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The Lie-Poisson Structure of the LAE-α Equation

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ABSTRACT. This paper shows that the time t map of the averaged Euler equations, with Dirichlet, Neumann, and mixed boundary conditions is canonical relative to a Lie-Poisson bracket constructed via a non-smooth reduction for the corresponding diffeomorphism groups. It is also shown that the geodesic spray for Neumann and mixed boundary conditions is smooth, a result already known for Dirichlet boundary conditions.

CONTENTS

1. Introduction

The role of Hamiltonian structures for evolutionary conservative equations in mathematical physics is well established. In the finite dimensional case, that is, the situation of ordinary differential Hamiltonian systems, classical symplectic and Poisson geometry and their Lagrangian counterparts form the framework in which the dynamics is formulated. When dealing with infinite dimensional systems, namely the case of partial differential equations, one is immediately confronted with serious technical and conceptual difficulties. The main issue is that, with the exception of certain equations in quantum mechanics, all these PDEs need to be formulated

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using a weak symplectic form. Also, for many equations, the time evolution is not smooth in the function spaces that are natural to the problem. If the system is linear, this corresponds to the fact that the right hand side of the evolutionary equation is given by an unbounded operator. Unfortunately, there is very little general theory dealing with the natural questions that arise when working with Hamiltonian PDEs. The first systematic attempt at such a devleopment can be found in [**7**] and more recently, motivated by questions regarding coherent states quantization, in [**16**]. The present paper adds to this literature, by presenting a precise Hamiltonian formulation of an equation appearing in fluid dynamics.

[**3**] has given a Hamiltonian formulation of the Euler equations for an incompressible homogeneous perfect fluid (see also [**4**], [**5**], [**13**]). [**8**] has shown that in appropriate Sobolev spaces, the Euler equations are the spatial representation of a geodesic spray that coincides with the dynamics of such a fluid in material representation and that this geodesic spray is a smooth vector field. In fact, this paper gives a rigorous explanation with all the analytical details on how one obtains the classical Euler equations as an Euler-Poincaré equation associated to the group of volume preserving diffoemorphisms; the derivative loss of the flow occuring in the passage from material to spatial representation is also explained in this paper. [**23**] has given a Hamiltonian formulation of the Euler equations by carefully analyzing the function spaces on which Poisson brackets are defined and carrying out a Lie-Poisson reduction that takes into account all analytical difficulties. They formulate an analytical precise sense in which the flow of the Euler equations are canonical. The remarkable fact is that the passage from the previous analytically rigorous Lagrangian formulation to this Hamiltonian picture is nontrivial, mainly due to the fact that the flow is not C^1 from the Sobolev space of the initial condition to itself. We shall comment below on the exact class of Sobolev spaces needed in this formulation. A similar analysis can be carried out for the incompressible non-homogeneous Euler equations due to the resuls of [**11**] which will involve semidirect product groups.

The first goal of this paper is to carry out the program outlined in [**23**], that is, a non-smooth Lie-Poisson reduction, for another equation appearing in fluid dynamics that has attracted a lot of attention lately, namely the averaged or α-Euler equation ([**9**]). It has been shown in [**14**], [**19**], [**20**] that these equations, either on boundaryless manifolds or with Dirichlet boundary conditions, have the same remarkable property, namely in Lagrangian formulation they are smooth geodesic sprays of H^1-like weak Riemannian metrics on appropriate diffeomorphism groups. These equations are intimately related to the Camassa-Holm equation ([**6**]) for which this program can also be carried out. We have chosen to work with the averaged Euler equations because they have certain technical difficulties not encountered for the homogeneous or inhomogeneous Euler equations or the Camassa-Holm equation; besides presenting more technical problems in several steps, there also appears a one derivative loss when formulating the precise sense in which they are a Lie-Poisson system and the flow is canonical.

The second goal of the paper is to show that the geodesic spray for Neumann (or free-slip) and mixed boundary conditions is also smooth. This completes the program outlined in [**14**], [**19**], [**20**] for these boundary conditions. This shows in a different way that the averaged Euler equations are well posed, a result due to [**21**] who uses one more derivative than the present paper. We need this result in order to achieve our third goal, namely to carry out a non-smooth Lie-Poisson reduction for the averaged Euler equations with mixed boundary conditions.

The plan of the paper is as follows. Section 2 recalls the relevant facts about the averaged Euler equations. Section 3 gives the formulation of the averaged Euler equations as a smooth geodesic spray of a weak Riemannian metric on an appropirate group of volume preserving diffeomeorphism. Section 4 gives the precise formulation of the Poisson bracket, explicitly defines the correct function spaces on which the Poisson bracket formula makes sense and satisfies the usual axioms. Section 5 shows that the averaged Euler equations are Hamiltonian relative to the Poisson bracket defined previously with Hamiltonian function given by the energy of the weak Riemannian metric. It is also shown in what function spaces the flow of these equations is a canonical map. The Lie-Poisson reduction is also carried out explicitly in this section. Section 6 proves the smoothness of the spray for the averaged Euler equations with mixed boundary conditions and generalizes to this case all the results previously obtained in for Dirichlet boundary conditions.

We close this introduction by presenting the geometric setting of this paper and briefly recalling some of the key facts about the Euler equations. Let (M, g) be a C^∞, compact, oriented, finite dimensional Riemannian manifold of dimension at least two with C^∞ boundary ∂M. The Riemannian volume form on M is denoted by μ and the induced volume form on ∂M by μ_∂. Let ∇ be the covariant derivative of the Levi-Civita connection on M.

Let N be another smooth boundaryless manifold. Recall that if $s > \frac{1}{2}\dim M$ then a map $\psi : M \to N$ is of class H^s if its local representative in any pair of charts is of class H^s as a map between open sets of $\mathbb{R}^{\dim M}$ and $\mathbb{R}^{\dim N}$ respectively. If $s \leq \frac{1}{2}\dim M$ then, in general, a map could be H^s in one pair of charts and fail to be H^s in another one. Denote by $H^s(M, N) := \{\psi : M \to N \mid \psi \text{ of class } H^s\}$ the space of H^s maps from M to N for $s > \frac{1}{2}\dim M$. The set $H^s(M, N)$ can be endowed with a smooth manifold structure (see, e.g., [**8, 17**]).

Let $\widetilde{M}$ denote the boundaryless double of M. Then if $s > \frac{1}{2}\dim M + 1$ the set

$$\mathcal{D}^s := \{\eta \in H^s(M, \widetilde{M}) \mid \eta : M \to M \text{ bijective, } \eta^{-1} \in H^s(M, \widetilde{M})\} \tag{1.1}$$

is a group and a smooth submanifold of $H^s(M, \widetilde{M})$. If $\partial M = \varnothing$, then $\mathcal{D}^s$ is an open subset of $H^s(M, M)$. By the Sobolev embedding theorem, $\eta \in \mathcal{D}^s$ and its inverse are necessarily of class C^1. Therefore, $\eta(\partial M) \subset \partial M$. The tangent space at the identity $T_e\mathcal{D}^s$ consists of the H^s class vector fields on M which are tangent to ∂M, denoted by $\mathfrak{X}^s_\parallel$. Let

$$\mathcal{D}^s_\mu := \{\eta \in \mathcal{D}^s \mid \eta^*\mu = \mu\} \tag{1.2}$$

be the subset of $\mathcal{D}^s$ whose elements preserve μ. As proven in [**8**], the set $\mathcal{D}^s_\mu$ is a subgroup and a smooth submanifold of $\mathcal{D}^s$. The tangent space $T_e\mathcal{D}^s_\mu$ at the identity equals $\mathfrak{X}^s_{\text{div},\parallel} := \{u \in \mathfrak{X}^s_\parallel \mid \operatorname{div} u = 0\}$, the vector space of all H^s divergence free vector fields tangent to the boundary. If $\dim M = 1$ each of its connected components is diffeomorphic to the circle S^1. Taking on S^1 the usual length function, we see that the volume preserving diffeomorphisms on the circle are rotations. So, in this case we have for each connected component $\mathcal{D}^s_\mu = S^1$, which is not an interesting case. Thus, since $\dim M \geq 2$ we always have $s > 2$.

On $\mathfrak{X}^s$ we can introduce the L^2 inner product

$$\langle u, v\rangle_0 := \int_M g(x)(u(x), v(x))\mu(x)$$

for any $u, v \in \mathfrak{X}^s$. This inner product on $\mathfrak{X}^s$ is the value at the identity of two distinct weak Riemannian metrics on $\mathcal{D}^s$, namely

$$\mathcal{G}^0(\eta)(u_\eta, v_\eta) := \langle u_\eta \circ \eta^{-1}, v_\eta \circ \eta^{-1}\rangle_0$$

and

$$\mathcal{G}(\eta)(u_\eta, v_\eta) := \int_M g(\eta(x))(u_\eta(x), v_\eta(x))\mu(x)$$

for any $u_\eta, v_\eta \in T_\eta\mathcal{D}^s$. Note that $\mathcal{G}^0$ is right invariant by construction, whereas $\mathcal{G}$ is not. Their pull backs to $\mathcal{D}^s_\mu$ coincide and yield a right invariant weak Riemannian metric on $\mathcal{D}^s_\mu$. The Euler equations

$$\begin{aligned} &\partial_t u(t) + \nabla_{u(t)} u(t) = -\operatorname{grad} p(t)\\ &u(t) \in \mathfrak{X}^s_{\mathrm{div},\|}\,, \qquad u(0) = u_0 \quad \text{given} \end{aligned}$$

are the spatial representation of the geodesic spray on $\mathcal{D}^s_\mu$ relative to this weak Riemannian metric on $\mathcal{D}^s_\mu$ and this geodesic spray is a smooth vector field on $T\mathcal{D}^s_\mu$ (see [**8**]). The averaged Euler equations will be presented in the next section.

2. The geometry of LAE-α equation

In this section we shall quickly review the results of [**20**] regarding the motion of the averaged Euler equations. For $s > 1 + \frac{1}{2}\dim M$ we define three subsets of $\mathcal{D}^s$ which correspond to various boundary conditions. The ***Dirichlet diffeomorphism group*** is defined by

$$\mathcal{D}^s_D := \{\eta \in \mathcal{D}^s \mid \eta_{|\partial M} = id_{\partial M}\}.$$

The ***Neumann diffeomorphism group*** is defined by

$$\mathcal{D}^s_N := \{\eta \in \mathcal{D}^s \mid (T\eta_{|\partial M} \circ n)^{tan} = 0 \text{ on } \partial M\},$$

where n denotes the outward-pointing unit normal vector field along the boundary ∂M, and $(\cdot)^{tan}$ denotes the tangential part to the boundary of a vector in $TM|\partial M$. The ***mixed diffeomorphism group*** is defined by

$$\mathcal{D}^s_{mix} := \{\eta \in \mathcal{D}^s \mid \eta \text{ leaves } \Gamma_i \text{ invariant},\ \eta_{|\Gamma_1} = id_{|\Gamma_1},\ (T\eta_{|\Gamma_2} \circ n)^{tan} = 0 \text{ on } \Gamma_2\},$$

where Γ_1 and Γ_2 are two disjoint subsets of ∂M such that $\partial M = \Gamma_1 \cup \Gamma_2$ and $\overline{\Gamma}_1 = \partial M \setminus \Gamma_2$; furthermore, we assume that for all $m \in \Gamma_i$ we can find a local chart U of M at m such that $\overline{U} \cap \partial M \subset \Gamma_i$.

The groups $\mathcal{D}^s_D, \mathcal{D}^s_N$ and $\mathcal{D}^s_{mix}$ are smooth Hilbert submanifolds and subgroups of $\mathcal{D}^s$. The corresponding tangent spaces at the identity are given by

$$\mathcal{V}^s_D := T_{id_M}\mathcal{D}^s_D = \{u \in \mathfrak{X}^s_\| \mid u_{|\partial M} = 0\},$$

$$\mathcal{V}^s_N := T_{id_M}\mathcal{D}^s_N = \{u \in \mathfrak{X}^s_\| \mid (\nabla_n u_{|\partial M})^{tan} + S_n(u) = 0 \text{ on } \partial M\},$$

$$\mathcal{V}^s_{mix} := T_{id_M}\mathcal{D}^s_{mix} = \{u \in \mathfrak{X}^s_\| \mid (\nabla_n u_{|\Gamma_1})^{tan} + S_n(u) = 0 \text{ on } \Gamma_1,\ u_{|\Gamma_2} = 0\},$$

where $S_n : T\partial M \to T\partial M$ is the Weingarten map defined by $S_n(u) := -\nabla_u n$. We can also form the corresponding sets $\mathcal{D}^s_{\mu,D}, \mathcal{D}^s_{\mu,N}$ and $\mathcal{D}^s_{\mu,mix}$ which have the volume-preserving constraint imposed. These sets are smooth Hilbert submanifolds and subgroups of $\mathcal{D}^s_\mu$ and $\mathcal{D}^s$. The corresponding tangent spaces at the identity are given by

$$\mathcal{V}^s_{\mu,D} := T_{id_M}\mathcal{D}^s_{\mu,D} = \{u \in \mathfrak{X}^s_{\mathrm{div},\|} \mid u_{|\partial M} = 0\},$$
$$\mathcal{V}^s_{\mu,N} := T_{id_M}\mathcal{D}^s_{\mu,N} = \{u \in \mathfrak{X}^s_{\mathrm{div},\|} \mid (\nabla_n u_{|\partial M})^{tan} + S_n(u) = 0 \text{ on } \partial M\},$$
$$\mathcal{V}^s_{\mu,mix} := T_{id_M}\mathcal{D}^s_{\mu,mix} = \{u \in \mathfrak{X}^s_{\mathrm{div},\|} \mid (\nabla_n u_{|\Gamma_1})^{tan} + S_n(u) = 0 \text{ on } \Gamma_1,\ u_{|\Gamma_2} = 0\}.$$

Note that, as vector spaces, $\mathcal{V}^r_D$ and $\mathcal{V}^r_{\mu,D}$ make sense for $r \geq 1$, and $\mathcal{V}^r_N, \mathcal{V}^r_{mix}, \mathcal{V}^r_{\mu,N}$ and $\mathcal{V}^r_{\mu,mix}$ make sense for $r \geq 2$ but it is only for $s > 1 + \frac{1}{2}\dim M$ that they are the tangent spaces at the identity to the corresponding diffeomorphism subgroups. If $1 \leq r < 2$ we set

$$\mathcal{V}^r_N := \mathfrak{X}^r_\|, \qquad \mathcal{V}^r_{mix} := \{u \in \mathfrak{X}^r_\| \mid u_{|\Gamma_2} = 0\}$$
$$\mathcal{V}^r_{\mu,N} := \mathfrak{X}^r_{\mathrm{div},\|}, \qquad \mathcal{V}^r_{\mu,mix} := \{u \in \mathfrak{X}^r_{\mathrm{div},\|} \mid u_{|\Gamma_2} = 0\}.$$

For an arbitrary constant $\alpha > 0$, consider on $\mathfrak{X}^1$ the inner product

$$\langle u, v\rangle_1 := \int_M \big(g(x)(u(x), v(x)) + 2\alpha^2 \overline{g}(x)(\mathrm{Def}(u)(x), \mathrm{Def}(v)(x))\big)\mu(x), \tag{2.1}$$

for all $u, v \in \mathfrak{X}^1$, where

$$\mathrm{Def}(u) := \frac{\nabla u + (\nabla u)^t}{2} \tag{2.2}$$

is the ***deformation tensor***. In this formula, $(\nabla u)^t$ denotes the transpose of the $(1,1)$-tensor ∇u relative to the metric g, that is, $g(\nabla_v u, w) = g(v, (\nabla u)^t(w))$, for all $u, v, w \in \mathfrak{X}^1$. The symbol $\overline{g}$ denotes the naturally induced inner product on $(1,1)$-tensors; in coordinates, if R, S are $(1,1)$-tensors then $\overline{g}(R, S) = g_{ik}g^{j\ell}R^i_j S^k_\ell = \mathrm{Tr}(R^t \cdot S)$. This inner product induces by right translations a right invariant weak Riemannian metric on $\mathcal{D}^s_{\mu,mix}$ given by

$$\mathcal{G}^1(\eta)(u_\eta, v_\eta) := \langle u_\eta \circ \eta^{-1}, v_\eta \circ \eta^{-1}\rangle_1 \tag{2.3}$$

for $u_\eta, v_\eta \in T_\eta \mathcal{D}^s_{\mu,mix}$.

We shall use throughout the paper the index lowering and raising operators $\flat : \mathfrak{X} \to \Omega^1$ and $\sharp := \flat^{-1} : \Omega^1 \to \mathfrak{X}$ induced by the metric g, that is, $u^\flat := g(u, \cdot)$ for any $u \in \mathfrak{X}$. Our conventions for the curvature and the Ricci tensor and operator are

$$\mathrm{R}(u, v) := \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}$$
$$\overline{\mathrm{R}}(u, v, w, z) := g(\mathrm{R}(u, v)w, z)$$
$$\mathrm{Ricci}(u, v) := \mathrm{Tr}(w \longmapsto \mathrm{R}(w, u)v)$$
$$g(\mathrm{Ric}(u), v) := \mathrm{Ricci}(u, v)$$

Let δ be the codifferential associated to g. We denote by

$$\Delta u = -[(d\delta + \delta d)u^\flat]^\sharp$$

the usual ***Hodge Laplacian*** on vector fields and let

$$\Delta_r := \Delta + 2\,\mathrm{Ric}$$

be the ***Ricci Laplacian***. We shall also need the operator

$$\mathcal{L} := \Delta_r + \operatorname{grad}\operatorname{div}.$$

wich appears in the following formula ([**20**])

$$\langle u, v\rangle_1 = \langle (1 - \alpha^2\mathcal{L})u, v\rangle_0 \text{ for all } u, v \in \mathcal{V}^2_{mix} \tag{2.4}$$

that will be used many times in this paper. For completeness we shall provide below a complete proof. Denote by $\mathfrak{X}^{C^2}(U)$ the C^2 vector fields on an open subset U of M. We begin with the following.

LEMMA 2.1. (*Weitzenböck formula*) *Let $\{e_i \mid i=1,\dots n\}$ be a local orthonormal frame on an open subset U of M. Then on $\mathfrak{X}^{C^2}(U)$ the following identity holds:*

$$\Delta = \nabla^2_{e_i,e_i} - \mathrm{Ric} \tag{2.5}$$

where $\nabla^2_{e_i,e_i} := \nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}$ is the second covariant derivative. In particular we remark that $\nabla^2_{e_i,e_i}$ does not depend on the local orthonormal frame and so can be defined globally on M.

Proof : We will use the formula $\delta\alpha = -\mathbf{i}_{e_i}(\nabla_{e_i}\alpha)$ where $\{e_i\}$ is a local orthonormal frame on an open subset U of M and α is a k-form (see [**18**]). We also need the identities $d\alpha(u,v) = (\nabla_u\alpha)(v) - (\nabla_v\alpha)(u)$ where α is a one-form and $\nabla_u v^\flat = (\nabla_u v)^\flat$ for any vector fields u, v on M. Let $u \in \mathfrak{X}^{C^2}(U)$ and recall that $\delta u^\flat = -\operatorname{div}(u)$. On U we have :

$$d(\delta u^\flat)(v) = -d(\operatorname{div}(u))(v) = -d(g(\nabla_{e_i}u, e_i))(v) = -g(\nabla_v\nabla_{e_i}u, e_i) - g(\nabla_{e_i}u, \nabla_v e_i).$$

We also have:

$$\begin{aligned}
\delta(du^\flat)(v) &= -\mathbf{i}_{e_i}(\nabla_{e_i}(du^\flat))(v) = -\nabla_{e_i}(du^\flat)(e_i,v)\\
&= -\nabla_{e_i}(du^\flat(e_i,v)) + du^\flat(\nabla_{e_i}e_i, v) + du^\flat(e_i, \nabla_{e_i}v)\\
&= -\nabla_{e_i}\left(\nabla_{e_i}u^\flat(v) - \nabla_v u^\flat(e_i)\right) + \nabla_{\nabla_{e_i}e_i}u^\flat(v) - \nabla_v u^\flat(\nabla_{e_i}e_i)\\
&\quad + \nabla_{e_i}u^\flat(\nabla_{e_i}v) - \nabla_{\nabla_{e_i}v}u^\flat(e_i)\\
&= -\nabla_{e_i}\left(g(\nabla_{e_i}u, v)\right) + \nabla_{e_i}\left(g(\nabla_v u, e_i)\right) + g\left(\nabla_{\nabla_{e_i}e_i}u, v\right) - g\left(\nabla_v u, \nabla_{e_i}e_i\right)\\
&\quad + g\left(\nabla_{e_i}u, \nabla_{e_i}v\right) - g\left(\nabla_{\nabla_{e_i}v}u, e_i\right)\\
&= -g(\nabla_{e_i}\nabla_{e_i}u, v) - g(\nabla_{e_i}u, \nabla_{e_i}v) + g(\nabla_{e_i}\nabla_v u, e_i) + g(\nabla_v u, \nabla_{e_i}e_i)\\
&\quad + g\left(\nabla_{\nabla_{e_i}e_i}u, v\right) - g\left(\nabla_v u, \nabla_{e_i}e_i\right) + g\left(\nabla_{e_i}u, \nabla_{e_i}v\right) - g\left(\nabla_{\nabla_{e_i}v}u, e_i\right)\\
&= -g\left(\nabla^2_{e_i,e_i}u, v\right) + g\left(\nabla_{e_i}\nabla_v u, e_i\right) - g\left(\nabla_{\nabla_{e_i}v}u, e_i\right).
\end{aligned}$$

Using the formula for the curvature R and the Ricci curvature we obtain

$$\begin{aligned}
-(d\delta + \delta d)u^\flat(v) &= g(\nabla^2_{e_i,e_i}u, v) - g(\mathrm{R}(e_i,v)u, e_i) + g(\nabla_{e_i}u, \nabla_v e_i) + g(\nabla_{\nabla_v e_i}u, e_i)\\
&= g(\nabla^2_{e_i,e_i}u, v) - \operatorname{Ricci}(u,v) + 0\\
&= g(\nabla^2_{e_i,e_i}u - \operatorname{Ric}(u), v).
\end{aligned}$$

The fact that $g(\nabla_{e_i}u, \nabla_v e_i) + g(\nabla_{\nabla_v e_i}u, e_i) = 0$ can be simply proved pointwise at $x \in M$, assuming $\nabla e_i(x) = 0$. (See [**18**] p.176/7 for a proof for a general local orthonormal frame). ■

LEMMA 2.2. *For all $u, v \in \mathfrak{X}^{C^2}(M)$ we have*

$$\operatorname{div}(\nabla_v u) = \operatorname{Tr}(\nabla u \cdot \nabla v) + \operatorname{Ricci}(u,v) + g(\operatorname{grad}\operatorname{div}(u), v)$$

Proof : We shall prove the identity at a fixed point $x \in M$ so we can choose a local orthonormal frame $\{e_i\}$ such that $\nabla e_i(x) = 0$. For the (1,1) tensor ∇u we shall use the notation $\nabla u(v) := \nabla_v u$. At x we have :

$$\begin{aligned}
\operatorname{Tr}(\nabla u \cdot \nabla v) + \operatorname{Ricci}(u,v) &= g\left(\nabla u\left(\nabla v(e_i)\right), e_i\right) + g\left(\mathrm{R}(e_i,v)u, e_i\right) \\
&= g\left(\nabla_{\nabla_{e_i} v} u, e_i\right) + g\left(\nabla_{e_i}\nabla_v u, e_i\right) - g\left(\nabla_v \nabla_{e_i} u, e_i\right) - g\left(\nabla_{[e_i,v]} u, e_i\right) \\
&= g(\nabla_{e_i}\nabla_v u, e_i) - g(\nabla_v\nabla_{e_i} u, e_i) \quad \text{because} \quad \nabla_{e_i} v = [e_i, v] \text{ at } x \\
&= g(\nabla_{e_i}\nabla_v u, e_i) - \nabla_v(g(\nabla_{e_i} u, e_i)) + g(\nabla_{e_i} u, \nabla_v e_i) \\
&= \operatorname{div}(\nabla_v u) - d(\operatorname{div}(u))(v) + 0 \\
&= \operatorname{div}(\nabla_v u) - g(\operatorname{grad}\operatorname{div}(u), v)
\end{aligned}$$

We can do that at each x so the identity is proved. ■

We shall denote below by $\Gamma^{L^2}(L(TM,TM))$ the L^2 sections of the vector bundle $L(TM,TM) \mapsto M$.

LEMMA 2.3. *Consider on* $\Gamma^{L^2}(L(TM,TM))$ *the* L^2 *inner product*

$$(R,S)_0 := \int_M \overline{g}(R,S)\mu.$$

Then the following identities hold:
(1) *For all* $u, v \in \mathfrak{X}^{C^2}(M)$ *:*

$$(\nabla u, \nabla v)_0 = \int_{\partial M} g(\nabla_n u, v)\mu_\partial - \langle(\Delta + \operatorname{Ric})(u), v\rangle_0$$

$$(\nabla u, (\nabla v)^t)_0 = \int_{\partial M} g(\nabla_v u, n)\mu_\partial - \langle(\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v\rangle_0.$$

(2) *For all* $u, v \in \mathfrak{X}^{C^2}_{\parallel}(M)$ *:*

$$-2(\operatorname{Def}(u), \operatorname{Def}(v))_0 = \langle\mathcal{L}(u), v\rangle_0 - \int_{\partial M} g\left((\nabla_n u)^{tan} + S_n(u), v\right)\mu_\partial.$$

Here n *denotes the outward-pointing unit normal vector field along the boundary* ∂M. *We let* $S_n : T\partial M \to T\partial M$ *be Weingarten map defined by* $S_n(u) := -\nabla_u n$. *The symbol* $(.)^{tan}$ *denotes the tangential part to the boundary of a vector in* $TM|\partial M$.

Proof : (1) Let $\{e_i\}$ be a local orthonormal frame on an open subset U of M. Recall the formula $\operatorname{div}(fu) = f\operatorname{div}(u) + df(u)$. On U we have :

$$\begin{aligned}
\overline{g}(\nabla u, \nabla v) &= \operatorname{Tr}((\nabla u)^t \cdot \nabla v) = g(\nabla_{e_i} u, \nabla_{e_i} v) \\
&= d(g(\nabla_{e_i} u, v))(e_i) - g(\nabla_{e_i}\nabla_{e_i} u, v) \\
&= \operatorname{div}(g(\nabla_{e_i} u, v)e_i) - g(\nabla_{e_i} u, v)\operatorname{div}(e_i) - g(\nabla_{e_i}\nabla_{e_i} u, v).
\end{aligned}$$

Using the relation $\nabla_{e_j} e_j = \sum_i g(\nabla_{e_j} e_j, e_i)e_i$ in the third equality below, we get

$$g(\nabla_{e_i} u, v)\operatorname{div}(e_i) = g(\nabla_{e_i} u, v)g(\nabla_{e_j} e_i, e_j) = -g(\nabla_{e_i} u, v)g(e_i, \nabla_{e_j} e_j) = g(\nabla_{\nabla_{e_j} e_j} u, v)$$

and hence we conclude

$$\begin{aligned}
\overline{g}(\nabla u, \nabla v) &= \operatorname{div}(g(\nabla_{e_i} u, v)e_i) - g(\nabla^2_{e_i,e_i} u, v) \\
&= \operatorname{div}(g(\nabla_{e_i} u, v)e_i) - g((\Delta + \operatorname{Ric})(u), v) \qquad (2.6)
\end{aligned}$$

because of formula (2.5). We remark that the vector field $g(\nabla_{e_i}u, v)e_i$ does not depend on the choice of the local orthonormal frame, so it defines a vector field on M. Denote by w this vector field. We obtain from (2.6) using Stokes' theorem:

$$\begin{aligned}(\nabla u, \nabla v)_0 &= \int_M \overline{g}(\nabla u, \nabla v) = \int_M \operatorname{div}(w)\mu - \int_M g((\Delta + \operatorname{Ric})u, v)\mu \\ &= \int_{\partial M} g(w, n)\mu_\partial - \langle(\Delta + \operatorname{Ric})(u), v\rangle_0.\end{aligned}$$

On U we have $g(w, n) = g(g(\nabla_{e_i}u, v)e_i, n) = g(\nabla_n u, v)$. So the first identity is proved.

We proceed similarly with the proof of the second identity. We have:

$$\begin{aligned}\overline{g}(\nabla u, (\nabla v)^t) &= \operatorname{Tr}(\nabla v \cdot \nabla u) = g(e_i, \nabla_{\nabla_{e_i}u} v) \\ &= d(g(e_i, v))(\nabla_{e_i}u) - g(\nabla_{\nabla_{e_i}u} e_i, v) \\ &= \operatorname{div}(g(e_i, v)\nabla_{e_i}u) - g(e_i, v)\operatorname{div}(\nabla_{e_i}u) - g(\nabla_{\nabla_{e_i}u} e_i, v).\end{aligned}$$

Using the formula $\operatorname{div}(\nabla_{e_i}u) = \operatorname{Tr}(\nabla u \cdot \nabla e_i) + \operatorname{Ricci}(u, e_i) + g(\operatorname{grad}\operatorname{div}(u), e_i)$ proved in Lemma 2.2, we obtain

$$\begin{aligned}g(e_i, v)\operatorname{div}(\nabla_{e_i}u) &= g(e_i, v)\operatorname{Tr}(\nabla u \cdot \nabla e_i) + g(e_i, v)g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), e_i)) \\ &= g(e_i, v)\operatorname{Tr}(\nabla e_i \cdot \nabla u) + g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v) \\ &= g(e_i, v)g(\nabla_{\nabla_{e_j}u} e_i, e_j) + g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v) \\ &= -g(e_i, v)g(e_i, \nabla_{\nabla_{e_j}u} e_j) + g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v) \\ &= -g(v, \nabla_{\nabla_{e_j}u} e_j) + g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v).\end{aligned}$$

Thus $\overline{g}(\nabla u, (\nabla v)^t) = \operatorname{div}(g(e_i, v)\nabla_{e_i}u) - g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v)$. As before, the vector field $w := g(e_i, v)\nabla_{e_i}u$ does not depend on the choice of the local orthonormal frame. We obtain :

$$\begin{aligned}(\nabla u, (\nabla v)^t)_0 &= \int_M \overline{g}(\nabla u, (\nabla v)^t) = \int_M \operatorname{div}(w)\mu - \int_M g((\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v)\mu \\ &= \int_{\partial M} g(w, n)\mu_\partial - \langle(\operatorname{Ric} + \operatorname{grad}\operatorname{div})(u), v\rangle_0\end{aligned}$$

by Stokes' theorem. On U we have $g(w, n) = g(g(e_i, v)\nabla_{e_i}u, n) = g(\nabla_v u, n)$. So the second identity is proved.

(2) Using the two formulas in part (1) and the defintions

$$\operatorname{Def} u = \frac{\nabla u + (\nabla u)^t}{2} \quad \text{and} \quad \mathcal{L} = \Delta + 2\operatorname{Ric} + \operatorname{grad}\operatorname{div}$$

a direct computation gives

$$-2(\operatorname{Def} u, \operatorname{Def} v)_0 = \langle\mathcal{L}(u), v\rangle - \int_{\partial M} g(\nabla_n u, v)\mu_\partial - \int_{\partial M} g(\nabla_v u, n)\mu_\partial.$$

If u, v are tangent to the boundary, then on ∂M we get the relations $g(\nabla_n u, v) = g((\nabla_n u)^{tan}, v)$ and $g(\nabla_v u, n) = d(g(u, n))(v) - g(u, \nabla_v n) = 0 + g(u, S_n(v)) = g(S_n(u), v)$. ■

Now we shall prove the following useful Lemma.

LEMMA 2.4. (1) *For* $r \geq 1$, $\mathcal{L} : \mathfrak{X}^r \to \mathfrak{X}^{r-2}$ *is a continuous linear map.*
(2) *For all* $u, v \in \mathcal{V}^r_{mix}$ *with* $r \geq 2$ *we have*

$$\langle u, v\rangle_1 = \langle (1-\alpha^2\mathcal{L})u, v\rangle_0.$$

Proof : The first part is a direct verification. To prove the second we use the preceding Lemma to obtain $\langle u, v\rangle_1 = \langle (1-\alpha^2\mathcal{L})u, v\rangle_0$ for all $u, v \in \mathcal{V}^{C^2}_{mix}$. By the Sobolev embedding theorem, the identity holds for all $u, v \in \mathcal{V}^s_{mix}, s > \frac{1}{2} dim M + 2$. Using the fact that $\mathcal{V}^s_{mix}$ is dense in $\mathcal{V}^2_{mix}$ with the H^2 topology, and the fact that $\langle\ ,\ \rangle_0$,$\langle\ ,\ \rangle_1$, and $\mathcal{L}$ are continuous on $\mathfrak{X}^2$, the identity holds for vector fields in $\mathcal{V}^2_{mix}$. ■

Using the previous lemma and solving a boundary value problem we can prove (see [**20**]) that for $r \geq 1$ the linear map

$$(1-\alpha^2\mathcal{L}) : \mathcal{V}^r_{mix} \longrightarrow \mathfrak{X}^{r-2}$$

is a continuous isomorphism with inverse

$$(1-\alpha^2\mathcal{L})^{-1} : \mathfrak{X}^{r-2} \longrightarrow \mathcal{V}^r_{mix}.$$

We recall from [**20**] the two principal results concerning the geometry of the Lagrangian averaged Euler equation (LAE-α).

THEOREM 2.5. *(Stokes decomposition) For* $r \geq 1$ *we have the following* $\langle\ ,\ \rangle_1-$ *orthogonal decomposition:*

$$\mathcal{V}^r_{mix} = \mathcal{V}^r_{\mu,mix} \oplus (1-\alpha^2\mathcal{L})^{-1}\,\mathrm{grad}\, H^{r-1}(M)$$

We denote by $\mathcal{P}_e : \mathcal{V}^r_{mix} \longrightarrow \mathcal{V}^r_{\mu,mix}$ *the projection onto the first factor (Stokes projector).*
Then

$$\overline{\mathcal{P}} : T\mathcal{D}^s_{mix}|\mathcal{D}^s_{\mu,mix} \longrightarrow T\mathcal{D}^s_{\mu,mix}$$

defined by $\overline{\mathcal{P}}(u_\eta) := [\mathcal{P}_e(u_\eta \circ \eta^{-1})] \circ \eta$, *is a* C^∞ *bundle map.*

THEOREM 2.6. *Let* $\eta(t) \in \mathcal{D}^s_{\mu,D}$ *be a curve in* $\mathcal{D}^s_{\mu,D}$ *and let* $u(t) := TR_{\eta(t)^{-1}}(\dot\eta(t)) = \dot\eta(t) \circ \eta(t)^{-1} \in \mathcal{V}^s_{\mu,D}$. *Then the following properties are equivalent :*
(1) $\eta(t)$ *is a geodesic of* $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$
(2) $u(t)$ *is a solution of LAE-α :*

$$(1-\alpha^2\Delta_r)\partial_t u(t) + \nabla_{u(t)}[(1-\alpha^2\Delta_r)u(t)] - \alpha^2\nabla u(t)^t \cdot \Delta_r u(t) = -\,\mathrm{grad}\, p(t)$$

(3) $u(t)$ *is a solution of :*

$$\partial_t u(t) + \mathcal{P}_e\left(\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\right) = 0 \tag{2.7}$$

where $\mathcal{F}^\alpha := \mathcal{U}^\alpha + \mathcal{R}^\alpha : \mathcal{V}^s_{\mu,D} \longrightarrow \mathcal{V}^s_D$ *with :*

$$\mathcal{U}^\alpha(u) := (1-\alpha^2\mathcal{L})^{-1}\alpha^2\,\mathrm{Div}(\nabla u\cdot\nabla u^t + \nabla u\cdot\nabla u - \nabla u^t\cdot\nabla u) \tag{2.8}$$

$$\begin{aligned}\mathcal{R}^\alpha(u) := (1-\alpha^2\mathcal{L})^{-1}\alpha^2\Big(&\,\mathrm{Tr}\left(\nabla.(\mathrm{R}(\cdot,u)u) + \mathrm{R}(\cdot,u)\nabla.u + \mathrm{R}(u,\nabla.u)\,\cdot\right)\\ &-(\nabla_u\,\mathrm{Ric})u - \nabla u^t\cdot\mathrm{Ric}(u)\Big)\end{aligned} \tag{2.9}$$

(4) $V(t) := \dot\eta(t)$ *(Lagrangian velocity) is a solution of :*

$$\dot V(t) = \mathcal{S}^1(V(t))$$

where $\mathcal{S}^1 \in \mathfrak{X}^{C^\infty}(T\mathcal{D}^s_{\mu,D})$ *is the geodesic spray of* $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$.

In part (3), Div denotes the divergence of a $(1,1)$-tensor :

$$\mathrm{Div}(S) := (\nabla_{e_i} S)(e_i)$$

for $\{e_i\}$ a local orthonormal frame. In the last section we will generalise the previous theorem to the case of Neumann and mixed boundary conditions.

3. Geodesic spray and connector of $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$

In this section we shall give the formula of the geodesic spray $\mathcal{S}^1$ and the connector K^1 of the weak Riemannian metric $\mathcal{G}^1$ on $\mathcal{D}^s_{\mu,D}$. Recall that the geodesic spray is the Lagrangian vector field on $T\mathcal{D}^s_{\mu,D}$ associated to the Lagrangian $L : T\mathcal{D}^s_{\mu,D} \longrightarrow \mathbb{R}$ given by $L(u_\eta) = \frac{1}{2}\mathcal{G}^1(\eta)(u_\eta, u_\eta)$, that is

$$\mathbf{i}_{\mathcal{S}^1}\, \Omega_L = dL$$

where Ω_L is the weak symplectic form associated to L, that is, the pull back by the Legendre transformation defined by L of the canonical weak symplectic form on $T^*\mathcal{D}^s_{\mu,D}$ (see, e.g. [**13**]). So the integral curves of the geodesic spray are $V(t) = \dot{\eta}(t)$ where $\eta(t)$ is a geodesic of $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$. Using that $u(t) := \dot{\eta}(t) \circ \eta(t)^{-1}$ is a solution of (2.7) we will prove the following lemma.

LEMMA 3.1. *The geodesic spray of* $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$ *is given by :*

$$\mathcal{S}^1(u_\eta) = T\overline{\mathcal{P}}\big(S \circ u_\eta - \mathrm{Ver}_{u_\eta}(\overline{\mathcal{F}}^\alpha(u_\eta))\big)$$

where $\overline{\mathcal{F}}^\alpha(u_\eta) := \mathcal{F}^\alpha(u_\eta \circ \eta^{-1}) \circ \eta$ *and* S *is the geodesic spray of* (M, g) *and* $\mathrm{Ver}_{u_\eta}(v_\eta) \in T_{u_\eta}(T\mathcal{D}^s_{\mu,D})$ *is the vertical lift of* $v_\eta \in T_\eta \mathcal{D}^s_{\mu,D}$ *at* $u_\eta \in T_\eta\mathcal{D}^s_{\mu,D}$*, that is,*

$$\mathrm{Ver}_{u_\eta}(v_\eta) = \left.\frac{d}{dt}\right|_{t=0} (u_\eta + t v_\eta).$$

Proof : Let $\eta(t)$ be a goedesic of $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$. Then $u(t) := \dot{\eta}(t) \circ \eta(t)^{-1}$ is a solution of

$$\partial_t u(t) + \mathcal{P}_e(\nabla_{u(t)} u(t) + \mathcal{F}^\alpha(u(t))) = 0.$$

We have $V(t) = \dot{\eta}(t) = u(t) \circ \eta(t)$. In the following computation we denote by $\dot{u}(t)$ the t-derivative of $u(t)$ thought of as a curve in $T\mathcal{D}^s_{\mu,D}$. However, $u(t) \in \mathcal{V}^s_{\mu,D}$ for all t and therefore, one can take the derivative $\partial_t u(t)$ of $u(t)$ as a curve in the Hilbert space $\mathcal{V}^s_{\mu,D}$. The relation between these two derivatives is $\dot{u}(t) = \mathrm{Ver}_{u(t)}(\partial_t u(t))$ using the standard identification between a vector space and its tangent space at a point. Differentiating $V(t)$ and using the preceding equation we obtain

$$\begin{aligned}
\dot{V}(t) &= T(u(t)) \circ \dot{\eta}(t) + \dot{u}(t) \circ \eta(t) \\
&= T(u(t)) \circ \dot{\eta}(t) + \mathrm{Ver}_{u(t)}(\partial_t u(t)) \circ \eta(t) \\
&= T(u(t)) \circ \dot{\eta}(t) - \mathrm{Ver}_{u(t)}(\mathcal{P}_e(\nabla_{u(t)} u(t) + \mathcal{F}^\alpha(u(t)))) \circ \eta(t).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathcal{S}^1(u_\eta) &= T(u_\eta \circ \eta^{-1}) \circ u_\eta - \mathrm{Ver}_{u_\eta \circ \eta^{-1}}\left(\mathcal{P}_e(\nabla_{u_\eta \circ \eta^{-1}}(u_\eta \circ \eta^{-1}) + \mathcal{F}^\alpha(u_\eta \circ \eta^{-1}))\right) \circ \eta \\
&= Tu \circ u \circ \eta - \mathrm{Ver}_u\left(\mathcal{P}_e(\nabla_u u + \mathcal{F}^\alpha(u))\right) \circ \eta \quad \text{where } u := u_\eta \circ \eta^{-1} \in \mathcal{V}^s_{\mu,D}.
\end{aligned}$$

Now it suffices to prove that for all $u \in \mathcal{V}^s_{\mu,D}$ we have :

(1) $T\overline{\mathcal{P}}(Tu \circ u) = Tu \circ u$ and
(2) $T\overline{\mathcal{P}}(Ver_u(\nabla_u u + \mathcal{F}^\alpha(u))) = Ver_u(\mathcal{P}_e(\nabla_u u + \mathcal{F}^\alpha(u)))$.

(1) Let $c(t)$ be a curve in $\mathcal{D}^s_{\mu,D}$ such that $c(0) = id_M$ and $\dot{c}(0) = u$. Let $d(t) := u \circ c(t)$. Then we have $d(0) = u$ and $\dot{d}(0) = Tu \circ u$. We get

$$\begin{aligned} T\overline{\mathcal{P}}(Tu \circ u) &= \left.\frac{d}{dt}\right|_{t=0} \overline{\mathcal{P}}(u \circ c(t)) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{P}_e(u) \circ c(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} u \circ c(t) = Tu \circ u. \end{aligned}$$

(2) Let $v := \nabla_u u + \mathcal{F}^\alpha(u) \in \mathcal{V}^{s-1}_D$. We get

$$\begin{aligned} T\overline{\mathcal{P}}(\mathrm{Ver}_u(v)) &= T\overline{\mathcal{P}}\Big(\left.\frac{d}{dt}\right|_{t=0} (u + tv)\Big) = \left.\frac{d}{dt}\right|_{t=0} \overline{\mathcal{P}}(u + tv) \\ &= \left.\frac{d}{dt}\right|_{t=0} (u + t\mathcal{P}_e(v)) = \mathrm{Ver}_u(\mathcal{P}_e(v)). \end{aligned}$$

So, using right-invariance of $T\overline{\mathcal{P}}$ in the second equality below and the expression of the spray S on (M, g), namely $S \circ u = Tu \circ u - \mathrm{Ver}_u(\nabla_u u)$, we obtain

$$\begin{aligned} \mathcal{S}^1(u_\eta) &= T\overline{\mathcal{P}}\big(Tu \circ u - \mathrm{Ver}_u(\nabla_u u + \mathcal{F}^\alpha(u))\big) \circ \eta \\ &= T\overline{\mathcal{P}}\big[(Tu \circ u - \mathrm{Ver}_u(\nabla_u u)) \circ \eta - \mathrm{Ver}_u(\mathcal{F}^\alpha(u)) \circ \eta\big] \\ &= T\overline{\mathcal{P}}\big(S \circ u \circ \eta - \mathrm{Ver}_{u_\eta}(\mathcal{F}^\alpha(u) \circ \eta)\big) \\ &= T\overline{\mathcal{P}}\big(S \circ u_\eta - \mathrm{Ver}_{u_\eta}(\overline{\mathcal{F}}^\alpha(u_\eta))\big). \ \blacksquare \end{aligned}$$

Recall that locally the expressions of the geodesic spray and the connector of a Riemannian manifold are given by

$$\mathcal{S}^1(\eta, u) = (\eta, u, u, -\Gamma^1(\eta)(u, u))$$

and

$$K^1(\eta, u, v, w) = (\eta, w + \Gamma^1(\eta)(u, v))$$

where the symetric bilinear map $\Gamma^1(\eta)$ is the Christoffel map of the Riemannian metric. Using these formula and the previous Lemma we obtain the global expression of K^1 below.

Lemma 3.2. *The connector $K^1 : TT\mathcal{D}^s_{\mu,D} \longrightarrow T\mathcal{D}^s_{\mu,D}$ of $(\mathcal{D}^s_{\mu,D}, \mathcal{G}^1)$ is given by :*

$$K^1(X_{u_\eta}) = \overline{\mathcal{P}}\Big(K \circ X_{u_\eta} + \overline{\mathfrak{F}}^\alpha\big(\pi_{T\mathcal{D}^s_{\mu,D}}(X_{u_\eta}), T\pi_{\mathcal{D}^s_{\mu,D}}(X_{u_\eta})\big)\Big),$$

where

$$\overline{\mathfrak{F}}^\alpha(u_\eta, v_\eta) := \frac{1}{2}\Big(\overline{\mathcal{F}}^\alpha(u_\eta + v_\eta) - \overline{\mathcal{F}}^\alpha(u_\eta) - \overline{\mathcal{F}}^\alpha(v_\eta)\Big),$$

$\pi_{\mathcal{D}^s_{\mu,D}} : T\mathcal{D}^s_{\mu,D} \to \mathcal{D}^s_{\mu,D}$ and $\pi_{T\mathcal{D}^s_{\mu,D}} : TT\mathcal{D}^s_{\mu,D} \to T\mathcal{D}^s_{\mu,D}$ are tangent bundle projections, and $K : TTM \longrightarrow TM$ is the connector of (M, g).

Proof : Let $\eta \in \mathcal{D}^s_{\mu,D}$, $u_\eta, v_\eta \in T_\eta\mathcal{D}^s_{\mu,D}$, and $w_\eta \in T_\eta\mathcal{D}_D$. We write $\mathcal{S}^0(u_\eta) := S \circ u_\eta$ (in case M has no boundary, $\mathcal{S}^0$ is the geodesic spray of $(\mathcal{D}^s, \mathcal{G}^0)$).

In local representation we have (with $(\eta, u), (\eta, v), (\eta, w)$ the local expressions of u_η, v_η, w_η):

$\mathcal{S}^i(\eta, u) = (\eta, u, u, -\Gamma^i(\eta)(u, u)), i = 1, 2$ where Γ^i are the Christoffel maps,

$\overline{\mathcal{F}}^\alpha(\eta, u) = (\eta, \overline{\mathcal{F}}^\alpha_{loc}(\eta, u))$ and $\overline{\mathfrak{F}}^\alpha((\eta, u), (\eta, v)) = (\eta, \overline{\mathfrak{F}}^\alpha_{loc}(\eta)(u, v))$,

$\mathrm{Ver}_{(\eta,u)}(\overline{\mathcal{F}}^\alpha(\eta, u)) = (\eta, u, 0, \overline{\mathcal{F}}^\alpha_{loc}(\eta, u))$,

$$\overline{\mathcal{P}}(\eta,u)=(\eta,\overline{\mathcal{P}}_{loc}(\eta,u)),$$
$$T\overline{\mathcal{P}}(\eta,u,v,w)=(\eta,\overline{\mathcal{P}}_{loc}(\eta,u),v,D\overline{\mathcal{P}}_{loc}(\eta,u)(v,w))=(\eta,u,v,\overline{\mathcal{P}}_{loc}(\eta,w)).$$

Thus we find

$$\begin{aligned}\mathcal{S}^1(\eta,u) &= T\overline{\mathcal{P}}(\mathcal{S}^0(\eta,u)-\mathrm{Ver}_{(\eta,u)}(\overline{\mathcal{F}}^{\alpha}(\eta,u)))\text{ by Lemma 3.1}\\ &= T\overline{\mathcal{P}}(\eta,u,u,-\Gamma^0(\eta)(u,u)-\overline{\mathcal{F}}^{\alpha}_{loc}(\eta,u))\\ &= \left(\eta,u,u,-\overline{\mathcal{P}}_{loc}\left(\Gamma^0(\eta)(u,u)+\overline{\mathcal{F}}^{\alpha}_{loc}(\eta,u)\right)\right).\end{aligned}$$

We deduce that $\Gamma^1(\eta)(u,u)=\overline{\mathcal{P}}_{loc}\left(\Gamma^0(\eta)(u,u)+\overline{\mathcal{F}}^{\alpha}_{loc}(\eta,u)\right)$ and then that

$$\Gamma^1(\eta)(u,v)=\overline{\mathcal{P}}_{loc}(\Gamma^0(\eta)(u,v)+\overline{\mathfrak{F}}^{\alpha}_{loc}(\eta)(u,v)).$$

Thus, with $u_\eta,v_\eta,w_\eta\in T_\eta\mathcal{D}^s_{\mu,D}$, we obtain

$$\begin{aligned}K^1(\eta,u,v,w) &= (\eta,w+\Gamma^1(\eta)(u,v))\\ &= \left(\eta,w+\overline{\mathcal{P}}_{loc}\left(\Gamma^0(\eta)(u,v)+\overline{\mathfrak{F}}^{\alpha}_{loc}(\eta)(u,v)\right)\right)\\ &= \overline{\mathcal{P}}\left((\eta,w+\Gamma^0(\eta)(u,v))+(\eta,\overline{\mathfrak{F}}^{\alpha}_{loc}(\eta)(u,v))\right)\\ &= \overline{\mathcal{P}}\left(K^0(\eta,u,v,w)+\overline{\mathfrak{F}}^{\alpha}((\eta,u),(\eta,v))\right),\end{aligned}$$

where $K^0(X_{u_\eta}):=K\circ X_{u_\eta}$ with $K:TTM\longrightarrow TM$ the connector of (M,g). A globalisation of the previous formula gives the result :

$$K^1(X_{u_\eta})=\overline{\mathcal{P}}\Big(K\circ X_{u_\eta}+\overline{\mathfrak{F}}^{\alpha}\big(\pi_{T\mathcal{D}^s_{\mu,D}}(X_{u_\eta}),T\pi_{\mathcal{D}^s_{\mu,D}}(X_{u_\eta})\big)\Big).\ \blacksquare$$

4. The Lie-Poisson structure of LAE-α equation

In this section we shall define a Lie-Poisson bracket on a certain class of functions on $\mathcal{V}^r_{\mu,D}$, if $r\geq s>\frac{1}{2}\dim M+1$ and shall specify precise sharp conditions on their smoothness class. In particular, we shall also determine the conditions under which the Jacobi identity holds.

Let $s>\frac{1}{2}\dim M+1$. Because of the existence of the geodesic spray $\mathcal{S}^1$ of the weak Riemannian Hilbert manifold $(\mathcal{D}^s_{\mu,D},\mathcal{G}^1)$ and the fact that the inclusion $\flat: T_\eta\mathcal{D}^s_{\mu,D}\longrightarrow T^*_\eta\mathcal{D}^s_{\mu,D}$ is dense, we can use the results of section 4 in [**23**]. Therefore, by those results, $T\mathcal{D}^s_{\mu,D}$ carries a Poisson structure in the precise sense given there. To give it explicitly in our case for the metric $\mathcal{G}^1$ we need a few preliminaries.

If $F:T\mathcal{D}^s_{\mu,D}\to\mathbb{R}$ is of class C^1 we define the ***horizontal derivative*** of F by

$$\frac{\partial F}{\partial\eta}:T\mathcal{D}^s_{\mu,D}\to T^*\mathcal{D}^s_{\mu,D}$$

by

$$\left\langle\frac{\partial F}{\partial\eta}(u_\eta),v_\eta\right\rangle:=\left.\frac{d}{dt}\right|_{t=0}F(\gamma(t)),$$

where $\langle\,,\rangle$ is the duality paring and $\gamma(t)\subset T\mathcal{D}^s_{\mu,D}$ is a smooth path defined in a neighborhood of zero, with base point denoted by $\eta(t)\subset\mathcal{D}^s_{\mu,D}$, satisfying the following conditions:

- $\gamma(0)=u_\eta$
- $\dot\eta(0)=v_\eta$

- γ is parallel, that is, its covariant derivative of the $\mathcal{G}^1$ Levi-Civita connection vanishes.

The ***vertical derivative***

$$\frac{\partial F}{\partial u} : T\mathcal{D}^s_{\mu,D} \to T^*\mathcal{D}^s_{\mu,D}$$

of F is defined as the usual fiber derivative, that is,

$$\left\langle \frac{\partial F}{\partial u}(u_\eta), v_\eta \right\rangle := \left.\frac{d}{dt}\right|_{t=0} F(u_\eta + tv_\eta).$$

These derivatives naturally induce corresponding functional derivatives relative to the weak Riemannian metric $\mathcal{G}^1$. The ***horizontal*** and ***vertical functional derivatives***

$$\frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u} : T\mathcal{D}^s_{\mu,D} \to T\mathcal{D}^s_{\mu,D}$$

are defined by the equalities

$$\mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta \eta}(u_\eta), v_\eta\right) = \left\langle \frac{\partial F}{\partial \eta}(u_\eta), v_\eta \right\rangle \quad \text{and} \quad \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta u}(u_\eta), v_\eta\right) = \left\langle \frac{\partial F}{\partial u}(u_\eta), v_\eta \right\rangle$$

for any $u_\eta, v_\eta \in T\mathcal{D}^s_{\mu,D}$. Note that due to the weak character of $\mathcal{G}^1$, the existence of the fucntional derivatives is not guaranteed. But if they exist, they are unique.

We define, for $k \geq 1$ and $r, t > \frac{1}{2}\dim M + 1$:

$$C^k_r(T\mathcal{D}^t_{\mu,D}) := \left\{ F \in C^k(T\mathcal{D}^t_{\mu,D}) \middle| \exists\, \frac{\delta F}{\delta \eta}, \frac{\delta F}{\delta u} : T\mathcal{D}^t_{\mu,D} \longrightarrow T\mathcal{D}^r_{\mu,D} \right\}.$$

With these definitions the Poisson bracket of $F, G \in C^k_r(T\mathcal{D}^t_{\mu,D})$ is given by

$$\{F,G\}^1(u_\eta) = \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta \eta}(u_\eta), \frac{\delta G}{\delta u}(u_\eta)\right) - \mathcal{G}^1(\eta)\left(\frac{\delta F}{\delta u}(u_\eta), \frac{\delta G}{\delta \eta}(u_\eta)\right) \tag{4.1}$$

As in the case of Euler equation (see [**23**]) we have the following result.

Proposition 4.1. *Let $\pi_R : T\mathcal{D}^s_{\mu,D} \longrightarrow \mathcal{V}^s_{\mu,D}$ be definied by $\pi_R(u_\eta) := u_\eta \circ \eta^{-1}$. Let F_t be the flow of $\mathcal{S}^1$ and $\widetilde{F}_t := \pi_R \circ F_t$. Then $\widetilde{F}_t$ is the flow of LAE-α equation. Moreover we have the following commutative diagram :*

$$\begin{array}{ccc} T\mathcal{D}^s_{\mu,D} & \xrightarrow{F_t} & T\mathcal{D}^s_{\mu,D} \\ \downarrow{\scriptstyle \pi_R} & & \downarrow{\scriptstyle \pi_R} \\ \mathcal{V}^s_{\mu,D} & \xrightarrow{\widetilde{F}_t} & \mathcal{V}^s_{\mu,D}. \end{array}$$

Proof : Let $u \in \mathcal{V}^s_{\mu,D}$ and $V(t) = F_t(u)$. Then V is an integral curve of $\mathcal{S}^1$ with initial condition u. Note that $\widetilde{F}_t(u) = \pi_R(V(t)) = V(t) \circ \eta(t)^{-1}$, where $\eta(t)$ is the base point of $V(t)$, which by Theorem 2.6 (1) is the geodesic of $\mathcal{S}^1$. Therefore, by Theorem 2.6 (2), $\widetilde{F}_t(u)$ is the integral curve of LAE-α with initial condition u.

We still need to show that $\widetilde{F}_t \circ \pi_R = \pi_R \circ F_t$. Indeed, since $\mathcal{S}^1$ is a right invariant vector field, its flow F_t is right equivariant and we conclude

$$\begin{aligned}(\widetilde{F}_t \circ \pi_R)(u_\eta) &= (\pi_R \circ F_t \circ \pi_R)(u_\eta) = (\pi_R \circ F_t \circ TR_{\eta^{-1}})(u_\eta) \\ &= (\pi_R \circ TR_{\eta^{-1}} \circ F_t)(u_\eta) = (\pi_R \circ F_t)(u_\eta). \ \blacksquare\end{aligned}$$

We shall need later the fact that $\pi_R \in C^k(T\mathcal{D}^{s+k}_{\mu,D}, \mathcal{V}^s_{\mu,D})$ so if $k = 0$ then π_R is only continuous.

Our goal is to first study the Lie-Poisson structure of $\mathcal{V}^s_{\mu,D}$ and secondly to show in what sense the maps $F_t, \pi_R, \tilde{F}_t$ are Poisson maps. We begin with the definition of some function spaces needed later when we introduce the relevant Poisson bracket.

Definition 4.2. Let $s > \frac{1}{2}\dim M + 1$.
(1) For $k, t \geq 1$ and $r \geq s$ define:

$$C^k_{r,t}(\mathcal{V}^s_{\mu,D}) := \{f \in C^k(\mathcal{V}^s_{\mu,D}) | \exists\, \delta f : \mathcal{V}^r_{\mu,D} \longrightarrow \mathcal{V}^t_{\mu,D}\} \quad \text{and} \quad C^k_t(\mathcal{V}^s_{\mu,D}) := C^k_{s,t}(\mathcal{V}^s_{\mu,D})$$

where δf is the ***functional derivative*** of f with respect to the inner product $\langle\ ,\ \rangle_1$:

$$\langle \delta f(u), v\rangle_1 = Df(u)(v), \quad \forall\, u, v \in \mathcal{V}^r_{\mu,D}$$

(2) For $k \geq 0$, $r \geq s$, and $t \geq 1$ define:

$$\mathcal{K}^k_{r,t}(\mathcal{V}^s_{\mu,D}) := \{f \in C^{k+1}_{r,t}(\mathcal{V}^s_{\mu,D}) | \delta f \in C^k(\mathcal{V}^r_{\mu,D}, \mathcal{V}^t_{\mu,D})\} \text{ and } \mathcal{K}^k(\mathcal{V}^s_{\mu,D}) := \mathcal{K}^k_{s,s}(\mathcal{V}^s_{\mu,D}).$$

(3) Let $k \geq 1$, $r \geq s$, and $t > \frac{1}{2}\dim M + 1$. The **Poisson bracket on** $C^k_{r,t}(\mathcal{V}^s_{\mu,D})$ is defined by:

$$\{f, g\}^1_+(u) := \langle u, [\delta g(u), \delta f(u)]\rangle_1, \quad \forall u \in \mathcal{V}^r_{\mu,D}. \tag{4.2}$$

Remark When $t > \frac{1}{2}\dim M + 2$ we have

$$\{f, g\}^1_+(u) = \langle u, [\delta g(u), \delta f(u)]^R_{Lie}\rangle_1$$

where $[\ ,\ ]^R_{Lie}$ is the right-Lie bracket on the "Lie-algebra" of $\mathcal{D}^s_{\mu,D}$. We recognize the classical Lie-Poisson bracket.

Theorems 4.6 and 4.7 will summarize the properties of this Poisson bracket. In the proofs we will use the three following Lemmas.

Lemma 4.3. *Let* $s > \frac{1}{2}\dim M + 1$.
(1) *Let* $u \in \mathfrak{X}^s_{\mathrm{div},\|}$ *and* $v, w \in \mathfrak{X}^s$. *Then:*

$$\langle v, \nabla_u w\rangle_0 = -\langle \nabla_u v, w\rangle_0$$

(2) *Let* $u, v \in \mathcal{V}^s_{\mu,D}$ *and* $w \in \mathcal{V}^s_D$. *Then:*

$$\langle v, \nabla_u w\rangle_1 = -\langle \nabla_u v + \mathcal{D}^\alpha(u, v), w\rangle_1$$

where $\mathcal{D}^\alpha : \mathcal{V}^s_{\mu,D} \times \mathcal{V}^s_{\mu,D} \longrightarrow \mathcal{V}^s_D$ *is the bilinear continuous map given by*

$$\begin{aligned}\mathcal{D}^\alpha(u, v) := \alpha^2(1 - \alpha^2\mathcal{L})^{-1}\Big(&\operatorname{Div}(\nabla v \cdot \nabla u^t + \nabla v \cdot \nabla u)\\ &+ \operatorname{Tr}\big(\nabla.(\mathrm{R}(\cdot, u)v) + \mathrm{R}(\cdot, u)\nabla.v\big)\\ &+ \operatorname{grad}\big(\operatorname{Tr}(\nabla u \cdot \nabla v) + \operatorname{Ricci}(u, v)\big) - (\nabla_u \operatorname{Ric})(v)\Big)\end{aligned}$$

Proof : The first part follows by an integration by parts argument which is justified since all vector fields are of class C^1 by the Sobolev embedding theorem.

Indeed, integrating the identity $\pounds_u(g(v,w)) = \nabla_u(g(v,w)) = g(\nabla_u v, w) + g(v, \nabla_u w)$ and using $\pounds_u \mu = (\operatorname{div} u)\mu = 0$ we get

$$\begin{aligned}\langle \nabla_u v, w\rangle_0 + \langle v, \nabla_u w\rangle_0 &= \int_M g\left(\nabla_u v, w\right)\mu + \int_M g\left(v, \nabla_u w\right)\mu \\ &= \int_M \pounds_u(g(v,w))\mu = \int_M \pounds_u(g(v,w)\mu) = \int_M d\mathbf{i}_u(g(v,w)\mu) \\ &= \int_{\partial M} \mathbf{i}_u(g(v,w)\mu) = \int_{\partial M} g(v,w)g(u,n)\mu_\partial = 0\end{aligned}$$

by the Stokes theorem and the hypothesis that $g(u,n) = 0$ on ∂M.

For the second part we will use the following formula (see Lemma 3 in [**20**]): for all $u \in \mathcal{V}^s_{\mu,D}$ and $v \in \mathcal{V}^r_{\mu,D}, r > \frac{1}{2}\dim M + 3$ we have:

$$(1 - \alpha^2\mathcal{L})^{-1}\nabla_u[(1-\alpha^2\Delta_r)v] = \nabla_u v + \mathcal{D}^\alpha(u,v) \tag{4.3}$$

Using Lemma 2.4, the first part, and formula (4.3) we obtain for $u \in \mathcal{V}^s_{\mu,D}, w \in \mathcal{V}^s_D$ and $v \in \mathcal{V}^r_{\mu,D}, r > \frac{1}{2}\dim M + 3$:

$$\begin{aligned}\langle v, \nabla_u w\rangle_1 &= \langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0 \\ &= -\langle \nabla_u[(1-\alpha^2\Delta_r)v], w\rangle_0 \\ &= -\langle (1-\alpha^2\mathcal{L})^{-1}\nabla_u[(1-\alpha^2\Delta_r)v], w\rangle_1 \\ &= -\langle \nabla_u v + \mathcal{D}^\alpha(u,v), w\rangle_1.\end{aligned}$$

Using the fact that $v \in \mathcal{V}^r_{\mu,D}, r > \frac{1}{2}\dim M + 3$ is dense in $\mathcal{V}^s_{\mu,D}$, and the fact that $\langle\, ,\rangle_1, \nabla$, and $\mathcal{D}^\alpha$ are continuous on $\mathcal{V}^s_{\mu,D}$ we obtain that

$$\langle v, \nabla_u w\rangle_1 = -\langle \nabla_u v + \mathcal{D}^\alpha(u,v), w\rangle_1, \text{ for all } u, v \in \mathcal{V}^s_{\mu,D} \text{ and } w \in \mathcal{V}^s_D. \blacksquare$$

LEMMA 4.4. *Let $s > \frac{1}{2}\dim M + 1$. Let $B^\alpha : \mathcal{V}^{s+1}_{\mu,D} \times \mathfrak{X}^s \longrightarrow \mathcal{V}^{s+1}_{\mu,D}$ the continuous bilinear map given by*

$$B^\alpha(v,w) := \mathcal{P}_e(1-\alpha^2\mathcal{L})^{-1}(\nabla w^t \cdot (1-\alpha^2\Delta_r)v).$$

Then we have

$$\langle v, \nabla_u w\rangle_1 = \langle B^\alpha(v,w), u\rangle_1$$

for all $u \in \mathcal{V}^r_{\mu,D}, r > \frac{1}{2}\dim M$, and for all $v \in \mathcal{V}^{s+1}_{\mu,D}$, and $w \in \mathfrak{X}^s$.

Proof : Using Lemma 2.4 and the Stokes decomposition (see Theorem 2.5), we obtain:

$$\begin{aligned}\langle v, \nabla_u w\rangle_1 &= \langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0 \\ &= \langle \nabla w^t \cdot (1-\alpha^2\Delta_r)v, u\rangle_0 \\ &= \langle (1-\alpha^2\mathcal{L})^{-1}(\nabla w^t \cdot (1-\alpha^2\Delta_r)v), u\rangle_1 \\ &= \langle \mathcal{P}_e(1-\alpha^2\mathcal{L})^{-1}(\nabla w^t \cdot (1-\alpha^2\Delta_r)v), u\rangle_1. \ \blacksquare\end{aligned}$$

LEMMA 4.5. *Let $s > \frac{1}{2}\dim M + 1$. Let $k \geq 1$ and $f \in C^k(\mathcal{V}^s_{\mu,D})$ be such that there exists $\delta f \in C^1(\mathcal{V}^r_{\mu,D}, \mathcal{V}^t_{\mu,D}), r \geq s, t \geq 1$. Then:*

$$\langle D\delta f(u)(v), w\rangle_1 = \langle D\delta f(u)(w), v\rangle_1, \forall\, u, v, w \in \mathcal{V}^r_{\mu,D}$$

Proof : The proof is similar to that of Lemma 5.5 in [**23**]. We have

$$\begin{aligned}\langle D\delta f(u)(v), w\rangle_1 &= \left.\frac{d}{dt}\right|_{t=0} \langle \delta f(v+tu), w\rangle_1 = \left.\frac{d}{dt}\right|_{t=0} Df(v+tu)(w)\\ &= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{ds}\right|_{s=0} f(v+tu+sw)\\ &= \left.\frac{d}{ds}\right|_{s=0} \left.\frac{d}{dt}\right|_{t=0} f(v+tu+sw)\\ &= \langle D\delta f(u)(w), v\rangle_1. \quad \blacksquare\end{aligned}$$

THEOREM 4.6. *Let* $s > \frac{1}{2}\dim M + 1$ *and* $k \geq 1$. *Then:*

$$\{\ ,\ \}^1_+ : \mathcal{K}^k(\mathcal{V}^s_{\mu,D}) \times \mathcal{K}^k(\mathcal{V}^s_{\mu,D}) \longrightarrow \mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,D})$$

and for all $u \in \mathcal{V}^{s+1}_{\mu,D}$ *we have*

$$\begin{aligned}\delta(\{f,g\}^1_+)(u) &= \mathcal{P}_e(\nabla_{\delta g(u)}\delta f(u) - \nabla_{\delta f(u)}\delta g(u))\\ &+ D\delta g(u)\left(\mathcal{P}_e\left(\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right) + B^\alpha(u, \delta f(u)\right)\\ &- D\delta f(u)\left(\mathcal{P}_e\left(\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right) + B^\alpha(u, \delta g(u)\right)\end{aligned}$$

Proof : Let $h := \{f,g\}^1_+$. We have to show that $h \in \mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,D})$.

• Let's show that $h \in C^k(\mathcal{V}^s_{\mu,D})$.

We have $h(u) = \langle u, \nabla_{\delta g(u)}\delta f(u)\rangle_1 - \langle u, \nabla_{\delta f(u)}\delta g(u)\rangle_1$. Using the facts that $\nabla : \mathcal{V}^s_{\mu,D} \times \mathcal{V}^s_{\mu,D} \longrightarrow \mathcal{V}^{s-1}_D$ and $\langle\ ,\ \rangle_1 : \mathcal{V}^{s-1}_D \times \mathcal{V}^{s-1}_D \longrightarrow \mathbb{R}$ are bilinear continuous maps, and that $\delta f, \delta g \in C^k(\mathcal{V}^s_{\mu,D}, \mathcal{V}^s_{\mu,D})$ by hypothesis, we obtain the result.

• Let's show that $h \in C^k(\mathcal{V}^s_{\mu,D})$ admits a functional derivative $\delta h \in C^{k-1}(\mathcal{V}^{s+1}_{\mu,D}, \mathcal{V}^{s-1}_{\mu,D})$.

Let $u, v \in \mathcal{V}^{s+1}_{\mu,D}$. Using Lemmas 4.4, 4.3, and 4.5 we obtain:

$$\begin{aligned}Dh(u)(v) &= \langle v, \nabla_{\delta g(u)}\delta f(u)\rangle_1 + \langle u, \nabla_{D\delta g(u)(v)}\delta f(u)\rangle_1\\ &\quad + \langle u, \nabla_{\delta g(u)} D\delta f(u)(v)\rangle_1 - (f \leftrightarrow g)\\ &= \langle v, \mathcal{P}_e(\nabla_{\delta g(u)}\delta f(u))\rangle_1 + \langle B^\alpha(u, \delta f(u)), D\delta g(u)(v)\rangle_1\\ &\quad - \langle \nabla_{\delta g(u)} u + \mathcal{D}^\alpha(\delta g(u), u), D\delta f(u)(v)\rangle_1 - (f \leftrightarrow g)\\ &= \langle v, \mathcal{P}_e(\nabla_{\delta g(u)}\delta f(u))\rangle_1 + \langle D\delta g(u)(B^\alpha(u, \delta f(u))), v\rangle_1\\ &\quad - \langle D\delta f(u)\left(\mathcal{P}_e\left(\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right)\right), v\rangle_1 - (f \leftrightarrow g).\end{aligned}$$

Thus we conclude that the functional derivative exists and equals

$$\begin{aligned}\delta h(u) &= \mathcal{P}_e(\nabla_{\delta g(u)}\delta f(u) - \nabla_{\delta f(u)}\delta g(u))\\ &+ D\delta g(u)\left(\mathcal{P}_e\left(\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right) + B^\alpha(u, \delta f(u)\right)\\ &- D\delta f(u)\left(\mathcal{P}_e\left(\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right) + B^\alpha(u, \delta g(u)\right)\end{aligned}$$

A meticulous analysis show that $\delta h \in C^{k-1}(\mathcal{V}^{s+1}_{\mu,D}, \mathcal{V}^{s-1}_{\mu,D})$. $\blacksquare$

With all these preparations we can now establish the precise sense in which (4.2) is a Lie-Poisson bracket.

THEOREM 4.7. *Let* $s, t > \frac{1}{2}\dim M + 1$, $r \geq s$, *and* $k \geq 1$.

(1) $\{\ ,\ \}^1_+$ *is* $\mathbb{R}$*-bilinear and anti-symmetric on* $C^k_{r,t}(\mathcal{V}^s_{\mu,D}) \times C^k_{r,t}(\mathcal{V}^s_{\mu,D})$.

(2) $\{\ ,\ \}^1_+$ *is a derivation in each factor:*

$$\{fg, h\}^1_+ = \{f,h\}^1_+ g + f\{g,h\}^1_+, \forall\, f, g, h \in C^k_{r,t}(\mathcal{V}^s_{\mu,D}).$$

(3) *If* $s > \frac{1}{2}\dim M + 2$, $\{\ ,\ \}_+^1$ *satisfies the Jacobi identity: For all* $f, g, h \in \mathcal{K}^k(\mathcal{V}^s_{\mu,D})$ *and* $u \in \mathcal{V}^{s+1}_{\mu,D}$ *we have:*

$$\{f, \{g, h\}_+^1\}_+^1(u) + \{g, \{h, f\}_+^1\}_+^1(u) + \{h, \{f, g\}_+^1\}_+^1(u) = 0$$

Proof: (1) This is obvious.
(2) A direct computation, using Lemma 4.3, the fact that for all $f, g \in C^k_{r,t}(\mathcal{V}^s_{\mu,D})$ we have $fg \in C^k_{r,t}(\mathcal{V}^s_{\mu,D})$, and the relation $\delta(fg)(u) = \delta f(u)g(u) + f(u)\delta g(u)$ proves the required identity.
(3) Let $f, g, h \in \mathcal{K}^k(\mathcal{V}^s_{\mu,D})$, and $u \in \mathcal{V}^{s+1}_{\mu,D}$. By Theorem 4.6 we obtain $\{g, h\}_+^1 \in \mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,D}) \subset C^k_{s+1,s-1}(\mathcal{V}^s_{\mu,D})$. Since $s - 1 > \frac{1}{2}\dim M + 1$ we can compute the expression $\{f, \{g, h\}_+^1\}_+^1(u)$. Using Lemmas 4.4, 4.3, and 4.5 we obtain:

$$\begin{aligned}
&\{f, \{g, h\}_+^1\}_+^1(u) \\
&\quad = \langle u, [\delta\{g, h\}_+^1(u), \delta f(u)]\rangle_1 \\
&\quad = \langle u, \nabla_{\delta\{g,h\}_+^1(u)}\delta f(u)\rangle_1 - \langle u, \nabla_{\delta f(u)}\delta\{g, h\}_+^1(u)\rangle_1 \\
&\quad = \langle B^\alpha(u, \delta f(u)), \delta\{g, h\}_+^1(u)\rangle_1 + \langle \nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u), \delta\{g, h\}_+^1(u)\rangle_1 \\
&\quad = \langle \delta\{g, h\}_+^1(u), B^\alpha(u, \delta f(u)) + \mathcal{P}_e\left(\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right)\rangle_1 \\
&\quad = \langle \delta\{g, h\}_+^1(u), B_f\rangle_1,
\end{aligned}$$

where we denoted, for convenience, $B_f := B^\alpha(u, \delta f(u)) + \mathcal{P}_e\left(\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right) \in \mathcal{V}^s_{\mu,D}$. Using the formula in Theorem 4.6 this equals

$$\begin{aligned}
&\langle \mathcal{P}_e(\nabla_{\delta h(u)}\delta g(u) - \nabla_{\delta g(u)}\delta h(u)), B_f\rangle_1 \\
&\qquad + \langle D\delta h(u)\left(\mathcal{P}_e\left(\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right) + B^\alpha(u, \delta g(u))\right), B_f\rangle_1 \\
&\qquad - \langle D\delta g(u)\left(\mathcal{P}_e\left(\nabla_{\delta h(u)}u + \mathcal{D}^\alpha(\delta h(u), u)\right) + B^\alpha(u, \delta h(u))\right), B_f\rangle_1 \\
&\quad = \langle [\delta h(u), \delta g(u)], B^\alpha(u, \delta f(u)) + \nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\rangle_1 + D_{hgf} - D_{ghf},
\end{aligned}$$

where we denote

$$D_{hgf} := \langle D\delta h(u)\left(\mathcal{P}_e\left(\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right) + B^\alpha(u, \delta g(u))\right), B_f\rangle_1.$$

Note that by Lemma 4.5, we have $D_{hgf} = D_{hfg}$. Using Lemma 4.4 and 4.3 this equals

$$\begin{aligned}
&\langle \nabla_{[\delta h(u), \delta g(u)]}\delta f(u), u\rangle_1 - \langle \nabla_{\delta f(u)}[\delta h(u), \delta g(u)], u\rangle_1 + D_{hgf} - D_{ghf} \\
&\quad = \langle [[\delta h(u), \delta g(u)], \delta f(u)], u\rangle_1 + D_{hgf} - D_{ghf} \\
&\quad = \langle [[\delta h(u), \delta g(u)], \delta f(u)], u\rangle_1 + D_{hgf} - D_{gfh}.
\end{aligned}$$

Using Jacobi identity for the Jacobi-Lie bracket of vector fields we obtain:

$$\begin{aligned}
&\{f, \{g, h\}_+^1\}_+^1(u) + \{g, \{h, f\}_+^1\}_+^1(u) + \{h, \{f, g\}_+^1\}_+^1(u) \\
&\quad = 0 + (D_{hgf} - D_{gfh}) + (D_{fhg} - D_{hgf}) + (D_{gfh} - D_{fhg}) = 0 \quad \blacksquare
\end{aligned}$$

5. Geometric Properties of the Flow of LAE-α

Now we will prove that the maps π_R, F_t, and $\tilde{F}_t$ in Proposition 4.1 are Poisson maps. As we shall see, the considerations below need the hypothesis that π_R be at least of class C^1. Note that $\pi_R : T\mathcal{D}^s_{\mu,D} \longrightarrow \mathcal{V}^s_{\mu,D}$ is only continuous. Later on we shall use the fact that $\pi_R \in C^k(T\mathcal{D}^{s+k}_{\mu,D}, \mathcal{V}^s_{\mu,D})$ for all $k \geq 0$ (see [**8**]). If $f \in C^k(\mathcal{V}^s_{\mu,D})$, we shall denote $f_R := f \circ \pi_R \in C^k(T\mathcal{D}^{s+k}_{\mu,D})$.

LEMMA 5.1. *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 1$ such that $s + k \geq r$. Let $f \in C_r^k(\mathcal{V}_{\mu,D}^s)$. Then the vertical functional derivative of f_R with respect to $\mathcal{G}^1$ exists and is given by:*

$$\frac{\delta f_R}{\delta u}(u_\eta) = TR_\eta(\delta f(\pi_R(u_\eta))) \in T\mathcal{D}_{\mu,D}^r, \qquad \forall\, u_\eta \in T\mathcal{D}_{\mu,D}^{s+k}$$

Proof: This is a direct computation using the chain rule, the right-invariance of $\mathcal{G}^1$, and the fact that the naturel isomorphism between a vector space and its tangent space at a point is the vertical-lift. Indeed, we have:

$$\begin{aligned}
\left\langle \frac{\partial f_R}{\partial u}(u_\eta), v_\eta \right\rangle &= \left.\frac{d}{dt}\right|_{t=0} f_R(u_\eta + tv_\eta) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \pi_R)(u_\eta + tv_\eta) \\
&= df(\pi_R(u_\eta)) \left(\left.\frac{d}{dt}\right|_{t=0} \Big(\pi_R(u_\eta) + t\pi_R(v_\eta) \Big) \right) \\
&= df(\pi_R(u_\eta)) \left(\mathrm{Ver}_{\pi_R(u_\eta)}(\pi_R(v_\eta)) \right) = Df(\pi_R(u_\eta))(\pi_R(v_\eta)) \\
&= \langle \delta f(\pi_R(v_\eta)), \pi_R(v_\eta) \rangle_1 = \mathcal{G}^1(\eta)\Big(TR_\eta(\delta f(\pi_R(v_\eta)), v_\eta \Big),
\end{aligned}$$

where in the fifth equlity D denotes the Fréchet derivative of f thought of as a function defined on the Hilbert space $\mathcal{V}_{\mu,D}^s$ and in the third equality d denotes the exterior derivative of f thought of as a function defined on the manifold $\mathcal{V}_{\mu,D}^s$.

So we conclude that the functional vertical covariant derivative exists and is given by

$$\frac{\delta f_R}{\delta u}(u_\eta) = TR_\eta(\delta f(\pi_R(u_\eta))).$$

Since $s + k \geq r$, it is an element of $T\mathcal{D}_{\mu,D}^r$. ■

The computation of the horizontal functional derivative of f_R will involve the connector and therefore the map $\mathcal{F}^\alpha$ defined in Theorem 2.6. The following Lemma gives a useful expression for $\mathcal{F}^\alpha$.

LEMMA 5.2. (1) *For all $u \in \mathcal{V}_{\mu,D}^s$ we have:*

$$\begin{aligned}
\nabla u^t \cdot \Delta_r u = \mathrm{Div}(\nabla u^t \cdot \nabla u) - \mathrm{Tr}(\mathrm{R}(u, \nabla.u)\cdot) \\
+ \nabla u^t \cdot \mathrm{Ric}\, u - \frac{1}{2}\,\mathrm{grad}(\mathrm{Tr}(g(\nabla.u, \nabla.u))).
\end{aligned}$$

This shows that $\nabla u^t \cdot \Delta_r u$ is in $\mathfrak{X}^{s-2}$.
(2) *For all $u \in \mathcal{V}_{\mu,D}^s$ we have:*

$$\mathcal{F}^\alpha(u) = \mathcal{D}^\alpha(u,u) - (1 - \alpha^2\mathcal{L})^{-1}\alpha^2\Big(\mathrm{grad}(F(u)) + \nabla u^t \cdot \Delta_r u\Big),$$

where $\mathcal{D}^\alpha$ was defined in Lemma 4.3 and $F \in C^\infty(\mathcal{V}_{\mu,D}^s)$ is given by

$$F(u) = \mathrm{Tr}(\nabla u \cdot \nabla u) + \mathrm{Ricci}(u,u) + \frac{1}{2}\,\mathrm{Tr}(g(\nabla.u, \nabla.u)).$$

Proof: (1) We shall prove the identity at a given point $x \in M$ so we can choose a local orthonormal frame $\{e_i\}$ such that $\nabla e_i(x) = 0$. The computation below is carried out at the point x and we shall not write this evaluation. We have

$$\begin{aligned}
\mathrm{Div}(\nabla u^t \cdot \nabla u) &= \nabla_{e_i}(\nabla u^t \cdot \nabla u)(e_i) = \nabla_{e_i}(\nabla u^t \cdot \nabla u(e_i)) \\
&= \nabla_{e_i}(g(\nabla u^t \cdot \nabla u(e_i), e_k)e_k) = \nabla_{e_i}(g(\nabla_{e_i} u, \nabla_{e_k} u))e_k \\
&= g(\nabla_{e_i}\nabla_{e_i} u, \nabla_{e_k} u)e_k + g(\nabla_{e_i} u, \nabla_{e_i}\nabla_{e_k} u)e_k \\
&= g(\nabla u^t \cdot \nabla_{e_i}\nabla_{e_i} u, e_k)e_k + g(\nabla_{e_i} u, \mathrm{R}(e_i, e_k)u)e_k + g(\nabla_{e_i} u, \nabla_{e_k}\nabla_{e_i} u)e_k \\
&= \nabla u^t \cdot \nabla_{e_i}\nabla_{e_i} u + g(\mathrm{R}(u, \nabla_{e_i} u)e_i, e_k)e_k + \frac{1}{2}\nabla_{e_k}(g(\nabla_{e_i} u, \nabla_{e_i} u))e_k \\
&= \nabla u^t \cdot \nabla_{e_i}\nabla_{e_i} u + \mathrm{R}(u, \nabla_{e_i} u)e_i + \frac{1}{2}d(g(\nabla_{e_i} u, \nabla_{e_i} u))(e_k)e_k \\
&= \nabla u^t \cdot \nabla_{e_i}\nabla_{e_i} u + \mathrm{Tr}(\mathrm{R}(u, \nabla . u)\cdot) + \frac{1}{2}g(\mathrm{grad}(g(\nabla_{e_i} u, \nabla_{e_i} u)), e_k)e_k \\
&= \nabla u^t \cdot \nabla_{e_i}\nabla_{e_i} u + \mathrm{Tr}(\mathrm{R}(u, \nabla . u)\cdot) + \frac{1}{2}\,\mathrm{grad}(g(\nabla_{e_i} u, \nabla_{e_i} u))
\end{aligned}$$

which, using the Weitzenböck formula in Lemma 2.1, proves the desired formula.

(2) Using the formulas (2.8) and (2.9), and part (1) above, we have:

$$\begin{aligned}
\mathcal{F}^\alpha(u) &= \mathcal{U}^\alpha(u) + \mathcal{R}^\alpha(u) \\
&= \mathcal{D}^\alpha(u,u) + (1-\alpha^2\mathcal{L})^{-1}\alpha^2\Big(- \mathrm{grad}\,(\mathrm{Tr}(\nabla u \cdot \nabla u) + \mathrm{Ricci}(u,u)) \\
&\qquad - \mathrm{Div}(\nabla u^t \cdot \nabla u) + \mathrm{Tr}(\mathrm{R}(u, \nabla . u)\cdot) - \nabla u^t \cdot \mathrm{Ric}\, u\Big) \\
&= \mathcal{D}^\alpha(u,u) - (1-\alpha^2\mathcal{L})^{-1}\alpha^2\Big(\nabla u^t \cdot \Delta_r u \\
&\qquad + \mathrm{grad}\,\big[\,\mathrm{Tr}(\nabla u \cdot \nabla u) + \mathrm{Ricci}(u,u) + \frac{1}{2}\,\mathrm{Tr}(g(\nabla . u, \nabla . u))\big]\Big). \ \blacksquare
\end{aligned}$$

LEMMA 5.3. *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$. Let $f \in C^k_r(\mathcal{V}^s_{\mu,D})$. Then the horizontal functional derivative of f_R with respect to $\mathcal{G}^1$ exists. It is given by:*

$$\frac{\delta f_R}{\delta \eta}(u_\eta) = \frac{1}{2}TR_h\Big[B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u))\big)\Big]$$

for all $u_\eta \in T\mathcal{D}^{s+k}_{\mu,D}$, where $u := \pi_R(u_\eta)$ and B^α was defined in Lemma 4.4. So we have:

$$\frac{\delta f_R}{\delta \eta}(u_\eta) \in T\mathcal{D}^r_{\mu,D}, \qquad \forall\, u_\eta \in T\mathcal{D}^{s+k}_{\mu,D}.$$

Proof : By Lemma 3.2, we will have the two following formula

$$K^1(Tu \circ v) = \mathcal{P}_e(\nabla_v u + \mathfrak{F}^\alpha(u,v)), \tag{5.1}$$

where, using part (2) in Lemma 5.2 and the definition of $\mathfrak{F}^\alpha$ in Lemma 3.2, we have

$$\begin{aligned}
\mathfrak{F}^\alpha(u,v) &= \frac{1}{2}\Big(\mathcal{F}^\alpha(u+v) - \mathcal{F}^\alpha(u) - \mathcal{F}^\alpha(v)\Big) \\
&= \frac{1}{2}\Big(\mathcal{D}^\alpha(u,v) + \mathcal{D}^\alpha(v,u) \\
&\qquad - (1-\alpha^2\mathcal{L})^{-1}\alpha^2\big(\,\mathrm{grad}(G(u,v)) + \nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\Big) \qquad (5.2)
\end{aligned}$$

denoting $G(u,v) := F(u+v) - F(u) - F(v)$.

Let $u_\eta, v_\eta \in T\mathcal{D}^{s+k}_{\mu,D}$ and $\gamma(t) \subset T\mathcal{D}^s_{\mu,D}$ a smooth path defined in a neighborhood of zero, with base point denoted by $\eta(t) \subset \mathcal{D}^s_{\mu,D}$, satisfying the following conditions:

- $\gamma(0) = u_\eta$
- $\dot\eta(0) = v_\eta$
- γ is parallel.

By definition we have:

$$\begin{aligned}
\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \pi_R)(\gamma(t)) \\
&= df(\pi_R(u_\eta)) \left(\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t)) \right) \\
&= df(\pi_R(u_\eta)) \left(\operatorname{Ver}_{\pi_R(u_\eta)} \left(K^1 \left(\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t)) \right) \right) \right) \\
&= Df(\pi_R(u_\eta)) \left(K^1 \left(\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t)) \right) \right).
\end{aligned}$$

For the third equality, it suffices to remark that $\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t))$ is a vertical vector field. To obtain the last equality it suffices to use the natural isomorphism between a vector space and its tangent space at a point.

Using the formulas for the derivative of the composition and inversion we have:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t)) &= \left.\frac{d}{dt}\right|_{t=0} \left(\gamma(t) \circ \eta(t)^{-1}\right) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta(0)^{-1} + T(\gamma(0)) \circ \left(\left.\frac{d}{dt}\right|_{t=0} \eta(t)^{-1} \right) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} - Tu_\eta \circ T(\eta(0)^{-1}) \circ \left(\left.\frac{d}{dt}\right|_{t=0} \eta(t) \right) \circ \eta(0)^{-1} \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} - Tu_\eta \circ T\eta^{-1} \circ v_\eta \circ \eta^{-1} \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} - T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1}).
\end{aligned}$$

Using the right-invariance of the connector, the definition of the covariant derivative D/dt, and the fact that $\gamma(t)$ is parallel we obtain:

$$\begin{aligned}
K^1 \left(\left.\frac{d}{dt}\right|_{t=0} \pi_R(\gamma(t)) \right) &= K^1 \left(\left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} \right) - K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1})) \\
&= K^1 \left(\left.\frac{d}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} - K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1})) \\
&= \left(\left.\frac{D}{dt}\right|_{t=0} \gamma(t) \right) \circ \eta^{-1} - K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1})) \\
&= 0 - K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1})).
\end{aligned}$$

Thus we obtain:

$$\begin{aligned}
\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle &= -Df(u_\eta \circ \eta^{-1})(K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1}))) \\
&= -Df(u)(K^1(Tu \circ v)) \qquad \text{where } u := u_\eta \circ \eta^{-1} \text{ and } v := v_\eta \circ \eta^{-1} \\
&= -Df(u)(\mathcal{P}_e(\nabla_v u + \mathfrak{F}^\alpha(u,v))) \qquad \text{by formula (5.1)} \\
&= -\langle \delta f(u), \mathcal{P}_e(\nabla_v u + \mathfrak{F}^\alpha(u,v))\rangle_1 \\
&= -\langle \delta f(u), \nabla_v u + \mathfrak{F}^\alpha(u,v)\rangle_1 \\
&= -\langle \delta f(u), \nabla_v u + \frac{1}{2}(\mathcal{D}^\alpha(u,v) + \mathcal{D}^\alpha(v,u))\rangle_1 \\
&\quad + \frac{1}{2}\langle \delta f(u), (1-\alpha^2\mathcal{L})^{-1}\alpha^2\big(\operatorname{grad}(G(u,v))\big)\rangle_1 \\
&\quad + \frac{1}{2}\langle \delta f(u), (1-\alpha^2\mathcal{L})^{-1}\alpha^2\big(\nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\rangle_1 \qquad \text{by formula (5.2).}
\end{aligned}$$

The second term is zero because of the Stokes decomposition (see Theorem 2.5). For the first term we have by Lemmas 4.3 and 4.4:

$$\begin{aligned}
-\langle \delta f(u), &\nabla_v u + \frac{1}{2}(\mathcal{D}^\alpha(u,v) + \mathcal{D}^\alpha(v,u))\rangle_1 \\
&= -\langle \delta f(u), \frac{1}{2}\nabla_v u + \frac{1}{2}\mathcal{D}^\alpha(u,v) + \frac{1}{2}(\mathcal{D}^\alpha(v,u) + \nabla_v u)\rangle_1 \\
&= -\frac{1}{2}\Big(\langle \delta f(u), \nabla_v u\rangle_1 + \langle \delta f(u), \mathcal{D}^\alpha(u,v)\rangle_1 - \langle \nabla_v \delta f(u), u\rangle_1\Big) \\
&= -\frac{1}{2}\Big(\langle B^\alpha(\delta f(u), u), v\rangle_1 + \langle \delta f(u), \mathcal{D}^\alpha(u,v) + \nabla_u v\rangle_1 \\
&\qquad - \langle \delta f(u), \nabla_u v\rangle_1 - \langle B^\alpha(u, \delta f(u)), v\rangle_1\Big) \\
&= -\frac{1}{2}\Big(\langle B^\alpha(\delta f(u), u), v\rangle_1 - \langle \nabla_u \delta f(u), v\rangle_1 \\
&\qquad + \langle \nabla_u \delta f(u) + \mathcal{D}^\alpha(u, \delta f(u)), v\rangle_1 - \langle B^\alpha(u, \delta f(u)), v\rangle_1\Big) \\
&= \frac{1}{2}\langle B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u)), v\rangle_1.
\end{aligned}$$

By Lemmas 2.4, 4.3, the third term becomes:

$$\begin{aligned}
\langle \delta f(u), &(1-\alpha^2\mathcal{L})^{-1}\alpha^2\big(\nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\rangle_1 \\
&= \langle \delta f(u), \alpha^2\big(\nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\rangle_0 \\
&= -\langle \delta f(u), \nabla u^t \cdot (1-\alpha^2\Delta_r)v\rangle_0 + \langle \delta f(u), \nabla u^t \cdot v\rangle_0 \\
&\qquad - \langle \delta f(u), \nabla v^t \cdot (1-\alpha^2\Delta_r)u\rangle_0 + \langle \delta f(u), \nabla v^t \cdot u\rangle_0 \\
&= -\langle \nabla_{\delta f(u)} u, v\rangle_1 + \langle \nabla_{\delta f(u)} u, v\rangle_0 \\
&\qquad - \langle \nabla_{\delta f(u)} v, u\rangle_1 + \langle \nabla_{\delta f(u)} v, u\rangle_0 \\
&= -\langle \nabla_{\delta f(u)} u, v\rangle_1 + \langle \nabla_{\delta f(u)} u, v\rangle_0 \\
&\qquad + \langle \nabla_{\delta f(u)} u + \mathcal{D}^\alpha(\delta f(u), u), v\rangle_1 - \langle v, \nabla_{\delta f(u)} u\rangle_0 \\
&= \langle \mathcal{D}^\alpha(\delta f(u), u), v\rangle_1.
\end{aligned}$$

So we obtain:

$$
\begin{aligned}
&\frac{\partial f_R}{\partial \eta}(u_\eta)(v_\eta) \\
&= \frac{1}{2}\langle B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u))\big), v\rangle_1 \\
&= \frac{1}{2}\mathcal{G}^1(\eta)\Big(TR_\eta\Big[B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) \\
&+ \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u))\big)\Big], v_\eta\Big).
\end{aligned}
$$

Therefore we obtain the existence of $\dfrac{\delta f_R}{\delta \eta}(u_\eta) \in T\mathcal{D}^r_{\mu,D}$, given by

$$
\frac{\delta f_R}{\delta \eta}(u_\eta) = \frac{1}{2}TR_h\Big[B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u))\big)\Big],
$$

where $u := \pi_R(u_\eta)$. ■

Lemmas 5.1 and 5.3 yield the following theorem:

Theorem 5.4. *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$. Let $f \in C^k_r(\mathcal{V}^s_{\mu,D})$. Then $f_R := f \circ \pi_R$ is in $C^k_r(T\mathcal{D}^{s+k}_{\mu,D})$.*

Theorem 5.5. (π_R *is a Poisson map*) *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$. Then :*

$$
\{f \circ \pi_R, g \circ \pi_R\}^1(u_\eta) = \left(\{f, g\}^1_+ \circ \pi_R\right)(u_\eta), \quad \forall\, f, g \in C^k_r(\mathcal{V}^s_{\mu,D}), \quad u_\eta \in T\mathcal{D}^{s+k}_{\mu,D}.
$$

Proof : Let $u_\eta \in T\mathcal{D}^{s+k}_{\mu,D}$ and $u := \pi_R(u_\eta)$. The proof is a direct computation using Lemmas 4.3, 4.4, 5.1 and 5.3. Indeed, formula (4.1):

$$
\{f \circ \pi_R, g \circ \pi_R\}^1(u_\eta) = \mathcal{G}^1(\eta)\left(\frac{\delta f_R}{\delta \eta}(u_\eta), \frac{\delta g_R}{\delta u}(u_\eta)\right) - \mathcal{G}^1(\eta)\left(\frac{\delta f_R}{\delta u}(u_\eta), \frac{\delta g_R}{\delta \eta}(u_\eta)\right).
$$

So it suffices to compute the first term:

$$
\begin{aligned}
&\mathcal{G}^1(\eta)\left(\frac{\delta f_R}{\delta \eta}(u_\eta), \frac{\delta g_R}{\delta u}(u_\eta)\right) \\
&\quad = \frac{1}{2}\mathcal{G}^1(\eta)\Big(TR_\eta\Big[B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) \\
&\quad - \mathcal{D}^\alpha(u, \delta f(u))\big)\Big], TR_\eta(\delta g(u))\Big) \\
&\quad = \frac{1}{2}\left\langle B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u)), \delta g(u)\right\rangle_1 \\
&\quad = \frac{1}{2}\Big(\langle B^\alpha(u, \delta f(u)), \delta g(u)\rangle_1 - \langle B^\alpha(\delta f(u), u), \delta g(u)\rangle_1 \\
&\qquad\qquad + \langle \mathcal{D}^\alpha(\delta f(u), u) + \nabla_{\delta f(u)} u, \delta g(u)\rangle_1 - \langle \nabla_{\delta f(u)} u, \delta g(u)\rangle_1 \\
&\qquad\qquad - \langle \mathcal{D}^\alpha(u, \delta f(u)) + \nabla_u \delta f(u), \delta g(u)\rangle_1 + \langle \nabla_u \delta f(u), \delta g(u)\rangle_1\Big) \\
&\quad = \frac{1}{2}\Big(\langle u, \nabla_{\delta g(u)} \delta f(u)\rangle_1 - \langle \delta f(u), \nabla_{\delta g(u)} u\rangle_1 - \langle u, \nabla_{\delta f(u)} \delta g(u)\rangle_1 \\
&\qquad\qquad - \langle \nabla_{\delta f(u)} u, \delta g(u)\rangle_1 + \langle \delta f(u), \nabla_u \delta g(u)\rangle_1 + \langle \nabla_u \delta f(u), \delta g(u)\rangle_1\Big),
\end{aligned}
$$

where we have used Lemmas 4.3 and 4.4 in the last equality. Finally, after cancellation of several terms we obtain:

$$\mathcal{G}^1(\eta)\left(\frac{\delta f_R}{\delta\eta}(u_\eta), \frac{\delta g_R}{\delta u}(u_\eta)\right) - \mathcal{G}^1(\eta)\left(\frac{\delta f_R}{\delta u}(u_\eta), \frac{\delta g_R}{\delta\eta}(u_\eta)\right)$$
$$= \langle u, \nabla_{\delta g(u)}\delta f(u)\rangle_1 - \langle u, \nabla_{\delta f(u)}\delta g(u)\rangle_1 = \left(\{f,g\}^1_+ \circ \pi_R\right)(u_\eta). \blacksquare$$

THEOREM 5.6. (F_t *is a Poisson map*) *Let* F_t *be the flow of* $\mathcal{S}^1$, $t_1, t_2 > \frac{1}{2}dimM + 1$ *such that* $t_1 \geq t_2$. *Then for all* $G, H \in C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$ *we have:*
(1) $G \circ F_t, H \circ F_t \in C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$
(2) $\{G \circ F_t, H \circ F_t\}^1 = \{G, F\}^1 \circ F_t$ *on* $T\mathcal{D}^{t_1}_{\mu,D}$.

Proof : This is done as in Proposition 5.12 of [**23**]. First of all we recall some general facts about weak Riemannian Banach-manifolds. Let $(Q, \langle\!\langle\, , \rangle\!\rangle)$ be a weak Riemannian Banach-manifold with smooth geodesic spray. We define:

$$\mathcal{K}^\infty(TQ) := \left\{F \in C^\infty(TQ) \middle| \exists \frac{\delta F}{\delta\eta}, \frac{\delta F}{\delta u} \in C^\infty(TQ, TQ)\right\}.$$

Here, $\partial F/\partial\eta$ and $\partial F/\partial u$ are the partial derivatives and $\delta F/\delta\eta$ and $\delta F/\delta u$ denote the horizontal and vertical functional derivatives relative to the given weak Riemannian metric on Q of $F \in C^\infty(TQ)$ as defined at the beginning of section 4.

Let F_t be the geodesic flow and $G \in \mathcal{K}^\infty(TQ)$. Then $G \circ F_t \in \mathcal{K}^\infty(TQ)$ and F_t is a Poisson map:

$$\{G \circ F_t, H \circ F_t\} = \{G, H\} \circ F_t,\ \forall\, G, H \in \mathcal{K}^\infty(TQ), \tag{5.3}$$

where $\{\, , \}$ is the Poisson bracket on $\mathcal{K}^\infty(TQ)$ induced by the weak Riemannian metric and the weak sympectic form on T^*Q (see (4.1)).

We will use the following formula for $G \in \mathcal{K}^\infty(TQ)$:

$$dG(u_\eta)(X_{u_\eta}) = \frac{\partial G}{\partial\eta}(u_\eta)(T\pi_Q(X_{u_\eta})) + \frac{\partial G}{\partial u}(u_\eta)(K(X_{u_\eta})), \tag{5.4}$$

where $\eta \in Q$, $u_\eta \in T_\eta Q$, $X_{u_\eta} \in T_{u_\eta}(TQ)$, $\pi_Q : TQ \to Q$ is the tangent bundle projection, and K is the connector of the given weak Riemannian metric on Q.

With these general preparations, let $Q = \mathcal{D}^{t_2}_{\mu,D}$ be endowed with the weak Riemannian metric $\mathcal{G}^1$.

(1) Let $G \in C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$, and $u_\eta \in T\mathcal{D}^{t_1}_{\mu,D}$. So we have:

$$\frac{\delta G}{\delta\eta}(F_t(u_\eta)), \frac{\delta G}{\delta u}(F_t(u_\eta)) \in T\mathcal{D}^{t_2}_{\mu,D}.$$

Let $\widetilde{G} \in \mathcal{K}^\infty(T\mathcal{D}^{t_2}_{\mu,D})$ be such that:

$$\frac{\delta G}{\delta\eta}(F_t(u_\eta)) = \frac{\delta\widetilde{G}}{\delta\eta}(F_t(u_\eta)) \ \text{ and } \ \frac{\delta G}{\delta u}(F_t(u_\eta)) = \frac{\delta\widetilde{G}}{\delta u}(F_t(u_\eta)).$$

This is possible since $\mathcal{D}^{t_2}_{\mu,D}$, and hence $T\mathcal{D}^{t_2}_{\mu,D}$, are Hilbert manifolds so they admit bump fuctions. Using (5.4) we find

$$\begin{aligned}\frac{\partial(G\circ F_t)}{\partial\eta}(u_\eta) &= dG(F_t(u_\eta))\left(\frac{\partial F_t}{\partial\eta}(u_\eta)\right)\\ &= \mathcal{G}^1(u_\eta)\left(\frac{\delta G}{\delta\eta}(F_t(u_\eta)), T\pi_{\mathcal{D}^s_{\mu,D}}\left(\frac{\partial F_t}{\partial\eta}(u_\eta)\right)\right)\\ &+ \mathcal{G}^1(u_\eta)\left(\frac{\delta G}{\delta u}(F_t(u_\eta)), K^1\left(\frac{\partial F_t}{\partial\eta}(u_\eta)\right)\right)\end{aligned}$$

and so we obtain
$$\frac{\partial(G\circ F_t)}{\partial\eta}(u_\eta) = \frac{\partial(\widetilde{G}\circ F_t)}{\partial\eta}(u_\eta).$$
Since $\widetilde{G}\in\mathcal{K}^\infty(T\mathcal{D}^{t_2}_{\mu,D})$, we obtain the existence of
$$\frac{\delta(G\circ F_t)}{\delta\eta}(u_\eta) = \frac{\delta(\widetilde{G}\circ F_t)}{\delta\eta}(u_\eta)\in T\mathcal{D}^{t_2}_{\mu,D}$$
and the same is true for the vertical partial covariant derivative. Doing this for all $u_\eta\in T\mathcal{D}^{t_1}_{\mu,D}$ we obtain that $G\circ F_t$ is in $C^k_{t_2}(T\mathcal{D}^{t_1}_{\mu,D})$.

(2) Let u_η be in $T\mathcal{D}^{t_1}_{\mu,D}$. By part one, $\{G\circ F_t, H\circ F_t\}^1(u_\eta)$ is well-defined and only depends on
$$\frac{\delta G}{\delta\eta}(F_t(u_\eta)), \frac{\delta G}{\delta u}(F_t(u_\eta)), \frac{\delta H}{\delta\eta}(F_t(u_\eta)), \frac{\delta H}{\delta u}(F_t(u_\eta)).$$
Choosing $\widetilde{G}$ and $\widetilde{H}$ as in part one, and using (5.3) we obtain the desired formula. ∎

Theorem 5.7. *($\tilde{F}_t$ is a Poisson map) Let $\tilde{F}_t = \pi_R\circ F_t$ be the flow of LAE-α equation. Then we have*
$$\{f\circ\tilde{F}_t, g\circ\tilde{F}_t\}^1_+(u) = \left(\{f,g\}^1_+\circ\tilde{F}_t\right)(u),\qquad \forall f,g\in C^k_r(\mathcal{V}^s_{\mu,D}),\qquad u\in\mathcal{V}^{s+2k}_{\mu,D},$$
where $k\geq 1$ and $r>\frac{1}{2}dimM+2$ such that $s+k\geq r$ (for example $k=1$).

Proof : Let $f\in C^k_r(\mathcal{V}^s_{\mu,D})$. We have $f\circ\pi_R\in C^k_r(T\mathcal{D}^{s+k}_{\mu,D})$ by Theorem 5.4. Therefore, by part (1) of Theorem 5.6 we get $f\circ\pi_R\circ F_t\in C^k_r(T\mathcal{D}^{s+k}_{\mu,D})$ and hence $f\circ\tilde{F}_t = f\circ\pi_R\circ F_t|_{\mathcal{V}^{s+k}_{\mu,D}}\in C^k_r(\mathcal{V}^{s+k}_{\mu,D})$. Since $\pi_R(u)=u$, we have

$$\begin{aligned}\{f\circ\tilde{F}_t, g\circ\tilde{F}_t\}^1_+(u) &= \{f\circ\tilde{F}_t\circ\pi_R, g\circ\tilde{F}_t\circ\pi_R\}^1(u) \text{ by Theorem 5.5}\\ &= \{f\circ\pi_R\circ F_t, g\circ\pi_R\circ F_t\}^1(u) \text{ by Proposition 4.1}\\ &= \{f\circ\pi_R, g\circ\pi_R\}^1(F_t(u)) \text{ by Theorem 5.6}\\ &= (\{f,g\}^1_+\circ\pi_R)(F_t(u)) \text{ by Theorem 5.5}\\ &= (\{f,g\}^1_+\circ\tilde{F}_t)(u) \text{ by Proposition 4.1.}\end{aligned}$$

Note that for the first equality we need $u\in\mathcal{V}^{(s+k)+k}_{\mu,D}$ by Theorem 5.5. ∎

The last Theorem gives the Poisson formulation of the LAE-α equation. We recall that an integral curve $u(t)$ of the LAE-α (or the Euler) equation is C^1 as a map in $\mathcal{V}^{s-1}_{\mu,D}$, but it is believed to be continuous but not differentiable as a map in $\mathcal{V}^s_{\mu,D}$.

THEOREM 5.8. *Let $u(t) \subset \mathcal{V}^s_{\mu,D}$ be a curve such that $u \in C^0(I,\mathcal{V}^s_{\mu,D}) \cap C^1(I,\mathcal{V}^{s-1}_{\mu,D})$. Then*

$$\frac{d}{dt}f(u(t)) = \{f,h\}^1_+(u(t)), \forall\, f \in C^1_s(\mathcal{V}^{s-1}_{\mu,D}) \Longleftrightarrow u(t) \text{ is a solution of LAE-}\alpha \text{ equation}$$

where $h(u) := \frac{1}{2}\langle u,u\rangle_1$ is the reduced Hamiltonian.

Proof : We remark that $h \in C^1_s(\mathcal{V}^s_{\mu,D})$ with $\delta h(u) = u$. We find:

$$\begin{aligned}\frac{d}{dt}f(u(t)) &= Df(u(t))(\partial_t u(t))\\ &= \langle \delta f(u(t)), \partial_t u(t)\rangle_1\end{aligned}$$

and, by Lemma 4.3,

$$\begin{aligned}\{f,h\}^1_+(u(t)) &= \langle u(t), \nabla_{u(t)}\delta f(u(t))\rangle_1 - \langle u(t), \nabla_{\delta f(u(t))}u(t)\rangle_1\\ &= -\langle \nabla_{u(t)}u(t) + \mathcal{D}^\alpha(u(t),u(t)), \delta f(u(t))\rangle_1 - \langle (1-\alpha^2\Delta_r)u(t), \nabla_{\delta f(u(t))}u(t)\rangle_0.\end{aligned}$$

Using the remarkable fact that $\nabla u^t \cdot \Delta_r u$ is in $\mathfrak{X}^{s-2}$ (Lemma 5.2), and the identity $\nabla u^t \cdot u = \mathrm{grad}(g(u,u))$, we obtain for the second term:

$$\begin{aligned}&\langle (1-\alpha^2\Delta_r)u(t), \nabla_{\delta f(u(t))}u(t)\rangle_0\\ &\quad= \langle \nabla u(t)^t \cdot (1-\alpha^2\Delta_r)u(t), \delta f(u(t))\rangle_0\\ &\quad= \langle (1-\alpha^2\mathcal{L})^{-1}\nabla u(t)^t \cdot (1-\alpha^2\Delta_r)u(t), \delta f(u(t))\rangle_1\\ &\quad= \langle (1-\alpha^2\mathcal{L})^{-1}\,\mathrm{grad}[g(u(t),u(t))] - (1-\alpha^2\mathcal{L})^{-1}\alpha^2\nabla u(t)^t \cdot \Delta_r u(t), \delta f(u(t))\rangle_1\\ &\quad= -\langle (1-\alpha^2\mathcal{L})^{-1}\alpha^2\nabla u(t)^t \cdot \Delta_r u(t), \delta f(u(t))\rangle_1 \quad \text{by the Stokes decomposition.}\end{aligned}$$

So we obtain by Lemma 5.2:

$$\begin{aligned}&\{f,h\}^1_+(u(t))\\ &\quad= \langle -\nabla_{u(t)}u(t) - \mathcal{D}^\alpha(u(t),u(t)) + (1-\alpha^2\mathcal{L})^{-1}\alpha^2\nabla u(t)^t \cdot \Delta_r u(t), \delta f(u(t))\rangle_1\\ &\quad= \langle -\nabla_{u(t)}u(t) - \mathcal{F}^\alpha(u(t)) - (1-\alpha^2\mathcal{L})^{-1}\alpha^2\,\mathrm{grad}(F(u(t))), \delta f(u(t))\rangle_1\\ &\quad= -\langle \mathcal{P}_e\left(\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\right), \delta f(u(t))\rangle_1.\end{aligned}$$

Thus $\dfrac{d}{dt}f(u(t)) = \{f,h\}^1_+(u(t)), \forall\, f \in C^1_s(\mathcal{V}^{s-1}_{\mu,D})$ is equivalent to:

$$\partial_t u(t) + \mathcal{P}_e\left(\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\right) = 0$$

which is LAE-α. ■

6. The case of free-slip and mixed boundary conditions

In this section we shall generalize all our results to the case of free-slip and mixed boundary conditions. Note that setting $\Gamma_1 = \varnothing$ in the mixed case, gives the free-slip case. The fundamental difference between these boundary conditions and the no-slip case we studied before is the following. For all vector fields u,v in $\mathcal{V}^s_D$, the vector field $\nabla_u v$ lies in $\mathcal{V}^{s-1}_D$. This is a fact we used several times in our previous computations. Unfortunately, for vector fields u,v in $\mathcal{V}^s_{mix}$ this is not true since $\nabla_u v$ may not be in $\mathcal{V}^{s-1}_{mix}$. In this case we will use that $\nabla_u v - \nabla_v u = [u,v]$ is in $\mathcal{V}^{s-1}_{mix}$. As a first consequence, the useful identity (4.3) for the no-slip case

$$(1-\alpha^2\mathcal{L})^{-1}\nabla_u[(1-\alpha^2\Delta_r)v] = \nabla_u v + \mathcal{D}^\alpha(u,v),$$

where u is in $\mathcal{V}^s_{\mu,D}$, $s > \frac{1}{2}\dim M + 1$, v is in $\mathcal{V}^r_{\mu,D}$, and $r > \frac{1}{2}\dim M + 3$, is replaced by

$$(1-\alpha^2\mathcal{L})^{-1}\nabla_u[(1-\alpha^2\Delta_r)v] = (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_u v + \mathcal{D}^\alpha(u,v) \tag{6.1}$$

if u is in $\mathcal{V}^s_{\mu,mix}$, $s > \frac{1}{2}\dim M + 1$, v is in $\in \mathcal{V}^r_{\mu,mix}$, and $r > \frac{1}{2}\dim M + 3$.

Recall that for $r \geq 1$, $(1-\alpha^2\mathcal{L})$ denotes the continuous linear map $(1-\alpha^2(\Delta + 2\operatorname{Ric} + \operatorname{grad}\operatorname{div})) : \mathfrak{X}^r \longrightarrow \mathfrak{X}^{r-2}$ acting on all H^r vector fields, and $(1-\alpha^2\mathcal{L})^{-1} : \mathfrak{X}^{r-2} \longrightarrow \mathcal{V}^r_{mix}$ denotes the inverse of the isomorphism $(1-\alpha^2\mathcal{L})_{|\mathcal{V}^r_{mix}}$. Formula (6.1) induces some changes in Lemmas 4.3 and 4.4 which must be replaced by the following.

Lemma 6.1. *Let $s > \frac{1}{2}\dim M + 1$. Let $u, v \in \mathcal{V}^s_{\mu,mix}$ and $w \in \mathcal{V}^s_{mix}$. Then:*

$$\langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0 = -\langle (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_u v + \mathcal{D}^\alpha(u,v), w\rangle_1$$

where $\mathcal{D}^\alpha : \mathcal{V}^s_{\mu,mix} \times \mathcal{V}^s_{\mu,mix} \longrightarrow \mathcal{V}^s_{mix}$ is the bilinear continuous map given by

$$\begin{aligned}\mathcal{D}^\alpha(u,v) := \alpha^2(1-\alpha^2\mathcal{L})^{-1}\Big(&\operatorname{Div}(\nabla v\cdot\nabla u^t + \nabla v\cdot\nabla u)\\ &+ \operatorname{Tr}\big(\nabla.(\mathrm{R}(\cdot,u)v) + \mathrm{R}(\cdot,u)\nabla.v\big)\\ &+ \operatorname{grad}\big(\operatorname{Tr}(\nabla u\cdot\nabla v) + \operatorname{Ricci}(u,v)\big) - (\nabla_u \operatorname{Ric})(v)\Big)\end{aligned}$$

Proof : Using the first part of Lemma 4.3 and formula (6.1) we obtain for $u \in \mathcal{V}^s_{\mu,mix}, w \in \mathcal{V}^s_{mix}$ and $v \in \mathcal{V}^r_{\mu,mix}, r > \frac{1}{2}\dim M + 3$:

$$\begin{aligned}\langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0 &= -\langle \nabla_u[(1-\alpha^2\Delta_r)v], w\rangle_0\\ &= -\langle (1-\alpha^2\mathcal{L})^{-1}\nabla_u[(1-\alpha^2\Delta_r)v], w\rangle_1\\ &= -\langle (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_u v + \mathcal{D}^\alpha(u,v), w\rangle_1.\end{aligned}$$

Using the fact that $\mathcal{V}^r_{\mu,mix}, r > \frac{1}{2}\dim M + 3$ is dense in $\mathcal{V}^s_{\mu,mix}$, and the fact that $\langle\,,\rangle_1, \nabla$, and $\mathcal{D}^\alpha$ are continuous on $\mathcal{V}^s_{\mu,D}$, and $(1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})$ is continuous on $\mathcal{V}^{s-1}_{\mu,D}$ we obtain the desired result. ∎

Lemma 6.2. *Let $s > \frac{1}{2}\dim M + 1$. Let $B^\alpha : \mathcal{V}^{s+1}_{\mu,mix} \times \mathfrak{X}^s \longrightarrow \mathcal{V}^{s+1}_{\mu,mix}$ the continuous bilinear map given by*

$$B^\alpha(v,w) := \mathcal{P}_e(1-\alpha^2\mathcal{L})^{-1}(\nabla w^t\cdot(1-\alpha^2\Delta_r)v).$$

Then we have

$$\langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0 = \langle B^\alpha(v,w), u\rangle_1$$

for all $u \in \mathcal{V}^r_{\mu,mix}, r > \frac{1}{2}\dim M$, and for all $v \in \mathcal{V}^{s+1}_{\mu,mix}$, and $w \in \mathfrak{X}^s$.

Proof : The proof is similar to that of Lemma 4.3. Note that $\langle (1-\alpha^2\Delta_r)v, \nabla_u w\rangle_0$ does not equal $\langle v, \nabla_u w\rangle_1$ since $\nabla_u w$ does not belong to $\mathcal{V}^r_{mix}$. ∎

In order to carry out the Lie-Poisson reduction procedure for the mixed boundary conditions, we have to establish the existence and the smoothness of the geodesic spray of the weak Riemannian manifold $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$. So we will need a reformulation of LAE-α similar to (2.7) in the case of mixed boundary conditions. This reformulation is given by the following proposition where we use the Euler-Poincaré reduction theorem.

PROPOSITION 6.3. *Let $\eta(t)$ be a curve in $\mathcal{D}^s_{\mu,mix}$, and let $u(t) := TR_{\eta(t)^{-1}}(\dot\eta(t)) = \dot\eta(t)\circ\eta(t)^{-1} \in \mathcal{V}^s_{\mu,mix}$. Then the following properties are equivalent :*
(1) *$\eta(t)$ is a geodesic of $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$*
(2) *$u(t)$ is a solution of :*

$$\partial_t u(t) + \mathcal{P}_e\left((1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\right) = 0 \tag{6.2}$$

Proof : By the the Euler-Poincaré reduction theorem, $\eta(t)$ is a geodesic of $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$ if and only if $u(t) := \dot\eta(t)\circ\eta(t)^{-1}$ is an extremum of the reduced action

$$s(u) = \frac{1}{2}\int_a^b \langle u(t), u(t)\rangle_1 dt$$

for variations of the form

$$\delta u(t) = \partial_t w(t) + [u(t), w(t)]$$

where $w(t) := \delta\eta(t)\circ\eta^{-1}(t)$ vanishes at the endpoints. Integrating by parts, using the fact that $[u(t), w(t)]$ is in $\mathcal{V}^{s-1}_{\mu,mix}$ and with Lemma 6.2 we find:

$$\begin{aligned}
&Ds(u)(\delta u) = \int_a^b \langle u(t), \delta u(t)\rangle_1 dt\\
&= \int_a^b \langle u(t), \partial_t w(t)\rangle_1 dt + \int_a^b \langle u(t), [u(t), w(t)]\rangle_1 dt\\
&= -\int_a^b \langle \partial_t u(t), w(t)\rangle_1 dt + \int_a^b \langle (1-\alpha^2\Delta_r)u(t), [u(t), w(t)]\rangle_0 dt\\
&= -\int_a^b \langle \partial_t u(t), w(t)\rangle_1 dt + \int_a^b \langle (1-\alpha^2\Delta_r)u(t), \nabla_{u(t)}w(t)\rangle_0 dt\\
&\quad -\int_a^b \langle (1-\alpha^2\Delta_r)u(t), \nabla_{w(t)}u(t)\rangle_0 dt\\
&= -\int_a^b \langle \partial_t u(t), w(t)\rangle_1 dt - \int_a^b \langle (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{u(t)}u(t)\\
&\quad + \mathcal{D}^\alpha(u(t), u(t)), w(t)\rangle_1 dt - \int_a^b \langle \nabla u(t)^t\cdot(1-\alpha^2\Delta_r)u(t), w(t)\rangle_0 dt.
\end{aligned}$$

With Lemma 4.3 (1), we have $\langle \nabla u(t)^t\cdot u(t), w(t)\rangle_0 = 0$, thus the last term equals

$$\alpha^2\int_a^b \langle \nabla u(t)^t\cdot\Delta_r u(t), w(t)\rangle_0 dt$$

and we obtain:

$$\begin{aligned}
Ds(u)(\delta u) = -\int_a^b &\big\langle \partial_t u(t) + (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{u(t)}u(t)\\
&+ \mathcal{D}^\alpha(u(t), u(t)) - \alpha^2(1-\alpha^2\mathcal{L})^{-1}\nabla u(t)^t\cdot\Delta_r u(t), w(t)\big\rangle_1 dt.
\end{aligned}$$

So by the Stokes decomposition theorem, $Ds(u)(\delta u) = 0$ for all δu, is equivalent to

$$\begin{aligned}
\mathcal{P}_e(\partial_t u(t) &+ (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{u(t)}u(t)\\
&+ \mathcal{D}^\alpha(u(t), u(t)) - \alpha^2(1-\alpha^2\mathcal{L})^{-1}\nabla u(t)^t\cdot\Delta_r u(t)) = 0
\end{aligned}$$

and, with Lemma 5.2 (which remains valid on $\mathcal{V}^s_{\mu,mix}$), this is equivalent to

$$\mathcal{P}_e\big(\partial_t u(t) + (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{u(t)}u(t) + \mathcal{F}^\alpha(u(t))\big) = 0. \blacksquare$$

Let $\eta \in \mathcal{D}^s_{mix}$, $r \geq 0$, and $H^r_\eta := \{u_\eta \in H^r(M, TM) | \pi \circ u = \eta\}$. We denote by $H^r_\eta \downarrow \mathcal{D}^s_{mix}$ the vector bundle over $\mathcal{D}^s_{mix}$, whose fiber at $\eta \in \mathcal{D}^s_{mix}$ is H^r_η. The proof of Proposition 5 in [**20**] shows that for $s > \frac{1}{2}\dim M + 1$, the map

$$\overline{(1-\alpha^2\mathcal{L})} : H^s_\eta \downarrow \mathcal{D}^s_{mix} \longrightarrow H^{s-2}_\eta \downarrow \mathcal{D}^s_{mix}$$

defined by $\overline{(1-\alpha^2\mathcal{L})}(u_\eta) := [(1-\alpha^2\mathcal{L})(u_\eta \circ \eta^{-1})] \circ \eta$ is a C^∞ bundle map. Furthermore,

$$\overline{(1-\alpha^2\mathcal{L})} : T\mathcal{D}^s_{mix} \longrightarrow H^{s-2}_\eta \downarrow \mathcal{D}^s_{mix}$$

is a bijection, whose inverse is denoted by

$$\overline{(1-\alpha^2\mathcal{L})}^{-1} : H^{s-2}_\eta \downarrow \mathcal{D}^s_{mix} \longrightarrow T\mathcal{D}^s_{mix}$$

With the same method and notations as in section 3, but using equation (6.2) instead of (2.7), we obtain the following lemma

LEMMA 6.4. *The geodesic spray of $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$ is given by:*

$$\mathcal{S}^1(u_\eta) = T\overline{\mathcal{P}}\left[T\left(\overline{(1-\alpha^2\mathcal{L})}^{-1} \circ \overline{(1-\alpha^2\mathcal{L})}\right)(S \circ u_\eta) - \mathrm{Ver}_{u_\eta}(\overline{\mathcal{F}}^\alpha(u_\eta))\right],$$

where S is the geodesic spray of (M, g).

The connector $K^1 : TT\mathcal{D}^s_{\mu,mix} \longrightarrow T\mathcal{D}^s_{\mu,mix}$ of $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$ is given by:

$$K^1(X_{u_\eta}) = \overline{\mathcal{P}}\bigg(\overline{(1-\alpha^2\mathcal{L})}^{-1} \circ \overline{(1-\alpha^2\mathcal{L})}(K \circ X_{u_\eta})$$
$$+ \overline{\mathfrak{F}}^\alpha\big(\pi_{T\mathcal{D}^s_{\mu,mix}}(X_{u_\eta}), T\pi_{\mathcal{D}^s_{\mu,mix}}(X_{u_\eta})\big)bigg),$$

where $K : TTM \longrightarrow TM$ is the connector of (M, g).

Because of the existence of the geodesic spray $\mathcal{S}^1 \in \mathfrak{X}^{C^\infty}(T\mathcal{D}^s_{\mu,mix})$ of the weak Riemannian manifold $(\mathcal{D}^s_{\mu,mix}, \mathcal{G}^1)$, we can define the sets $C^k_r(T\mathcal{D}^t_{\mu,mix})$, the Poisson bracket $\{\ ,\ \}^1$ on $C^k_r(T\mathcal{D}^t_{\mu,mix})$, the sets $C^k_{r,t}(\mathcal{V}^s_{\mu,mix})$ and $\mathcal{K}^k_{r,t}(\mathcal{V}^s_{\mu,mix})$, and the Poisson bracket $\{\ ,\ \}^1_+$ on $C^k_{r,t}(\mathcal{V}^s_{\mu,mix})$ exactly in the same way we did in the case of no-slip boundary conditions.

As we shall see, all the properties of the Poisson bracket $\{\ ,\ \}^1_+$ on $C^k_{r,t}(\mathcal{V}^s_{\mu,mix})$ (Theorem 4.6 and 4.7) are still true in the mixed case but since the Levi-Civita connection does not preserve the boundary conditions, the computations in the proofs are more subtle.

THEOREM 6.5. *Let $s > \frac{1}{2}\dim M + 1$ and $k \geq 1$. Then:*

$$\{\ ,\ \}^1_+ : \mathcal{K}^k(\mathcal{V}^s_{\mu,mix}) \times \mathcal{K}^k(\mathcal{V}^s_{\mu,D}) \longrightarrow \mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,mix})$$

and for all $u \in \mathcal{V}^{s+1}_{\mu,mix}$ we have

$$\begin{aligned}\delta(\{f,g\}^1_+)(u) &= \mathcal{P}_e(\nabla_{\delta g(u)}\delta f(u) - \nabla_{\delta f(u)}\delta g(u)) \\ &+ D\delta g(u)\left(\mathcal{P}_e\left((1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right) + B^\alpha(u, \delta f(u)\right) \\ &- D\delta f(u)\left(\mathcal{P}_e\left((1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right) + B^\alpha(u, \delta g(u)\right).\end{aligned}$$

Proof : Let $h := \{f,g\}^1_+$. We have to show that $h \in \mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,mix})$. As in Theorem 4.6 we obtain that $h \in C^k(\mathcal{V}^s_{\mu,mix})$, so we can compute $Dh(u)(v)$. Let $u, v \in \mathcal{V}^{s+1}_{\mu,mix}$. Using Lemmas 2.4, 6.1, 6.2, and 4.5 (still valid in the mixed case) we obtain:

$$
\begin{aligned}
&Dh(u)(v)\\
&= \langle v, \nabla_{\delta g(u)}\delta f(u)\rangle_1 + \langle u, \nabla_{D\delta g(u)(v)}\delta f(u)\rangle_1 + \langle u, \nabla_{\delta g(u)} D\delta f(u)(v)\rangle_1\\
&\quad - \langle v, \nabla_{\delta f(u)}\delta g(u)\rangle_1 - \langle u, \nabla_{D\delta f(u)(v)}\delta g(u)\rangle_1 - \langle u, \nabla_{\delta f(u)} D\delta g(u)(v)\rangle_1\\
&= \langle v, [\delta g(u), \delta f(u)]\rangle_1 + \langle u, [D\delta g(u)(v), \delta f(u)]\rangle_1 + \langle u, [\delta g(u), D\delta f(u)(v)]\rangle_1\\
&= \langle v, [\delta g(u), \delta f(u)]\rangle_1 + \langle (1-\alpha^2\Delta_r)u, [D\delta g(u)(v), \delta f(u)]\rangle_0\\
&\quad + \langle (1-\alpha^2\Delta_r)u, [\delta g(u), D\delta f(u)(v)]\rangle_0\\
&= \langle v, [\delta g(u), \delta f(u)]\rangle_1 + \langle (1-\alpha^2\Delta_r)u, \nabla_{D\delta g(u)(v)}\delta f(u)\rangle_0\\
&\quad - \langle (1-\alpha^2\Delta_r)u, \nabla_{\delta f(u)(v)} D\delta g(u)(v)\rangle_0 + \langle (1-\alpha^2\Delta_r)u, \nabla_{\delta g(u)} D\delta f(u)(v)\rangle_0\\
&\quad - \langle (1-\alpha^2\Delta_r)u, \nabla_{D\delta f(u)(v)}\delta g(u)\rangle_0\\
&= \langle v, [\delta g(u), \delta f(u)]\rangle_1 + \langle B^\alpha(u, \delta f(u)), D\delta g(u)(v)\rangle_1\\
&\quad + \langle (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u), D\delta g(u)(v)\rangle_1\\
&\quad - \langle (1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u), D\delta f(u)(v)\rangle_1\\
&\quad - \langle B^\alpha(u, \delta g(u)), D\delta f(u)(v)\rangle_1\\
&= \langle v, [\delta g(u), \delta f(u)]\rangle_1 + \langle D\delta g(u)\left(B^\alpha(u, \delta f(u))\right), v\rangle_1\\
&\quad + \langle D\delta g(u)\left(\mathcal{P}_e\left((1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta f(u)}u + \mathcal{D}^\alpha(\delta f(u), u)\right)\right), v\rangle_1\\
&\quad - \langle D\delta f(u)\left(\mathcal{P}_e\left((1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta g(u)}u + \mathcal{D}^\alpha(\delta g(u), u)\right)\right), v\rangle_1\\
&\quad - \langle D\delta f(u)\left(B^\alpha(u, \delta g(u))\right), v\rangle_1.
\end{aligned}
$$

Now the result follows as in Theorem 4.6. ■

THEOREM 6.6. *Let $s, t > \frac{1}{2}\dim M + 1$, $r \geq s$, and $k \geq 1$.*
(1) *$\{\ ,\ \}^1_+$ is $\mathbb{R}$-bilinear and anti-symmetric on $C^k_{r,t}(\mathcal{V}^s_{\mu,mix}) \times C^k_{r,t}(\mathcal{V}^s_{\mu,mix})$.*
(2) *$\{\ ,\ \}^1_+$ is a derivation in each factor:*

$$\{fg, h\}^1_+ = \{f,h\}^1_+ g + f\{g,h\}^1_+, \forall\, f,g,h \in C^k_{r,t}(\mathcal{V}^s_{\mu,mix}).$$

(3) *If $s > \frac{1}{2}\dim M + 2$, $\{\ ,\ \}^1_+$ satisfies the Jacobi identity:*
For all $f, g, h \in \mathcal{K}^k(\mathcal{V}^s_{\mu,mix})$ and $u \in \mathcal{V}^{s+1}_{\mu,mix}$ we have:

$$\{f, \{g,h\}^1_+\}^1_+(u) + \{g, \{h,f\}^1_+\}^1_+(u) + \{h, \{f,g\}^1_+\}^1_+(u) = 0$$

Proof: (1) This is obvious.
(2) Let $f,g,h\in C^k_{r,t}(\mathcal{V}^s_{\mu,mix})$, and $u\in\mathcal{V}^r_{\mu,mix}$. Using Lemmas 2.4 and 6.1 we find:

$$\begin{aligned}
\{fg,h\}^1_+(u) &= \langle u,[\delta(fg)(u),\delta h(u)]\rangle_1\\
&= \langle(1-\alpha^2\Delta_r)u,\nabla_{\delta(fg)(u)}\delta h(u)\rangle_0-\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta h(u)}\delta(fg)(u)\rangle_0\\
&= \langle(1-\alpha^2\Delta_r)u,\nabla_{\delta f(u)}\delta h(u)\rangle_0 g(u)+\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta g(u)}\delta h(u)\rangle_0 f(u)\\
&\qquad+\langle(1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta h(u)}u+\mathcal{D}^\alpha(\delta h(u),u),\delta(fg)(u)\rangle_1\\
&= \langle(1-\alpha^2\Delta_r)u,\nabla_{\delta f(u)}\delta h(u)\rangle_0 g(u)+\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta g(u)}\delta h(u)\rangle_0 f(u)\\
&\qquad+\langle(1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta h(u)}u+\mathcal{D}^\alpha(\delta h(u),u),\delta f(u)\rangle_1 g(u)\\
&\qquad+\langle(1-\alpha^2\mathcal{L})^{-1}(1-\alpha^2\mathcal{L})\nabla_{\delta h(u)}u+\mathcal{D}^\alpha(\delta h(u),u),\delta g(u)\rangle_1 f(u)\\
&= \langle(1-\alpha^2\Delta_r)u,\nabla_{\delta f(u)}\delta h(u)\rangle_0 g(u)+\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta g(u)}\delta h(u)\rangle_0 f(u)\\
&\qquad-\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta h(u)}\delta f(u)\rangle_0 g(u)-\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta h(u)}\delta g(u)\rangle_0 f(u)\\
&= \langle(1-\alpha^2\Delta_r)u,[\delta f(u),\delta h(u)]\rangle_0 g(u)+\langle(1-\alpha^2\Delta_r)u,[\delta g(u),\delta h(u)]\rangle_0 f(u)\\
&= \langle u,[\delta f(u),\delta h(u)]\rangle_1 g(u)+\langle u,[\delta g(u),\delta h(u)]\rangle_1 f(u)\\
&= \{f,h\}^1_+(u)g(u)+f(u)\{g,h\}^1_+(u).
\end{aligned}$$

(3) Let $f,g,h\in\mathcal{K}^k(\mathcal{V}^s_{\mu,mix})$, and $u\in\mathcal{V}^{s+1}_{\mu,D}$. By Theorem 4.6 we obtain $\{g,h\}^1_+\in\mathcal{K}^{k-1}_{s+1,s-1}(\mathcal{V}^s_{\mu,mix})\subset C^k_{s+1,s-1}(\mathcal{V}^s_{\mu,mix})$. Since $s-1>\frac{1}{2}\dim M+1$ we can compute the expression $\{f,\{g,h\}^1_+\}^1_+(u)$. We have:

$$\begin{aligned}
&\{f,\{g,h\}^1_+\}^1_+(u)\\
&\quad= \langle u,[\delta\{g,h\}^1_+(u),\delta f(u)]\rangle_1\\
&\quad= \langle(1-\alpha^2\Delta_r)u,\nabla_{\delta\{g,h\}^1_+(u)}\delta f(u)\rangle_1-\langle(1-\alpha^2\Delta_r)u,\nabla_{\delta f(u)}\delta\{g,h\}^1_+(u)\rangle_1
\end{aligned}$$

So we can use Lemmas 6.1 and 6.2 and then the expression for $\delta\{g,h\}^1_+(u)$ in Theorem 6.5. Doing exactly the same computation as in Theorem 4.7 and using analogous notations we find

$$\begin{aligned}
&\{f,\{g,h\}^1_+\}^1_+(u)\\
&= \langle\nabla_{[\delta h(u),\delta g(u)]}\delta f(u),(1-\alpha^2\Delta_r)u\rangle_0-\langle\nabla_{\delta f(u)}[\delta h(u),\delta g(u)],(1-\alpha^2\Delta_r)u\rangle_0\\
&+D_{hgf}-D_{ghf}=\langle[[\delta h(u),\delta g(u)],\delta f(u)],(1-\alpha^2\Delta_r)u\rangle_0+D_{hgf}-D_{ghf}\\
&= \langle[[\delta h(u),\delta g(u)],\delta f(u)],u\rangle_1+D_{hgf}-D_{gfh}.
\end{aligned}$$

Using the Jacobi identity for the Jacobi-Lie bracket of vector fields we obtain the desired result. ■

As in section 5, for $f\in C^k_r(\mathcal{V}^s_{\mu,mix})$, we shall denote $f_R:=f\circ\pi_R\in C^k(T\mathcal{D}^{s+k}_{\mu,mix})$. The proof of Lemma 5.1 remains valid in the mixed case, so if $f\in C^k_r(\mathcal{V}^s_{\mu,mix})$, $k\geq 1$ and $r>\frac{1}{2}\dim M+1$ are such that $s+k\geq r$, then the vertical functional derivative of f_R with respect to $\mathcal{G}^1$ exists and is given by:

$$\frac{\delta f_R}{\delta u}(u_\eta)=TR_\eta(\delta f(\pi_R(u_\eta)))\in T\mathcal{D}^r_{\mu,D},\qquad\forall\,u_\eta\in T\mathcal{D}^{s+k}_{\mu,mix}.$$

Lemma 5.3 about the horizontal functional derivative remains valid in the mixed case but some computations in the proof should be adapted to this case. These computations are given below.

Lemma 6.7. *Let $k \geq 1$ and $r > \frac{1}{2}\dim M + 2$ such that $s + k \geq r$. Let $f \in C_r^k(\mathcal{V}^s_{\mu,mix})$. Then the horizontal functional derivative of f_R with respect to $\mathcal{G}^1$ exists. It is given by:*

$$\frac{\delta f_R}{\delta \eta}(u_\eta) = \frac{1}{2} TR_h\Big[B^\alpha(u, \delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{P}_e\big(\mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u))\big)\Big]$$

for all $u_\eta \in T\mathcal{D}^{s+k}_{\mu,mix}$, where $u := \pi_R(u_\eta)$ and B^α was defined in Lemma 6.2. So we have:

$$\frac{\delta f_R}{\delta \eta}(u_\eta) \in T\mathcal{D}^r_{\mu,mix}, \qquad \forall\, u_\eta \in T\mathcal{D}^{s+k}_{\mu,mix}.$$

Proof : As in Lemma 5.3, we find for $u_\eta, v_\eta \in T\mathcal{D}^{s+k}_{\mu,D}$:

$$\begin{aligned}\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle &= -Df(u_\eta \circ \eta^{-1})(K^1(T(u_\eta \circ \eta^{-1}) \circ (v_\eta \circ \eta^{-1}))) \\ &= -Df(u)(K^1(Tu \circ v)) \qquad \text{where } u := u_\eta \circ \eta^{-1} \text{ and } v := v_\eta \circ \eta^{-1}.\end{aligned}$$

With the formula for the connector in Lemma 6.4 we obtain the following identity:

$$K^1(Tu \circ v) = \mathcal{P}_e((1 - \alpha^2\mathcal{L})^{-1}(1 - \alpha^2\mathcal{L})\nabla_v u + \mathfrak{F}^\alpha(u, v)).$$

So using formula (5.2) (still valid in the mixed case), and the notation $L^\alpha := (1 - \alpha^2\mathcal{L})^{-1}(1 - \alpha^2\mathcal{L})$ we find:

$$\begin{aligned}\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle &= -\langle \delta f(u), L^\alpha(\nabla_v u) + \frac{1}{2}(\mathcal{D}^\alpha(u, v) + \mathcal{D}^\alpha(v, u))\rangle_1 \\ &\qquad + \frac{1}{2}\langle \delta f(u), (1 - \alpha^2\mathcal{L})^{-1}\alpha^2\big(\operatorname{grad}(G(u, v))\big)\rangle_1 \\ &\qquad + \frac{1}{2}\langle \delta f(u), (1 - \alpha^2\mathcal{L})^{-1}\alpha^2\big(\nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\rangle_1 \\ &= -\langle \delta f(u), L^\alpha(\nabla_v u) + \frac{1}{2}(\mathcal{D}^\alpha(u, v) + \mathcal{D}^\alpha(v, u))\rangle_1 \\ &\qquad + \frac{1}{2}\langle \delta f(u), \alpha^2\big(\nabla u^t \cdot \Delta_r v + \nabla v^t \cdot \Delta_r u\big)\rangle_0 \\ &= -\langle \delta f(u), \frac{1}{2}L^\alpha(\nabla_v u) + \frac{1}{2}\mathcal{D}^\alpha(u, v) + \frac{1}{2}(\mathcal{D}^\alpha(v, u) + L^\alpha(\nabla_v u))\rangle_1 \\ &\qquad - \frac{1}{2}\langle \delta f(u), \nabla u^t \cdot (1 - \alpha^2\Delta_r)v\rangle_0 + \frac{1}{2}\langle \delta f(u), \nabla u^t \cdot v\rangle_0 \\ &\qquad - \frac{1}{2}\langle \delta f(u), \nabla v^t \cdot (1 - \alpha^2\Delta_r)u\rangle_0 + \frac{1}{2}\langle \delta f(u), \nabla v^t \cdot u\rangle_0.\end{aligned}$$

Using Lemmas 6.1 and 6.2, we obtain:

$$
\begin{aligned}
&2\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle \\
&= -\langle \delta f(u), L^\alpha(\nabla_v u)\rangle_1 - \langle \delta f(u), \mathcal{D}^\alpha(u,v)\rangle_1 + \langle (1-\alpha^2\Delta_r)u, \nabla_v \delta f(u)\rangle_0 \\
&\qquad - \langle \nabla_{\delta f(u)} u, (1-\alpha^2\Delta_r)v\rangle_0 \\
&- \langle \nabla_{\delta f(u)} v, (1-\alpha^2\Delta_r)u\rangle_0 \\
&= -\langle \delta f(u), L^\alpha(\nabla_v u)\rangle_1 - \langle \delta f(u), L^\alpha(\nabla_u v) + \mathcal{D}^\alpha(u,v)\rangle_1 + \langle \delta f(u), L^\alpha(\nabla_u v)\rangle_1 \\
&\qquad + \langle B^\alpha(u,\delta f(u)), v\rangle_1 - \langle \nabla_{\delta f(u)} u, (1-\alpha^2\Delta_r)v\rangle_0 + \langle L^\alpha(\nabla_{\delta f(u)} u) + \mathcal{D}^\alpha(\delta f(u), u), v\rangle_1 \\
&= -\langle \delta f(u), L^\alpha(\nabla_v u)\rangle_1 + \langle (1-\alpha^2\Delta_r)v, \nabla_u \delta f(u)\rangle_0 + \langle \delta f(u), L^\alpha(\nabla_u v)\rangle_1 \\
&\qquad + \langle B^\alpha(u,\delta f(u)), v\rangle_1 - \langle \nabla_{\delta f(u)} u, (1-\alpha^2\Delta_r)v\rangle_0 + \langle L^\alpha(\nabla_{\delta f(u)} u), v\rangle_1 \\
&\qquad + \langle \mathcal{D}^\alpha(\delta f(u), u), v\rangle_1 \\
&= \langle \delta f(u), L^\alpha(\nabla_u v - \nabla_v u)\rangle_1 + \langle (1-\alpha^2\Delta_r)v, \nabla_u \delta f(u) - \nabla_{\delta f(u)} u\rangle_0 \\
&\qquad + \langle L^\alpha(\nabla_{\delta f(u)} u), v\rangle_1 + \langle B^\alpha(u,\delta f(u)), v\rangle_1 + \langle \mathcal{D}^\alpha(\delta f(u), u), v\rangle_1 .
\end{aligned}
$$

Since the Jacobi-Lie bracket of vector fields preserves the mixed boundary condition we have:

$$
\begin{aligned}
&\langle \delta f(u), L^\alpha(\nabla_u v - \nabla_v u)\rangle_1 + \langle (1-\alpha^2\Delta_r)v, \nabla_u \delta f(u) - \nabla_{\delta f(u)} u\rangle_0 \\
&\qquad = \langle \delta f(u), \nabla_u v - \nabla_v u\rangle_1 + \langle v, \nabla_u \delta f(u) - \nabla_{\delta f(u)} u\rangle_1 \\
&\qquad = \langle (1-\alpha^2\Delta_r)\delta f(u), \nabla_u v - \nabla_v u\rangle_0 + \langle v, L^\alpha(\nabla_u \delta f(u) - \nabla_{\delta f(u)} u)\rangle_1 \\
&\qquad = \langle (1-\alpha^2\Delta_r)\delta f(u), \nabla_u v\rangle_0 - \langle (1-\alpha^2\Delta_r)\delta f(u), \nabla_v u\rangle_0 \\
&\qquad\qquad + \langle v, L^\alpha(\nabla_u \delta f(u)) - \langle v, L^\alpha(\nabla_{\delta f(u)} u)\rangle_1 \\
&\qquad = -\langle L^\alpha(\nabla_u \delta f(u)) + \mathcal{D}^\alpha(u, \delta f(u), v\rangle_1 - \langle B^\alpha(\delta f(u), u), v\rangle_1 \\
&\qquad\qquad + \langle v, L^\alpha(\nabla_u \delta f(u)) - \langle v, L^\alpha(\nabla_{\delta f(u)} u)\rangle_1 \\
&\qquad = -\langle \mathcal{D}^\alpha(u, \delta f(u), v\rangle_1 - \langle B^\alpha(\delta f(u), u), v\rangle_1 - \langle v, L^\alpha(\nabla_{\delta f(u)} u)\rangle_1 .
\end{aligned}
$$

So we obtain

$$
2\left\langle \frac{\partial f_R}{\partial \eta}(u_\eta), v_\eta \right\rangle = \langle B^\alpha(u,\delta f(u)) - B^\alpha(\delta f(u), u) + \mathcal{D}^\alpha(\delta f(u), u) - \mathcal{D}^\alpha(u, \delta f(u)), v\rangle_1
$$

and the result follow. ■

We conclude that Theorem 5.4 remains valid in the mixed case. For proving that π_R is a Poisson map (in the sense of Theorem 5.5) it suffices to use Lemmas 6.1 and 6.2 insteed of Lemmas 4.3 and 4.4 in the proof of Theorem 5.5. So Theorems 5.6, 5.7, and 5.8 are also valid in this case.

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