the equivalence of problem (5.16) (aka (2.19)) and problem (5.1) (aka (2.20)), it suffices to show that problem (5.1) has a solution $(\varphi_{\sigma, \xi}, \gamma)$ as stated in this theorem.

According to Theorem 3.6, if we set

$$(5.18) \quad h = S[W] + P(W, \varphi),$$

then problem (5.1) is to find a solution $\varphi \in L_\mu^\infty$ of the equation

$$(5.19) \quad \varphi = \mathcal{J}(h) = \mathcal{J}(S[W] + P(W, \varphi)),$$

and then $\gamma = \Pi(h)$ can be determined, where the bounded linear operator \mathcal{J} on L_μ^∞ is well-defined and (3.80) holds. Define

$$(5.20) \quad Q(\varphi) \triangleq \mathcal{J}(S[W] + P(W, \varphi)), \quad \text{for } \varphi \in L_\mu^\infty.$$

Then problem (5.1) is reduced to finding a fixed point $\varphi \in L_\mu^\infty$ of the mapping Q , i.e.,

$$\varphi = Q(\varphi).$$

Let B be the closed, convex set in L_μ^∞ defined by

$$(5.21) \quad B = \left\{ \varphi \in L_\mu^\infty : \begin{array}{l} \|\varphi\|_\mu \leq 2\Gamma_0 C_\mu \sigma^{1-\beta-\mu/2} \\ \langle \varphi, W_j \rangle = 0, \quad j = 1, \dots, k, \\ \varphi(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty \end{array} \right\}.$$

We use the contraction mapping argument to prove that Q has a fixed point $\varphi_{\sigma, \xi}$ in B .

First we show $Q(B) \subset B$. For any $\varphi \in B$, by the description of problem (3.13) and the definition of \mathcal{J} , $Q(\varphi)$ satisfies automatically the two conditions in (5.21),

$$\langle Q(\varphi), W_j \rangle = 0, \quad j = 1, \dots, k, \text{ and } Q(\varphi)(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.$$

By Theorem 4.3 and Lemma 5.1, there is a small constant $\check{\sigma} > 0$, such that for $0 < \sigma \leq \check{\sigma}$,

$$(5.22) \quad \begin{aligned} \|Q(\varphi)\|_\mu &\leq \|\mathcal{J}\|_{\mathcal{L}(L_\mu^\infty)} \{ \|S(W)\|_\mu + \|P(W, \varphi)\|_\mu \} \\ &\leq \Gamma_0 \left\{ C_\mu \sigma^{1-\beta-\mu/2} + C \left[\|\varphi\|_\mu + O(\sigma^{1-\beta} |\log \sigma|) + O(\sigma^{\frac{1}{2}}) \right] \|\varphi\|_\mu \right\} \\ &\leq 2\Gamma_0 C_\mu \sigma^{1-\beta-\mu/2}, \quad \text{for any } \varphi \in B. \end{aligned}$$

Indeed, $\check{\sigma} > 0$ can be chosen to ensure that for $0 < \sigma \leq \check{\sigma}$, in (5.22),

$$(5.23) \quad C \left[2\Gamma_0 C_\mu \sigma^{1-\beta-\frac{\mu}{2}} + O(\sigma^{1-\beta} |\log \sigma|) + O(\sigma^{\frac{1}{2}}) \right] \leq \frac{1}{8},$$

so that (5.22) holds. This implies $Q(B) \subset B$, for $0 < \sigma \leq \check{\sigma}$.

Next, for any $\varphi_1, \varphi_2 \in B$, we can estimate

$$(5.24) \quad \begin{aligned} \|Q(\varphi_1) - Q(\varphi_2)\|_\mu &\leq \|\mathcal{J}\| \|P(W, \varphi_1) - P(W, \varphi_2)\|_\mu \\ &\leq \Gamma_0 \left\{ \|P_1(W, \varphi_1) - P_1(W, \varphi_2)\|_\mu + \|P_2(W, \varphi_1) - P_2(W, \varphi_2)\|_\mu \right\} \end{aligned}$$

through the similar steps as we have gone in (5.5) through (5.10), for $x \in I_\xi$. Then a corresponding estimate can be made by the similar steps as (5.11) through (5.14), for $x \in \mathbb{R} \setminus I_\xi$. The details are omitted here. We can conclude that for $0 < \sigma \leq \check{\sigma}$, where $\check{\sigma} > 0$ is the constant chosen in accordance with (5.23), it holds that

$$(5.25) \quad \|Q(\varphi_1) - Q(\varphi_2)\|_\mu \leq \frac{1}{2} \|\varphi_1 - \varphi_2\|_\mu, \quad \text{for any } \varphi_1, \varphi_2 \in B.$$

Therefore, Q is a contraction mapping on the set B .

By the Schauder fixed point theorem, there exists a unique fixed point $\varphi = \varphi_{\sigma, \xi}$ of Q in the set B , which is a solution of problem (5.1) and of the original problem (5.16), and $\gamma = \gamma_{\sigma, \xi}$ can be determined by solving (3.16). The property (5.17) follows from (5.21). \square

Let $\varphi_{\sigma,\xi}$ be the solution shown in Theorem 5.2. In order to prepare for the final stage, we need to know about

$$\begin{aligned}
 (5.26) \quad & \int_{\mathbb{R}} S[W + \varphi_{\sigma,\xi}](x) U'(x - \xi_\ell) dx \\
 &= \int_{\mathbb{R}} S[W](x) U'(x - \xi_\ell) dx \\
 &\quad + \int_{\mathbb{R}} [S[W + \varphi_{\sigma,\xi}](x) - S[W](x)] U'(x - \xi_\ell) dx,
 \end{aligned}$$

for $\ell = 1, \dots, k$. The information about $\int_{\mathbb{R}} S[W](x) U'(x - \xi_\ell) dx$ is known in Theorem 4.3. Here we study the second part. Denote by S' the Fréchet derivative of the operator S . We have

$$\begin{aligned}
 (5.27) \quad & \int_{\mathbb{R}} [S[W + \varphi_{\sigma,\xi}](x) - S[W](x)] U'(x - \xi_\ell) dx \\
 &= \int_{\mathbb{R}} S'[W](\varphi_{\sigma,\xi}) U'(x - \xi_\ell) dx + O\left(\sigma^{2(1-\beta-\mu/2)}\right) \\
 &= \int_{\mathbb{R}} \left[\varphi''_{\sigma,\xi} - \varphi_{\sigma,\xi} + \frac{2W\varphi_{\sigma,\xi}}{V} - \frac{2W^2}{V^2} T[W\varphi_{\sigma,\xi}] \right] U'(x - \xi_\ell) dx + O\left(\sigma^{2(1-\beta-\mu/2)}\right),
 \end{aligned}$$

and according to Lemma 1.4, by integration by parts, we have

$$\begin{aligned}
 & \int_{\mathbb{R}} [\varphi''_{\sigma,\xi} - \varphi_{\sigma,\xi} + 2U(x - \xi_\ell)\varphi_{\sigma,\xi}] U'(x - \xi_\ell) dx \\
 &= \int_{\mathbb{R}} \varphi_{\sigma,\xi} [U'''(x - \xi_\ell) - U'(x - \xi_\ell) + 2U(x - \xi_\ell)U'(x - \xi_\ell)] dx = 0.
 \end{aligned}$$

Hence we get

$$\begin{aligned}
 (5.28) \quad & \int_{\mathbb{R}} [S[W + \varphi_{\sigma,\xi}] - S[W]] U'(x - \xi_\ell) dx - O\left(\sigma^{2(1-\beta-\mu/2)}\right) \quad (\text{in (5.27)}) \\
 &= \int_{\mathbb{R}} \left(\frac{2W(x)}{V(x)} - 2U(x - \xi_\ell) \right) \varphi_{\sigma,\xi} U'(x - \xi_\ell) dx \\
 &\quad - \int_{\mathbb{R}} \frac{2W^2}{V^2} T[W\varphi_{\sigma,\xi}] U'(x - \xi_\ell) dx \\
 &= g_1(W, \varphi_{\sigma,\xi}, \ell) + g_2(W, \varphi_{\sigma,\xi}, \ell),
 \end{aligned}$$

where

$$\begin{aligned}
|g_1(W, \varphi_{\sigma, \xi}, \ell)| &\leq 2 \int_{\mathbb{R}} \left| W \left(\frac{1}{V} - 1 \right) \varphi_{\sigma, \xi}(x) U'(x - \xi_\ell) \right| dx \\
&\quad + 2 \sum_{j \neq \ell} \int_{\mathbb{R}} U(x - \xi_j) |\varphi_{\sigma, \xi}(x)| |U'(x - \xi_\ell)| dx \\
&\leq O(\sigma^{1-\beta} |\log \sigma|) \int_{\mathbb{R}} \sum_{j=1}^k e^{-|x-\xi_j|} e^{-|x-\xi_\ell|} e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} dx \|\varphi_{\sigma, \xi}\|_\mu \\
&\quad + 72 \sum_{j \neq \ell} \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} dx \|\varphi_{\sigma, \xi}\|_\mu \\
(5.29) \quad &\leq O(\sigma^{1-\beta} |\log \sigma|) \sum_{j=1}^k \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} dx \|\varphi_{\sigma, \xi}\|_\mu \\
&\quad + 72 \sum_{j \neq \ell} \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} dx \|\varphi_{\sigma, \xi}\|_\mu \\
&\leq O(\sigma^{1-\beta} |\log \sigma|) O(1) 2\Gamma_0 C_\mu \sigma^{1-\beta-\mu/2} \\
&\quad + 72 O(\sigma^{1-\beta} |\log \sigma|) 2\Gamma_0 C_\mu \sigma^{1-\beta-\mu/2} \\
&= O(\sigma^{2(1-\beta)-\mu/2} |\log \sigma|) + O(\sigma^{2(1-\beta)-\mu/2} |\log \sigma|) \\
&= O(\sigma^{2(1-\beta)-\mu/2} |\log \sigma|),
\end{aligned}$$

in which Lemmas 1.2 and 1.3 are used to claim

$$\sum_{j=1}^k \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} dx = O(1)$$

and

$$\sum_{j \neq \ell} \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} dx = O(\sigma^{1-\beta} |\log \sigma|).$$

Next, we have

$$\begin{aligned}
(5.30) \quad g_2(W, \varphi_{\sigma, \xi}, \ell) &= - \int_{\mathbb{R}} \frac{2W^2}{V^2} T[W\varphi_{\sigma, \xi}] U'(x - \xi_\ell) dx \\
&= - \int_{\mathbb{R}} 2U^2(x - \xi_\ell) U'(x - \xi_\ell) T[W\varphi_{\sigma, \xi}](x) dx \\
&\quad - \sum_{\substack{i, j \\ \text{not both } \ell}} \int_{\mathbb{R}} 2U(x - \xi_i) U(x - \xi_j) U'(x - \xi_\ell) T[W\varphi_{\sigma, \xi}](x) dx + \text{Rmd},
\end{aligned}$$

where the remainder

$$\text{Rmd} = \int_{\mathbb{R}} 2W^2 \left(1 - \frac{1}{V^2} \right) T[W\varphi_{\sigma, \xi}] U'(x - \xi_\ell) dx$$

is a small amount of higher order in comparison with the first two parts on the right-hand side of (5.30). Note that

$$\begin{aligned} \frac{d}{dx} T [W\varphi_{\sigma,\xi}] (x) &= \frac{1}{\omega} \frac{d}{dx} \int_{\mathbb{R}} e^{-\sigma|x-y|} W(y) \varphi_{\sigma,\xi}(y) dy \\ &= \frac{\sigma}{\omega} \left[\int_x^\infty e^{-\sigma|x-y|} W(y) \varphi_{\sigma,\xi}(y) dy - \int_{-\infty}^x e^{-\sigma|x-y|} W(y) \varphi_{\sigma,\xi}(y) dy \right]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} (5.31) \quad \left| \frac{d}{dx} T [W\varphi_{\sigma,\xi}] (x) \right| &\leq C\sigma^{1+\beta} \|\varphi_{\sigma,\xi}\|_\mu \int_{-\infty}^\infty e^{-\sigma|x-y|} \sum_{j=1}^k e^{-|y-\xi_j|} e^{-\mu \min_{1 \leq i \leq k} |x-\xi_i|} dy \\ &\leq C\sigma^{1+\beta} \sum_{j=1}^k \left[e^{-\sigma|x-\xi_j|} \left(\int_{\mathbb{R}} e^{-(1-\sigma)|y-\xi_j|} dy \right) \right] \|\varphi_{\sigma,\xi}\|_\mu \\ &\leq C\sigma^{1+\beta} O(\sigma^{-\beta}) \|\varphi_{\sigma,\xi}\|_\mu \leq C\sigma^{1+\beta} O(\sigma^{-\beta}) 2\Gamma_0 C_\mu \sigma^{1-\beta-\mu/2} \\ &= O(\sigma^{2-\beta-\mu/2}). \end{aligned}$$

Hence, we have

$$\begin{aligned} (5.32) \quad &\left| - \int_{\mathbb{R}} 2U^2(x-\xi_\ell) U'(x-\xi_\ell) T [W\varphi_{\sigma,\xi}] (x) dx \right| \\ &= \left| \frac{2}{3} \int_{\mathbb{R}} U^3(x-\xi_\ell) T' [W\varphi_{\sigma,\xi}] (x) dx \right| \leq C \int_{\mathbb{R}} e^{-3|x-\xi_\ell|} |T' [W\varphi_{\sigma,\xi}] (x)| dx \\ &= O(\sigma^{2-\beta-\mu/2}). \end{aligned}$$

On the other hand, since it is shown that $T [W\varphi_{\sigma,\xi}] = O(\sigma^\beta \|\varphi_{\sigma,\xi}\|_\mu)$ in (5.6), which is valid for $x \in I_\xi$ or $x \in \mathbb{R} \setminus I_\xi$, but the details for $x \in \mathbb{R} \setminus I_\xi$ are omitted, here we can get

$$\begin{aligned} (5.33) \quad &\left| - \sum_{\substack{i,j \\ \text{not both } \ell}} \int_{\mathbb{R}} 2U(x-\xi_i) U(x-\xi_j) U'(x-\xi_\ell) T [W\varphi_{\sigma,\xi}] (x) dx \right| \\ &\leq O(\sigma^\beta \|\varphi_{\sigma,\xi}\|_\mu) \sum_{\substack{i,j \\ \text{not both } \ell}} \int_{\mathbb{R}} e^{-|x-\xi_i|} e^{-|x-\xi_j|} e^{-|x-\xi_\ell|} dx \\ &\leq O(\sigma^\beta \|\varphi_{\sigma,\xi}\|_\mu) \sum_{j=1}^k \sum_{i(\neq \ell)=1}^k \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_i-\xi_j|} e^{-|x+\xi_i-\xi_\ell|} dx \\ &\leq O(\sigma^\beta \|\varphi_{\sigma,\xi}\|_\mu) \sum_{j=1}^k \sum_{i(\neq \ell)=1}^k e^{-|\xi_i-\xi_\ell|} \int_{\mathbb{R}} e^{-|x+\xi_i-\xi_j|} dx \\ &\leq O(\sigma^\beta \|\varphi_{\sigma,\xi}\|_\mu) 2k C\sigma^{1-\beta} = O(\sigma^\beta \sigma^{1-\beta-\mu/2}) O(\sigma^{-\beta}) O(\sigma^{1-\beta}) \\ &\leq O(\sigma^{2(1-\beta)-\mu/2}). \end{aligned}$$

Substituting (5.29), (5.32) and (5.33) into (5.28), we reach the following estimate, which is stated as a lemma. Here $O(\sigma^{2(1-\beta)-\mu/2} |\log \sigma|)$ is dominated by $O(\sigma^{2(1-\beta)-\mu})$.

LEMMA 5.3. *Under the same assumption as in Theorem 5.2, it holds that*

$$(5.34) \quad \int_{\mathbb{R}} (S[W + \varphi_{\sigma, \xi}] - S[W]) U'(x - \xi_\ell) dx = O(\sigma^{2(1-\beta)-\mu}).$$

Now we prove the following key result.

THEOREM 5.4. *Under the same assumption as in Theorem 5.2, for σ sufficiently small, $\xi \in \Lambda$, let $(\varphi_{\sigma, \xi}, \gamma_{\sigma, \xi}) \in L_\mu^\infty \times \mathbb{R}^k$, where $\gamma_{\sigma, \xi} = (\gamma_1, \dots, \gamma_k)$, be the solution of problem (5.16). Then, for $\ell = 1, \dots, k$,*

$$(5.35) \quad \begin{aligned} & \sum_{j=1}^k \gamma_j \int_{\mathbb{R}} U'(x - \xi_j) U'(x - \xi_\ell) dx \\ &= \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O(\sigma^{2(1-\beta)-\mu}), \end{aligned}$$

where $F(r)$ is given by (4.54), and

$$(5.36) \quad \begin{aligned} & \langle U'_j, U'_k \rangle \triangleq \int_{\mathbb{R}} U'(x - \xi_j) U'(x - \xi_\ell) dx \\ &= \begin{cases} 6/5, & \text{for } j = \ell, \\ O(\sigma^{|j-\ell|(1-\beta)} |j-\ell| |\log \sigma|), & \text{for } j \neq \ell. \end{cases} \end{aligned}$$

PROOF. Combining (4.53) in Theorem 4.3 and (5.34) in Lemma 5.3, we get from (5.16) the following equality, for $\ell = 1, \dots, k$,

$$\begin{aligned} \sum_{j=1}^k \gamma_j \int_{\mathbb{R}} U'(x - \xi_j) U'(x - \xi_\ell) dx &= \int_{\mathbb{R}} S[W] U'(x - \xi_\ell) dx \\ &+ \int_{\mathbb{R}} (S[W + \varphi_{\sigma, \xi}] - S[W]) U'(x - \xi_\ell) dx \\ &= \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O(\sigma^{2(1-\beta)} |\log \sigma|^2) + O(\sigma^{2(1-\beta)-\mu}) \\ &= \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O(\sigma^{2(1-\beta)-\mu}), \end{aligned}$$

where the term $O(\sigma^{2(1-\beta)} |\log \sigma|^2)$ is absorbed by the last term. Moreover, we have

$$\langle U'_j, U'_j \rangle = \int_{\mathbb{R}} |U'(x)|^2 dx = 6/5, \quad j = 1, \dots, k,$$

by Lemma 1.1. If $j \neq \ell$, then

$$\begin{aligned} \langle U'_j, U'_\ell \rangle &\leq 36 \int_{\mathbb{R}} e^{-|x|} e^{-|x+\xi_j-\xi_\ell|} dx \quad (\text{by (1.7)}) \\ &= 36 (1 + |\xi_j - \xi_\ell|) e^{-|\xi_j-\xi_\ell|} \\ &\leq 36 (1 + |j - \ell| |\log \sigma|) \sigma^{|j-\ell|(1-\beta)} \\ &= O\left(\sigma^{|j-\ell|(1-\beta)} |j - \ell| |\log \sigma|\right). \end{aligned}$$

The proof is completed. \square

6. Solution to the Reduced Finite-Dimensional Problem

In this section, we shall take the final step of the Lyapunov-Schmidt method to solve the reduced finite-dimensional problem: To find a set of points $\xi = (\xi_1, \dots, \xi_k) \in \Lambda$ such that the unique solution $\gamma_{\sigma, \xi} = (\gamma_1, \dots, \gamma_k) = 0 \in \mathbb{R}^k$ for (5.16), provided that σ is sufficiently small. If we achieve this, then (5.16) implies that there exists a $\varphi_{\sigma, \xi} \in L_\mu^\infty$, which satisfies

$$\begin{aligned} (6.1) \quad &S[W + \varphi_{\sigma, \xi}](x) = 0, \quad x \in \mathbb{R}, \\ &\langle \varphi_{\sigma, \xi}, U'(\cdot - \xi_j) \rangle = 0, \quad j = 1, \dots, k, \\ &\varphi_{\sigma, \xi}(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty. \end{aligned}$$

This means $u = W + \varphi_{\sigma, \xi}$ is a solution of the original nonlocal problem (2.5), i.e.,

$$\begin{aligned} (6.2) \quad &S[u] = u'' - u + \frac{u^2}{T[u^2]} = 0 \text{ in } \mathbb{R}, \\ &u > 0 \text{ and } u \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{aligned}$$

The key term in (5.35) is

$$\begin{aligned} (6.3) \quad &\frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] = -\frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} d(\sigma) \sigma^{1+\beta} |\xi_j - \xi_\ell| \right] \\ &\quad - \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \right] \\ &= \sum_{j \neq \ell} d(\sigma) \sigma^{1+\beta} \operatorname{sgn}(\xi_j - \xi_\ell) + \sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_\ell - \xi_j) \\ &= d(\sigma) \sigma^{1+\beta} (k + 1 - 2\ell) - \sum_{j \neq \ell} 36 e^{|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_j - \xi_\ell), \quad \ell = 1, \dots, k. \end{aligned}$$

Choose k points ξ_1^0, \dots, ξ_k^0 such that the following conditions are satisfied

$$(6.4a) \quad \sum_{j=1}^k \xi_j^0 = 0,$$

and

$$(6.4b) \quad \xi_j^0 - \xi_{j+1}^0 = -|\xi_j^0 - \xi_{j+1}^0| = (1 + \beta) \log \sigma + \log[j(k - j)] + \log \frac{d(\sigma)}{36},$$

for $j = 1, \dots, k - 1$.

Denote these points by $\xi^0 = (\xi_1^0, \dots, \xi_k^0)$. Since $k = C\sigma^{-\beta}$, we have

$$(6.5) \quad \max_{1 \leq j \leq k-1} j(k-j) \leq \frac{k^2}{4} = \left(\frac{C}{2}\right)^2 \sigma^{-2\beta}$$

so that

$$(6.6) \quad \begin{aligned} |\xi_j^0 - \xi_{j+1}^0| &= -(1+\beta) \log \sigma - \log [j(k-j)] - \log \frac{d(\sigma)}{36} \\ &\geq (1-\beta) |\log \sigma| - 2 \log \frac{C}{2} - \log \frac{d(\sigma)}{36}. \end{aligned}$$

By (4.12) and (4.41), for σ sufficiently small,

$$(6.7) \quad -2 \log \frac{C}{2} - \log \frac{d(\sigma)}{36} = \log \frac{144}{C^2 d(\sigma)} > \log \left(\frac{144}{C^2} \cdot \frac{5 \times 6 C}{12 \times 40} \right) = \log \left(\frac{9}{C} \right).$$

If k satisfies

$$(6.8) \quad k = 9 \sigma^{-\beta},$$

then we have, for $j = 1, \dots, k$,

$$(6.9) \quad \begin{aligned} (1-\beta) |\log \sigma| &< |\xi_j^0 - \xi_{j+1}^0| \leq -(1+\beta) \log \sigma - \log \frac{(k-1)d(\sigma)}{36} \\ &\approx -(1+\beta) \log \sigma - \log \frac{kd(\sigma)}{36} \quad (\text{by (4.12) and (4.41)}) \\ &\approx |\log \sigma| - \log \frac{12 \times 36}{5 \times 6 \times 36} = |\log \sigma| + \log \frac{5}{2} \\ &< |\log \sigma| + 1. \end{aligned}$$

It is seen from (6.9) that $\xi^0 \in \Lambda(\sigma, \beta)$. Moreover, if $\eta > 0$ is a sufficiently small number, then

$$(6.10) \quad \Lambda_\eta = \left\{ \xi = (\xi_1, \dots, \xi_k) : \|\xi - \xi^0\| < \eta, \sum_{j=0}^k \xi_j = 0 \right\} \subset \Lambda(\sigma, \beta).$$

In fact, this choice of ξ^0 implies that for σ sufficiently small and for $\xi \in \Lambda_\eta$,

$$(6.11) \quad \begin{aligned} -\sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_j - \xi_\ell) &= -\sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_j^0 - \xi_\ell^0) \\ &= 36 \left[e^{-|\xi_{\ell-1} - \xi_\ell|} - e^{|\xi_{\ell+1} - \xi_\ell|} \right] + O\left(\sigma^{2(1-\beta)}\right) \\ &= 36 \left[e^{-|\xi_{\ell-1}^0 - \xi_\ell^0 + \hat{\xi}_{\ell-1} - \hat{\xi}_\ell|} - e^{-|\xi_{\ell+1}^0 - \xi_\ell^0 + \hat{\xi}_{\ell+1} - \hat{\xi}_\ell|} \right] + O\left(\sigma^{2(1-\beta)}\right), \end{aligned}$$

where

$$(6.12) \quad \xi_j = \xi_j^0 + \hat{\xi}_j, \quad j = 1, \dots, k, \quad \text{and} \quad \sum_{j=0}^k \hat{\xi}_j = 0.$$

By Taylor expansions, we get

$$(6.13) \quad \begin{aligned} e^{-|\xi_j^0 - \xi_\ell^0 + \hat{\xi}_j - \hat{\xi}_\ell|} &= e^{-|\xi_j^0 - \xi_\ell^0|} + e^{-|\xi_j^0 - \xi_\ell^0|} (\operatorname{sgn}(\xi_\ell - \xi_j)) (\hat{\xi}_j - \hat{\xi}_\ell) \\ &\quad + O\left(\sigma \left| \hat{\xi}_j - \hat{\xi}_\ell \right|^2\right), \end{aligned}$$

for $|j - \ell| = 1$. Substituting (6.12) into (6.11), we obtain

$$(6.14) \quad \begin{aligned} & - \sum_{j \neq \ell} 36 e^{-|\xi_j - \xi_\ell|} \operatorname{sgn}(\xi_j - \xi_\ell) = 36 \left[e^{-|\xi_{\ell-1}^0 - \xi_\ell^0|} - e^{-|\xi_{\ell+1}^0 - \xi_\ell^0|} \right] \\ & + 36 \left[e^{-|\xi_{\ell-1}^0 - \xi_\ell^0|} \left(\hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right) + e^{-|\xi_{\ell+1}^0 - \xi_\ell^0|} \left(\hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right) \right] \\ & + O \left(\sigma \left[\left| \hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right|^2 + \left| \hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right|^2 \right] \right) + O \left(\sigma^{2(1-\beta)} \right), \end{aligned}$$

where

$$(6.15) \quad \begin{aligned} 36 \left[e^{-|\xi_{\ell-1}^0 - \xi_\ell^0|} - e^{-|\xi_{\ell+1}^0 - \xi_\ell^0|} \right] &= d(\sigma) \sigma^{1+\beta} [(\ell-1)(k-\ell+1) - \ell(k-\ell)] \\ &= -d(\sigma) \sigma^{1+\beta} (k+1-2\ell), \quad \ell = 1, \dots, k. \end{aligned}$$

From (6.3), (6.14) and (6.15), it is seen that (5.35) becomes

$$(6.16) \quad \begin{aligned} \sum_{j=1}^k \gamma_j \langle U'_j, U'_\ell \rangle &= \frac{\partial}{\partial \xi_\ell} \left[\sum_{j \neq \ell} F(|\xi_j - \xi_\ell|) \right] + O \left(\sigma^{2(1-\beta)-\mu} \right) \\ &= 36 \left[e^{-|\xi_{\ell-1}^0 - \xi_\ell^0|} \left(\hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right) + e^{-|\xi_{\ell+1}^0 - \xi_\ell^0|} \left(\hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right) \right] \\ &+ O \left(\sigma \left[\left| \hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right|^2 + \left| \hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right|^2 \right] \right) + O \left(\sigma^{2(1-\beta)-\mu} \right), \end{aligned}$$

where $\langle U'_j, U'_\ell \rangle$, $j, \ell = 1, \dots, k$, satisfy (5.36). For notational convenience, set

$$(6.17) \quad m(i, j) = e^{-|\xi_i^0 - \xi_j^0|}, \quad i, j = 1, \dots, k.$$

The system of equations (6.16) for $\gamma = (\gamma_1, \dots, \gamma_k)$ can be written as

$$(6.18) \quad \begin{aligned} & \begin{pmatrix} \sum_{j=1}^k \gamma_j \langle U'_j, U'_1 \rangle \\ \vdots \\ \sum_{j=1}^k \gamma_j \langle U'_j, U'_k \rangle \end{pmatrix} = M \begin{pmatrix} \hat{\xi}_1 \\ \vdots \\ \hat{\xi}_k \end{pmatrix} \\ & + O \left(\sigma \left[\left| \hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right| + \left| \hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right| \right] \right) + O \left(\sigma^{2(1-\beta)-\mu} \right), \end{aligned}$$

where M is the $k \times k$ matrix as follows,

$$(6.19) \quad M = 36 \begin{pmatrix} -m(1,2) & m(1,2) & 0 & \cdots \\ m(1,2) & -(m(1,2) + m(2,3)) & m(2,3) & \cdots \\ & 0 & 0 & \ddots \\ & 0 & 0 & \cdots & m(j-1, j) \\ & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ -(m(j-1, j) + m(j, j+1)) & m(j, j+1) & \cdots & \cdots & 0 \\ & \ddots & & & \\ & & \ddots & & \\ \cdots & \cdots & m(k-1, k) & -m(k-1, k) & \end{pmatrix}.$$

LEMMA 6.1. *The symmetric matrix M has an eigenvalue $\lambda_1 = 0$, whose eigenspace N_1 is one-dimensional,*

$$(6.20) \quad N_1 = \left\{ \hat{\xi} = (s, s, \dots, s) \in \mathbb{R}^k \mid s \neq 0 \right\}.$$

On the orthogonal complement subspace

$$(6.21) \quad N_1^\perp = \left\{ \hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_k) \in \mathbb{R}^k \mid \sum_{j=1}^k \hat{\xi}_j = 0 \right\},$$

the matrix M is invertible and the operator norm of the inverse matrix M^{-1} on N_1^\perp satisfies

$$(6.22) \quad \|M^{-1}\| \leq \tilde{C} \sigma^{-1},$$

for some uniform constant $\tilde{C} > 0$.

PROOF. In the matrix $\lambda I - M$, adding all the 2nd through k th rows to the first row, we get

$$\det(\lambda I - M) = \lambda \det M_1(\lambda),$$

where $M_1(\lambda)$ is another λ -matrix. This shows $\lambda_1 = 0$ is an eigenvalue of M . It can be shown directly that its associated eigenspace N_1 is characterized by (6.20). For any $\hat{\xi} \in N_1^\perp$,

$$\sum_{j=1}^k s \hat{\xi}_j = 0 \quad \text{so that (6.21) holds,}$$

and vice-versa. Since M is symmetric, beside $\lambda_1 = 0$, all the other $k - 1$ eigenvalues must be real numbers.

Let λ_{\min} be a nonzero eigenvalue of M on N_1^\perp , which has the smallest absolute value. Then we know that

$$|\lambda_{\min}| = \inf_{\substack{x \in N_1^\perp \\ \|x\|=1}} \{|\langle Mx, x \rangle|\}.$$

For any $x = \text{col}(x_1, \dots, x_k) \in \mathbb{R}^k$ with $\|x\| = 1$, we have

$$\begin{aligned}
(6.23) \quad \frac{1}{36} \langle Mx, x \rangle &= -m(1, 2)(x_1 - x_2)x_1 + m(1, 2)(x_1 - x_2)x_2 \\
&\quad -m(2, 3)(x_2 - x_3)x_2 + m(2, 3)(x_2 - x_3)x_3 \\
&\quad \dots \dots \\
&\quad -m(j-1, j)(x_{j-1} - x_j)x_{j-1} + m(j-1, j)(x_{j-1} - x_j)x_j \\
&\quad \dots \dots \\
&\quad -m(k-1, k)(x_{k-1} - x_k)x_{k-1} + m(k-1, k)(x_{k-1} - x_k)x_k \\
&= -\sum_{j=1}^{k-1} m(j, j+1)(x_j - x_{j+1})^2 \leq 0,
\end{aligned}$$

here we know that

$$m(j, j+1) = e^{-|\xi_j^0 - \xi_{j+1}^0|} > 0.$$

Hence, by (6.4b) and (6.23), we have

$$\begin{aligned}
(6.24) \quad |\lambda_{\min}| = -\lambda_{\min} &= \inf \left\{ 36 \sum_{j=1}^{k-1} m(j, j+1)(x_j - x_{j+1})^2 : x \in N_1^\perp, \|x\| = 1 \right\} \\
&= 36 m(1, 2) = 36 m(k-1, k) = d(\sigma) \sigma^{1+\beta} (9\sigma^{-\beta} - 1) \geq C\sigma,
\end{aligned}$$

for some constant $C > 0$. Thus there is a constant $\tilde{C} > 0$, such that on the subspace N_1^\perp , the norm of the inverse operator M^{-1} satisfies

$$\|M^{-1}\| = \frac{1}{|\lambda_{\min}|} \leq \tilde{C}\sigma^{-1}.$$

□

Note that for $k = 2$, the matrix M has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -12m(1, 2)$. But this pattern is not true for $k \geq 3$ in general. For $k = 3$, the three eigenvalues are $\lambda_1 = 0$ and

$$\lambda_2, \lambda_3 = m(1, 2) + m(2, 3) \pm \sqrt{m(1, 2)^2 - m(1, 2)m(2, 3) + m(2, 3)^2}.$$

However, (6.22) always holds.

THEOREM 6.2. *Under the assumption that*

$$(6.25) \quad k = 9\sigma^{-\beta}, \text{ with } \beta < \frac{1}{2} \text{ and arbitrarily close to } \frac{1}{2},$$

there is a constant $\sigma^ > 0$, such that for any given $0 < \sigma \leq \sigma^*$, there exists a $\xi = (\xi_1, \dots, \xi_k) \in \Lambda = \Lambda(\sigma, \beta)$ such that the problem (5.16) has a unique solution $(\varphi_{\sigma, \xi}, \gamma_{\sigma, \xi})$, where*

$$(6.26) \quad \gamma_{\sigma, \xi} = (\gamma_1, \dots, \gamma_k) = 0.$$

PROOF. According to Theorem 5.4, the unique solution $(\varphi_{\sigma, \xi}, \gamma_{\sigma, \xi})$ to the problem (5.16) satisfies Eq. (5.35), which is equivalently reduced to the system (6.18). By (5.36), we can show that the coefficient matrix on the left-hand side of (6.18),

$$\begin{pmatrix} \langle U'_1, U'_1 \rangle & \dots & \langle U'_k, U'_1 \rangle \\ \dots & \dots & \dots \\ \langle U'_1, U'_k \rangle & \dots & \langle U'_k, U'_k \rangle \end{pmatrix} = \frac{6}{5} I + A,$$

is the same coefficient matrix in (3.16) and it is nonsingular. We can solve (6.18) for $\gamma_{\sigma, \xi}$ and get

$$(6.27) \quad \gamma_{\sigma, \xi} = \left(\frac{6}{5} I + A \right)^{-1} M \left\{ \begin{pmatrix} \hat{\xi}_1 \\ \vdots \\ \hat{\xi}_k \end{pmatrix} - \Psi_{\sigma}(\hat{\xi}) \right\},$$

where the mapping Ψ_{σ} is defined by

$$(6.28) \quad \Psi_{\sigma}(\hat{\xi}) = -M^{-1} \left\{ O \left(\sigma \left[|\hat{\xi}_{\ell-1} - \hat{\xi}_{\ell}|^2 + |\hat{\xi}_{\ell+1} - \hat{\xi}_{\ell}|^2 \right] \right) + O \left(\sigma^{2(1-\beta)-\mu} \right) \right\},$$

in which, of course, the two terms inside $\{\dots\}$ are k -dimensional column vectors whose ℓ th components are as shown, respectively.

In order to show (6.26), it suffices to prove that there is a small number $\eta > 0$ such that

$$(6.29) \quad \Psi_{\sigma} \text{ has a fixed point } \hat{\xi} \in \Lambda_{\eta} (\subset N_1^{\perp}).$$

Now we use the argument of contraction mapping to prove it. Take a set

$$Z = \left\{ \hat{\xi} \in N_1^{\perp} : \|\hat{\xi}\| \leq \eta = |\log \sigma|^{-\nu} \right\},$$

where $\nu > 0$ is a small positive number and let ν be fixed. For any fixed β such that

$$0 < \beta < \frac{1}{2}, \text{ arbitrarily close to } \frac{1}{2},$$

a value of the parameter μ can be chosen such that

$$(6.30) \quad 0 < \mu < \min \left\{ \frac{1}{3}, 1 - 2\beta \right\},$$

which implies that

$$(6.31) \quad \mu < \frac{1-2\beta}{1-\beta} \text{ so that (3.63) is satisfied: } \beta < \frac{1-\mu}{2-\mu}.$$

Let μ be fixed in such a way. Then we have, by Lemma 6.1 and (6.22),

$$(6.32) \quad \begin{aligned} \|\Psi_{\sigma}(\hat{\xi})\|_{\mathbb{R}^k} &\leq \|M^{-1}\| \left\{ \left\| O \left(\sigma \left[|\hat{\xi}_{\ell-1} - \hat{\xi}_{\ell}|^2 + |\hat{\xi}_{\ell+1} - \hat{\xi}_{\ell}|^2 \right] \right) \right\| \right. \\ &\quad \left. + \left\| O \left(\sigma^{2(1-\beta)-\mu} \right) \right\| \right\} \leq C^* \left\{ \|\hat{\xi}\|^2 + \sigma^{1-2\beta-\mu} \right\} \\ &\leq C^* \left\{ |\log \sigma|^{-\nu} + \sigma^{1-2\beta-\mu} |\log \sigma|^{\nu} \right\} |\log \sigma|^{-\nu}. \end{aligned}$$

Then there exists a constant $\sigma_1 > 0$ such that, whenever $0 < \sigma \leq \sigma_1$, in (6.32) we have

$$(6.33) \quad C^* \left\{ |\log \sigma|^{-\nu} + \sigma^{1-2\beta-\mu} |\log \sigma|^{\nu} \right\} < 1.$$

Therefore, for any $0 < \sigma \leq \sigma_1$, we have

$$(6.34) \quad \Psi_{\sigma}(Z) \subset Z.$$

Next, for any $\hat{\xi}$ and $\hat{\zeta}$ in Z , we have

$$\begin{aligned}
(6.35) \quad & \left\| \Psi_\sigma(\hat{\xi}) - \Psi_\sigma(\hat{\zeta}) \right\|_{\mathbb{R}^k} \leq C^* \left\| \operatorname{col} \left(\left| \hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right|^2 - \left| \hat{\zeta}_{\ell-1} - \hat{\zeta}_\ell \right|^2 \right. \right. \\
& \quad \left. \left. + \left| \hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right|^2 - \left| \hat{\zeta}_{\ell+1} - \hat{\zeta}_\ell \right|^2 \right) \right\|_{\mathbb{R}^k} \\
& \leq 2C^* \left(\|\hat{\xi}\| + \|\hat{\zeta}\| \right) \left\| \operatorname{col} \left(\left| \hat{\xi}_{\ell-1} - \hat{\xi}_\ell \right| - \left| \hat{\zeta}_{\ell-1} - \hat{\zeta}_\ell \right| \right. \right. \\
& \quad \left. \left. + \left| \hat{\xi}_{\ell+1} - \hat{\xi}_\ell \right| - \left| \hat{\zeta}_{\ell+1} - \hat{\zeta}_\ell \right| \right) \right\| \\
& \quad \text{(by the triangle inequalities)} \\
& \leq 2C^* \left(\|\hat{\xi}\| + \|\hat{\zeta}\| \right) \left\| \operatorname{col} \left(\left| \hat{\xi}_{\ell-1} - \hat{\zeta}_{\ell-1} \right| + 2 \left| \hat{\xi}_\ell - \hat{\zeta}_\ell \right| + \left| \hat{\xi}_{\ell+1} - \hat{\zeta}_{\ell+1} \right| \right) \right\| \\
& \leq 8C^* \left(\|\hat{\xi}\| + \|\hat{\zeta}\| \right) \|\hat{\xi} - \hat{\zeta}\| \\
& \leq 16C^* |\log \sigma|^{-\nu} \|\hat{\xi} - \hat{\zeta}\|.
\end{aligned}$$

There exists a small constant $\sigma_2 > 0$ such that, whenever $0 < \sigma \leq \sigma_2$, in (6.35) we have

$$(6.36) \quad 16C^* |\log \sigma|^{-\nu} \leq \frac{1}{2},$$

so that

$$(6.37) \quad \left\| \Psi_\sigma(\hat{\xi}) - \Psi_\sigma(\hat{\zeta}) \right\| \leq \frac{1}{2} \|\hat{\xi} - \hat{\zeta}\|, \text{ for any } \hat{\xi}, \hat{\zeta} \in Z.$$

Therefore, by (6.34) and (6.37), Ψ_σ is a contraction mapping on the set Z , provided that

$$(6.38) \quad 0 < \sigma < \min \{ \sigma_1, \sigma_2 \}.$$

By Brouwer's fixed point theorem, there exists a fixed point

$$(6.39) \quad \hat{\xi}_\sigma \in Z \text{ such that } \hat{\xi}_\sigma = \Psi_\sigma(\hat{\xi}_\sigma),$$

and consequently, this $\hat{\xi}_\sigma$ yields

$$(6.40) \quad \gamma_{\sigma, \xi_\sigma} = (\gamma_1, \dots, \gamma_k) = 0,$$

in (6.27), with $\xi_\sigma = \xi^0 + \hat{\xi}_\sigma$. Thus, Theorem 6.2 is proved with

$$(6.41) \quad \sigma^* = \min \{ \sigma_1, \sigma_2 \}.$$

The proof is completed. \square

CONCLUSION (The completion of the proof of Theorem 2.1). Let

$$(6.42) \quad \sigma_0 = \min \{ \hat{\sigma}, \check{\sigma}, \sigma^* \},$$

where $\hat{\sigma} \leq \frac{1}{2}$, $\check{\sigma}$, and σ^* are determined in Theorem 3.6, Theorem 5.2, and Theorem 6.2, respectively. For any $0 < \sigma \leq \sigma_0$, all the results shown in Sections 3 through 6 are valid. Therefore, there is a multi-spike solution (u, v) to the Gierer-Meinhardt system (2.3), in which

$$(6.43) \quad u = W + \varphi_{\sigma, \xi_\sigma} \quad \text{and} \quad V = T[u^2],$$

where $W(x)$ is given by (1.17), $\varphi_{\sigma, \xi_\sigma} \in L_\mu^\infty$ is the solution of problem (5.16) together with $\gamma_{\sigma, \xi_\sigma} = 0$. Hence the property (2.8) is valid. The property (2.9) for the v component is valid because

$$\begin{aligned} v &= T[u^2] = T[W^2] + 2T[W\varphi_{\sigma, \xi_\sigma}] + T[\varphi_{\sigma, \xi_\sigma}^2] \\ &= V(x) + 2T[W\varphi_{\sigma, \xi_\sigma}] + T[\varphi_{\sigma, \xi_\sigma}^2] \end{aligned}$$

and by Lemmas 4.1 and 4.2, (5.6) and (5.7), with a little extension, we have

$$\begin{aligned} |v(\xi_j + x) - 1| &\leq |V(\xi_j + x) - V(\xi_j)| \\ &\quad + |V(\xi_j) - 1| + 2|T[W\varphi_{\sigma, \xi_\sigma}]| + |T[\varphi_{\sigma, \xi_\sigma}^2]| \\ &= |V(\xi_j + x) - V(\xi_j)| + |V(\xi_j) - 1| + O(\sigma^\beta) \|\varphi_{\sigma, \xi_\sigma}\|_\mu \\ &\quad + O(1) \|\varphi_{\sigma, \xi_\sigma}\|_\mu^2 \rightarrow 0 \end{aligned}$$

as $\sigma \rightarrow 0$, uniformly on any compact set of x . Thus Theorem 2.1 is proved. \square

Appendix A

We can solve the nonlinear ODE problem:

$$\begin{aligned} u'' - u + u^2 &= 0 \quad \text{in } \mathbb{R}, \\ 0 < u(x) &\rightarrow 0, \text{ as } |x| \rightarrow \infty, \end{aligned}$$

to get the unique explicit solution

$$u(x) = 6 \operatorname{sech}^2\left(\frac{x}{2}\right) = \frac{6e^x}{(1+e^x)^2}.$$

By multiplying u' on both sides of the equation and integrating it, we have

$$(u')^2 = u^2 - \frac{2}{3}u^3,$$

so that

$$u' = -u\sqrt{1 - \frac{2}{3}u}.$$

The sign is chosen according to the condition $0 < u(x) \rightarrow 0$, as $|x| \rightarrow \infty$. Let $y = \log u$. Then

$$\int \frac{dy}{\sqrt{1 - \frac{2}{3}e^y}} = -x,$$

where we do not involve an arbitrary constant also because of the asymptotical boundary condition. Another substitution $z = \sqrt{1 - \frac{2}{3}e^y}$, $dz = \frac{-1/3 e^y dy}{\sqrt{1 - \frac{2}{3}e^y}}$, ren-

ders the above equality to

$$\int \frac{-2dz}{1-z^2} = (-1) \int \left(\frac{1}{1-z} + \frac{1}{1+z} \right) dz = -x,$$

or

$$x = \log \frac{1+z}{1-z}, \quad \text{i.e.,} \quad z = \frac{e^x - 1}{e^x + 1} = \sqrt{1 - \frac{2}{3}u}.$$

Then solve this algebraic equation for u , finally we obtain the unique solution

$$u(x) = \frac{6e^x}{(1+e^x)^2}, \quad x \in \mathbb{R}.$$

Appendix B

In this Appendix we compute the integrals in Lemma 1.1. First,

$$\begin{aligned} \int_{\mathbb{R}} U^2(x) dx &= \int_{\mathbb{R}} \frac{36e^{-2|x|}}{(1+e^{-|x|})^4} dx = 72 \int_{-\infty}^0 \frac{e^{2x}}{(1+e^x)^4} dx \quad (\text{let } y = 1+e^x) \\ &= 72 \int_1^2 \frac{y-1}{y^4} dy = 72 \left\{ \frac{-1}{2y^2} + \frac{1}{3y^3} \right\} \Big|_1^2 = 72 \left[\frac{1}{2} \left(1 - \frac{1}{4} \right) + \frac{1}{3} \left(\frac{1}{8} - 1 \right) \right] \\ &= 6. \end{aligned}$$

Next,

$$\begin{aligned} \int_{\mathbb{R}} U^3(x) dx &= \int_{\mathbb{R}} \frac{6^3 e^{-3|x|}}{(1+e^{-|x|})^6} dx = 432 \int_{-\infty}^0 \frac{e^{3x}}{(1+e^x)^6} dx \quad (\text{let } y = 1+e^x) \\ &= 432 \int_1^2 \frac{(y-1)^2}{y^6} dy = 432 \int_1^2 \left[\frac{1}{y^4} - \frac{2}{y^5} + \frac{1}{y^6} \right] dy \\ &= 432 \left[-\frac{1}{3y^3} + \frac{1}{2y^4} - \frac{1}{5y^5} \right] \Big|_1^2 = 432 \left(\frac{7}{24} - \frac{15}{32} + \frac{31}{160} \right) = \frac{36}{5}. \end{aligned}$$

Now we compute $\int_{\mathbb{R}} |U'(x)|^2 dx$.

$$\begin{aligned} \int_{\mathbb{R}} |U'(x)|^2 dx &= 72 \int_{-\infty}^0 \frac{e^{2x} (1-e^x)^2}{(1+e^x)^6} dx \quad (\text{by substitution } y = 1+e^x) \\ &= 72 \int_1^2 \frac{(y-1)(y-2)^2}{y^6} dy = 72 \int_1^2 \frac{1}{y^6} (y^3 - 5y^2 + 8y - 4) dy \\ &= 72 \int_1^2 (y^{-3} - 5y^{-4} + 8y^{-5} - 4y^{-6}) dy \\ &= 72 \left[-\frac{1}{2} y^{-2} + \frac{5}{3} y^{-3} - 2y^{-4} + \frac{4}{5} y^{-5} \right] \Big|_1^2 \\ &= 72 \left(\frac{3}{8} - \frac{35}{24} + \frac{15}{8} - \frac{31}{40} \right) = \frac{6}{5}. \end{aligned}$$

Then we have

$$\begin{aligned} \int_{\mathbb{R}} |x| U(x) dx &= 2 \int_0^{\infty} x U(x) dx = 12 \int_0^{\infty} \frac{x e^x}{(1+e^x)^2} dx \\ &= 12 \left[\frac{-x}{1+e^x} \Big|_0^{\infty} + \int_0^{\infty} \frac{dx}{1+e^x} \right] = 12 \int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx \\ &= -12 \log(1+e^{-x}) \Big|_0^{\infty} = 12 \log 2. \end{aligned}$$

Next,

$$\begin{aligned} \int_{\mathbb{R}} |x| U^2(x) dx &= 72 \int_0^{\infty} \frac{x e^{-2x}}{(1+e^{-x})^4} dx = 72 \int_0^{\infty} \frac{x e^{2x}}{(1+e^x)^4} dx \\ &= 72 \left\{ -\frac{1}{3} \frac{x e^x}{(1+e^x)^3} \Big|_0^{\infty} + \frac{1}{3} \int_0^{\infty} \frac{(1+x)e^x}{(1+e^x)^3} dx \right\} \\ &= 72 \left\{ -\frac{1}{6} \frac{1+x}{(1+e^x)^2} \Big|_0^{\infty} + \frac{1}{6} \int_0^{\infty} \frac{dx}{(1+e^x)^2} \right\} \\ &= 12 \left\{ \frac{1}{4} + \int_0^{\infty} \frac{e^{-2x}}{(1+e^{-x})^2} dx \right\} = 3 + 12 \left\{ \frac{e^{-x}}{1+e^{-x}} \Big|_0^{\infty} + \int_0^{\infty} \frac{e^{-x}}{1+e^{-x}} dx \right\} \\ &= 3 + 12 \left\{ -\frac{1}{2} - \log(1+e^{-x}) \Big|_0^{\infty} \right\} \\ &= 3 - 6 + 12 \log 2 = 12 \log 2 - 3. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}} e^{-|x|} U^2(x) dx &= 2 \int_0^{\infty} e^{-|x|} U^2(x) dx = 72 \int_0^{\infty} \frac{e^{-x} e^{2x}}{(1+e^x)^4} dx \\ &= 72 \int_0^{\infty} \frac{e^x}{(1+e^x)^4} dx = -\frac{72}{3} \frac{1}{(1+e^x)^3} \Big|_0^{\infty} = 3. \end{aligned}$$

Finally,

$$\int_{\mathbb{R}} e^{-x} U^2(x) dx = \int_{-\infty}^{\infty} 36 \frac{e^{-x} e^{2x}}{(1+e^x)^4} dx = -\frac{36}{3} \frac{1}{(1+e^x)^3} \Big|_{-\infty}^{\infty} = 12.$$

Appendix C

Here we give the detailed proof of Lemma 1.2. The first statement in (1.7),

$$p_r(x) \leq e^{-|r|}, \quad x \in \mathbb{R},$$

simply follows from the triangle inequality. Now we show the second statement in (1.7) by calculation for both cases $r \geq 0$ and $r < 0$. For $r \geq 0$, by the explicit expression of $p_r(x)$,

$$\begin{aligned} \int_{\mathbb{R}} e^{-|x|} e^{-|x+r|} dx &= \int_{-\infty}^{-r} e^{2x+r} dx + \int_{-r}^0 e^{-r} dx + \int_0^{\infty} e^{-(2x+r)} dx \\ &= \frac{1}{2} e^{-r} + r e^{-r} + \frac{1}{2} e^{-r} = (1+|r|) e^{-|r|}. \end{aligned}$$

For $r < 0$,

$$\begin{aligned} \int_{\mathbb{R}} e^{-|x|} e^{-|x+r|} dx &= \int_{-\infty}^0 e^{2x+r} dx + \int_0^{-r} e^r dx + \int_{-r}^{\infty} e^{-(2x+r)} dx \\ &= \frac{1}{2} e^r - r e^r + \frac{1}{2} e^r = (1+|r|) e^{-|r|}. \end{aligned}$$

Therefore, (1.7) holds.

To save space, we show (1.8), (1.9), and (1.10) only for the case $r \geq 0$. For the case $r < 0$, the verification is similar and is omitted. Since for $r \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}} |x| e^{-|x|} e^{-|x+r|} dx &= \int_{-\infty}^{-r} -x e^{2x+r} + \int_{-r}^0 -x e^{-r} + \int_0^{\infty} x e^{-(2x+r)} \\ &= \left(-\frac{x}{2} e^{2x+r} + \frac{1}{4} e^{2x+r} \right) \Big|_{-\infty}^{-r} - \frac{x^2}{2} e^{-r} \Big|_{-r}^0 - \left(\frac{x}{2} e^{-(2x+r)} + \frac{1}{4} e^{-(2x+r)} \right) \Big|_0^{\infty} \\ &= e^{-r} \left(\frac{r}{2} + \frac{1}{4} \right) + \frac{r^2}{2} e^{-r} + \frac{1}{4} e^{-r} = \frac{1}{2} e^{-|r|} (1+|r|+r^2), \end{aligned}$$

so (1.8) is valid.

Next we show (1.9). For $r \geq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}} |x|^2 e^{-2|x|} e^{-2|x+r|} dx &= \int_{-\infty}^{-r} x^2 e^{2x} e^{2x+2r} + \int_{-r}^0 x^2 e^{-2r} dx \\
&\quad + \int_0^{\infty} x^2 e^{-2x} e^{-2x-2r} dx \\
&= e^{2r} \left(\frac{x^2}{4} e^{4x} - \frac{x}{8} e^{4x} + \frac{1}{32} e^{4x} \right) \Big|_{-\infty}^{-r} + \frac{x^3}{3} e^{-2r} \Big|_{-r}^0 \\
&\quad + e^{-2r} \left(-\frac{x^2}{4} e^{-4x} - \frac{x}{8} e^{-4x} - \frac{1}{32} e^{-4x} \right) \Big|_0^{\infty} \\
&= e^{-2r} \left\{ \left(\frac{r^2}{4} + \frac{|r|}{8} + \frac{1}{32} \right) + \frac{|r|^3}{3} + \frac{1}{32} \right\} \\
&= e^{-2|r|} \left(\frac{1}{16} + \frac{|r|}{8} + \frac{r^2}{4} + \frac{|r|^3}{3} \right).
\end{aligned}$$

Therefore, (1.9) is valid.

Finally we show (1.10). For $r \geq 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}} |x| e^{-|x|} e^{-2|x+r|} dx &\leq e^{-|r|} \int_{\mathbb{R}} |x| e^{-|x+r|} dx \quad (\text{by (1.7)}) \\
&= e^{-|r|} \left\{ \int_{-\infty}^{-r} -x e^{x+r} + \int_{-r}^0 -x e^{-(x+r)} + \int_0^{\infty} x e^{-(x+r)} \right\} \\
&= e^{-|r|} \left\{ (-x e^{x+r} + e^{x+r}) \Big|_{-\infty}^{-r} + (x e^{-(x+r)} + e^{-(x+r)}) \Big|_{-r}^0 \right. \\
&\quad \left. + (-x e^{-(x+r)} - e^{-(x+r)}) \Big|_0^{\infty} \right\} \\
&= e^{-|r|} (r + 1 + r + e^{-r} - 1 + e^{-r}) \\
&= e^{-|r|} (2r + 2e^{-r}) = 2e^{-|r|} (|r| + e^{-|r|}).
\end{aligned}$$

Therefore, (1.10) is valid. The proof is completed.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH FLORIDA, TAMPA, FL 33620-5700,
USA

E-mail address: you@math.usf.edu