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Anti-Periodic Forced Oscillations of Damped Beams on Elastic Bearings

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ABSTRACT. We show the existence of anti-periodic solutions for certain damped linear beam equations with anti-periodic forcing terms and resting on nonlinear elastic bearings.

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1. Introduction

In this paper, we consider an anti-periodically forced and damped beam resting on two different bearings with purely elastic responses. The length of the beam is $\pi/4$. The equation of vibrations is as follows

$$
\begin{aligned}
&u_{tt} + u_{xxxx} + \delta u_t + h_1(x,t) = 0\,,\\
&u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,,\\
&u_{xxx}(0,\cdot) = -f(u(0,\cdot)) - h_2(t)\,,\\
&u_{xxx}(\pi/4,\cdot) = g(u(\pi/4,\cdot)) + h_3(t)\,,
\end{aligned}
\tag{1.1}
$$

where $u = u(x,t)$, $\delta > 0$ is a constant, $f, g \in C(\mathbb{R},\mathbb{R})$ are odd functions and $h_1 \in X$, h_2, $h_3 \in Y$ are anti-periodic forcing terms. Here X and Y are the following Banach

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spaces

$$X := \Big\{ h \in C([0,\pi/4]\times\mathbb{R},\mathbb{R}) \mid h(x,t+T) = -h(x,t) \quad \text{for any} \quad (x,t)\in[0,\pi/4]\times\mathbb{R} \Big\},$$
$$Y := \Big\{ h \in C(\mathbb{R},\mathbb{R}) \mid h(t+T) = -h(t) \quad \text{for any} \quad t\in\mathbb{R} \Big\}$$

endowed with the usual sup norms $\|\cdot\|$ for a fixed $T>0$.

Recently, we investigated the existence of periodic solutions of (1.1) for general non-odd functions f, g and T-periodic $h_1(x,t)$ with $h_2(t)=h_3(t)=0$. In [**9**] and [**10**], we proved existence and non-existence results for T-periodic solutions of (1.1) depending on the forcing function $h_1(x,t)$. Chaotic solutions for equations similar to (1.1) are considered in [**2**] and [**3**]. The existence of free vibrations of undamped and unforced equations like (1.1) is studied in [**8**] and [**11**] by using variational methods.

Now we study the existence of anti-periodic (weak) solutions $u\in X$ of (1.1). The plan of the paper is as follows. In Section 2, we formulate the notion of a weak T-anti-periodic solution of (1.1). We also recall some well-known results on the corresponding linear eigenvalue problem. Then in Section 3, we study linear problems and certain Poincaré type inequalities related to (1.1). Section 4 contains the main existence result for weak T-anti-periodic solutions of (1.1), when in addition $h_2,h_3\in W^{1,2}(0,T)$. The approach relies on topological degree arguments. Some results are also presented for semilinear problems by assuming only $h_2,h_3\in Y$. In the final Section 5, we extend the main result (Theorem 4.2) of Section 4 to a discontinuous/multivalued case (cf. (5.2)). There we suppose that the functions f and g are upper semicontinuous with compact interval values.

Finally we note (cf. [**11**]) that equation (1.1) is a simple analogue of a more complicated shaft dynamics model introduced in [**5**] and [**6**].

2. Setting of the problem

By a weak T-anti-periodic solution of (1.1), we mean any $u\in X$ satisfying the identity

$$\begin{aligned}\int_0^T\int_0^{\pi/4}\Big[u(x,t)\Big\{v_{tt}(x,t)+v_{xxxx}(x,t)-\delta v_t(x,t)\Big\}+h_1(x,t)v(x,t)\Big]\,dx\,dt\\ +\int_0^T\Big\{\Big(f(u(0,t))+h_2(t)\Big)v(0,t)+\Big(g(u(\pi/4,t))+h_3(t)\Big)v(\pi/4,t)\Big\}\,dt=0\end{aligned}\tag{2.1}$$

for any $v\in X^\infty$ with

$$X^\infty := \Big\{v\in X\cap C^\infty([0,\pi/4]\times\mathbb{R}) \mid v_{xx}(0,\cdot)=v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot)=v_{xxx}(\pi/4,\cdot)=0\Big\}$$

The eigenvalue problem

$$\begin{aligned}&w_{xxxx}(x)=\mu^4 w(x)\,,\\ &w_{xx}(0)=w_{xx}(\pi/4)=0,\\ &w_{xxx}(0)=w_{xxx}(\pi/4)=0\end{aligned}$$

is known [**11**] to possess a sequence of eigenvalues μ_k, $k=-1,0,1,\cdots$ with

$$\mu_{-1}=\mu_0=0$$

and

$$\cos(\mu_k \pi/4)\cosh(\mu_k \pi/4) = 1, \quad k = 1, 2, \cdots . \tag{2.2}$$

The corresponding orthonormal system of eigenvectors in $L^2(0, \pi/4)$ is

$$w_{-1}(x) = \frac{2}{\sqrt{\pi}}, \quad w_0(x) = \frac{16}{\pi}\left(x - \frac{\pi}{8}\right)\sqrt{\frac{3}{\pi}}$$
$$w_k(x)\frac{4}{\sqrt{\pi}W_k}\Big[\cosh(\mu_k x) + \cos(\mu_k x) - \frac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\big(\sinh(\mu_k x) + \sin(\mu_k x)\big)\Big]$$

where the constants W_k are given by the formulas

$$W_k = \cosh(\xi_k) + \cos\xi_k - \frac{\cosh\xi_k - \cos\xi_k}{\sinh\xi_k - \sin\xi_k}\big(\sinh\xi_k + \sin\xi_k\big)$$

for $\xi_k = \mu_k \pi/4$. From (2.2) we get the asymptotic formulas

$$1 < \mu_k = 2(2k+1) + r(k) \quad \forall k \geq 1$$

along with

$$|r(k)| \leq \bar{c}_1 e^{-\bar{c}_2 k} \quad \forall k \geq 1\,,$$

where $\bar{c}_1$, $\bar{c}_2$ are positive constants. Moreover, the eigenfunctions $\{w_i\}_{i=-1}^{\infty}$ are uniformly bounded in $C[0, \pi/4]$.

3. Linear Equations

Let $H_1 \in X$, H_2, $H_3 \in Y$. In order to solve (2.1), we consider the equation

$$\begin{aligned} &\int_0^T \int_0^{\pi/4} \Big[u(x,t)\Big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\Big\} + H_1(x,t)v(x,t)\Big]\,dx\,dt \\ &+ \int_0^T \Big\{H_2(t)v(0,t) + H_3(t)v(\pi/4,t)\Big\}\,dt = 0 \end{aligned} \tag{3.1}$$

for any $v \in X^\infty$. We look for $u(x,t)$ in the form

$$u(x,t) = \sum_{i=-1}^{\infty} z_i(t) w_i(x)\,. \tag{3.2}$$

We formally put (3.2) into (3.1) to get a system of ordinary differential equations

$$\ddot{z}_i(t) + \delta\dot{z}_i(t) + \mu_i^4 z_i(t) = h_i(t)\,, \tag{3.3}$$

where

$$h_i(t) = -\Big(\int_0^{\pi/4} H_1(x,t)w_i(x)\,dx + H_2(t)w_i(0) + H_3(t)w_i(\pi/4)\Big)\,. \tag{3.4}$$

Clearly $h_i \in Y$ for any $i \geq -1$. Since $\mu_i > 0$ for $i \geq 1$, we reason as in [**10**] to conclude that equation (3.3) has a unique T-anti-periodic solution $z_i \in Y$, namely:

(i): for $2\mu_i^2 > \delta$, z_i is given by

$$z_i(t) = \frac{2}{\bar{\omega}_i} \int_{-\infty}^{t} e^{-\delta(t-s)/2} \sin\Big(\frac{\bar{\omega}_i}{2}(t-s)\Big) \times h_i(s)\,ds\,, \tag{3.5}$$

where $\bar{\omega}_i = \sqrt{4\mu_i^4 - \delta^2}$;

(ii): for $2\mu_i^2 = \delta$, z_i is given by

$$z_i(t) = \int_{-\infty}^{t} e^{-\delta(t-s)/2}(t-s) \times h_i(s)\,ds\,; \tag{3.6}$$

(iii): for $2\mu_i^2 < \delta$, z_i is given by

$$z_i(t) = \int_{-\infty}^{t} \frac{1}{\tilde{\omega}_i}\Big(e^{(-\delta+\tilde{\omega}_i)(t-s)/2} - e^{(-\delta-\tilde{\omega}_i)(t-s)/2}\Big) \times h_i(s)\,ds\,, \tag{3.7}$$

where $\tilde{\omega}_i = \sqrt{\delta^2 - 4\mu_i^4}$.

Like in [**10**], from (3.5)-(3.7) we get

$$\begin{aligned} \|z_i\| &\le \frac{1}{\mu_i^2}\Big(1 + \frac{4}{\delta}\Big)\|h_i\| \\ \|\dot{z}_i\| &\le \Big(\frac{4}{\delta} + \delta\Big)\|h_i\|\,, \end{aligned} \tag{3.8}$$

for any $i \ge 1$. Since $\mu_i = 0$ for $i = -1$ and $i = 0$, we see that (3.3) has a unique solution $z_i \in Y$ for $i = -1, 0$ on $[0, T]$ given by

$$\begin{aligned} z_i(t) &= \frac{1}{\delta}\int_0^t h_i(s)\,ds - \frac{1}{2\delta}\int_0^T h_i(s)\,ds \\ &-\frac{1}{\delta}\int_0^t h_i(s)h_i(s)\,ds + \frac{\int_0^T e^{-\delta(T+t-s)}h_i(s)\,ds}{\delta(1+e^{-\delta T}}\,, \end{aligned}$$

which yields

$$\begin{aligned} \|z_i\| &\le \Big(\frac{3T}{2\delta} + \frac{3}{\delta^2}\Big)\|h_i\|\,, \\ \|\dot{z}_i\| &\le \frac{2}{\delta}\|h_i\|\,. \end{aligned} \tag{3.9}$$

From (3.4) we get

$$\|h_i\| \le M_1\Big(\frac{\pi}{4}\|H_1\| + \|H_2\| + \|H_3\|\Big) \tag{3.10}$$

for

$$M_1 := \sup_{i\ge 1, x} |w_i(x)|\,.$$

Plugging (3.5)-(3.7) into (3.2) and using (3.8)-(3.10), we obtain

$$\begin{aligned}\|u\| &\le \sum_{i=-1}^{\infty} \|z_i\|\,\|w_i\| \le M_1^2\Big\{\frac{3T}{\delta}+\frac{6}{\delta^2}\\ &+\Big(1+\frac{4}{\delta}\Big)\sum_{i=1}^{\infty}\frac{1}{\mu_i^2}\Big\}\Big(\frac{\pi}{4}\|H_1\|+\|H_2\|+\|H_3\|\Big)\\ &\le M_2\Big(\|H_1\|+\|H_2\|+\|H_3\|\Big)\end{aligned}$$

for

$$M_2 := M_1^2\Big\{\frac{3T}{\delta}+\frac{6}{\delta^2}+\Big(1+\frac{4}{\delta}\Big)\sum_{i=1}^{\infty}\frac{1}{\mu_i^2}\Big\}\,.$$

We note that $\sum\limits_{i=1}^{\infty}\frac{1}{\mu_i^2}<\infty$. Summarizing the above results, we arrive at:

Proposition 3.1. *For any given functions $H_1 \in X$, $H_2, H_3 \in Y$, there is a unique solution $L(H_1, H_2, H_3) := u(x,t) \in X$ of equation (3.1). The linear mapping $L : X \times Y \times Y \to X$ is compact with the norm $\|L\| \le M_2$ when the norm on $V := X \times Y \times Y$ is given by $\|v\| := \|H_1\| + \|H_2\| + \|H_3\|$, $v = (H_1, H_2, H_3) \in V$.*

Now let us fix $n \in \mathbb{N}$ and consider an approximating linear problem to (3.1), namely

$$\begin{aligned}&\int_0^T\int_0^{\pi/4}\Big[u_n(x,t)\Big\{v_{tt}(x,t)+v_{xxxx}(x,t)-\delta v_t(x,t)\Big\}+H_1(x,t)v(x,t)\Big]\,dx\,dt\\ &+\int_0^T\Big\{H_2(t)v(0,t)+H_3(t)v(\pi/4,t)\Big\}\,dt=0\end{aligned}\tag{3.11}$$

for any $v \in X_n^\infty$ with

$$X_n^\infty := \Big\{v \in X^\infty \mid v(x,t)=\sum_{i=-1}^{n} v_i(t)w_i(x) \quad \text{for} \quad v_i \in Y \cap C^\infty(\mathbb{R},\mathbb{R})\Big\}\,.$$

Here wee look for $u_n(x,t)$ in the form

$$u_n(x,t)=\sum_{i=-1}^{n} z_i(t)w_i(x)\,.$$

By repeating the above approach to (3.1) for (3.11), we arrive at the following result.

Proposition 3.2. *For any given functions $H_1 \in X$, $H_2, H_3 \in Y$, equation (3.11) has a unique solution $u_n \in X$ of the form*

$$u_n(x,t)=\sum_{i=-1}^{n} z_i(t)w_i(x)\,.$$

Such a solution satisfies the following conditions:

(a):

$$\max_{-1\le i\le n}\|z_i\|(i^2+1)\le M_3(\|H_1\|+\|H_2\|+\|H_3\|)$$

$$\max_{-1\le i\le n}\|\dot z_i\|\le M_3(\|H_1\|+\|H_2\|+\|H_3\|)$$

for

$$M_3 := \sup_{i\geq 1}\Big\{\frac{i^2+1}{\mu_i^2}\Big\}\Big(1+\frac{4}{\delta}\Big)+\frac{3T}{\delta}+\frac{6}{\delta^2}+\frac{4}{\delta}+\delta\,.$$

(b): *The linear mapping* $L_n : X\times Y\times Y\to X$ *defined by* $L_n(H_1,H_2,H_3) := u_n(x,t)$ *is compact.*

Now we recall the following result from [**10**]:

Proposition 3.3. *A sequence* $\{u_n\}_{n=1}^\infty \subset X$ *is precompact if there is a constant* $M>0$ *such that*

$$\sup_{i\geq -1, n\geq 1}\|z_{i,n}\|(i^2+1)<M,\qquad \sup_{i\geq -1,n\geq 1}\|\dot z_{i,n}\|<M\,,$$

where $u_n(x,t)=\sum\limits_{i=-1}^{\infty} z_{i,n}(t)w_i(x)$.

We end this section with two Poincaré inequalities.

Proposition 3.4. *The following Poincaré inequality holds*

$$\|w\|\leq \frac{\pi^{3/2}}{8\sqrt{105}}\sqrt{\int_0^{\pi/4} w_{xx}(x)^2\,dx}$$

for any $w\in W^{2,2}(0,\pi/4)$ *satisfying*

$$\int_0^{\pi/4} w(x)\,dx=\int_0^{\pi/4} xw(x)\,dx=0\,. \tag{3.12}$$

Proof of Proposition 3.4. For any $h\in L^2(0,\pi/4)$, the solution $w(x)$ of the differential equation

$$w_{xx}(x)=h(x)$$

which satisfies conditions (3.12) is given by

$$w(x)=\int_0^{\pi/4} G(x,s)h(s)\,ds$$

for a Green function G defined by

$$G(x,s)=\begin{cases}\Big(\frac{48}{\pi^2}s^2-\frac{128}{\pi^3}s^3\Big)x+\frac{16}{\pi^2}s^3-\frac{8}{\pi}s^2 & \text{for}\quad 0\leq s\leq x\leq \frac{\pi}{4}\\[2mm] \Big(\frac{48}{\pi^2}s^2-\frac{128}{\pi^3}s^3-1\Big)x+\frac{16}{\pi^2}s^3-\frac{8}{\pi}s^2+s & \text{for}\quad 0\leq x\leq s\leq \frac{\pi}{4}\,.\end{cases}$$

Thus for any $x\in[0,\pi/4]$ we have

$$|w(x)|\leq \max_{x\in[0,\pi/4]}\sqrt{\int_0^{\pi/4} G(x,s)^2\,ds}\sqrt{\int_0^{\pi/4} h(s)^2\,ds}\,.$$

By using *Mathematica*, we compute

$$\int_0^{\pi/4} G(x,s)^2\,ds = \frac{1}{6720\pi^3}\Big(\pi^6 - 44\pi^5 x + 624\pi^4 x^2 - 2240\pi^3 x^3 - 8960\pi^2 x^4 + 64512\pi x^5 - 86016 x^6\Big)\,, \tag{3.13}$$

and check that the maximum of the right-hand side of (3.13) on the interval $[0,\pi/4]$ is $\pi^3/6720$, which is attained at the end points $x=0$ and $x=\pi/4$. Consequently, we obtain

$$|w(x)| \le \max_{x\in[0,\pi/4]} \sqrt{\int_0^{\pi/4} G(x,s)^2\,ds}\sqrt{\int_0^{\pi/4} h(s)^2\,ds} \le \frac{\pi^{3/2}}{8\sqrt{105}}\sqrt{\int_0^{\pi/4} h(s)^2\,ds}\,.$$

The proof is complete.

PROPOSITION 3.5. *Let $\widetilde{X}$ be a Banach space with a norm $|\cdot|$. Then the following Poincaré inequality holds*

$$\max_{t\in[0,T]} |h(t)| \le \sqrt{T}\sqrt{\int_0^T |\dot h(t)|^2\,dt}$$

for any T-anti-periodic function $h \in W^{1,2}(0,T;\widetilde{X})$.

For the proof of Proposition 3.5, see [**1**].

4. Nonlinear Equations

First, we suppose that in addition to the conditions listed in the Introduction, $h_2, h_3 \in W^{1,2}(0,T)$. Now we use the Bubnov-Galerkin approximation method. So we put the form

$$u_n(x,t) = \sum_{i=-1}^{n} z_i(t) w_i(x) \tag{4.1}$$

into (1.1) to derive the system of ordinary differential equations

$$\begin{aligned} &\ddot z_i(t) + \delta \dot z_i(t) + \mu_i^4 z_i(t) + h_{1,i}(t) \\ &+ f\Big(\sum_{i=-1}^{n} z_i(t) w_i(0)\Big) w_i(0) + g\Big(\sum_{i=-1}^{n} z_i(t) w_i(\pi/4)\Big) w_i(\pi/4) \\ &+ h_2(t) w_i(0) + h_3(t) w_i(\pi/4) = 0\,, \end{aligned} \tag{4.2}$$

where

$$h(x,t) = \sum_{i=-1}^{\infty} h_{1,i}(t) w_i(x)\,.$$

The system (4.2) is a Bubnov-Galerkin approximation of (1.1). Now we solve (4.2). For this purpose, we consider a Banach space $Z_n = Y^{n+2}$ with the norm

$$\|z\|_n := \|u_n\|\,,$$

where $u_n(x,t)$ is defined by (4.1) for $z = \big(z_{-1}(t), z_0(t), z_1(t), \cdots, z_n(t)\big) \in Z_n$. Next we introduce the following nonlinear operator

$$\begin{aligned}
&F_n : Z_n \to Z_n\\
&F_n(z) := \big(\widetilde{z}_{-1}(t), \widetilde{z}_0(t), \widetilde{z}_1(t), \cdots, \widetilde{z}_n(t)\big)\\
&\sum_{i=-1}^{n} \widetilde{z}_i(t) w_i(x) := L_n\Bigg(h_1, f\Big(\sum_{i=-1}^{n} z_i(t)w_i(0)\Big) + h_2(t), g\Big(\sum_{i=-1}^{n} z_i(t)w_i(\pi/4)\Big) + h_3(t)\Bigg)\\
&z = \big(z_{-1}(t), z_0(t), z_1(t), \cdots, z_n(t)\big)\,.
\end{aligned}$$

We note that according to Proposition 3.2, the operator F_n is compact. Then (4.2) is equivalent to the fixed point problem

$$z = F_n(z)\,. \tag{4.3}$$

In order to solve (4.3) uniformly for $n \in \mathbb{N}$, by the Leray-Schauder degree theory for maps [**4**], [**12**], it is enough to show that there is a constant $c_1 > 0$ such that for any $\lambda \geq 1$ and $n \in \mathbb{N}$, every solution of the equation

$$\lambda z = F_n(z) \tag{4.4}$$

satisfies $\|z\|_n \leq c_1$. But this means that we must find an a-priori bound for the T-anti-periodic solutions of the system

$$\begin{aligned}
&\lambda \ddot{z}_i(t) + \lambda\delta \dot{z}_i(t) + \lambda \mu_i^4 z_i(t) + h_{1,i}(t)\\
&+ f\Big(\sum_{i=-1}^{n} z_i(t)w_i(0)\Big)w_i(0) + g\Big(\sum_{i=-1}^{n} z_i(t)w_i(\pi/4)\Big)w_i(\pi/4)\\
&+ h_2(t)w_i(0) + h_3(t)w_i(\pi/4) = 0\,,
\end{aligned} \tag{4.5}$$

for any $\lambda \geq 1$.

To do this, we first multiply (4.5) by $\dot{z}_i(t)$, integrate the result from 0 to T, and then sum up these equations to obtain

$$\begin{aligned}
&\sum_{i=-1}^{n} \lambda \int_0^T \ddot{z}_i(t)\dot{z}_i(t)\,dt + \sum_{i=-1}^{n} \lambda\delta \int_0^T \dot{z}_i(t)^2\,dt + \lambda \sum_{i=-1}^{n} \mu_i^4 \int_0^T z_i(t)\dot{z}_i(t)\,dt\\
&+ \int_0^T f\Big(\sum_{i=-1}^{n} z_i(t)w_i(0)\Big)\Big(\sum_{i=-1}^{n} \dot{z}_i(t)w_i(0)\Big)\,dt +\\
&\int_0^T g\Big(\sum_{i=-1}^{n} z_i(t)w_i(\pi/4)\Big)\Big(\sum_{i=-1}^{n} \dot{z}_i(t)w_i(\pi/4)\Big)\,dt\\
&+ \int_0^T \sum_{i=-1}^{n} h_{1,i}(t)\dot{z}_i(t)\,dt + \int_0^T \sum_{i=-1}^{n} h_2(t)\dot{z}_i(t)w_i(0)\,dt +\\
&\int_0^T \sum_{i=-1}^{n} h_3(t)\dot{z}_i(t)w_i(\pi/4)\,dt = 0\,.
\end{aligned}$$

This implies

$$\begin{aligned}
&\lambda\delta \sum_{i=-1}^{n} \int_0^T \dot{z}_i(t)^2\,dt + \int_0^T \sum_{i=-1}^{n} h_{1,i}(t)\dot{z}_i(t)\,dt\\
&- \sum_{i=-1}^{n} \int_0^T \dot{h}_2(t)z_i(t)w_i(0)\,dt - \sum_{i=-1}^{n} \int_0^T \dot{h}_3(t)z_i(t)w_i(\pi/4)\,dt = 0\,,
\end{aligned}$$

and hence

$$\lambda\delta\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt \le \sqrt{\int_0^T\sum_{i=-1}^{n} h_{1,i}(t)^2\,dt}\sqrt{\int_0^T\sum_{i=-1}^{n}\dot z_i(t)^2\,dt}+$$
$$+M_1\Bigg(\sqrt{\int_0^T \dot h_2(t)^2\,dt}+\sqrt{\int_0^T \dot h_3(t)^2\,dt}\Bigg)\sqrt{2+\sum_{i=1}^{n}\frac{1}{\mu_i^4}}$$
$$\sqrt{\int_0^T\Big(z_{-1}(t)^2+z_0(t)^2+\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\Big)\,dt}$$
$$\le \widetilde K\Bigg(\sqrt{\int_0^T\sum_{i=-1}^{n}\dot z_i(t)^2\,dt}+\sqrt{\int_0^T\Big(z_{-1}(t)^2+z_0(t)^2+\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\Big)\,dt}\Bigg)$$
$$\le \widetilde K\Bigg(\sqrt{\int_0^T\sum_{i=-1}^{n}\dot z_i(t)^2\,dt}+\sqrt{\int_0^T\big(z_{-1}(t)^2+z_0(t)^2\big)\,dt}+\sqrt{\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt}\Bigg).$$

The Poincaré inequality of Proposition 3.5 gives

$$\lambda\delta\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt \le \widetilde K\Bigg((1+T)\sqrt{\int_0^T\sum_{i=-1}^{n}\dot z_i(t)^2\,dt}+\sqrt{\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt}\Bigg). \tag{4.6}$$

Next, we multiply (4.5) by $z_i(t)$, integrate from 0 to T, and then sum up these equations to arrive at

$$\sum_{i=-1}^{n}\lambda\int_0^T \ddot z_i(t)z_i(t)\,dt+\sum_{i=-1}^{n}\lambda\delta\int_0^T \dot z_i(t)z_i(t)\,dt+\lambda\sum_{i=-1}^{n}\mu_i^4\int_0^T z_i(t)^2\,dt$$
$$+\int_0^T f\Big(\sum_{i=-1}^{n} z_i(t)w_i(0)\Big)\Big(\sum_{i=-1}^{n} z_i(t)w_i(0)\Big)\,dt+$$
$$\int_0^T g\Big(\sum_{i=-1}^{n} z_i(t)w_i(\pi/4)\Big)\Big(\sum_{i=-1}^{n} z_i(t)w_i(\pi/4)\Big)\,dt$$
$$+\int_0^T\sum_{i=-1}^{n} h_{1,i}(t)z_i(t)\,dt+\int_0^T\sum_{i=-1}^{n} h_2(t)z_i(t)w_i(0)\,dt+$$
$$\int_0^T\sum_{i=-1}^{n} h_3(t)z_i(t)w_i(\pi/4)\,dt=0\,.$$

This gives

$$
\begin{aligned}
&\lambda \sum_{i=-1}^{n} \mu_i^4 \int_0^T z_i(t)^2\,dt = \lambda \sum_{i=-1}^{n} \int_0^T \dot z_i(t)^2\,dt \\
&- \int_0^T f\Big(\sum_{i=-1}^{n} z_i(t) w_i(0)\Big)\Big(\sum_{i=-1}^{n} z_i(t) w_i(0)\Big)\,dt - \\
&\int_0^T g\Big(\sum_{i=-1}^{n} z_i(t) w_i(\pi/4)\Big)\Big(\sum_{i=-1}^{n} z_i(t) w_i(\pi/4)\Big)\,dt \\
&- \int_0^T \sum_{i=-1}^{n} h_{1,i}(t) z_i(t)\,dt - \int_0^T \sum_{i=-1}^{n} h_2(t) z_i(t) w_i(0)\,dt - \int_0^T \sum_{i=-1}^{n} h_3(t) z_i(t) w_i(\pi/4)\,dt\,.
\end{aligned}
$$

We impose the following condition

- (H) There are non-negative constants α_f and α_g, with

$$(\alpha_f + \alpha_g) M_1 \Big(2 + \sum_{i=1}^{\infty} \frac{1}{\mu_i^4}\Big) < 1 \tag{4.7}$$

  and a non-negative constant β such that

$$f(u)u \geq -\alpha_f u^2 - \beta, \quad g(u)u \geq -\alpha_g u^2 - \beta \tag{4.8}$$

  for any $u \in \mathbb{R}$.

As a result, we obtain

$$
\begin{aligned}
&\sum_{i=-1}^{n} \mu_i^4 \int_0^T z_i(t)^2\,dt \le \lambda \sum_{i=-1}^{n} \int_0^T \dot z_i(t)^2\,dt \\
&+\sqrt{\int_0^T \sum_{i=-1}^{n} h_{1,i}(t)^2\,dt}\sqrt{\int_0^T \sum_{i=-1}^{n} z_i(t)^2\,dt + 2\beta} \\
&+(\alpha_f+\alpha_g)M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)\Big(T^2\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt + \int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt\Big) \\
&+M_1\bigg(\sqrt{\int_0^T h_2(t)^2\,dt}+\sqrt{\int_0^T h_3(t)^2\,dt}\bigg)\sqrt{2+\sum_{i=1}^{n}\frac{1}{\mu_i^4}} \\
&\times\sqrt{\int_0^T\Big(z_{-1}(t)^2+z_0(t)^2+\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\Big)\,dt} \\
&\le \bar K_1\bigg((\lambda+1)\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt + \sqrt{\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt + 1} \\
&+\sqrt{\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt}\bigg) + (\alpha_f+\alpha_g)M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt \\
&\le \bar K\bigg(\lambda\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt + \sqrt{\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt + 1} \\
&+\sqrt{\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt}\bigg) + (\alpha_f+\alpha_g)M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt\,.
\end{aligned}
\tag{4.9}
$$

Here $\bar K_1$ and $\bar K$ are positive constants which are independent of $z_i(t)$ and n. By denoting

$$
A_n := \sqrt{\sum_{i=-1}^{n}\int_0^T \dot z_i(t)^2\,dt}, \quad B_n := \sqrt{\int_0^T\sum_{i=1}^{n}\mu_i^4 z_i(t)^2\,dt}\,,
$$

inequalities (4.6) and (4.9) take the forms

$$
\begin{aligned}
&\lambda A_n^2 \le \hat K(A_n+B_n) \\
&B_n^2 \le \hat K(\lambda A_n^2 + A_n + 1 + B_n) + (\alpha_f+\alpha_g)M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)B_n^2
\end{aligned}
\tag{4.10}
$$

for a constant $\hat K$ depending on the constants α_f, α_g, β, δ, T and functions $h_1(x,t)$, $h_2(t)$, $h_3(t)$. By putting

$$
\gamma := 1-(\alpha_f+\alpha_g)M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)
$$

from (4.10) we get

$$\begin{aligned}\lambda A_n^2 &\le \hat{K}(A_n+B_n)\\ B_n^2 &\le \frac{\hat{K}}{\gamma}(\lambda A_n^2+A_n+1+B_n)\,.\end{aligned} \tag{4.11}$$

Now, (4.11) implies

$$\begin{aligned}A_n^2 &\le \hat{K}(A_n+B_n)\\ B_n^2 &\le \Big(\frac{\hat{K}}{\gamma}+\frac{\hat{K}^2}{\gamma}\Big)(A_n+B_n)+\frac{\hat{K}}{\gamma}\end{aligned}$$

and then

$$\big(A_n+B_n\big)^2/2 \le A_n^2+B_n^2 \le \Big(\hat{K}+\frac{\hat{K}}{\gamma}+\frac{\hat{K}^2}{\gamma}\Big)(A_n+B_n)+\frac{\hat{K}}{\gamma}\,,$$

which gives

$$A_n+B_n \le \Gamma \tag{4.12}$$

for

$$\Gamma := \Big(\hat{K}+\frac{\hat{K}}{\gamma}+\frac{\hat{K}^2}{\gamma}\Big)+\sqrt{\Big(\hat{K}+\frac{\hat{K}}{\gamma}+\frac{\hat{K}^2}{\gamma}\Big)^2+\frac{2\hat{K}}{\gamma}}\,.$$

Now from Section 2, we immediately obtain

$$\int_0^{\pi/4} w_{i,xx}(x)^2\,dx=\mu_i^4,\quad \int_0^{\pi/4} w_{i,xx}(x)w_{j,xx}(x)\,dx=0 \quad \text{for} \quad i\ne j\,. \tag{4.13}$$

Then (4.1), (4.12) and (4.13) imply

$$\int_0^{\pi/4} u_{n,xx}(x,t)^2\,dx=\sum_{i=1}^n \mu_i^4 z_i(t)^2 \le \Gamma^2$$

for any $t\in\mathbb{R}$. Hence the Poincaré inequality of Proposition 3.4 gives

$$|\bar{u}_n(x,t)| \le \frac{\pi^{3/2}}{8\sqrt{105}}\Gamma \tag{4.14}$$

for any $(x,t)\in[0,\pi/4]\times\mathbb{R}$ and

$$\bar{u}_n(x,t)=\sum_{i=1}^n z_i(t)w_i(x)\,.$$

On the other hand, the Poincaré inequality of Proposition 3.5 and estimate (4.12) imply

$$|z_{-1}(t)w_{-1}(x)+z_0(t)w_0(x)| \le 2M_1\sqrt{T}\Gamma\,. \tag{4.15}$$

Consequently, estimates (4.14) and (4.15) give

$$\|u_n\| \le \Theta := \Big(2M_1\sqrt{T}+\frac{\pi^{3/2}}{8\sqrt{105}}\Big)\Gamma\,. \tag{4.16}$$

Summarizing, we obtain the following result.

PROPOSITION 4.1. *For any $n \geq 1$, every solution*

$$u_n(x,t) = \sum_{i=-1}^{n} z_i(t) w_i(x)$$

of (4.2) *satisfies* (4.16).

Now returning to equations (4.3) and (4.4), and applying Proposition 4.1, we obtain the next theorem.

THEOREM 4.1. *For any $n \geq 1$, there is a solution*

$$u_n(x,t) = \sum_{i=-1}^{n} z_i(t) w_i(x)$$

of (4.2) *satisfying* (4.16).

Remark that

$$|h_{1,i}(t)| = \Big| \int_0^{\pi/4} h_1(x,t) w_i(x)\, dx \Big| \leq \frac{\pi}{4} \|h_1\| M_1 ,$$

$$\Big| f\Big(\sum_{i=-1}^{n} z_i(t) w_i(0) \Big) w_i(0) \Big| + \Big| g\Big(\sum_{i=-1}^{n} z_i(t) w_i(\pi/4) \Big) w_i(\pi/4) \Big| \leq 2K_4 M_1 ,$$

for

$$K_4 := \max_{|z|\leq \Theta} \Big\{ |f(z)|, |g(z)| \Big\} .$$

Then Proposition 3.2 ensures the existence of a constant $K_5 > 0$ such that the solutions

$$u_n(x,t) = \sum_{i=-1}^{n} z_{i,n}(t) w_i(x)$$

of (4.2) from Theorem 4.1 satisfy

$$\sup_{i\geq -1, n\geq 1} \|z_{i,n}\|(i^2+1) \leq K_5, \qquad \sup_{i\geq -1, n\geq 1} \|\dot{z}_{i,n}\| \leq K_5 .$$

Then according to Proposition 3.3, there is a subsequence $\{u_{n_i}(x,t)\}_{i=1}^{\infty}$ of $\{u_n(x,t)\}_{n=1}^{\infty}$ which is uniformly convergent to a function $u \in X$ on $[0,\pi/4] \times [0,T]$. On the other hand, equation (4.2) implies that $u_n(x,t)$ solves the following approximating equation

$$\int_0^T \int_0^{\pi/4} \Big[u_n(x,t) \Big\{ v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t) \Big\} + h_1(x,t) v(x,t) \Big]\, dx\, dt$$
$$+ \int_0^T \Big\{ \Big(f(u_n(0,t)) + h_2(t) \Big) v(0,t) + \Big(g(u_n(\pi/4,t)) + h_3(t) \Big) v(\pi/4,t) \Big\}\, dt = 0$$

for any $v \in X_n^\infty$. But then clearly the limit function $u(x,t)$ satisfies (2.1) for any $v \in X_n^\infty$ and any $n \in \mathbb{N}$. Since $\cup_{n\in\mathbb{N}} X_n^\infty$ is dense in X^∞ with respect to the topology of X, we see that $u(x,t)$ satisfies (2.1) for any $v \in X^\infty$; in other words, u is a weak solution of (1.1). Summarizing, we obtain the following result:

THEOREM 4.2. *If f, $g \in C(\mathbb{R},\mathbb{R})$ are odd functions satisfying condition (H), and $h_1 \in X$, h_2, $h_3 \in Y \cap W^{1,2}(0,T)$, then equation* (1.1) *possesses a weak T-anti-periodic solution.*

Theorem 4.2 is an improvement of results in [**9**] and [**10**], since we assume in [**9**] that the function $h_1(x,t)$ is small T-periodic, while in [**10**], we consider only functions $f(u)$ and $g(u)$ with at most linear growth at infinity. We also studied (1.1) in [**9**] and [**10**] with $h_2(t) = h_3(t) = 0$. Of course, now both functions $f(u)$ and $g(u)$ are odd and the forcing terms are T-anti-periodic. For instance, condition (H) holds (see inequalities (4.8)) if

$$\liminf_{u\to+\infty} f(u) + \liminf_{u\to+\infty} g(u) > -\infty\,. \tag{4.17}$$

As an example, if

$$f(u) = f_1 u + f_2 u^3, \quad g(u) = g_1 u + g_2 u^3$$

for constants f_1, f_2, g_1 and g_2, then (4.17) holds if $f_2 > 0$ and $g_2 > 0$, so that both springs are hard at the ends of the beam.

Now let us consider the case when both functions $f(u)$ and $g(u)$ are linear, i.e.

$$f(u) = f_1 u, \quad g(u) = g_1 u\,.$$

Then Theorem 4.2 is applicable if

$$\big(\max\{-f_1, 0\} + \max\{-g_1, 0\}\big) M_1 \Big(2 + \sum_{i=1}^{\infty} \frac{1}{\mu_i^4}\Big) < 1\,. \tag{4.18}$$

Then by using Proposition 3.1, we can rewrite (2.1) as the linear equation

$$Mu = h_4 \tag{4.19}$$

for

$$\begin{aligned} Mu &= u - L\big(0, f_1 u(0,\cdot), g_1 u(\pi/4,\cdot)\big)\,,\\ h_4 &= L(h_1, h_2, h_3)\,. \end{aligned}$$

Clearly M is a linear bounded Fredholm operator from X to X with index 0. Theorem 4.2 ensures that $X^\infty \subset R(M)$ - the range of M. Indeed, for any $v \in X^\infty$, we take $h_1(x,t) = -v_{tt} - v_{xxxx} - \delta v_t$, $h_2(t) = 0$ and $h_3(t) = 0$. Then $h_4 = L(h_1, 0, 0) = v(x,t)$. On the other hand, Theorem 4.2 implies the existence of $u \in X$ such that $M(u) = h_4$. Hence $v \in R(M)$, i.e. $X^\infty \subset R(M)$. Since M is of Fredholm type, $R(M)$ is closed. Since X^∞ is dense in X and $X^\infty \subset R(M)$, we get $R(M) = X$ and then $N(M) = \{0\}$ - the kernel of M. Consequently, M is a linear isomorphism from X to X. So we obtain the following result.

THEOREM 4.3. *Let $f(u) = f_1 u$ and $g(u) = g_1 u$ for constants f_1, g_1 satisfying* (4.18). *Then equation* (1.1) *possesses a unique weak T-anti-periodic solution $\widetilde{L}(h_1, h_2, h_3) := u \in X$ for any $h_1 \in X$ and h_2, $h_3 \in Y$. In addition, the linear mapping $\widetilde{L} : X \times Y \times Y \to X$ is compact.*

The implicit function theorem together with Theorem 4.3 yields:

THEOREM 4.4. *If f, $g \in C^1(\mathbb{R}, \mathbb{R})$ are odd functions and $f_1 = f'(0)$, $g_1 = g'(0)$ satisfy* (4.18)*, then there are positive constants K_1, ε_0 such that for any given functions $h_1 \in X$, h_2, $h_3 \in Y$ with $\|h_1\| + \|h_2\| + \|h_3\| < \varepsilon_0$, equation* (1.1) *possesses a unique small weak T-anti-periodic solution $u \in X$ satisfying $\|u\| \le K_1(\|h_1\| + \|h_2\| + \|h_3\|)$.*

Furthermore, by using Schauder's fixed point theorem [**4**] along with Theorem 4.3 and adapting the arguments of [**10**], we obtain the following result.

THEOREM 4.5. *Let* $f(u) = f_1 u + \widetilde{f}(u)$ *and* $g(u) = g_1 u + \widetilde{g}(u)$ *with odd functions* $\widetilde{f}, \widetilde{g} \in C(\mathbb{R}, \mathbb{R})$ *and constants* f_1, g_1 *satisfying* (4.18). *If there are positive constants* $c_{11}, c_{12}, c_{21}, c_{22}$ *where*

$$c_{12} + c_{22} < 1/\|\widetilde{L}\|$$

and such that

$$\begin{aligned} |\widetilde{f}(u)| &\le c_{11} + c_{12}|u|, \quad \forall u \in \mathbb{R} \\ |\widetilde{g}(u)| &\le c_{21} + c_{22}|u|, \quad \forall u \in \mathbb{R}, \end{aligned}$$

then for any given functions $h_1 \in X$, h_2, $h_3 \in Y$, *equation* (1.1) *possesses a weak* T*-anti-periodic solution* $u \in X$.

Of course, when $\widetilde{f}, \widetilde{g}$ have sublinear growth at infinity:

$$\lim_{|u| \to \infty} \widetilde{f}(u)/u = 0, \quad \lim_{|u| \to \infty} \widetilde{g}(u)/u = 0,$$

then the assumptions of Theorem 4.5 hold and equation (1.1) possesses a weak T-anti-periodic solution $u \in C([0, \pi/4] \times S^T)$ for any $h_1 \in X$, h_2, $h_3 \in Y$.

Theorems 4.3, 4.4 and 4.5 are improvements of similar results in [**10**]. We note that in Theorem 4.2 we have more general odd functions f, g than in Theorems 4.3, 4.4, 4.5, but on the other hand, we suppose in Theorem 4.2 that $h_2, h_3 \in W^{1,2}(0, T)$.

Finally, we numerically estimate from above the constant

$$M_1 \Big(2 + \sum_{i=1}^{\infty} \frac{1}{\mu_i^4} \Big)$$

from condition (H). We know from [**3**] that

$$M_1 \le 4.763953413 . \tag{4.20}$$

Now we evaluate the sum

$$\sum_{i=1}^{\infty} \frac{1}{\xi_i^4} .$$

We have from [**3**] that

$$\Big| \xi_i - \frac{\pi(2i+1)}{2} \Big| \le \frac{e^{-\pi i}}{2}$$

for any $i \ge 1$. Since $\xi_i \ge 4$ and $\pi(2i+1)/2 \ge 4$ for all $i \ge 1$, we have

$$\Big| \frac{1}{\xi_i^4} - \frac{16}{\pi^4 (2i+1)^4} \Big| \le \frac{1}{256} \Big| \xi_i - \frac{\pi(2i+1)}{2} \Big| \le \frac{e^{-\pi i}}{512} . \tag{4.21}$$

By solving the equations

$$\cos \xi_i \cosh(\xi_i) = 1, \quad i = 1, 2, \cdots$$

with the help of *Mathematica*, we get

$$\begin{aligned} &\xi_1 = 4.730040744, \quad \xi_2 = 7.853204624, \\ &\xi_3 = 10.995607838, \quad \xi_4 = 14.137165491 \\ &\xi_5 = 17.278759657, \quad \xi_6 = 20.420352245 , \end{aligned} \tag{4.22}$$

By using

$$\sum_{i=0}^{\infty} \frac{1}{(2i+1)^4} = \frac{\pi^4}{96} ,$$

(4.21) and (4.22), we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{\xi_i^4} &\le \sum_{i=1}^{6}\frac{1}{\xi_i^4}+\sum_{i=7}^{\infty}\Big|\frac{1}{\xi_i^4}-\frac{16}{\pi^4(2i+1)^4}\Big|+\sum_{i=1}^{\infty}\frac{16}{\pi^4(2i+1)^4}\\ &-\sum_{i=1}^{6}\frac{16}{\pi^4(2i+1)^4}\le \sum_{i=1}^{6}\frac{1}{\xi_i^4}+\frac{e^{-7\pi}}{512(1-e^{-\pi})}+\frac{1}{6}-\frac{16}{\pi^4}\\ &-\frac{16}{\pi^4}\sum_{i=1}^{6}\frac{1}{(2i+1)^4}=0.002381090\,.\end{aligned}\tag{4.23}$$

Consequently, from (4.20) and (4.23), we derive

$$M_1\Big(2+\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\Big)=M_1\Big(2+\frac{\pi^4}{256}\sum_{i=1}^{\infty}\frac{1}{\xi_i^4}\Big)\le 9.532223039\,.$$

Hence, the inequality (4.7) of condition (H) holds, if

$$\alpha_f+\alpha_g<1/9.532223039=0.104907322$$

and similarly, the inequality (4.18) holds, if

$$\big(\max\{-f_1,0\}+\max\{-g_1,0\}\big)<0.104907322\,.$$

5. Multivalued Equations

In this section, we study (1.1) when f and g are multivalued, i.e., we suppose:

- **:** (C1) $f,g:\mathbb{R}\to 2^{\mathbb{R}}\setminus\emptyset$ are odd and upper semicontinuous mappings with compact interval values.
- **:** (C2) There are non-negative constants α_f and α_g satisfying (4.7), and a non-negative constant β such that

$$\begin{aligned}vu&\ge-\alpha_f u^2-\beta,\quad \forall\, v\in f(u)\\ vu&\ge-\alpha_g u^2-\beta,\quad \forall\, v\in g(u)\end{aligned}$$

  for any $u\in\mathbb{R}$.

Remark 5.1. i) According to [**7**], condition (C1) is equivalent to the existence of lower semicontinuous functions $f_-,g_-:\mathbb{R}\to\mathbb{R}$ and upper semicontinuous functions $f_+,g_+:\mathbb{R}\to\mathbb{R}$ such that

$$\begin{aligned}&f_-(u)\le f_+(u),\quad g_-(u)\le g_+(u)\,,\\ &f(u)=[f_-(u),f_+(u)],\quad g(u)=[g_-(u),g_+(u)]\end{aligned}$$

for any $u\in\mathbb{R}$. Moreover, the oddness of f and g implies

$$-f_+(u)=f_-(-u),\quad -g_+(u)=g_-(-u)\tag{5.1}$$

for any $u\in\mathbb{R}$.

ii) Since (5.1) holds, condition (C2) is equivalent to the assumption that the functions f_- and g_- satisfy condition (H) with constants α_f, α_g and β, respectively.

Now the equation of vibrations is as follows

$$\begin{aligned}&u_{tt}+u_{xxxx}+\delta u_t+h_1(x,t)=0\,,\\ &u_{xx}(0,\cdot)=u_{xx}(\pi/4,\cdot)=0\,,\\ &u_{xxx}(0,\cdot)\in-f(u(0,\cdot))-h_2(t)\,,\\ &u_{xxx}(\pi/4,\cdot)\in g(u(\pi/4,\cdot))+h_3(t)\,.\end{aligned}\tag{5.2}$$

By a weak T-anti-periodic solution of (5.2), we mean any $u \in X$ satisfying the identity

$$\begin{aligned}\int_0^T \int_0^{\pi/4} \Big[u(x,t)\Big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\Big\} + h_1(x,t)v(x,t)\Big]\, dx\, dt \\ + \int_0^T \Big\{f_1(t)v(0,t) + g_1(t)v(\pi/4,t)\Big\}\, dt = 0\end{aligned} \tag{5.3}$$

for any $v \in X^\infty$ and some $f_1, g_1 \in L^2(0,T)$ with

$$\begin{aligned} f_1(t) &\in f(u(0,t)) + h_2(t)\,, \\ g_1(t) &\in g(u(\pi/4,t)) + h_3(t)\end{aligned}$$

for a.a. $t \in (0,T)$.

We take $\widetilde{\alpha}_f = \alpha_{f_-}$, $\widetilde{\alpha}_g = \alpha_{g_-}$ and $\widetilde{\beta} = 2\beta$ and we find the corresponding constant Θ from (4.16) for these $\widetilde{\alpha}_f$, $\widetilde{\alpha}_g$ and $\widetilde{\beta}$. Then it is not difficult to modify the proof of Proposition 1.1 (d) of [**7**], p. 7 to show that for any ε, $0 < \varepsilon < 1/2$ there are continuous and odd functions

$$f_\varepsilon,\, g_\varepsilon : [-\Theta - 1, \Theta + 1] \to \mathbb{R}$$

such that

$$\begin{aligned} f_\varepsilon(u) &\in f((u-\varepsilon, u+\varepsilon)) + (-\varepsilon,\varepsilon)\,, \\ g_\varepsilon(u) &\in g((u-\varepsilon, u+\varepsilon)) + (-\varepsilon,\varepsilon)\end{aligned} \tag{5.4}$$

for any $u \in [-\Theta - 1, \Theta + 1]$. Then there is a small $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the functions $f_\varepsilon(u)$ and $g_\varepsilon(u)$ satisfy condition (H) with constants $\widetilde{\alpha}_f$, $\widetilde{\alpha}_g$ and $\widetilde{\beta}$ on $[-\Theta-1, \Theta+1]$. Now we extend f_ε and g_ε to the whole $\mathbb{R}$ so that they are continuous and odd, and they satisfy condition (H) with constants $\widetilde{\alpha}_f$, $\widetilde{\alpha}_g$ and $\widetilde{\beta}$ on $\mathbb{R}$. Then we apply Theorem 4.2 to get a function $u_\varepsilon \in X$ satisfying

$$\begin{aligned}\int_0^T \int_0^{\pi/4} \Big[u_\varepsilon(x,t)\Big\{v_{tt}(x,t) + v_{xxxx}(x,t) - \delta v_t(x,t)\Big\} + h_1(x,t)v(x,t)\Big]\, dx\, dt \\ + \int_0^T \Big\{\Big(f_\varepsilon(u_\varepsilon(0,t)) + h_2(t)\Big)v(0,t) + \Big(g_\varepsilon(u_\varepsilon(\pi/4,t)) + h_3(t)\Big)v(\pi/4,t)\Big\}\, dt = 0\end{aligned} \tag{5.5}$$

for any $v \in X^\infty$. We note that according to the choice of the constant Θ, $u_\varepsilon(x,t)$ satisfies (4.16), so it is a solution for nonextended f_ε and g_ε. For this reason, (5.5) holds. Moreover, we know that the sequence $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is precompact in X. Then $\sup\limits_{0<\varepsilon<\varepsilon_0} \|u_\varepsilon\| < \infty$. From (5.4) and (C1) we see that $\sup\limits_{0<\varepsilon<\varepsilon_0} \|f_\varepsilon(u_\varepsilon)\| < \infty$. So the sequences $\{f_\varepsilon(u_\varepsilon(0,\cdot))\}_{0<\varepsilon<\varepsilon_0}$ and $\{g_\varepsilon(u_\varepsilon(\pi/4,\cdot))\}_{0<\varepsilon<\varepsilon_0}$ are bounded in $L^2(0,T)$. Summarizing, we can find a subsequence $\{u_{\varepsilon_i}\}_{i=1}^\infty$ of $\{u_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ such that

$$\begin{aligned} &\varepsilon_i \to 0\,, \\ &u_{\varepsilon_i} \to u \quad \text{in} \quad X\,, \\ &f_{\varepsilon_i}(u_{\varepsilon_i}(0,\cdot)) \to f_0(t) \quad \text{weakly in} \quad L^2(0,T)\,, \\ &g_{\varepsilon_i}(u_{\varepsilon_i}(\pi/4,\cdot)) \to g_0(t) \quad \text{weakly in} \quad L^2(0,T)\end{aligned} \tag{5.6}$$

as $i \to \infty$ for some $u \in X$ and $f_0, g_0 \in L^2(0,T)$. Now we take $\zeta > 0$. Then the upper semicontinuity of f and g, and (5.4) imply that there is an i_0 such that for any $i > i_0$, one has

$$f_{\varepsilon_i}(u_{\varepsilon_i}(0,\cdot)) \in f(u(0,t)) + [-\zeta,\zeta] \\ g_{\varepsilon_i}(u_{\varepsilon_i}(\pi/4,\cdot)) \in g(u(\pi/4,t)) + [-\zeta,\zeta] \tag{5.7}$$

for any $t \in [0,T]$. On the other hand, it is obvious that the sets

$$\Big\{s \in L^2(0,T) \mid s(t) \in f(u(0,t)) + [-\zeta,\zeta] \quad \text{for a.a.} \quad t \in (0,T)\Big\}$$
$$\Big\{s \in L^2(0,T) \mid s(t) \in g(u(\pi/4,t)) + [-\zeta,\zeta] \quad \text{for a.a.} \quad t \in (0,T)\Big\}$$

are closed and convex in the Hilbert space $L^2(0,T)$. Consequently, they are also weakly closed, so that

$$f_0(t) \in f(u(0,t)) + [-\zeta,\zeta]\,, \\ g_0(t) \in g(u(\pi/4,t)) + [-\zeta,\zeta] \tag{5.8}$$

for a.a. $t \in (0,T)$. Since $\zeta > 0$ is arbitrarily small and condition (C1) holds, from (5.8) we get

$$f_0(t) \in f(u(0,t)), \quad g_0(t) \in g(u(\pi/4,t))$$

for a.a. $t \in (0,T)$.

Now by passing to the limit as $i \to \infty$ with $\varepsilon = \varepsilon_i$ in (5.5) for a fixed $v \in X^\infty$, we see that the function $u(x,t)$ from (5.6) satisfies (5.3) with

$$f_1(t) = f_0(t) + h_2(t), \quad g_1(t) = g_0(t) + h_3(t)$$

for any $v \in X^\infty$. Hence such $u(x,t)$ is a weak T-anti-periodic solution of (5.2). Summarizing, we obtain the following result:

THEOREM 5.1. *If f, $g : \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ satisfy conditions $(C1)$, $(C2)$ and $h_1 \in X$, h_2, $h_3 \in Y \cap W^{1,2}(0,T)$, then equation* (5.2) *possesses a weak T-anti-periodic solution.*

Theorem 5.1 is certainly applicable to the simplest multivalued mappings $f(u) = g(u) = \operatorname{sgn}(u)$ (cf. [**7**]) with

$$\operatorname{sgn}(u) = \begin{cases} -1 & \text{for} \quad u < 0\,, \\ [-1,1] & \text{for} \quad u = 0\,, \\ 1 & \text{for} \quad u > 0\,. \end{cases}$$

The multivalued problem (5.2) was not studied in the papers mentioned in the Introduction.

References

[1] AIZICOVICI, S. and PAVEL, N. H. Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space, *J. Funct. Anal.* **99** (1991), 387-408.

[2] BATTELLI, F. and FEČKAN, M. Chaos in the beam equation, *preprint* (2003).

[3] BATTELLI, F. and FEČKAN, M. Homoclinic orbits of slowly periodically forced and weakly damped beams resting on weakly elastic bearings, *Adv. Differential Equations* **8** (2003), 1043-1080.

[4] BERGER, M. S. Nonlinearity and Functional Analysis, Academic Press, New York 1977.

[5] CAPRIZ, G. Self-excited vibrations of rotors, *Int. Union Theor. Appl. Mech., Symp. Lyngby/Denmark 1974*, Springer-Verlag 1975.

[6] CAPRIZ, G. and LARATTA, A. Large amplitude whirls of rotors, *Vibrations in Rotating Machinery*, Churchill College, Cambridge 1976.

[7] DEIMLING, K. Multivalued Differential Equations, W. De Gruyter, Berlin 1992.

[8] FEČKAN, M. Free vibrations of beams on bearings with nonlinear elastic responses, *J. Differential Equations* **154** (1999), 55-72.

[9] FEČKAN, M. Periodically forced damped beams resting on nonlinear elastic bearings, *Mathematica Slovaca* (to appear).

[10] FEČKAN, M. Forced oscillations of beams on elastic bearings, *Electr. J. Qualitative Theor. Differential Equations* (to appear)

[11] FEIREISEL, E. Nonzero time periodic solutions to an equation of Petrovsky type with nonlinear boundary conditions: Slow oscillations of beams on elastic bearings, *Ann. Scuola Norm. Sup. Pisa* **20** (1993), 133-146.

[12] MAWHIN, J. Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces, *J. Differential Equations* **12** (1972), 610-636.

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