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Integrable Nonlinear Schrödinger Systems and their Soliton Dynamics

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Abstract. Nonlinear Schrödinger (NLS) systems are important examples of physically-significant nonlinear evolution equations that can be solved by the inverse scattering transform (IST) method. In fact, the IST for discrete and continuous, as well as scalar and vector, NLS systems all fit into the same framework, which is reviewed here. The parallel presentation of the IST for each of these systems not only clarifies the common structure of the IST, but also highlights the key variations. Importantly, these variations manifest themselves in the dynamics of the solutions. With the IST approach, one can explicitly construct the soliton solutions of each of these systems, as well as formulas from which one can determine the dynamics of soliton interaction. In particular, vector solitons, both continuous and discrete, are partially characterized by a polarization vector, which is shifted by soliton interaction. Here, we give a complete account of the nature of this polarization shift. The polarization vector can be used to encode the value of a binary digit ("bit") and the soliton interaction arranged so as to effect logical computations.

Contents

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Key words and phrases. Nonlinear Schrödinger systems, inverse scattering transform, soliton interaction.

	Continuous	**Discrete**
Scalar	NLS (1.1) $iq_t = q_{xx} + 2\left\|q\right\|^2 q$	IDNLS (1.3) $i\frac{d}{d\tau}Q_n = \Delta Q_n + \left\|Q_n\right\|^2 (Q_{n-1} + Q_{n+1})$
Vector	VNLS (1.2) $i\mathbf{q}_t = \mathbf{q}_{xx} + 2\left\|\mathbf{q}\right\|^2 \mathbf{q}$	IDVNLS (1.5) $i\frac{d}{d\tau}\mathbf{Q}_n = \Delta \mathbf{Q}_n + \left\|\mathbf{Q}_n\right\|^2 (\mathbf{Q}_{n-1} + \mathbf{Q}_{n+1})$

FIGURE 1. Integrable nonlinear Schrödinger systems

1. Introduction

1.1. Overview. The nonlinear Schrödinger equation (NLS)

$$iq_t = q_{xx} \pm 2\left|q\right|^2 q \tag{1.1}$$

describes the evolution of generic small-amplitude, slowly varying wave packets in a nonlinear medium [**22**]. For example, NLS has been derived in the modeling of deep water waves [**23, 90**], plasmas [**91**], nonlinear optical fibers [**47, 48**], and magnetostatic spin waves [**87, 52, 95**]. In addition to its importance as a model of physical systems, NLS has been the subject of extensive attention by mathematicians due to its rich mathematical structure. In particular, NLS can be solved via the inverse-scattering transform (IST) and has soliton solutions (localized traveling waves that interact elastically) [**92**]. Moreover, as is typical of evolution equations solvable by the IST, NLS has a Hamiltonian structure with an infinite number of conserved quantities.

We note that the NLS equation (1.1) with a "minus" sign in front of the nonlinear term is sometimes referred to as the "defocusing" case and the equation with a "plus" sign is referred to as the "focusing" case. In contrast to the focusing NLS, the defocusing NLS equation does not admit soliton solutions that vanish at infinity. However, the defocusing NLS does admit soliton solutions which have a nontrivial background intensity, often referred to as "dark solitons" [**48, 93**]. For a discussion of dark solitons see [**24, 56, 58, 83, 94**]. In this review, we only consider solutions that decay at infinity and, therefore, we only consider the bright solitons of focusing NLS and the corresponding focusing forms of the related nonlinear Schrödinger systems.

The mathematical theory of the IST can be profitably extended to related nonlinear Schödinger systems. Specifically, in this review, we consider both the spatial discretization and vector generalization of NLS. Thus, we consider three systems in addition to NLS (1.1): the vector generalization of NLS (i.e. VNLS), the integrable discretization (IDNLS) and the integrable discretization of the vector system (IDVNLS). See Figure 1.1. The parallel presentation of the IST for each of these systems (in Sections 2–5) clarifies, in a fundamental way, both the overall similarities and important differences between these systems.

Making concrete use of the IST, we derive explicit expressions for the soliton solutions of all four systems. Further, we present explicit formulas, also derived with the machinery of the IST, that describe the dynamics of soliton interaction for each of the systems. This approach was used by Zakharov and Shabat [**92**] to analyze the soliton interaction in NLS and by Manakov [**64**] to derive the equivalent formulae for VNLS. In Section 6 we extend the method to the analysis of discrete-soliton interactions. Moreover, in Section 7, we show how these formulas can be

$$
\begin{array}{ccccc}
q(x,0) & \xrightarrow{\text{Integral Equation}} & N(x,k,0) & \xrightarrow{\text{Explicit Formula}} & \left\{\rho(k,0), \quad \{k_j, C_j(0)\}_{j=1}^{J}\right\} \\
 & & & & \big\downarrow \text{Simple} \\
q(x,t) & \xleftarrow{\text{Explicit Formula}} & N(x,k,t) & \xleftarrow{\text{Riemann-Hilbert}} & \left\{\rho(k,t), \quad \{k_j, C_j(t)\}_{j=1}^{J}\right\}
\end{array}
$$

FIGURE 2. Scheme of the Inverse Scattering Transform

applied to the construction of logic gates with vector solitons (both continuous and discrete).

1.2. Inverse Scattering Transform. Beyond its applications to nonlinear Schrödinger systems, the IST method allows one to solve a broad class nonlinear evolution equations (cf., e.g., [**2, 13, 28, 34, 38, 70**]). While the details of the IST depend on the particular evolution equation studied, there is a common framework. Moreover, many of the details of the IST are similar for all the nonlinear Schrödinger systems considered here.

An essential precondition for solving a nonlinear evolution equation via the IST is the association of a pair of linear problems (Lax pair) such that the given equation results as compatibility condition between them: an associated (generalized) eigenvalue problem (or spectral problem), and an auxiliary spectral problem fixing the time dependence.

The operation of the IST is analogous to the Fourier Transform solution of linear evolution equations (see Figure 1.2.) At a fixed time (say $t = 0$), one applies a transformation of the solution from a function of the spatial variable (x) to a function of the spectral variable (k), the "forward" or "direct" problem. In the case of the IST, the solution of the direct problem includes the reflection coefficient $\rho(k)$, which is a function of the spectral variable, and the eigenvalues of the associated scattering problem along with additional constants, the "norming constants". This function of the spectral variable, along with the eigenvalues and the constants, are referred to collectively as the "scattering data". The time evolution of the scattering data is simple (in particular, the eigenvalues are independent of the time evolution). The solution at a later time is obtained by an inverse transformation, the "inverse" problem.

The Inverse Scattering Transform is, however, more involved than the Fourier transform. In the IST, both the forward and inverse transformations require an intermediate step in which one computes a function that depends on both the spatial variable and the spectral variable. Moreover, the solution of the forward problem involves the solution of a linear integral equation and the solution of the inverse problem requires the solution of a Riemann-Hilbert boundary-value problem.

While both the forward and inverse problems are necessary steps in the solution of the initial-value problem, one can use the inverse problem on its own to generate special solutions of the evolution equation. We obtain a special solution by specifying the scattering data and then solving the inverse problem with the time-dependence included. In particular, the solitons of nonlinear Schrödinger systems (as well as many other nonlinear evolution equations) correspond to scattering data that consist of only eigenvalues and their associated constants (i.e., no continuous spectrum). We note that, for nonlinear Schrödinger systems, when there is no continuous spectrum, the Riemann-Hilbert boundary-value problem reduces to a finite-dimensional linear system and we obtain explicit soliton solutions.

1.3. Nonlinear Schrödinger Systems. The system

$$iq_t^{(1)} = q_{xx}^{(1)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right) q^{(1)}$$
$$iq_t^{(2)} = q_{xx}^{(2)} + 2\left(|q^{(1)}|^2 + |q^{(2)}|^2\right) q^{(2)},$$

sometimes referred to in the literature as the coupled NLS equation, was posited by Manakov [**64**] as a model for the propagation of the electric field in a waveguide. In Manakov's formulation, each equation governs the evolution of one of the component of the field transverse to the direction of propagation. More generally, this system can be derived as a model for wave propagation under conditions similar to those where NLS applies and there are two wavetrains moving with nearly the same group velocity [**73, 88**]. In recent years, this system was derived as a key model for light-wave propagation in optical fibers (cf. [**37, 65, 66, 85**]).

We refer to the N-component vector generalization of the two-component system,

$$i\mathbf{q}_t = \mathbf{q}_{xx} + 2\left\|\mathbf{q}\right\|^2 \mathbf{q}, \tag{1.2}$$

where $\mathbf{q}$ is an N-component vector and $\|\cdot\|$ is the Euclidean norm, as vector NLS (VNLS). The vector system can be derived, with some additional conditions, as an asymptotic model of the interaction of N wavetrains in a weakly nonlinear, conservative medium (cf. [**73**]).

Like the scalar NLS (1.1), VNLS (1.2) is integrable by the IST ([**61, 63**]. While Manakov first formulated the IST for two-component system, the extension to the N-component system is straightforward. Moreover, little adaptation is required to extend Manakov's method to the system

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} + 2\mathbf{Q}\mathbf{Q}^H\mathbf{Q}$$

where $\mathbf{Q}$ is an $N \times M$ matrix and H denotes the Hermitian (conjugate) transpose [**10**]. Indeed, as we show by our presentation here, the IST for the matrix system parallels the IST for NLS in a step-by-step manner.

The vector-soliton solutions of VNLS are the counterpart of the scalar solitons of NLS. A vector soliton is characterized, in part, by a polarization vector, which has no counterpart in the soliton solution of the scalar equation. Moreover, vector-soliton interactions can induce a shift in the polarization of the individual vector solitons. This polarization shift is the key feature that distinguishes the dynamics of the solitons of VNLS (vector solitons) from the solitons of NLS (scalar solitons). Although described accurately by Manakov [**64**], the polarization shift has been the subject of continuing attention (e.g., [**72, 54**]). In particular, the polarization shift has been investigated as a mechanism for computation [**76**]. Also, Vector NLS solitons and their properties have been investigated experimentally [**18, 19, 30, 53, 59**].

Spatial discretizations of both NLS (1.1) and VNLS (1.2) have been studied extensively, not only as the basis of numerical schemes for the solution of the respective PDEs, but also as models of spatially discrete physical systems (e.g., Davydov [**31**]-[**33**], Eilbeck *et al.* [**35**], Claude *et al.* [**27**], Christodoulides and Joseph [**29**], Its *et al.* [**49**] Aceves *et al* [**14, 15**], Eisenberg *et al.* [**36**], Morandotti *et al* [**67, 68**], Vakhnenko *et al.* [**80, 81**]). However, among the discretizations of NLS, the system

$$i\frac{d}{d\tau}Q_n = \Delta Q_n + |Q_n|^2 \left(Q_{n-1} + Q_{n+1}\right), \tag{1.3}$$

where

$$\Delta Q_n = Q_{n-1} - 2Q_n + Q_{n+1},$$

plays a very special role: it is solvable via the IST, has soliton solutions and infinitely-many conserved quantities as well as a Hamiltonian structure [**5**]-[**8**], [**60**], [**40**]-[**44**]. Because (1.3) is solvable by the IST, we refer to it as the integrable discrete NLS or IDNLS. (In the literature, eq. (1.3) is sometimes referred to as the Ablowitz-Ladik system.) Unlike other discretizations of NLS, IDNLS reproduces the characteristic soliton-interaction dynamics of NLS. Moreover, with the change of variables $Q_n = hq_n = hq(nh)$ and $\tau = h^{-2}t$, IDNLS takes the form

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n-1} - 2q_n + q_{n+1}\right) + |q_n|^2 \left(q_{n+1} + q_{n-1}\right), \tag{1.4}$$

whose solutions (as well as soliton interaction formulas) converge to solutions of NLS in the continuum limit ($h \to 0$).

We note that, while IDNLS reproduces the soliton dynamics of NLS, the discrete system (i.e., IDNLS) has additional traveling "breather" solutions that have no counterpart in the the PDE. These solutions are discussed in Section 3.

For IDNLS, we present not only the soliton interaction formulae, but also the IST of (1.3) in a manner that parallels as closely as possible the IST for NLS. In particular, unlike the earlier presentation of Ablowitz and Ladik [**5**]-[**8**], we present the inverse problem as a Riemann-Hilbert boundary-value problem so as to mirror our treatment of the inverse problem for NLS.

Given the integrable discretization of NLS (i.e., IDNLS (1.3)) and the integrable vector generalization of NLS (i.e., VNLS (1.2)), it is natural to propose the semi-discrete system

$$i\frac{d}{d\tau}\mathbf{Q}_n = \Delta \mathbf{Q}_n + \|\mathbf{Q}_n\|^2 \left(\mathbf{Q}_{n-1} + \mathbf{Q}_{n+1}\right), \tag{1.5}$$

where $\mathbf{Q}$ is an N-component vector,

$$\Delta \mathbf{Q}_n = \mathbf{Q}_{n-1} - 2\mathbf{Q}_n + \mathbf{Q}_{n+1},$$

and

$$\|\mathbf{Q}_n\|^2 = |Q_n^{(1)}|^2 + \cdots + |Q_n^{(N)}|^2,$$

as an integrable discretization of VNLS. Indeed, as detailed in Section 5, eq. (1.5) is integrable via the IST and has soliton solutions. Moreover, as in the scalar case, the discrete system reproduces the soliton interaction dynamics of its continuous limit, in this case VNLS (1.2).

Despite the apparent parallel between the integrable discretization of NLS and the integrable discretization of VNLS, the vector case has some novel aspects. Specifically, the discrete vector system (1.5) has traveling-wave solutions, which we refer to as "composite" solitons, that have no counterpart in the other nonlinear Schrödinger systems. However, these composite solitons are related to the discrete breather solutions of IDNLS. This relation is best understood by comparison of these solutions in the context of IST.

Surprisingly, in order to construct (via the IST) the discrete, composite solitons as well as the discrete "fundamental" solitons (the counterpart of the vector-solitons of the continuous limit i.e., VNLS), we must consider (1.5) as the reduction of a discrete *matrix* system. This, more than anything else, is the motivation for our consideration of the continuous matrix system in Section 4. The requirement that we consider a discrete matrix system is not merely a technical hurdle in the formulation of the IST for the discrete vector system in Section 5. Rather, the existence of the composite soliton solutions of IDVNLS is a manifestation of a necessary additional symmetry reduction. This additional symmetry is required in order to obtain the discrete vector equation (1.5) from the discrete matrix system described in Section 5.

2. Scalar Nonlinear Schrödinger equation (NLS)

2.1. Compatibility condition. For the purpose of the IST, it is convenient to consider the scalar, focusing nonlinear Schrödinger equation (1.1) in the equivalent form

$$iq_t = q_{xx} - 2rq^2 \tag{2.1a}$$

$$-ir_t = r_{xx} - 2qr^2 \tag{2.1b}$$

where $r = -q^*$. Note that the defocusing equation is equivalent to the same system with $r = q^*$.

The linear eigenvalue problem associated with the system (2.1a)–(2.1b) is

$$v_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v, \tag{2.2}$$

where v is a 2-component vector, $v(x,t) = \left(v^{(1)}(x,t), v^{(2)}(x,t)\right)^T$ (cf. [**13, 28, 70**]). (This linear system is often referred to as the AKNS scattering problem [**4**].) The association of the scattering problem and the evolution equation proceeds as follows: if the evolution of v is governed by

$$v_t = \begin{pmatrix} 2ik^2 + iqr & -2kq - iq_x \\ -2kr + ir_x & -2ik^2 - iqr \end{pmatrix} v, \tag{2.3}$$

then, the evolution equations (2.1a)–(2.1b) are equivalent to the statement that $v_{xt} = v_{tx}$, i.e. the mixed derivatives are equal.

2.2. Direct Scattering Problem. When the potentials $q, r \to 0$ rapidly as $x \to \pm\infty$, solutions of the scattering problem (2.2) can be defined by the following boundary conditions

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \qquad \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \qquad \text{as } x \to -\infty \tag{2.4a}$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \qquad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \qquad \text{as } x \to +\infty. \tag{2.4b}$$

However, in the following, we find it convenient to consider related functions with constant boundary conditions (the so-called "Jost functions"):

$$M(x,k) = e^{ikx}\phi(x,k), \qquad \bar{M}(x,k) = e^{-ikx}\bar{\phi}(x,k), \tag{2.5a}$$

$$N(x,k) = e^{-ikx}\psi(x,k), \qquad \bar{N}(x,k) = e^{ikx}\bar{\psi}(x,k). \tag{2.5b}$$

These Jost functions are solutions of the integral equations

$$M(x,k) = \begin{pmatrix}1\\0\end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_+(x-x',k)\hat{\mathbf{Q}}(x',k)M(x',k)\,dx' \tag{2.6a}$$

$$N(x,k) = \begin{pmatrix}0\\1\end{pmatrix} + \int_{-\infty}^{+\infty} \tilde{\mathbf{G}}_+(x-x',k)\hat{\mathbf{Q}}(x',k)N(x',k)\,dx' \tag{2.6b}$$

$$\bar{M}(x,k) = \begin{pmatrix}0\\1\end{pmatrix} + \int_{-\infty}^{+\infty} \tilde{\mathbf{G}}_-(x-x',k)\hat{\mathbf{Q}}(x',k)\bar{M}(x',k)\,dx' \tag{2.6c}$$

$$\bar{N}(x,k) = \begin{pmatrix}1\\0\end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_-(x-x',k)\hat{\mathbf{Q}}(x',k)\bar{N}(x',k)\,dx', \tag{2.6d}$$

where

$$\hat{\mathbf{Q}} = \begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$$

and the Green's functions are

$$\mathbf{G}_\pm(x,k) = \pm\theta(\pm x)\begin{pmatrix}1 & 0\\ 0 & e^{2ikx}\end{pmatrix}, \qquad \tilde{\mathbf{G}}_\pm(x,k) = \mp\theta(\mp x)\begin{pmatrix}e^{-2ikx} & 0\\ 0 & 1\end{pmatrix},$$

where

$$\theta(x) = \begin{cases}1, & x>0\\ 0, & x<0\end{cases}$$

is the Heaviside function.

Eqs. (2.6a)–(2.6d) are Volterra integral equations from which we can determine properties of the Jost functions. One can show, by the method of iteration, that if $q,r \in L^1(\mathbb{R})$, then the Neumann series of the integral equations for $M(x,k)$ and $N(x,k)$ converge absolutely and uniformly (in both x and k) in the upper half k-plane (i.e., $\operatorname{Im} k \geq 0$). Similarly, the Neumann series of the integral equations for $\bar{M}(x,k)$ and $\bar{N}(x,k)$ converge absolutely and uniformly (in x and k) in the lower half k-plane (i.e., $\operatorname{Im} k \leq 0$). (See, e.g., [**10**] for detailed proofs.) Consequently, the Jost functions $M(x,k)$ and $N(x,k)$ are analytic functions of the complex variable k in the region $\operatorname{Im} k > 0$ and continuous in the region $\operatorname{Im} k \geq 0$. Moreover, $\bar{M}(x,k), \bar{N}(x,k)$ are analytic functions of k in the region $\operatorname{Im} k < 0$ and continuous in the region $\operatorname{Im} k \leq 0$.

With the integral equations (2.6a)–(2.6d) one can derive asymptotic expansions functions of the Jost functions:

$$M(x,k) = \begin{pmatrix} 1 - \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x)\end{pmatrix} + O(k^{-2}) \tag{2.7a}$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 + \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx' \\ -\frac{1}{2ik}r(x)\end{pmatrix} + O(k^{-2}) \tag{2.7b}$$

$$N(x,k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 - \frac{1}{2ik}\int_{x}^{+\infty} q(x')r(x')dx'\end{pmatrix} + O(k^{-2}) \tag{2.7c}$$

$$\bar{M}(x,k) = \begin{pmatrix} \frac{1}{2ik}q(x) \\ 1 + \frac{1}{2ik}\int_{-\infty}^{x} q(x')r(x')dx'\end{pmatrix} + O(k^{-2}), \tag{2.7d}$$

which are valid for large k.

We now use the solutions of the scattering problem to define scattering data that are independent of the spatial variable, x, but depend on the complex scattering variable, k.

The solutions $\phi(x,k)$ and $\bar{\phi}(x,k)$ of the scattering problem (both with boundary conditions specified as $x \to -\infty$) are linearly independent. Because the trace of the matrix in the scattering problem vanishes, the Wronskian of any two solutions is independent of x. That is,

$$W\left(\phi,\bar{\phi}\right) = \lim_{x\to-\infty} W\left(\phi(x,k),\bar{\phi}(x,k)\right) = 1,$$

where, for any u,v

$$W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}.$$

Similarly,

$$W\left(\psi,\bar{\psi}\right) = \lim_{x\to+\infty} W\left(\psi(x,k),\bar{\psi}(x,k)\right) = -1. \tag{2.8}$$

We conclude that both $\left\{\phi(x,k),\bar{\phi}(x,k)\right\}$ and $\left\{\psi(x,k),\bar{\psi}(x,k)\right\}$ are sets of linearly-independent solutions of (2.2). Therefore, because the scattering problem is a second-order system of linear ODEs, the left eigenfunctions, $\phi(x,k)$ and $\bar{\phi}(x,k)$, are linear combinations of $\psi(x,k)$ and $\bar{\psi}(x,k)$ and vice-versa.

The coefficients of the linear combinations depend on k. Hence, the relations

$$\phi(x,k) = b(k)\psi(x,k) + a(k)\bar{\psi}(x,k) \tag{2.9a}$$

$$\bar{\phi}(x,k) = \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k) \tag{2.9b}$$

hold for any k such that all four functions exist. In particular, (2.9a)–(2.9b) are valid on the line $\operatorname{Im} k = 0$ and define the scattering coefficients $a(k)$,$\bar{a}(k)$,$b(k)$ and $\bar{b}(k)$ there.

Using both (2.9a)–(2.9b) and the x-invariance of the Wronskian to compute $W\left(\phi(x,k),\bar{\phi}(x,k)\right)$ in the limit $x \to \infty$, we observe that the scattering data satisfy the so-called unitarity relation

$$a(k)\bar{a}(k) - b(k)\bar{b}(k) = 1. \tag{2.10}$$

Similarly, by examining the limits $x \to \pm\infty$, one can verify the following formulas for the scattering data in terms of the Jost functions:

$$a(k) = W\left(M(x,k),N(x,k)\right), \qquad b(k) = -W\left(M(x,k),\bar{N}(x,k)\right)e^{-2ikx}, \tag{2.11a}$$

$$\bar{a}(k) = -W\left(\bar{M}(x,k),\bar{N}(x,k)\right), \qquad \bar{b}(k) = W\left(\bar{M}(x,k),N(x,k)\right)e^{-2ikx}. \tag{2.11b}$$

Alternatively, one can derive the following integral relationships for the scattering coefficients

$$a(k) = 1 + \int_{-\infty}^{+\infty} q(x')M^{(2)}(x',k)dx' \tag{2.12a}$$

$$b(k) = \int_{-\infty}^{+\infty} e^{-2ikx'}r(x')M^{(1)}(x',k)dx' \tag{2.12b}$$

$$\bar{a}(k) = 1 + \int_{-\infty}^{+\infty} r(x')\bar{M}^{(1)}(x',k)dx' \tag{2.12c}$$

$$\bar{b}(k) = \int_{-\infty}^{+\infty} e^{2ikx'}q(x')\bar{M}^{(2)}(x',k)dx' \tag{2.12d}$$

where $M^{(j)}$, $\bar{M}^{(j)}$ for $j = 1, 2$ denote the j-th component of vectors M and $\bar{M}$ respectively (cf. [**2**]).

The analyticity properties of the Jost functions and the formulae for the scattering data imply that $a(k)$ can be extended analytically in the region $\operatorname{Im} k > 0$ of the complex k-plane, while $\bar{a}(k)$ can be extended analytically in the region $\operatorname{Im} k < 0$. However, in general, one cannot conclude that $b(k)$ and $\bar{b}(k)$ can be extended off the real k-axis.

From the integral representations (2.12a) and (2.12c) and the asymptotic expansions (2.7a), (2.7d) we obtain the expansions

$$a(k) = 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} q(x')r(x')dx' + O(k^{-2}) \qquad \operatorname{Im} k > 0 \tag{2.13a}$$

$$\bar{a}(k) = 1 + \frac{1}{2ik}\int_{-\infty}^{+\infty} q(x')r(x')dx' + O(k^{-2}) \qquad \operatorname{Im} k < 0, \tag{2.13b}$$

which are valid for large k.

To obtain the scattering data in a form that is convenient for the inverse problem, we rewrite eqs. (2.9a)-(2.9b) as

$$\mu(x,k) = \bar{N}(x,k) + \rho(k)e^{2ikx}N(x,k) \tag{2.14a}$$

$$\bar{\mu}(x,k) = N(x,k) + \bar{\rho}(k)e^{-2ikx}\bar{N}(x,k) \tag{2.14b}$$

where

$$\mu(x,k) = M(x,k)a^{-1}(k), \qquad \bar{\mu}(x,k) = \bar{M}(x,k)\bar{a}^{-1}(k) \tag{2.15}$$

are meromorphic in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively, and

$$\rho(k) = b(k)a^{-1}(k), \qquad \bar{\rho}(k) = \bar{b}(k)\bar{a}^{-1}(k), \tag{2.16}$$

which we refer to as the "reflection coefficients", are, in general, only defined on $\operatorname{Im} k = 0$.

The scattering problem (2.2) may include proper eigenvalues. A proper eigenvalue, $k_j = \xi_j + i\eta_j$, in the upper k-plane (i.e., $\eta_j > 0$) occurs precisely where $a(k_j) = 0$. If $a(k_j) = 0$, it follows from (2.11a) that $\phi(x,k_j)$ and $\psi(x,k_j)$ are linearly-dependent functions of x That is, there exists a complex constant, b_j, such that

$$\phi(x,k_j) = b_j\psi(x,k_j).$$

Consequently,

$$M(x,k_j) = b_j e^{2ik_jx}N(x,k_j).$$

Similarly, the eigenvalues in the region $\operatorname{Im} k < 0$ are the zeros of $\bar{a}(k)$. These zeros, denoted $\bar{k}_j = \bar{\xi}_j + i\bar{\eta}_j$, where $\bar{\eta}_j < 0$, are such that

$$\bar{\phi}(x,\bar{k}_j) = \bar{b}_j\bar{\psi}(x,\bar{k}_j)$$

and

$$\bar{M}(x,\bar{k}_j) = \bar{b}_j e^{-2i\bar{k}_jx}\bar{N}(x,\bar{k}_j).$$

for some complex constant $\bar{b}_j$.

Because the eigenvalues, k_j, are the zeroes of $a(k)$, they correspond to the poles (in k) of $\mu(x,k)$ (in the region $\operatorname{Im} k > 0$). For each simple pole we have

$$\operatorname{Res}\{\mu; k_j\} = \frac{b_j}{a'(k_j)}e^{2ik_jx}N(x,k_j) = C_j e^{2ik_jx}N(x,k_j) \tag{2.17a}$$

where the last equality defines the "norming constant", C_j, corresponding to the eigenvalue k_j and $'$ denotes derivative with respect to k. Similarly, the eigenvalues in the region $\operatorname{Im} k < 0$, denoted $\bar{k}_j$, are the zeroes of $\bar{a}(k)$, and correspond to the poles (in k) of $\bar{\mu}(x,k)$. Again, for each simple pole, we have

$$\operatorname{Res}\{\bar{\mu};\bar{k}_j\} = \frac{\bar{b}_j}{\bar{a}'(\bar{k}_j)} e^{-2i\bar{k}_j x} \bar{N}(x,\bar{k}_j) = \bar{C}_j e^{-2i\bar{k}_j x} \bar{N}(x,\bar{k}_j) \tag{2.17b}$$

where the last equality defines the norming constant, $\bar{C}_j$, corresponding to the eigenvalue $\bar{k}_j$.

As stated above, the focusing NLS (1.1) is a special case of the system (2.1a)–(2.1b) under the symmetry reduction $r = -q^*$. This symmetry in the potential induces a symmetry in the scattering data. Indeed, if $v(x,k) = (v^{(1)}(x,k), v^{(2)}(x,k))^T$ satisfies eq. (2.2) and the symmetry holds, then

$$\hat{v}(x,k) = (v^{(2)}(x,k^*), -v^{(1)}(x,k^*))^H$$

also satisfies the scattering problem. Therefore, because the solutions of the scattering problem are uniquely determined by their respective boundary conditions, (2.4a))–(2.4b), we obtain the symmetry relations:

$$\bar{\psi}(x,k) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi^*(x,k^*), \qquad \bar{\phi}(x,k) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \phi^*(x,k^*).$$

It follows that

$$\bar{a}(k) = a^*(k^*), \qquad \bar{b}(k) = -b^*(k^*).$$

Consequently,

$$\bar{\rho}(k) = -\rho^*(k)$$

on the line $\operatorname{Im} k = 0$. Moreover, it follows that k_j is a zero of $a(k)$ in the the region $\operatorname{Im} k > 0$ if, and only if, k_j^* is a zero for $\bar{a}(k)$ where $\operatorname{Im} \bar{k}_j < 0$. Therefore, with this symmetry between the potentials q and r, the eigenvalues appear in complex-conjugate pairs. Finally, one can show that the norming constants satisfy the condition

$$\bar{C}_j = -C_j^*$$

where $\bar{k}_j = k_j^*$.

Recall that, if $r = q^*$, then the system (2.1a)–(2.1b) is equivalent to the defocusing NLS. In this case, the operator in the scattering problem (2.2) is Hermitian and, therefore, the spectrum lies on the real axis. Moreover,

$$\bar{\psi}(x,k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi^*(x,k^*), \qquad \bar{\phi}(x,k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi^*(x,k^*)$$

and it follows that, as before,

$$\bar{a}(k) = a^*(k^*)$$

while

$$\bar{b}(k) = b^*(k^*).$$

Hence, the unitarity relation becomes

$$|a(k)|^2 - |b(k)|^2 = 1 \tag{2.18}$$

on $\operatorname{Im} k = 0$. We conclude that $|a(k)| > 0$ on $\operatorname{Im} k = 0$. Also, for $r = q^*$, the problem is self-adjoint. Hence, when $r = q^*$ and $q \to 0$ sufficiently rapidly as $|x| \to \infty$, there are no discrete eigenvalues.

2.3. Inverse Scattering Problem. The inverse problem consists of constructing a map from the scattering data back to the potentials. Specifically, the scattering data consists of:

- reflection coefficients $\rho(k)$ and $\bar{\rho}(k)$ defined on $\operatorname{Im} k = 0$,
- eigenvalues and norming constants $\{k_j, C_j\}_{j=1}^{J}$ and $\{\bar{k}_j, \bar{C}_j\}_{j=1}^{\bar{J}}$ in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively.

First, we use these data to recover the Jost functions. Then, we recover the potentials in terms of the Jost functions.

Implicit in our description of the scattering data, is the assumption that the potential is such that the scattering coefficients $a(k)$ and $\bar{a}(k)$ have a finite number of simple zeros in the regions $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, respectively. As we have shown, these simple zeroes correspond to the eigenvalues given in the scattering data. (If the eigenvalues are not simple zeros, one can study the problem by considering the coalescence of simple poles [**92**] and Section 2.5 below.) We further assume that $a(k) \neq 0$, $\bar{a}(k) \neq 0$ on $\operatorname{Im} k = 0$.

The equations (2.14a)–(2.14b) can be considered the the jump conditions of a Riemann-Hilbert boundary-value problem (with poles) for the to-be-determined, sectionally meromorphic (in k) functions, $\mu(x,k)$ and $\bar{\mu}(x,k)$, and sectionally analytic functions, $N(x,k)$ and $\bar{N}(x,k)$. To recover these functions from the scattering data, we convert the Riemann-Hilbert problem to a system of linear integral equations with the use of projection operators and Plemelj formula (cf. [**3**]). Taking into account the asymptotics (2.7a)–(2.7d) and (2.17a)-(2.17b), we obtain:

(2.19a)
$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x}}{(k-k_j)} N(x,k_j) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\kappa) e^{2i\kappa x}}{\kappa-(k-i0)} N(x,\kappa)\, d\kappa$$

(2.19b)
$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2i\bar{k}_j x}\bar{N}(x,\bar{k}_j)}{(k-\bar{k}_j)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\bar{\rho}(\kappa) e^{-2i\kappa x}}{\kappa-(k+i0)} \bar{N}(x,\kappa) d\kappa,$$

where $k+i0$ indicates the limit $\epsilon \to 0$ of $k+i\epsilon$, with $\epsilon > 0$, and, similarly, $k-i0$ indicates the limit $\epsilon \to 0$ of $k-i\epsilon$, with $\epsilon > 0$. Note that (2.19a) is valid for k such that $\operatorname{Im} k \leq 0$, while (2.19b) is valid for k such that $\operatorname{Im} k \geq 0$. In the absence of eigenvalues (poles), the system (2.19a)–(2.19b) reduces to a pair of coupled, linear integral equations on the line $\operatorname{Im} k = 0$.

On the other hand, in the presence of poles, we must evaluate eq. (2.19a) at the poles $k = \bar{k}_j$, $j = 1, \dots, \bar{J}$ and (2.19b) at the poles $k = k_j$, $j = 1, \dots, J$ in order to close the system. These evaluations yield:

(2.20a)
$$\bar{N}(x,\bar{k}_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\ell=1}^{J} \frac{C_\ell e^{2ik_\ell x}}{(\bar{k}_j-k_\ell)} N(x,k_\ell) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\kappa) e^{2i\kappa x}}{\kappa-\bar{k}_j} N(x,\kappa)\, d\kappa$$

$$N(x,k_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{\ell=1}^{\bar{J}} \frac{\bar{C}_\ell e^{-2i\bar{k}_\ell x}}{(k_j - \bar{k}_\ell)} \bar{N}(x,\bar{k}_\ell) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\kappa) e^{-2i\kappa x}}{\kappa - k_j} \bar{N}(x,\kappa)\, d\kappa. \tag{2.20b}$$

Together, equations (2.19a)–(2.19b) and (2.20a)–(2.20b) constitute a linear system of algebraic-integral equations that determines the Jost function $N(x,k)$ in the upper-half k-plane, $\operatorname{Im} k \geq 0$, and the Jost function $\bar{N}(x,k)$ in the lower-half k-plane, $\operatorname{Im} k \leq 0$.

To recover the potentials from the Jost functions, we compare the large-k asymptotic expansions of the right-hand sides of (2.19a) and (2.19b) to the expansions (2.7b) and (2.7c), respectively. These comparisons yield

$$r(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j N^{(2)}(x,k_j) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(\kappa) e^{2i\kappa x} N^{(2)}(x,\kappa)\, d\kappa \tag{2.21a}$$

$$q(x) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j \bar{N}^{(1)}(x,\bar{k}_j) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{\rho}(\kappa) e^{-2i\kappa x} \bar{N}^{(1)}(x,\kappa)\, d\kappa \tag{2.21b}$$

where, as before, the superscript (ℓ) denotes the ℓ-th component of the corresponding vector. These relations are explicit expressions for q and r in terms of the Jost functions and the scattering data and, therefore, complete the formulation of the inverse problem.

Alternatively, it is possible to recover the potentials with Gel'fand-Levitan-Marchenko (GLM) integral equations. In this approach, we represent the Jost functions in terms of triangular kernels

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^{+\infty} K(x,s) e^{-ik(x-s)} ds \qquad s > x, \quad \operatorname{Im} k > 0 \tag{2.22a}$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^{+\infty} \bar{K}(x,s) e^{ik(x-s)} ds \qquad s > x, \quad \operatorname{Im} k < 0. \tag{2.22b}$$

With these representations, (2.19a)-(2.19b) may be rewritten as the GLM equations:

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{\infty} K(x,s)\, F(s+y)\, ds = 0$$

$$K(x,y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) + \int_x^{\infty} \bar{K}(x,s)\, \bar{F}(s+y)\, ds = 0$$

where

$$F(x) = -i \sum_{j=1}^{J} C_j e^{ik_j x} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \rho(\kappa) e^{i\kappa x}\, d\kappa$$

$$\bar{F}(x) = i \sum_{j=1}^{\bar{J}} \bar{C}_j e^{-i\bar{k}_j x} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \bar{\rho}(\kappa) e^{-i\kappa x}\, d\kappa.$$

The GLM equations constitute a coupled system of integral equations that determine the kernels K and $\bar{K}$ in terms of the scattering data.

We obtain explicit expressions for the potentials in terms of the GLM kernels by substituting the representations (2.22a)–(2.22b) into eqs. (2.21a)–(2.21b). The resulting expressions are

$$q(x) = -2K^{(1)}(x,x), \qquad r(x) = -2\bar{K}^{(2)}(x,x)$$

where, as before, $K^{(j)}$ and $\bar{K}^{(j)}$ for $j = 1,2$ denote the j-th component of the vectors K and $\bar{K}$ respectively.

In our formulation of the inverse problem we have focused on the concrete formulae. In particular, we have not been concerned with the existence and uniqueness of solutions for the linear systems that we obtained. Existence and uniqueness of solutions of the inverse problem have been considered in a more general context in [**20**]-[**21**]. Also, in [**10**], it was shown that the inverse problem admits a unique solution if the GLM integral equations are Fredholm. For instance, if the potentials are in the Schwartz class, then the kernels F and $\bar{F}$ decay sufficiently rapidly to ensure that the GLM equations are indeed Fredholm.

2.4. Time evolution. The auxiliary problem (2.3) determines the evolution of the scattering data. Specifically, one can show that (cf. [**10**])

$$a(k,t) = a(k,0), \qquad \bar{a}(k,t) = \bar{a}(k,0) \tag{2.24a}$$

$$b(k,t) = e^{-4ik^2t}b(k,0), \qquad \bar{b}(k,t) = e^{4ik^2t}\bar{b}(k,0). \tag{2.24b}$$

It follows immediately from (2.24a) that the eigenvalues of the scattering problem, which are equivalent to the zeros of $a(k)$ and $\bar{a}(k)$, are constant in time. Not only the number of eigenvalues, but also their locations are fixed. Thus, the eigenvalues are time-independent discrete states of the evolution. In fact, the persistence of solitons of NLS is a manifestation of the underlying invariance of the eigenvalues.

The evolution of the reflection coefficients is also determined by (2.24a)–(2.24b). Specifically,

$$\rho(k,t) = e^{-4ik^2t}\rho(k,0), \qquad \bar{\rho}(k,t) = e^{4ik^2t}\bar{\rho}(k,0).$$

We remark that $\rho(k,t) = 0$ for all t if, and only if, $\rho(k,0) = 0$, and similarly for $\bar{\rho}(k,t)$.

Finally, the evolution of the norming constants is

$$C_j(t) = e^{-4ik_j^2t}C_j(0), \qquad \bar{C}_j(t) = e^{4i\bar{k}_j^2t}\bar{C}_j(0). \tag{2.25}$$

2.5. Soliton Solution . In the case where the scattering data comprise proper eigenvalues (and their norming constants) and

$$\rho(k) = 0 \quad \text{and} \quad \bar{\rho}(k) = 0$$

for all $k \in \mathbb{R}$, the system (2.20a)–(2.20b) reduces to finite-dimensional linear algebraic system for

$$\{N(x,k_j)\}_{j=1}^{J} \qquad \text{and} \qquad \left\{\bar{N}(x,\bar{k}_j)\right\}_{j=1}^{\bar{J}}.$$

Moreover, in this case, eqs. (2.21a)-(2.21b) express q and r in terms $\bar{N}(x,\bar{k}_j)$ and $N(x,k_j)$. Therefore, given such "reflectionless" data, we can obtain an explicit expression for the solution.

Recall that, when $r = -q^*$, the eigenvalues appear in complex conjugate pairs. The one-soliton solution of focusing NLS corresponds to a single pair of complex-conjugate eigenvalues (i.e., $J = \bar{J} = 1$). Specifically, the one-soliton solution corresponding to the eigenvalue pair

$$k_1 = \xi + i\eta, \qquad \bar{k_1} = k_1^* = \xi - i\eta$$

is

$$q(x,t) = -ie^{-i(2\xi x - 4(\xi^2-\eta^2)t+\psi)} 2\eta \operatorname{sech}(2\eta x - 8\xi\eta t - \delta), \tag{2.26}$$

where

$$e^{\delta} = \frac{|C_1(0)|}{2\eta}, \qquad e^{i\psi} = \frac{C_1(0)}{|C_1(0)|} \tag{2.27}$$

and $C_1(0)$ is the norming constant (associated with the eigenvalue, k_1) at $t = 0$. The velocity of the sech envelope –described by $|q(x,t)|$– is 4ξ and its amplitude is 2η. In fact, the envelope velocity is proportional to the wavenumber (inverse wavelength) of the complex modulation. We note that, unlike the solitons of the Korteweg-de Vries equation, the amplitude and velocity of NLS solitons are independent.

A J-soliton solution is constructed from scattering data composed of J complex-conjugate pairs of eigenvalues and the associated norming constants. Because, generically, the solitons' speeds are unequal, the individual solitons become well-separated in the long-time limits ($t \to \pm\infty$). When the solitons are well-separated, the eigenvalue pair associated with each individual soliton determines the amplitude and velocity of that soliton, just as in the one-soliton case. The details of soliton interactions are described in Section 6.1.

If one considers scattering data that include only simple poles, it is not possible construct solutions in which two solitons have both the same amplitude and the same velocity. However, in contrast to the generic case described in the previous paragraph, one can construct a solution from scattering data in which two (or more) pairs of eigenvalues have the same imaginary part, but different real parts. In the resulting solution, the envelope peaks travel with the same velocity and, consequently, the distance between peaks does not increase in the long-time limit. However, in this special case, the amplitudes of the individual peaks oscillate periodically [**92**].

Solutions that correspond to scattering data with poles that are not simple can be derived by the coalescence of simple poles. For instance, we consider the coalescence of two pairs eigenvalues where the coalescence is parallel to the imaginary k-axis. We denote the two eigenvalues in the upper-half k-plane as

$$k_1 = \xi + i\eta, \qquad k_2 = k_1 + i\epsilon$$

and the corresponding norming constants as

$$C_1(0) = \epsilon^{-1}C, \qquad C_2(0) = -C_1(0).$$

In the limit $\epsilon \to 0$, one obtains the solution

$$q(x,t) = -iC^* e^{i\zeta(x,t)} \frac{A(x,t)}{B(x,t)}$$

where

$$A(x,t) = \left[\frac{2}{\eta}\theta(x,t) + 16i\eta t\right] e^{-\theta(x,t)} - \frac{|C|^2}{8\eta^5}\left[\theta(x,t) + 8i\eta^2 t + 2\right] e^{-3\theta(x,t)}$$

$$B(x,t) = 1 + \frac{|C|^2}{4\eta^4}\left[(\theta(x,t)+1)^2 + 64\eta^4 t^2 + \frac{1}{2}\right] e^{-2\theta(x,t)} + \frac{|C|^4}{256\eta^8} e^{-4\theta(x,t)}$$

and

$$\theta(x,t) = 2\eta x - 8\xi\eta t, \qquad \zeta(x,t) = 2\xi x - 4(\xi^2 - \eta^2)t.$$

While this solution appears ungainly, if we treat the terms in square brackets as constants (with respect to θ) we obtain a pair of sech-like envelopes that both travel with velocity 4ξ. The θ-dependence in the square-bracketed terms causes a spatial modulation of the sech envelopes, which is, for large θ, overwhelmed by the exponential decay in θ. On the other hand, the explicit t-dependence in square-bracketed terms causes these peaks to separate with distance $O(\log|t|)$ in the long-time limit.

3. Integrable Discrete Nonlinear Schrödinger equation (IDNLS)

3.1. Compatibility Condition. As in the continuous case, it is convenient to consider the system

$$i\frac{d}{d\tau}Q_n = Q_{n-1} - 2Q_n + Q_{n+1} - Q_n R_n\left(Q_{n-1} + Q_{n+1}\right) \tag{3.1a}$$

$$-i\frac{d}{d\tau}R_n = R_{n-1} - 2R_n + R_{n+1} - Q_n R_n\left(R_{n-1} + R_{n+1}\right), \tag{3.1b}$$

which reduces to focusing IDNLS (1.3) under the reduction $R_n = -Q_n^*$. (The defocusing IDNLS is equivalent to the reduction $R_n = Q_n^*$.)

The counterpart of the equality of the mixed derivatives in compatibility condition for NLS (cf. Section 2.1) is the discrete compatibility condition

$$\frac{d}{d\tau}v_{n+1} = \left(\frac{d}{d\tau}v_m\right)_{m=n+1},$$

where the discrete scattering problem is

$$v_{n+1} = \begin{pmatrix} z & Q_n \\ R_n & z^{-1} \end{pmatrix} v_n \tag{3.2a}$$

and the time-dependence is

$$\frac{d}{d\tau}v_n = \begin{pmatrix} iQ_nR_{n-1} - \frac{i}{2}(z-z^{-1})^2 & -i(zQ_n - z^{-1}Q_{n-1}) \\ i(z^{-1}R_n - zR_{n-1}) & -iR_nQ_{n-1} + \frac{i}{2}(z-z^{-1})^2 \end{pmatrix} v_n. \tag{3.2b}$$

The scattering problem (3.2a) is sometimes referred to as Ablowitz-Ladik scattering problem.

With the change of variables

$$\begin{aligned} &Q_n(\tau) \to hq_n(t) = hq(nh,t), \qquad R_n(\tau) \to hr_n(t) = hr(nh,t), \\ &\tau \to h^{-2}t, \qquad z \to e^{-ikh}, \end{aligned} \tag{3.3}$$

the discrete scattering problem (3.2a) converges to the scattering problem for NLS (2.2) in the limit $h \to 0$, $nh \to x$. Similarly, in this limit, the time-dependence equation (3.2b) converges to the time dependence equation for NLS (2.3). Moreover, this is precisely the transformation under which (3.1a)–(3.1b) becomes (2.1a)–(2.1b) and, consequently, IDNLS (1.3) converges to NLS (1.1). In fact, with this transformation, many of the formulas for the discrete IST given in the following sections converge to the corresponding formula in the IST for NLS.

3.2. Direct Scattering Problem. As in the direct scattering problem associated with NLS, we define scattering data, which are independent of the spatial variable, from the potentials $\{Q_n, R_n\}_{n=-\infty}^{\infty}$. However, in this section, the spatial variable is the discrete index n instead of the continuous variable x. Also, as before, the scattering data are determined in terms of Jost functions that depend on both n and the scattering parameter, in this case z.

For potentials Q_n, R_n such that $|Q_n|, |R_n| \to 0$ sufficiently rapidly as $n \to \pm\infty$, the solutions of the scattering problem (3.2a) are characterized by the boundary conditions

$$\phi_n(z) \sim z^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \bar{\phi}_n(z) \sim z^{-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{as } n \to -\infty \tag{3.4a}$$

$$\psi_n(z) \sim z^{-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \bar{\psi}_n(z) \sim z^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{as } n \to +\infty. \tag{3.4b}$$

As in the continuous case, it is also convenient to consider the related Jost functions with constant boundary conditions:

$$M_n(z) = z^{-n}\phi_n(z), \qquad \bar{M}_n(z) = z^n \bar{\phi}_n(z), \tag{3.5a}$$

$$N_n(z) = z^n \psi_n(z), \qquad \bar{N}_n(z) = z^{-n}\bar{\psi}_n(z). \tag{3.5b}$$

The Jost functions are solutions of the summation equations:

$$M_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{\ell}_{n-k}(z)\hat{\mathbf{Q}}_k M_k(z) \tag{3.6a}$$

$$\bar{N}_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{r}_{n-k}(z)\hat{\mathbf{Q}}_k \bar{N}_k(z) \tag{3.6b}$$

$$\bar{M}_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{\ell}_{n-k}(z)\hat{\mathbf{Q}}_k \bar{M}_k(z) \tag{3.6c}$$

$$N_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{r}_{n-k}(z)\hat{\mathbf{Q}}_k N_k(z) \tag{3.6d}$$

where

$$\hat{\mathbf{Q}}_n = \begin{pmatrix} 0 & Q_n \\ R_n & 0 \end{pmatrix}$$

and the respective Green's functions are

$$\mathbf{G}_n^{\ell}(z) = z^{-1}\theta(n-1)\begin{pmatrix} 1 & 0 \\ 0 & z^{-2(n-1)} \end{pmatrix}$$
$$\bar{\mathbf{G}}_n^{r}(z) = -z^{-1}\theta(-n)\begin{pmatrix} 1 & 0 \\ 0 & z^{-2(n-1)} \end{pmatrix}$$
$$\bar{\mathbf{G}}_n^{\ell}(z) = z\theta(n-1)\begin{pmatrix} z^{2(n-1)} & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{G}_n^{r}(z) = -z\theta(-n)\begin{pmatrix} z^{2(n-1)} & 0 \\ 0 & 1 \end{pmatrix},$$

where is $\theta(n)$ is the discrete version of the Heaviside function:

$$\theta(n) = \sum_{k=-\infty}^{n} \delta_{0,k} = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}. \tag{3.8}$$

These summation equations may be derived by the Fourier transform method or by the application of a summation factor.

In analogy with the continuous case, one can prove that, if

$$\|Q\|_1 = \sum_{-\infty}^{+\infty} |Q_n| < \infty \qquad \text{and} \qquad \|R\|_1 = \sum_{-\infty}^{+\infty} |R_n| < \infty,$$

then $M_n(z)$,$N_n(z)$, defined by (3.6a) and (3.6d) respectively, are analytic functions of z for $|z| > 1$ and continuous for $|z| \geq 1$. Similarly, $\bar{M}_n(z)$,$\bar{N}_n(z)$ defined by (3.6b) and (3.6c), respectively, are analytic functions of z for $|z| < 1$ and continuous for $|z| \leq 1$. Moreover, the solutions of the summation equations (3.6a)–(3.6d) are unique in the space of bounded functions (cf. [**10**]).

Because the Jost functions $\bar{M}_n(z)$ and $\bar{N}_n(z)$ are analytic in the region $|z| < 1$, they have convergent power series expansions about $z = 0$. From the summation equations, we determine that these expansions are of the form

$$\bar{M}_n(z) = \begin{pmatrix} zQ_{n-1} + O(z^3, \text{odd}) \\ 1 + O(z^2, \text{even}) \end{pmatrix} \tag{3.9a}$$

$$\bar{N}_n(z) = \begin{pmatrix} c_n^{-1} + O(z^2, \text{even}) \\ -zc_n^{-1}R_n + O(z^3, \text{odd}) \end{pmatrix} \tag{3.9b}$$

where "even" indicates that the higher-order terms are even powers of z^{-1}, "odd" indicates that the higher-order terms are odd powers and

$$c_n = \prod_{k=n}^{+\infty} (1 - Q_k R_k).$$

Note that the products c_n converge absolutely for all n if $\|Q\|_1, \|R\|_1 < \infty$, the same condition that guarantees the well-posedness of the summation equations.

Similarly, because the Jost functions $M_n(z)$ and $N_n(z)$ are analytic in the region $|z| > 1$, they have convergent Laurent series expansions in powers of z^{-1}. From the respective summation equations, we determine that

$$M_n(z) = \begin{pmatrix} 1 + O(z^{-2}, \text{even}) \\ z^{-1}R_{n-1} + O(z^{-3}, \text{odd}) \end{pmatrix} \tag{3.9c}$$

$$N_n(z) = \begin{pmatrix} -z^{-1}c_n^{-1}Q_n + O(z^{-3}, \text{odd}) \\ c_n^{-1} + O(z^{-2}, \text{even}). \end{pmatrix} \tag{3.9d}$$

The scattering problem (3.2a) is a linear, second-order difference equation. Therefore, for a given value of z, all of the solutions can be expressed as linear combinations of any two linearly-independent solutions. In order to evaluate the linear independence of solutions, say u_n and v_n, we consider their Wronskian:

$$W(u_n, v_n,) = \det |u_n, v_n| = u_n^{(1)}v_n^{(2)} - u_n^{(2)}v_n^{(1)},$$

which satisfies the relation

$$W(u_{n+1}, v_{n+1}) = (1 - R_nQ_n)\, W(u_n, v_n).$$

In particular, the boundary conditions (3.4a)–(3.4b) imply that

$$W\left(\phi_n(z), \bar{\phi}_n(z)\right) = \prod_{k=-\infty}^{n-1}(1 - R_kQ_k) \tag{3.10a}$$

$$W\left(\bar{\psi}_n(z), \psi_n(z)\right) = \prod_{k=n}^{+\infty}(1 - R_kQ_k)^{-1}. \tag{3.10b}$$

If, as we have previously assumed, $\|Q_n\|_1, \|R_n\| < \infty$, and also either $R_n = -Q_n^*$ or $|Q_n|, |R_n| < 1$ for all n, then the products are nonzero for all n and the above infinite products are convergent .

The nonvanishing of the Wronskians implies that, for a given z, both $\left\{\phi_n(z), \bar{\phi}_n(z)\right\}$ and $\left\{\bar{\psi}_n(z), \psi_n(z)\right\}$ span the space of solutions to the scattering problem. Consequently, one can express $\phi_n(z)$ and $\bar{\phi}_n(z)$ as linear combinations of $\psi_n(z)$ and $\bar{\psi}_n(z)$, or vice-versa. The coefficients of these linear combinations depend on z. To be concrete, the relations

$$\phi_n(z) = b(z)\psi_n(z) + a(z)\bar{\psi}_n(z) \tag{3.11a}$$

$$\bar{\phi}_n(z) = \bar{a}(z)\psi_n(z) + \bar{b}(z)\bar{\psi}_n(z) \tag{3.11b}$$

hold for any z such that all four eigenfunctions $\phi_n(z), \bar{\phi}_n(z), \psi_n(z)$ and $\bar{\psi}_n(z)$ exist. In particular, these relations are valid and well-defined on the unit circle $|z| = 1$. Therefore, eqs. (3.11a)–(3.11b) define the n-independent scattering coefficients, $a(z)$,$\bar{a}(z)$,$b(z)$ and $\bar{b}(z)$, on $|z| = 1$.

By evaluating Wronskians in the limits as $n \to \pm\infty$, one obtains explicit expressions for the scattering coefficients in terms of the Jost functions:

$$b(z) = z^{2n}c_nW\left(\bar{N}_n(z), M_n(z)\right) \tag{3.12a}$$

$$\bar{b}(z) = z^{-2n}c_nW\left(\bar{M}_n(z), N_n(z)\right) \tag{3.12b}$$

$$a(z) = c_nW\left(M_n(z), N_n(z)\right) \tag{3.12c}$$

$$\bar{a}(z) = c_nW\left(\bar{N}_n(z), \bar{M}_n(z)\right) \tag{3.12d}$$

where, as before,

$$c_n = \prod_{k=n}^{+\infty}(1 - R_kQ_k).$$

Alternatively, the scattering coefficients may be expressed as explicit sums of the Jost functions and the potentials (cf. [**10**]):

$$a(z) = 1 + \sum_{k=-\infty}^{+\infty} z^{-1} Q_k M_k^{(2)}(z) \tag{3.13a}$$

$$b(z) = \sum_{k=-\infty}^{+\infty} z^{2k+1} R_k M_k^{(1)}(z) \tag{3.13b}$$

$$\bar{a}(z) = 1 + \sum_{k=-\infty}^{+\infty} z R_k \bar{M}_k^{(1)}(z) \tag{3.13c}$$

$$\bar{b}(z) = \sum_{k=-\infty}^{+\infty} z^{-2k-1} Q_k \bar{M}_k^{(2)}(z). \tag{3.13d}$$

Comparing the limits of $W\left(\phi_n(z), \bar{\phi}_n(z)\right)$ as $n \to \pm\infty$ we obtain the relation

$$a(z)\bar{a}(z) - b(z)\bar{b}(z) = c_{-\infty} \tag{3.14}$$

where

$$c_{-\infty} = \lim_{n \to -\infty} c_n = \prod_{k=-\infty}^{+\infty} (1 - R_k Q_k). \tag{3.15}$$

We remark that the above relation stands in contrast to (2.10), its counterpart in Zakharov-Shabat scattering problem associated with NLS equation; eq. (2.10) does not depend on the potentials.

It follows immediately from the analytic properties of the Jost functions and (3.12c)-(3.12d) that $a(z)$ has an analytic extension in the region $|z| > 1$ while $\bar{a}(z)$ has an analytic extension in the region $|z| < 1$. Substituting the z expansions of the Jost functions, (3.9c), (3.9d) and (3.9b)–(3.9a), into (3.12c)–(3.12d), we obtain the expansions

$$a(z) = 1 + O(z^{-2}, \text{even}) \qquad |z| > 1 \tag{3.16a}$$

$$\bar{a}(z) = 1 + O(z^{2}, \text{even}) \qquad |z|\, z < 1, \tag{3.16b}$$

where, as before, "even" indicates that the higher-order terms contain only even powers. Moreover, both $a(z)$ and $\bar{a}(z)$ are are continuous up to $|z| = 1$.

Depending on the potentials, there may exist values of the spectral parameter, z, for which there is a solution of the scattering problem that vanishes as $n \to \pm\infty$. Such values of z are the discrete eigenvalues of the scattering problem. The eigenvalues in the region $|z| > 1$ correspond to the zeroes of $a(z)$ in that region while the eigenvalues in the region $|z| < 1$ correspond to the zeroes of $\bar{a}(z)$. Consider z_j such that $|z_j| > 1$ and $a(z_j) = 0$. Then,

$$W(\phi_n(z_j), \psi_n(z_j)) = 0$$

and the vanishing of the Wronskian implies that $\phi_n(z_j)$ and $\psi_n(z_j)$ linearly dependent. Equivalently,

$$\phi_n(z_j) = b_j \psi_n(z_j)$$

for some complex constant b_j. In this case,

$$\phi_n(z_j) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as $n \to \pm\infty$ and z_j is an eigenvalue. Conversely, if z_j is an eigenvalue, then the Wronskian must vanish and, therefore $a(z_j) = 0$. Similarly, $\bar{a}(\bar{z}_j) = 0$ for $\bar{z}_j$ such that $|\bar{z}_j| < 1$, if, and only if,

$$\bar{\phi}_n(\bar{z}_j) = \bar{b}_j \bar{\psi}_n(\bar{z}_j)$$

and, therefore, $\bar{z}_j$ is an eigenvalue of the scattering problem. In terms of the Jost functions, we have the relations

$$M_n(z_j) = b_j z_j^{-2n} N_n(z_j), \qquad \bar{M}_n(\bar{z}_j) = \bar{b}_j \bar{z}_j^{2n} \bar{N}_n(\bar{z}_j), \tag{3.17}$$

where the first relation holds if, and only if, z_j is an eigenvalue such that $|z_j| > 1$ and the second holds if, and only if, $\bar{z}_j$ is an eigenvalue such that $|\bar{z}_j| < 1$.

In the formulation of the inverse problem, it is convenient to introduce the functions

$$\mu_n(z) = \frac{M_n(z)}{a(z)} = \begin{pmatrix} 1 + O(z^{-2}) \\ z^{-1} R_{n-1} + O(z^{-3}) \end{pmatrix} \tag{3.18a}$$

$$\bar{\mu}_n(z) = \frac{\bar{M}_n(z)}{\bar{a}(z)} = \begin{pmatrix} z Q_{n-1} + O(z^3) \\ 1 + O(z^2) \end{pmatrix} \tag{3.18b}$$

as well as the reflection coefficients

$$\rho(z) = \frac{b(z)}{a(z)}, \qquad \bar{\rho}(z) = \frac{\bar{b}(z)}{\bar{a}(z)}. \tag{3.19}$$

In terms of these new functions, the conditions (3.11a)–(3.11b) may be restated as

$$\mu_n(z) - \bar{N}_n(z) = z^{-2n} \rho(z) N_n(z) \tag{3.20a}$$

$$\bar{\mu}_n(z) - N_n(z) = z^{2n} \bar{\rho}(z) \bar{N}_n(z). \tag{3.20b}$$

We remark that $\mu_n(z)$ is meromorphic in the region $|z| > 1$ with poles corresponding to the zeros of $a(z)$, while $\bar{\mu}_n(z)$ is meromorphic in the region $|z| < 1$ with poles at the zeros of $\bar{a}(z)$.

If $a(z)$ has J simple zeros

$$\{z_j : \ |z_j| > 1\}_{j=1}^{J}$$

and $\bar{a}(z)$ has $\bar{J}$ simple zeros at the points

$$\{\bar{z}_j : \ |\bar{z}_j| < 1\}_{j=1}^{\bar{J}},$$

then

$$\operatorname{Res}(\mu_n; z_j) = \frac{M_n(z_j)}{a'(z_j)} = \frac{b_j}{a'(z_j)} z_j^{-2n} N_n(z_j) = z_j^{-2n} C_j N_n(z_j) \tag{3.21a}$$

$$\operatorname{Res}(\bar{\mu}_n; \bar{z}_\ell) = \frac{\bar{M}_n(\bar{z}_\ell)}{\bar{a}'(\bar{z}_\ell)} = \frac{\bar{b}_\ell}{\bar{a}'(\bar{z}_\ell)} \bar{z}_\ell^{2n} \bar{N}_n(\bar{z}_\ell) = \bar{z}_\ell^{2n} \bar{C}_\ell \bar{N}_n(\bar{z}_\ell) \tag{3.21b}$$

where $'$ denotes derivative with respect to the spectral parameter, z. As in the continuous scattering problem, we refer to the constant C_j, defined by (3.21a), as the norming constant associated with the eigenvalue z_j and to the constant $\bar{C}_j$, defined by (3.21b), as the norming constant associated with the eigenvalue $\bar{z}_\ell$.

Because the scattering coefficient $a(z)$ is an even function of z, we have $a(-z) = 0$ if, and only if, $a(z) = 0$. Consequently, the eigenvalues in the region $|z| > 1$ appear in pairs $\pm z_j$. Moreover, the norming constant associated with the eigenvalue $-z_j$ is equal to the norming constant associated with the eigenvalue z_j. Similarly, as $\bar{a}(z)$ is an even function of z, the eigenvalues in the region $|z| < 1$ appear in pairs $\pm\bar{z}_j$, where the norming constant associated with the eigenvalue $-\bar{z}_j$ is equal to the norming constant associated with the eigenvalue $\bar{z}_j$. We remark that this symmetry in the eigenvalues does not depend on any symmetry in the potentials and has no counterpart in the continuous scattering problem.

The focusing IDNLS (1.3) is a special case of the compatibility condition (3.2a)–(3.2b) in which $R_n = -Q_n^*$. This symmetry in the potentials induces a symmetry in the scattering coefficients:

$$\bar{a}(z) = a^*(1/z^*) \tag{3.22a}$$

$$\bar{b}(z) = -b^*(1/z^*) \tag{3.22b}$$

Consequently, the reflection coefficients satisfy the symmetry

$$\bar{\rho}(z) = -\rho^*(1/z^*). \tag{3.23}$$

Furthermore, the symmetry (3.22a) implies that for each eigenvalue z_j such that $|z_j| > 1$, there is an eigenvalue $\bar{z}_j = 1/z_j^*$ with $|\bar{z}_j| < 1$ and vice-versa. It follows that the number of eigenvalues outside the unit circle equals that of the eigenvalues inside. Thus, taking into account the two symmetries, we conclude that the discrete spectrum is made up of quartets of eigenvalues

$$\left\{z_j, -z_j, 1/z_j^*, -1/z_j^*\right\}_{j=1}^{J}.$$

Finally, the symmetry (3.22a) induces a symmetry in the corresponding norming constants. Specifically,

$$\bar{C}_j = \frac{\bar{b}_j}{\bar{a}'(\bar{z}_j)} = \frac{-b_j^*}{-(z_j^2 a'(z_j))^*} = (z_j^*)^{-2} C_j^*. \tag{3.24}$$

Recall that the defocusing IDNLS is equivalent to (3.1a)–(3.1b) under the symmetry reduction $R_n = Q_n^*$. Under this symmetry, there are no discrete eigenvalues with $|z_j| \neq 1$. Indeed, like in the continuous case, the scattering problem with this reduction is self-adjoint. Moreover, from (3.14) and the symmetry relations (3.22a)–(3.22b) it follows that, on $|z| = 1$,

$$|a(z)|^2 - |b(z)|^2 = \prod_{k=-\infty}^{+\infty} (1 - |Q_k|)^2.$$

Therefore, if $|Q_n| < 1$ for all n, we have $|a(z)|^2 > 0$ on $|z| = 1$. Hence, in this case there are no discrete eigenvalues for the associated scattering problem.

3.3. Inverse Scattering Problem. The inverse problem consists of reconstructing the potentials in terms of the scattering data

$$\{\rho(z), \bar{\rho}(z) \;\text{ for }\; |z| = 1\} \cup \{\pm z_j, C_j\}_{j=1}^{J} \cup \left\{\pm\bar{z}_j, \bar{C}_j\right\}_{j=1}^{\bar{J}}.$$

As before, this process proceeds in two steps: first, we reconstruct the Jost functions from the scattering data; then, we recover the potentials from the Jost functions.

In the solution of the inverse problem, we assume all properties of the Jost functions and scattering data that we derived in the previous section. In particular, the to-be-determined functions $\mu_n(z)$ and $\bar{\mu}_n(z)$ are meromorphic in the regions $|z| > 1$ and $|z| < 1$, respectively, while the functions $N_n(z)$ and $\bar{N}_n(z)$ are analytic in the regions $|z| > 1$ and $|z| < 1$, respectively.

We make the additional assumption that $a(z), \bar{a}(z) \neq 0$ on $|z| = 1$. Consequently, we are guaranteed that there are finitely-many eigenvalues. Finally, we assume that the all eigenvalues correspond to simple zeros of $a(z)$ and $\bar{a}(z)$. (If the eigenvalues are not simple zeros, one can study the situation by the coalescence of simple poles, in analogy with the continuous case as in Section 3.5.)

The equations (3.20a)–(3.20b) can be considered to define the jump conditions for a Riemann-Hilbert boundary-value problem (with poles) for the functions $\mu_n(z)$, $\bar{\mu}_n(z)$, $N_n(z)$ and $\bar{N}_n(z)$, where the regions of analyticity are bounded by the unit circle $|z| = 1$ [**26**]. However, in the previous section, we determined that the boundary condition for $N_n(z)$ on the unit circle $|z| = 1$ depends on Q_k and R_k for all $k \geq n$, where Q_n and R_n are unknowns in the inverse problem. Therefore, in order to remove this dependence, we introduce the modified functions

$$N'_n = \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix} N_n = \begin{pmatrix} -z^{-1}c_n^{-1}Q_n \\ 1 \end{pmatrix} + O(z^{-2}) \tag{3.25a}$$

$$\mu'_n = \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix} \mu_n = \begin{pmatrix} 1 \\ z^{-1}c_n R_{n-1} \end{pmatrix} + O(z^{-2}) \tag{3.25b}$$

$$\bar{N}'_n = \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix} \bar{N}_n = \begin{pmatrix} c_n^{-1} \\ -zR_n \end{pmatrix} + O(z^2) \tag{3.25c}$$

$$\bar{\mu}'_n = \begin{pmatrix} 1 & 0 \\ 0 & c_n \end{pmatrix} \bar{\mu}_n = \begin{pmatrix} zQ_{n-1} \\ c_n \end{pmatrix} + O(z^2) \tag{3.25d}$$

where the z-expansions in (3.25a)–(3.25b) are valid in the region $|z| > 1$ and the z-expansions in (3.25c)–(3.25d) are valid in the region $|z| < 1$.

These modified functions satisfy modified jump conditions

$$\mu'_n(z) - \bar{N}'_n(z) = z^{-2n}\rho(z)N'_n(z) \tag{3.26a}$$

$$\bar{\mu}'_n(z) - N'_n(z) = z^{2n}\bar{\rho}(z)\bar{N}'_n(z) \tag{3.26b}$$

on $|z| = 1$. Furthermore, the poles and norming constants of $\mu'_n(z)$ and $\bar{\mu}'_n(z)$ are the same as the poles and norming constants of $\mu_n(z)$ and $\bar{\mu}_n(z)$.

To solve the Riemman-Hilbert problem, we apply the Plemelj formula where the regions of analyticity (meromorphicity) are the regions $|z| > 1$ and $|z| < 1$. The resulting equations are:

$$\bar{N}'_n(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} C_j z_j^{-2n} \left[\frac{1}{z - z_j} N'_n(z_j) + \frac{1}{z + z_j} N'_n(-z_j) \right] - \lim_{\substack{\zeta \to z \\ |\zeta|<1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{-2n}\rho(w)}{w - \zeta} N'_n(w)\, dw \tag{3.27a}$$

$$(3.27\text{b})\quad N_n'(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \bar{C}_j \bar{z}_j^{2n} \left[\frac{1}{z - \bar{z}_j} \bar{N}_n'(\bar{z}_j) + \frac{1}{z + \bar{z}_j} \bar{N}_n'(-\bar{z}_j) \right] + \lim_{\substack{\zeta \to z \\ |\zeta|>1}} \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n} \bar{\rho}(w)}{w - \zeta} \bar{N}_n'(w)\, dw,$$

where $N_n'(z_j)$ is $N_n'(z)$ evaluated at the eigenvalue z_j, $N_n'(-z_j)$ is $N_n'(z)$ evaluated at the eigenvalue $-z_j$ and similarly for $\bar{N}_n'(\bar{z}_j)$ and $\bar{N}_n'(-\bar{z}_j)$. Here we have explicitly accounted for the fact that the eigenvalues arise in pairs $\pm z_j$ in $|z| > 1$ and $\pm \bar{z}_j$ in $|z| < 1$ and the symmetry in the corresponding norming constants satisfy. Equations (3.27a)–(3.27b) constitute a linear system of integral equations for $N_n'(z)$ and $\bar{N}_n'(z)$ on $|z| = 1$.

In the absence of poles, equations (3.27a)–(3.27b), in principle, determine the Jost functions. However, in the presence of poles, the system depends on the vectors

$$\{N_n'(z_j), N_n'(-z_j)\}_{j=1}^{J} \qquad \text{and} \qquad \left\{\bar{N}_n'(\bar{z}_j), \bar{N}_n'(-\bar{z}_j)\right\}_{j=1}^{\bar{J}}.$$

To close the system, we obtain expressions for these vectors by evaluating (3.27a) at the points $\pm \bar{z}_j$ and (3.27b) at the points $\pm z_j$. These evaluations yield the equations:

$$(3.27\text{c})\quad \bar{N}_n'(\bar{z}_j) = \begin{pmatrix} 1 \end{pmatrix} + \sum_{k=1}^{J} C_k z_k^{-2n} \left[\frac{1}{\bar{z}_j - z_k} N_n'(z_k) + \frac{1}{\bar{z}_j + z_k} N_n'(-z_k) \right] - \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{-2n} \rho(w)}{w - \bar{z}_j} N_n'(w)\, dw$$

$$(3.27\text{d})\quad \bar{N}_n'(-\bar{z}_j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=1}^{J} C_k z_k^{-2n} \left[\frac{1}{\bar{z}_j + z_k} N_n'(z_k) + \frac{1}{\bar{z}_j - z_k} N_n'(-z_k) \right] - \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{-2n} \rho(w)}{w + \bar{z}_j} N_n'(w)\, dw$$

$$(3.27\text{e})\quad N_n'(z_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=1}^{\bar{J}} \bar{C}_k \bar{z}_k^{2n} \left[\frac{1}{z_j - \bar{z}_k} \bar{N}_n'(\bar{z}_k) + \frac{1}{z_j + \bar{z}_k} \bar{N}_n'(-\bar{z}_k) \right] + \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n} \bar{\rho}(w)}{w - z_j} \bar{N}_n'(w)\, dw$$

$$(3.27\text{f})\quad N_n'(-z_j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=1}^{\bar{J}} \bar{C}_k \bar{z}_k^{2n} \left[\frac{1}{z_j + \bar{z}_k} \bar{N}_n'(\bar{z}_k) + \frac{1}{z_j - \bar{z}_k} \bar{N}_n'(-\bar{z}_k) \right] + \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n} \bar{\rho}(w)}{w + z_j} \bar{N}_n'(w)\, dw$$

where (3.27c)–(3.27d) hold for each eigenvalue $\{\bar{z}_j\}_{j=1}^{\bar{J}}$ and (3.27e)–(3.27f) hold for each eigenvalue $\{z_j\}_{j=1}^{J}$. Together, eqs. (3.27a)–(3.27f) constitute a linear algebraic-integral system for the modified Jost functions $N_n'(z)$ and $\bar{N}_n'(z)$.

The Plemelj formula also yields

$$\bar{\mu}'_n(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \bar{C}_j \bar{z}_j^{2n} \left[\frac{1}{z - \bar{z}_j} \bar{N}'_n(\bar{z}_j) + \frac{1}{z + \bar{z}_j} \bar{N}'_n(-\bar{z}_j) \right] + \frac{1}{2\pi i} \oint_{|w|=1} \frac{w^{2n} \bar{\rho}(w)}{w - z} \bar{N}'_n(w)\, dw, \tag{3.28}$$

which is an explicit expression for $\bar{\mu}'_n(z)$ in terms of the modified Jost function $\bar{N}'_n(z)$. A similar formula exists for the modified Jost function $\mu'_n(z)$, but this formula is not necessary for the reconstruction of the potentials.

To recover the potentials from the Jost functions we compare the power-series expansions (in z) of the right-hand side of (3.27a) with (3.25c) and the power-series expansion of the right-hand side of (3.28) with (3.25d). These comparisons yield the explicit expressions:

$$R_n = 2 \sum_{j=1}^{J} C_j z_j^{-2(n+1)} N_n'^{(2)}(z_j) + \frac{1}{2\pi i} \oint_{|w|=1} w^{-2(n+1)} \rho(w) N_n'^{(2)}(w)\, dw \tag{3.29a}$$

$$Q_{n-1} = -2 \sum_{j=1}^{\bar{J}} \bar{C}_j \bar{z}_j^{2(n-1)} \bar{N}_n'^{(1)}(\bar{z}_j) + \frac{1}{2\pi i} \oint_{|w|=1} w^{2(n-1)} \bar{\rho}(w) \bar{N}_n'^{(1)}(w)\, dw. \tag{3.29b}$$

As in the continuous case, one can also reconstruct the potentials by means of a GLM equations. To obtain these equations, we represent the eigenfunctions in terms of triangular kernels

$$\psi_n(z) = \sum_{j=n}^{+\infty} z^{-j} K(n,j) \qquad |z| > 1 \tag{3.30a}$$

$$\bar{\psi}_n(z) = \sum_{j=n}^{+\infty} z^{j} \bar{K}(n,j) \qquad |z| < 1. \tag{3.30b}$$

The kernels satisfy the coupled equations

$$\bar{K}(n,m) + \sum_{j=n}^{+\infty} K(n,j) F(m+j) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta_{m,n} \qquad m \geq n \tag{3.31a}$$

$$K(n,m) + \sum_{j=n}^{+\infty} \bar{K}(n,j) \bar{F}(m+j) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta_{m,n} \qquad m \geq n \tag{3.31b}$$

where

$$F(n) = \sum_{j=1}^{J} z_j^{-n-1} C_j + \frac{1}{2\pi i} \oint_{|z|=1} z^{-n-1} \rho(z)\, dz$$

$$\bar{F}(n) = -\sum_{j=1}^{\bar{J}} \bar{z}_j^{n-1} \bar{C}_j + \frac{1}{2\pi i} \oint_{|z|=1} z^{n-1} \bar{\rho}(z)\, dz.$$

It is more convenient to write the equations (3.31a)-(3.31b) as forced summation equations. To obtain such equations, we introduce $\kappa\,(n,m)$ and $\bar{\kappa}(n,m)$ such that

$$\kappa(n,n) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \bar{\kappa}(n,n) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and, for $m > n$,

$$\begin{aligned} K(n,m) &= \prod_{j=n}^{+\infty} (1 - R_j Q_j)\ \kappa(n,m) \\ \bar{K}(n,m) &= \prod_{j=n}^{+\infty} (1 - R_j Q_j)\ \bar{\kappa}(n,m). \end{aligned}$$

Then, eqs. (3.31a)–(3.31b) are equivalent to

$$\begin{aligned} \bar{\kappa}(n,m) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(m+n) + \sum_{j=n+1}^{+\infty} \kappa(n,j) F(m+j) &= 0 \qquad m > n \\ \kappa(n,m) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(m+n) + \sum_{j=n+1}^{+\infty} \bar{\kappa}(n,j) \bar{F}(m+j) &= 0 \qquad m > n \end{aligned}$$

which constitute a coupled system for $\kappa(n,m)$ and $\bar{\kappa}(n,m)$. Finally, the potentials are obtained from $\kappa(m,n)$ and $\bar{\kappa}(m,n)$ by the relations:

$$Q_n = -\kappa^{(1)}(n,n+1), \qquad R_n = -\bar{\kappa}^{(2)}(n,n+1).$$

3.4. Time evolution. The operator (3.2b) determines the evolution of the Jost functions. From this, one deduces the time evolution of the scattering data. The evolution of scattering coefficients is:

$$\begin{aligned} b(z,\tau) &= b(z,0)e^{2i\omega\tau} \qquad & a(z,\tau) = a(z,0) \\ \bar{a}(z,\tau) &= \bar{a}(z,0) \qquad & \bar{b}(z,\tau) = \bar{b}(z,0)e^{-2i\omega\tau}. \end{aligned}$$

where

$$\omega = \frac{1}{2}\left(z - z^{-1}\right)^2.$$

It follows immediately that the eigenvalues (i.e., the zeros of $a(z)$ and $\bar{a}(z)$) are constant as the solution evolves. As in the continuous scattering problem, not only the number of eigenvalues, but also their locations are fixed.

As determined by the evolution of the scattering coefficients, the evolution of the reflection coefficients is given by:

$$\begin{aligned} \rho(z,\tau) &= e^{2i\omega\tau}\rho(z,0) \\ \bar{\rho}(z,\tau) &= e^{-2i\omega\tau}\bar{\rho}(z,0). \end{aligned}$$

In particular, if $\rho(z,0)=0$, then $\rho(z,\tau)=0$ for all τ, and similarly for $\bar{\rho}(z,\tau)$. Finally, the evolution of the norming constants is

$$C_j(\tau)=C_j(0)e^{2i\omega_j\tau}, \qquad \bar{C}_j(\tau)=\bar{C}_j(0)e^{-2i\bar{\omega}_j\tau} \tag{3.33}$$

where

$$\omega_j=\frac{1}{2}\left(z_j-z_j^{-1}\right)^2, \qquad \bar{\omega}_j=\frac{1}{2}\left(\bar{z}_j-\bar{z}_j^{-1}\right)^2.$$

3.5. Soliton Solution. The one-soliton of IDNLS corresponds to scattering data composed of a single quartet of eigenvalues

$$\left\{\pm z_1, \pm\frac{1}{z_1^*}\right\},$$

and the associated norming constant C_1 where $\rho(z)=\bar{\rho}(z)=0$ for $z=1$. Note that we need to only specify the norming constant associated with the eigenvalue z_1 because the symmetry $R_n=-Q_n^*$ fixes the the norming constant associated with the eigenvalue $1/z_1^*$.

We write the eigenvalue z_1 as

$$z_1=e^{\alpha+i\beta},$$

where, without loss of generality, $\alpha>0$ and $-\frac{\pi}{2}<\beta\le\frac{\pi}{2}$. With this scattering data, eqs. (3.27c)–(3.27f) reduce to a finite-dimensional linear system that can be solved explicitly. Taking into account the time-dependence of the norming constant (3.33), we obtain the IDNLS soliton:

$$Q_n(\tau)=-e^{i(2\beta(n+1)+2w\tau-\psi)}\sinh(2\alpha)\operatorname{sech}\left(2\alpha(n+1)-2v\tau-d\right) \tag{3.34}$$

where

$$v=-\sinh(2\alpha)\sin(2\beta), \qquad w=1-\cosh(2\alpha)\cos(2\beta), \tag{3.35a}$$

$$e^d=\frac{|C_1(0)|}{\sinh(2\alpha)}, \qquad e^{i\psi}=\frac{C_1(0)}{|C_1(0)|}. \tag{3.35b}$$

Like the soliton solution of NLS (cf. (2.26)), the soliton solution of IDNLS is a traveling sech envelope, modulated by a complex carrier phase.

The one-soliton solution of IDNLS is more easily compared to the one-soliton solution of NLS if we apply the transformation (3.3), which gives the one-soliton solution of eq. (1.4), the version of IDNLS that explicitly contains the grid size, h. The one-soliton solution of (1.4) is

$$q_n(t)=-e^{-i(2\xi(n+1)h-2wt+\psi)}\frac{\sinh(2\eta h)}{h}\operatorname{sech}\left(2\eta(n+1)h-2vt-d\right) \tag{3.36}$$

where

$$v=\frac{\sinh(2\eta h)\sin(2\xi h)}{h^2} \qquad w=\frac{1-\cosh(2\eta h)\cos(2\xi h)}{h^2},$$
$$e^d=\frac{|C_1(0)|}{\sinh(2\eta h)}, \qquad ,e^{i\psi}=\frac{C_1(0)}{|C_1(0)|}$$

and

$$z_1=e^{\alpha+i\beta}=e^{\eta h-i\xi h}.$$

For a fixed t, the solution, $q_n(t)$, converges to the one soliton solution of NLS in the limit $h\to 0$, $nh\to x$, with an error of $O\left(h^2\right)$.

Like the PDE case (i.e., NLS), in IDNLS, the amplitude of the of the sech envelope is determined by the parameter η. However, for the discrete soliton, the velocity of the sech envelope depends on both ξ and η. In contrast, for the continuous soliton, the velocity depends only on ξ, which, in both cases, is proportional to the spatial frequency of the complex modulation. In the discrete case, the spatial frequency of the complex modulation is limited by h, the distance between the lattice points.

Combining the preceding observations, we deduce a qualitative difference between discrete solitons and their continuous counterparts: for an IDNLS soliton,

$$|v| \leq h^{-1}A,$$

where v is the velocity of the sech envelope and $A = \frac{\sinh(2\eta h)}{h}$ is its amplitude. There is, however, no bound on the envelope velocity of NLS solitons of any amplitude.

A second qualitative difference between the solitons of NLS and IDNLS is that, on the lattice, the relation between the frequency of the complex modulation and the velocity of the sech envelope is not one-to-one. This can be most easily seen in terms of the parameter α, which determines the amplitude of the discrete soliton (cf. (3.34)) and the parameter β, which determines the complex modulation. The soliton corresponding to the parameter pair $\{\alpha, \beta'\}$ where $0 \leq \beta \leq \frac{\pi}{2}$ has the same velocity as the soliton corresponding to the pair $\{\alpha, \beta'\}$, where

$$\beta' = \frac{\pi}{2} - \beta.$$

A similar formula holds for $-\frac{\pi}{2} < \beta < 0$. More generally, for a fixed eigenvalue $z_1 = e^{\alpha + i\beta}$, there is a curve (in the complex z-plane) of eigenvalues $z_1' = e^{\alpha' + i\beta'}$ such that

$$\sinh(2\alpha')\sin(2\beta') = \sinh(2\alpha)\sin(2\beta)$$

where $\beta' \neq \beta$ for any α. Thus, a soliton corresponding to the eigenvalue z_1' has a sech envelope with the same velocity as the envelope of a soliton corresponding to the eigenvalue z_1 even though the frequencies of complex modulation are unequal.

As a consequence of the property described in the preceding paragraph, IDNLS has a class of breather solutions that has no counterpart in NLS [**12**]. These breathers correspond to reflectionless scattering data composed of two quartets of eigenvalues, where z_1 from the first quartet and z_2 from the second quartet are as in the preceding paragraph. The envelopes of these breathers travel with the velocity of the simple soliton associated with the eigenvalue z_1 (or, equivalently, z_2) and oscillate periodically in time. For any fixed τ, these breathers decay exponentially as $n \to \pm\infty$.

The solution of IDNLS corresponding to a double pole in the discrete spectrum can be obtained by considering the coalescence of two simple poles. For example, we consider the poles

$$z_1 = e^{a+ib}, \qquad z_2 = e^{a+i(b+\epsilon)}$$

in the limit $\epsilon \to 0$. If the corresponding norming constants are chosen such that

$$C_1(0) = \frac{1}{\epsilon}C, \qquad C_2(0) = -C_1(0) = \frac{1}{\epsilon}C,$$

one obtains the solution

$$Q_n(\tau) = 2iC^* e^{i\zeta(n,\tau)} \frac{A_n(\tau)}{B_n(\tau)}$$

where

$$A_n(\tau) = [f(n,\tau) - 2i\sinh(2\alpha)\cos(2\beta)\tau]\, e^{-\theta(n,\tau)} + \frac{|C|^2}{\sinh^6(2\alpha)}\,[f(n,\tau) - 2i\sinh(2\alpha)\cos(2\beta)\tau - 4\coth(2\alpha)]\, e^{-3\theta(n,\tau)}$$

$$B_n(\tau) = 1 + \frac{64|C|^2}{\sinh^2(2\alpha)}\Big[(\sinh(2\alpha)f(n,\tau) - 2\cosh(2\alpha))^2 + 2e^{-2\alpha}\sinh^3(2\alpha)\cos^2(2\beta)\tau^2 + 2\Big]\, e^{-2\theta(n,\tau)} + \frac{4|C|^4}{\sinh^6(2\alpha)\cosh^4(2\alpha)} e^{-4\theta(n,\tau)}$$

and

$$\begin{aligned}\theta(n,\tau) &= 2\alpha(n+1) + \sinh 2\alpha \sin 2\beta\tau,\\ \zeta(n,\tau) &= 2(n+1)\beta + (1 - \cosh 2\alpha\cos 2\beta)\,\tau\\ f(n,\tau) &= n + 1 - 2\cosh(2\alpha)\sin(2\beta)\tau.\end{aligned}$$

That is, as for NLS, the coalescence solution is the ratio of polynomials in the phase variable $\theta(n,\tau)$, with a complex modulation due to $\zeta(n,\tau)$. Moreover, the qualitative behavior is similar to that of the coalescence solution obtained for NLS. If the terms in square brackets are treated as constants, with respect to $\theta(n,\tau)$, the solution can be rewritten as a pair of travelling sech-like envelopes in the phase variable $\theta(n,\tau)$. The term $f(n,\tau)$ generates a spatial distortion. However, the strength of this distortion diminishes as $|\tau|$ gets large. Moreover, for large $\theta(n,\tau)$, the spatial distortion is overwhelmed by the spatial exponential decay. Finally, the τ-dependence in the square-bracketed terms, both explicit and through $f(n,\tau)$, causes the sech-envelope peaks to separate with distance $O(\log|\tau|)$ in the long-time limits.

4. Vector Nonlinear Schrödinger (VNLS) equation

4.1. Compatibility Condition. In this section, we describe the IST for the coupled matrix system

$$i\mathbf{Q}_t = \mathbf{Q}_{xx} - 2\mathbf{QRQ} \tag{4.1a}$$

$$-i\mathbf{R}_t = \mathbf{R}_{xx} - 2\mathbf{RQR}, \tag{4.1b}$$

where $\mathbf{Q}$ is an $N \times M$ matrix and $\mathbf{R}$ is an $M \times N$ matrix. Eqs. (4.1a)–(4.1b) reduce to VNLS (1.2) when

$$\mathbf{R} = -\mathbf{Q}^H,$$

where the superscript H denotes the Hermitian (conjugate) transpose, and $N = 1$ or $M = 1$. This coupled matrix system reduces to the coupled scalar system (2.1a)–(2.1b), which is related to NLS, when $N = M = 1$. The IST for (4.1a)–(4.1b) is a straightforward generalization of the IST for the two-component VNLS (1.2) that was developed by Manakov [**64**].

The compatibility condition for the matrix nonlinear Schrödinger equation is a direct matrix generalization of the compatibility condition for the scalar NLS. That is, substituting the matrix functions $\mathbf{Q}$ and $\mathbf{R}$ for the scalar functions q and r in (2.2)–(2.3), we obtain

$$\mathbf{v}_x = \begin{pmatrix} -ik\mathbf{I}_N & \mathbf{Q} \\ \mathbf{R} & ik\mathbf{I}_M \end{pmatrix} \mathbf{v} \tag{4.2a}$$

$$\mathbf{v}_t = \begin{pmatrix} 2ik^2\mathbf{I}_N + i\mathbf{QR} & -2k\mathbf{Q} - i\mathbf{Q}_x \\ -2k\mathbf{R} + i\mathbf{R}_x & -2ik^2\mathbf{I}_M - i\mathbf{RQ} \end{pmatrix} \mathbf{v} \tag{4.2b}$$

where $\mathbf{v}$ is an $N+M$-component column vector and $\mathbf{I}_N$, $\mathbf{I}_M$ are, respectively, the $N \times N$ and $M \times M$ identity matrices. With this pair, the compatibility condition $\mathbf{v}_{xt} = \mathbf{v}_{tx}$ is equivalent to evolution equations (4.1a)–(4.1b).

4.2. Direct Scattering Problem. When $\mathbf{Q}, \mathbf{R} \to 0$ sufficiently rapidly as $x \to \pm\infty$, the solutions of of the scattering problem (4.2a) are characterized by the boundary conditions

$$\boldsymbol{\phi}(x,k) \sim \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} e^{-ikx}, \qquad \bar{\boldsymbol{\phi}}(x,k) \sim \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} e^{ikx} \qquad \text{as} \quad x \to -\infty \tag{4.3a}$$

$$\boldsymbol{\psi}(x,k) \sim \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} e^{ikx}, \qquad \bar{\boldsymbol{\psi}}(x,k) \sim \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} e^{-ikx}, \qquad \text{as} \quad x \to +\infty \tag{4.3b}$$

where $\boldsymbol{\phi}, \bar{\boldsymbol{\phi}}, \boldsymbol{\psi}, \bar{\boldsymbol{\psi}}$ are matrix-valued functions with the following dimensions:

$$\begin{aligned} \boldsymbol{\phi}(x,k) &: \quad (N+M) \times N, \qquad \bar{\boldsymbol{\phi}}(x,k): \quad (N+M) \times M \\ \boldsymbol{\psi}(x,k) &: \quad (N+M) \times M, \qquad \bar{\boldsymbol{\psi}}(x,k): \quad (N+M) \times N. \end{aligned}$$

As in the preceding sections, it is convenient to consider functions with constant boundary conditions. Hence, we define the Jost functions as follows:

$$\mathbf{M}(x,k) = e^{ikx}\boldsymbol{\phi}(x,k), \qquad \bar{\mathbf{M}}(x,k) = e^{-ikx}\bar{\boldsymbol{\phi}}(x,k), \tag{4.4a}$$

$$\mathbf{N}(x,k) = e^{-ikx}\boldsymbol{\psi}(x,k), \qquad \bar{\mathbf{N}}(x,k) = e^{ikx}\bar{\boldsymbol{\psi}}(x,k). \tag{4.4b}$$

These functions are solutions of the linear integral equations

$$\mathbf{M}(x,k) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_+(x-x',k)\tilde{\mathbf{Q}}(x')\mathbf{M}(x',k)\,dx' \tag{4.5a}$$

$$\mathbf{N}(x,k) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \int_{-\infty}^{+\infty} \tilde{\mathbf{G}}_+(x-x',k)\tilde{\mathbf{Q}}\mathbf{N}(x',k)\,dx' \tag{4.5b}$$

$$\bar{\mathbf{M}}(x,k) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \int_{-\infty}^{+\infty} \tilde{\mathbf{G}}_-(x-x',k)\tilde{\mathbf{Q}}(x')\bar{\mathbf{M}}(x',k)\,dx' \tag{4.5c}$$

$$\bar{\mathbf{N}}(x,k) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \int_{-\infty}^{+\infty} \mathbf{G}_-(x-x',k)\tilde{\mathbf{Q}}(x')\bar{\mathbf{N}}(x',k)\,dx', \tag{4.5d}$$

where

$$\tilde{\mathbf{Q}}(x) = \begin{pmatrix} \mathbf{0} & \mathbf{Q}(x) \\ \mathbf{R}(x) & \mathbf{0} \end{pmatrix}$$

and the Green's functions are

$$\mathbf{G}_\pm(x,k) = \pm\theta(\pm x) \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & e^{2ikx}\mathbf{I}_M \end{pmatrix},$$

$$\tilde{\mathbf{G}}_\pm(x,k) = \mp\theta(\mp x \begin{pmatrix} e^{-2ikx}\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M \end{pmatrix},$$

where $\theta(x)$ is the Heavyside step function.

The proof of existence and uniqueness of the Jost functions associated with the NLS scattering problem (2.2) (cf. Section 2.2) can be generalized to the block-matrix scattering problem. Specifically, we hypothesize that $\mathbf{Q}$,$\mathbf{R} \in L^1(\mathbb{R})$ with respect to any matrix norm. That is,

$$\begin{aligned} \|\mathbf{Q}\|_1 &= \int_{-\infty}^{+\infty} \|\mathbf{Q}\|_a (x) dx < \infty \\ \|\mathbf{R}\|_1 &= \int_{-\infty}^{+\infty} \|\mathbf{R}\|_a (x) dx < \infty, \end{aligned}$$

where $\|\cdot\|_a$ is any matrix norm. Under this condition, the Jost functions $\mathbf{M}(x,k)$ and $\mathbf{N}(x,k)$ are analytic functions of k in the region $\operatorname{Im} k > 0$, while $\bar{\mathbf{M}}(x,k)$ and $\bar{\mathbf{N}}(x,k)$ are analytic functions of k in the region $\operatorname{Im} k < 0$. All four Jost functions are continuous up to the line $\operatorname{Im} k = 0$. Moreover, these solutions of the integral equations are unique in the space of continuous functions. (See, e.g., [**10**] for details.)

From the integral equations (4.5a)–(4.5d) we obtain the asymptotic expansion, for large k, of the Jost functions:

$$\mathbf{M}(x,k) = \begin{pmatrix} \mathbf{I}_N - \frac{1}{2ik}\int_{-\infty}^{x} \mathbf{Q}(x')\mathbf{R}(x')\,dx' \\ -\frac{1}{2ik}\mathbf{R}(x) \end{pmatrix} + O(k^{-2}) \tag{4.6a}$$

$$\bar{\mathbf{N}}(x,k) = \begin{pmatrix} \mathbf{I}_N + \frac{1}{2ik}\int_{x}^{+\infty} \mathbf{Q}(x')\mathbf{R}(x')\,dx' \\ -\frac{1}{2ik}\mathbf{R}(x) \end{pmatrix} + O(k^{-2}) \tag{4.6b}$$

$$\mathbf{N}(x,k) = \begin{pmatrix} \frac{1}{2ik}\mathbf{Q}(x) \\ \mathbf{I}_M - \frac{1}{2ik}\int_{x}^{+\infty} \mathbf{R}(x')\mathbf{Q}(x')\,dx' \end{pmatrix} + O(k^{-2}) \tag{4.6c}$$

$$\bar{\mathbf{M}}(x,k) = \begin{pmatrix} \frac{1}{2ik}\mathbf{Q}(x) \\ \mathbf{I}_M + \frac{1}{2ik}\int_{-\infty}^{x} \mathbf{R}(x')\mathbf{Q}(x')\,dx' \end{pmatrix} + O(k^{-2}). \tag{4.6d}$$

The matrix functions $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}$ constitute, together, $N+M$ linearly-independent vector solutions of the scattering problem (4.2a). Similarly, $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ constitute another collection of $N+M$ linearly-independent vector solutions. To see that these solutions are linearly independent, we consider the Wronskian

$$W(\mathbf{u},\mathbf{v}) = \det|\mathbf{u},\mathbf{v}|,$$

where $\mathbf{u}$ and $\mathbf{v}$ are matrices with $N+M$ rows and a total of $N+M$ columns. If $\mathbf{u}(x,k)$ and $\mathbf{v}(x,k)$ are solutions of the scattering problem, then

$$\partial_x W(\mathbf{u}(x,k),\mathbf{v}(x,k)) = i(M-N)kW(\mathbf{u}(x,k),\mathbf{v}(x,k)).$$

Therefore, the linear independence of each set of solutions is demonstrated by considering the respective Wronskians the limits $x \to \pm\infty$.

Because $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ together constitute a complete set of linearly-independent solutions of the scattering problem, we can write

$$\boldsymbol{\phi}(x,k) = \boldsymbol{\psi}(x,k)\mathbf{b}(k) + \bar{\boldsymbol{\psi}}(x,k)\mathbf{a}(k) \tag{4.7a}$$

$$\bar{\boldsymbol{\phi}}(x,k) = \boldsymbol{\psi}(x,k)\bar{\mathbf{a}}(k) + \bar{\boldsymbol{\psi}}(x,k)\bar{\mathbf{b}}(k), \tag{4.7b}$$

where $\mathbf{a}(k)$ and $\bar{\mathbf{a}}(k)$ are square matrices of dimension $N \times N$ and $M \times M$, respectively, while $\mathbf{b}(k)$ is an $M \times N$ matrix and $\bar{\mathbf{b}}(k)$ is an $N \times M$ matrix. The relations (4.7a)–(4.7b) hold for any k such that all four eigenfunctions exist. Moreover, these relations define the scattering coefficients: $\mathbf{a}(k)$, $\bar{\mathbf{a}}(k)$, $\mathbf{b}(k)$ and $\bar{\mathbf{b}}(k)$. Similarly, because $\boldsymbol{\phi}$ and $\bar{\boldsymbol{\phi}}$ together also constitute a complete set of linearly-independent solutions of the scattering problem, we have:

$$\boldsymbol{\psi}(x,k) = \boldsymbol{\phi}(x,k)\mathbf{d}(k) + \bar{\boldsymbol{\phi}}(x,k)\mathbf{c}(k) \tag{4.8a}$$

$$\bar{\boldsymbol{\psi}}(x,k) = \boldsymbol{\phi}(x,k)\bar{\mathbf{c}}(k) + \bar{\boldsymbol{\phi}}(x,k)\bar{\mathbf{d}}(k), \tag{4.8b}$$

which defines a second set of k-dependent scattering coefficients. The preceding relations imply that

$$W(\boldsymbol{\phi}(x,k), \boldsymbol{\psi}(x,k)) = \det \mathbf{a}(k) \tag{4.9a}$$

$$W(\bar{\boldsymbol{\psi}}(x,k), \bar{\boldsymbol{\phi}}(x,k)) = \det \bar{\mathbf{a}}(k). \tag{4.9b}$$

and

$$\det |\mathbf{a}(k)| = \det \mathbf{c}(k) \tag{4.10a}$$

$$\det \bar{\mathbf{a}}(k) = \det \bar{\mathbf{c}}(k). \tag{4.10b}$$

The scattering coefficients can be written as explicit integrals of the potentials and the Jost functions. The expressions are:

$$\mathbf{a}(k) = \mathbf{I}_N + \int_{-\infty}^{+\infty} \mathbf{Q}(x)\mathbf{M}^{(\mathrm{dn})}(x,k)\, dx \tag{4.11a}$$

$$\mathbf{b}(k) = \int_{-\infty}^{+\infty} e^{-2ikx}\mathbf{R}(x)\mathbf{M}^{(\mathrm{up})}(x,k)\, dx \tag{4.11b}$$

$$\bar{\mathbf{a}}(k) = \mathbf{I}_M + \int_{-\infty}^{+\infty} \mathbf{R}(x)\bar{\mathbf{M}}^{(\mathrm{up})}(x,k)\, dx \tag{4.11c}$$

$$\bar{\mathbf{b}}(k) = \int_{-\infty}^{+\infty} e^{2ikx}\mathbf{Q}(x)\bar{\mathbf{M}}^{(\mathrm{dn})}(x,k)\, dx \tag{4.11d}$$

and

$$\bar{\mathbf{c}}(k) = \mathbf{I}_N - \int_{-\infty}^{+\infty} \mathbf{Q}(x)\bar{\mathbf{N}}^{(\mathrm{dn})}(x,k)\, dx \tag{4.12a}$$

$$\bar{\mathbf{d}}(k) = -\int_{-\infty}^{+\infty} e^{-2ikx}\mathbf{R}(x)\bar{\mathbf{N}}^{(\mathrm{up})}(x,k)\, dx \tag{4.12b}$$

$$\mathbf{c}(k) = \mathbf{I}_M - \int_{-\infty}^{+\infty} \mathbf{R}(x)\mathbf{N}^{(\mathrm{up})}(x,k)\, dx \tag{4.12c}$$

$$\mathbf{d}(k) = -\int_{-\infty}^{+\infty} e^{2ikx}\mathbf{Q}(x)\mathbf{N}^{(\mathrm{dn})}(x,k)\, dx. \tag{4.12d}$$

In the above expressions, we introduce the superscript notation (up) and (dn) to indicate, respectively, the top N rows and the bottom M rows of a matrix. That is, $\mathbf{A}^{(\mathrm{up})}$ is the $N \times J$ upper block and $\mathbf{A}^{(\mathrm{dn})}$ is the lower $M \times J$ block of the $(N+M) \times J$ matrix $\mathbf{A}$.

Given the integral representation (4.11a) and the fact that, for suitable potentials, $\mathbf{Q}(x)$ and $\mathbf{R}(x)$, the Jost function $\mathbf{M}(x,k)$ is an analytic function of k in the region $\operatorname{Im} k > 0$, we conclude that $\mathbf{a}(k)$ is also an analytic function of k in the the upper-half k-plane. Similarly, from expression (4.11c), we conclude that $\bar{\mathbf{a}}(k)$ is analytic in the lower-half k-plane. Moreover, both $\mathbf{a}(k)$ and $\bar{\mathbf{a}}(k)$ are continuous up to the line $\operatorname{Im} k = 0$. In contrast, the scattering coefficients $\mathbf{b}(k)$ and $\bar{\mathbf{b}}(k)$ cannot, in general, be continued off the real k-axis.

From the integral representations (4.11a) and (4.11c) and the asymptotic expansions (4.6a) and (4.6d), we obtain the asymptotic expansions

$$\mathbf{a}(k) = \mathbf{I}_N - \frac{1}{2ik}\int_{-\infty}^{+\infty}\mathbf{Q}(x)\mathbf{R}(x)\,dx + O(k^{-2}) \tag{4.13a}$$

$$\bar{\mathbf{a}}(k) = \mathbf{I}_M + \frac{1}{2ik}\int_{-\infty}^{+\infty}\mathbf{R}(x)\mathbf{Q}(x)\,dx + O(k^{-2}), \tag{4.13b}$$

which are valid for large k.

For a more convenient formulation of the inverse problem, we rewrite (4.7a)-(4.7b) as

$$\boldsymbol{\mu}(x,k) = \bar{\mathbf{N}}(x,k) + e^{2ikx}\mathbf{N}(x,k)\boldsymbol{\rho}(k) \tag{4.14a}$$

$$\bar{\boldsymbol{\mu}}(x,k) = \mathbf{N}(x,k) + e^{-2ikx}\bar{\mathbf{N}}(x,k)\bar{\boldsymbol{\rho}}(k), \tag{4.14b}$$

where

$$\boldsymbol{\mu}(x,k) = \mathbf{M}(x,k)\mathbf{a}^{-1}(k), \qquad \bar{\boldsymbol{\mu}}(x,k) = \bar{\mathbf{M}}(x,k)\bar{\mathbf{a}}^{-1}(k)$$

and the reflection coefficients are

$$\boldsymbol{\rho}(k) = \mathbf{b}(k)\mathbf{a}^{-1}(k), \qquad \bar{\boldsymbol{\rho}}(k) = \bar{\mathbf{b}}(k)\bar{\mathbf{a}}^{-1}(k).$$

It follows immediately from the above that the function $\boldsymbol{\mu}(x,k)$ is meromorphic, in k, in the region $\operatorname{Im} k > 0$ and the poles of $\boldsymbol{\mu}(x,k)$ correspond to the zeroes of $\det|\mathbf{a}(k)|$. Similarly, the function $\bar{\boldsymbol{\mu}}(x,k)$ is meromorphic, in k, in the region $\operatorname{Im} k < 0$ and the poles of $\bar{\boldsymbol{\mu}}(x,k)$ correspond to the zeroes of $\det|\bar{\mathbf{a}}(k)|$.

As before, we define an eigenvalue of the scattering problem (4.2a) to be a (complex) value of k for which there is a bounded solution that decays as $x \to \pm\infty$. From eq. (4.9a) we conclude that the eigenvalues in the region $\operatorname{Im} k > 0$ are exactly the points $k = k_j$ such that $\det|\mathbf{a}(k_j)| = 0$. On the other hand, from eq. (4.9b) we conclude that the eigenvalues in the region $\operatorname{Im} k < 0$ are the exactly the points $\bar{k}_j$ such that $\det|\bar{\mathbf{a}}(\bar{k}_j)| = 0$.

The eigenvalues in the region $\operatorname{Im} k > 0$ correspond to the poles of $\boldsymbol{\mu}(x,k)$. If k_j is a simple pole of $\boldsymbol{\mu}(x,k)$ and (4.14a) is defined in a neighborhood of k_j, then

$$\begin{aligned}\operatorname{Res}\{\boldsymbol{\mu}; k_j\} &= e^{2ik_jx}\mathbf{N}(x,k_j)\frac{1}{a'(k_j)}\mathbf{b}(k_j)\boldsymbol{\alpha}(k_j)\\ &= e^{2ik_jx}\mathbf{N}(x,k_j)\,\mathbf{C}_j\end{aligned} \tag{4.15a}$$

where $a(k) = \det|\mathbf{a}(k)|$, $\boldsymbol{\alpha}(k)$ is the cofactor matrix of $\mathbf{a}(k)$ and $'$ denotes the derivative with respect to k. The last line defines the matrix-valued norming constant $\mathbf{C}_j$ associated with the eigenvalue k_j. Similarly, if $\bar{k}_j$ (with $\operatorname{Im}\bar{k}_j < 0$) is a simple pole of $\bar{\boldsymbol{\mu}}(x,k)$ and (4.14b) is defined in a neighborhood of $\bar{k}_j$, then

$$\operatorname{Res}\left\{\bar{\boldsymbol{\mu}}; \bar{k}_j\right\} = e^{-2i\bar{k}_jx}\bar{\mathbf{N}}\left(x,\bar{k}_j\right)\bar{\mathbf{C}}_j \tag{4.15b}$$

where $\bar{a}(k) = \det\bar{\mathbf{a}}(k)$ and $\bar{\boldsymbol{\alpha}}(k)$ is the cofactor matrix of $\bar{\mathbf{a}}(k)$.

The focusing vector NLS is a special case of the matrix system (4.1a)-(4.1b) under the symmetry $\mathbf{R} = -\mathbf{Q}^H$. This symmetry in the potentials induces a symmetries in the scattering data, even in the matrix system (i.e., $M > 1$ and $N > 1$). Specifically,

$$\bar{\boldsymbol{\rho}}^H(k) = -\boldsymbol{\rho}(k) \tag{4.16}$$

on the line $\operatorname{Im} k = 0$ and

$$\bar{\mathbf{a}}^H(k^*) = \mathbf{c}(k) \tag{4.17a}$$

$$\bar{\mathbf{c}}^H(k^*) = \mathbf{a}(k). \tag{4.17b}$$

The symmetries (4.17a)-(4.17b), together with (4.10a)-(4.10b), imply that

$$\det|\mathbf{a}(k_j)| = \det|\bar{\mathbf{a}}(k_j^*)|.$$

Therefore, under the symmetry $\mathbf{R} = -\mathbf{Q}^H$, a point k_j in the region $\operatorname{Im} k > 0$ is an eigenvalue if, and only if, $\bar{k}_j = k_j^*$ is an eigenvalue in the region $\operatorname{Im} k < 0$. Moreover, one can show that

$$\bar{\mathbf{C}}_j = \mathbf{C}_j^H, \tag{4.18}$$

where $\mathbf{C}_j$ is the norming constant corresponding to the eigenvalue k_j and $\bar{\mathbf{C}}_j$ is the norming constant corresponding to the eigenvalue $\bar{k}_j = k_j^*$.

We remark that, like the scalar case, when $\mathbf{R} = \mathbf{Q}^H$ the scattering operator (4.2a) is Hermitian and therefore, in this reduction, the scattering problem (4.2a) with potentials decaying rapidly enough as $x \to \pm\infty$ does not admit eigenvalues, k_j, with $\operatorname{Im} k_j \neq 0$.

4.3. Inverse Scattering Problem. As in the preceding sections, the inverse problem proceeds in two steps: first, the reconstruction of the Jost functions from the scattering data and, second, the recovery of the potentials from the Jost functions. Also, as before, the scattering data are composed of the reflection coefficients, $\boldsymbol{\rho}(k)$ and $\bar{\boldsymbol{\rho}}(k)$ defined for $\operatorname{Im} k = 0$ and the eigenvalues and norming constants

$$\{k_j, \mathbf{C}_j\}_{j=1}^{J} \qquad \text{and} \qquad \{\bar{k}_j, \bar{\mathbf{C}}_j\}_{j=1}^{\bar{J}}$$

where $\operatorname{Im} k_j > 0$ and $\operatorname{Im} \bar{k}_j < 0$. In our formulation of the inverse problem, we assume that all eigenvalues correspond to simple poles of the unknown (in the inverse problem) meromorphic Jost functions.

The equations (4.14a)–(4.14b), defined on $\operatorname{Im} k = 0$ are the jump conditions of a Riemann-Hilbert problem for the matrix-valued functions $\mathbf{N}(x,k)$ and $\boldsymbol{\mu}(x,k)$, which are, respectively, analytic and meromorphic in the region $\operatorname{Im} k > 0$ and $\bar{\mathbf{N}}(x,k)$ and $\boldsymbol{\mu}(x,k)$, which are, respectively, analytic and meromorphic in the region $\operatorname{Im} k < 0$.

With the Plemelj formula, we obtain the the integral equations

$$\bar{\mathbf{N}}(x,k) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \sum_{j=1}^{J} \frac{e^{2ik_j x}}{(k-k_j)} \mathbf{N}(x,k_j)\mathbf{C}_j + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{2i\kappa x}}{\kappa-(k-i0)} \mathbf{N}(x,\kappa)\boldsymbol{\rho}(\kappa)\, d\kappa \tag{4.19a}$$

$$\mathbf{N}(x,k) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{e^{-2i\bar{k}_j x}}{(k-\bar{k}_j)} \bar{\mathbf{N}}(x,\bar{k}_j)\bar{\mathbf{C}}_j - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-2i\kappa x}}{\kappa-(k-i0)} \bar{\mathbf{N}}(x,\kappa)\bar{\boldsymbol{\rho}}(\kappa)\, d\kappa, \tag{4.19b}$$

where (4.19a) is defined for k such that $\operatorname{Im} k \leq 0$, while (4.19b) is defined for k such that $\operatorname{Im} k \geq 0$. In the absence of poles (eigenvalues), eqs. (4.19a)– (4.19b) constitute a coupled system of (matrix) integral equations defined on $\operatorname{Im} k = 0$.

In order to close the system (4.19a)– (4.19b) –when the scattering data include poles– we evaluate eq. (4.19a) at the poles $k = \bar{k}_j$, $j = 1, \dots, \bar{J}$ and (4.19b) at the poles $k = k_j$, $j = 1, \dots, J$. These evaluations yield:

$$\text{(4.19c)} \quad \bar{\mathbf{N}}(x, \bar{k}_j) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \sum_{\ell=1}^{J} \frac{e^{2ik_\ell x}}{(\bar{k}_j - k_\ell)} \mathbf{N}(x, k_j)\mathbf{C}_j + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{2i\kappa x}}{\kappa - \bar{k}_\ell} \mathbf{N}(x, \kappa)\boldsymbol{\rho}(\kappa)\, d\kappa$$

$$\text{(4.19d)} \quad \mathbf{N}(x, k_j) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \sum_{\ell=1}^{\bar{J}} \frac{e^{-2i\bar{k}_\ell x}}{(k_j - \bar{k}_\ell)} \bar{\mathbf{N}}(x, \bar{k}_\ell)\bar{\mathbf{C}}_\ell - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-2i\kappa x}}{\kappa - k_j} \bar{\mathbf{N}}(x, \kappa)\bar{\boldsymbol{\rho}}(\kappa)\, d\kappa.$$

Together, eqs. (4.19a)–(4.19d) constitute a linear algebraic-integral system of equations that determine the Jost functions $\mathbf{N}(x, k)$ and $\bar{\mathbf{N}}(x, k)$ in terms of the scattering data.

In order to recover the potential from the Jost functions, we compare the asymptotic expansions, for large k, of the right-hand sides of (4.19a) and (4.19b) with, respectively, the expansions (4.6b) and (4.6c), which are also valid for large k. The comparisons yield:

$$\text{(4.20a)} \quad \mathbf{R}(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} \mathbf{N}_j^{(\mathrm{dn})}(x, k_j)\mathbf{C}_j + \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{2i\kappa x} \mathbf{N}^{\mathrm{dn})}(x, \kappa)\bar{\boldsymbol{\rho}}(\kappa)\, d\kappa$$

$$\text{(4.20b)} \quad \mathbf{Q}(x) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{\mathbf{N}}^{(\mathrm{up})}(x, \bar{k}_j)\bar{\mathbf{C}}_j + \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{-2i\kappa x} \bar{\mathbf{N}}^{(\mathrm{up})}(x, \kappa)\bar{\boldsymbol{\rho}}(\kappa)\, d\kappa,$$

which are explicit expressions for the potentials in terms of the Jost functions and the scattering data.

One can also restate the inverse problem in terms of the GLM integral equations. In analogy with the scalar case, we represent the Jost functions as integrals of triangular kernels:

$$\text{(4.21a)} \quad \mathbf{N}(x, k) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \int_x^{+\infty} \mathbf{K}(x, s) e^{-ik(x-s)} ds \qquad s > x, \quad \operatorname{Im} k > 0$$

$$\text{(4.21b)} \quad \bar{\mathbf{N}}(x, k) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \int_x^{+\infty} \bar{\mathbf{K}}(x, s) e^{ik(x-s)} ds \qquad s > x, \quad \operatorname{Im} k < 0$$

where $\mathbf{K}$ and $\bar{\mathbf{K}}$ are, respectively, $(N+M)\times M$ and $(N+M)\times N$ matrices. These kernels satisfy the integral equations

$$\bar{\mathbf{K}}(x,y) + \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} \mathbf{F}(x+y) + \int_x^{+\infty} \mathbf{K}(x,s)\mathbf{F}(s+y)\,ds = 0 \tag{4.22a}$$

$$\mathbf{K}(x,y) + \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} \bar{\mathbf{F}}(x+y) + \int_x^{+\infty} \bar{\mathbf{K}}(x,s)\bar{\mathbf{F}}(s+y)\,ds = 0, \tag{4.22b}$$

where

$$\mathbf{F}(x) = -i\sum_{j=1}^{J} e^{ik_j x}\mathbf{C}_j + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\kappa x}\boldsymbol{\rho}(\kappa)\,d\kappa \tag{4.23a}$$

$$\bar{\mathbf{F}}(x) = i\sum_{j=1}^{\bar{J}} e^{-i\bar{k}_j x}\bar{\mathbf{C}}_j + \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\kappa x}\bar{\boldsymbol{\rho}}(\kappa)\,d\kappa. \tag{4.23b}$$

Inserting the representations (4.21a)–(4.21b) for the Jost functions into the eqs. (4.20a)–(4.20b) one obtains expressions for the potentials in terms of the kernels of GLM equations:

$$\mathbf{Q}(x) = -2\mathbf{K}^{(\mathrm{dn})}(x,x), \qquad \mathbf{R}(x) = -2\bar{\mathbf{K}}^{(\mathrm{up})}(x,x).$$

The examination of existence and uniqueness for the inverse problem follows the same lines as for the inverse scattering problem associated with the scalar NLS. In particular, the inverse problem has a unique solution if the GLM integral equations are Fredholm. Moreover the GLM equations are indeed Fredholm, if the potentials are in Schwartz class [**10**].

4.4. Time evolution. Eq. (4.2b), which specifies the evolution of the eigenfunctions, also determines the evolution of the scattering data. Precisely, it can be shown that

$$\mathbf{a}(k,t) = \mathbf{a}(k,0), \qquad \bar{\mathbf{a}}(k,t) = \bar{\mathbf{a}}(k,0) \tag{4.24}$$

$$\mathbf{b}(k,t) = e^{-4ik^2t}\mathbf{b}(k,0), \qquad \bar{\mathbf{b}}(k,t) = e^{4ik^2t}\bar{\mathbf{b}}(k,0). \tag{4.25}$$

From (4.24) we obtain the familiar result that the eigenvalues are constant as the solutions evolve. As before, not only the number of eigenvalues, but also their locations are fixed. On the other hand, it follows from (4.24)-(4.25) that the evolution of the norming constants is given by:

$$\mathbf{C}_j(t) = \mathbf{C}_j(0)e^{-4ik_j^2t}, \qquad \bar{\mathbf{C}}_j(t) = \bar{\mathbf{C}}_j(0)e^{4i\bar{k}_j^2t}. \tag{4.26}$$

Similarly, the evolution of the reflection coefficients is given by:

$$\boldsymbol{\rho}(k,t) = \boldsymbol{\rho}(k,0)e^{-4ik^2t}, \qquad \bar{\boldsymbol{\rho}}(k,t) = \bar{\boldsymbol{\rho}}(k,0)e^{4ik^2t}.$$

4.5. Soliton Solutions. As for the NLS scattering problem, we obtain pure soliton solutions from scattering data composed of complex-conjugate pairs of proper eigenvalues and

$$\boldsymbol{\rho}(k) = \bar{\boldsymbol{\rho}}(k) = 0$$

for all $k \in \mathbb{R}$. With such data, the algebraic-integral system (4.19a)–(4.19d) reduces to a system of linear algebraic equations and can be solved explicitly.

We obtain the pure one-soliton solution when the scattering data are composed of a single pair of poles, $k_1 = \xi + i\eta$ and $\bar{k}_1 = k_1^*$, and associated norming constants,

$\mathbf{C}_1(t)$, and $\bar{\mathbf{C}}_1(t) = \mathbf{C}_1^*(t)$. Taking into account the time dependence of the norming constants (4.26), we obtain

$$\mathbf{Q}(x,t) = \mathbf{p}\ 2\eta\ e^{-2i\xi x + 4i(\xi^2-\eta^2)t - i\pi/2}\,\mathrm{sech}(2\eta x - 8\xi\eta t - 2\delta)$$

where

$$e^{2\delta} = \frac{\|\mathbf{C}_1(0)\|}{2\eta} \qquad \mathbf{p} = \frac{\mathbf{C}_1^H(0)}{\|\mathbf{C}_1(0)\|}.$$

Note that, when $N = 1$ or $M = 1$ (i.e., vector, not matrix, NLS), $\mathbf{p}$ is a unit vector. In the vector case, we refer to $\mathbf{p}$ as the "polarization" of the vector soliton.

We remark that the one-soliton solution of VNLS – a vector soliton– is of the form

$$\mathbf{q}(x,t) = \mathbf{p}\ q(x,t)$$

where $q(x,t)$ is the one-soliton solution of scalar NLS (cf. (2.26)). Therefore, an individual soliton of VNLS is, on its own, fundamentally governed by the scalar NLS. All properties particular to VNLS are embedded in the polarization vector. More generally, for any solution of this form, $q(x,t)$ is a solution of scalar NLS and vice-versa.

As for NLS, distinct solitons (corresponding to distinct pairs of poles/eigenvalues in the scattering data) will, generically have different speeds and will therefore separate spatially with a distance that grows linearly in time as $t \to \pm\infty$. However, when solitons with different polarizations collide, the interaction is governed by VNLS. Typically, the polarizations of the individual solitons shift as a result of the collision. These vector-soliton interactions are described in Section 6.3.

5. Integrable Discrete Vector Nonlinear Schrödinger equation (IDVNLS)

5.1. Compatibility Condition. The integrable matrix generalization of IDVNLS (1.5) is

$$\text{(5.1a)}\quad i\frac{d}{d\tau}\mathbf{Q}_n = \mathbf{Q}_{n-1} - 2\mathbf{Q}_n + \mathbf{A}\mathbf{Q}_n + \mathbf{Q}_n\mathbf{B} + \mathbf{Q}_{n+1} - \mathbf{Q}_n\mathbf{R}_n\mathbf{Q}_{n-1} - \mathbf{Q}_{n+1}\mathbf{R}_n\mathbf{Q}_n$$

$$\text{(5.1b)}\quad -i\frac{d}{d\tau}\mathbf{R}_n = \mathbf{R}_{n-1} - 2\mathbf{R}_n + \mathbf{B}\mathbf{R}_n + \mathbf{R}_n\mathbf{A} + \mathbf{R}_{n+1} - \mathbf{R}_n\mathbf{Q}_n\mathbf{R}_{n-1} - \mathbf{R}_{n+1}\mathbf{Q}_n\mathbf{R}_n$$

where $\mathbf{Q}_n$ and $\mathbf{R}_n$ are, respectively, $N \times M$ and $M \times N$ matrices. This system of evolution equations is equivalent to the compatibility condition of the discrete scattering problem

$$\text{(5.2a)}\qquad \mathbf{v}_{n+1} = \begin{pmatrix} z\mathbf{I}_N & \mathbf{Q}_n \\ \mathbf{R}_n & z^{-1}\mathbf{I}_M \end{pmatrix}\mathbf{v}_n$$

and the time dependence

$$\text{(5.2b)}\quad \frac{d}{d\tau}\mathbf{v}_n = \begin{pmatrix} i\mathbf{Q}_n\mathbf{R}_{n-1} - \frac{i}{2}\left(z - z^{-1}\right)^2\mathbf{I}_N - i\mathbf{A} & -iz\mathbf{Q}_n + iz^{-1}\mathbf{Q}_{n-1} \\ iz^{-1}\mathbf{R}_n - iz\mathbf{R}_{n-1} & -i\mathbf{R}_n\mathbf{Q}_{n-1} + \frac{i}{2}\left(z - z^{-1}\right)^2\mathbf{I}_M + i\mathbf{B} \end{pmatrix}\mathbf{v}_n,$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix and $\mathbf{I}_M$ is the $M \times M$ identity matrix [**9, 77**]. We remark that the IST for an equivalent scattering problem is formulated in [**40, 41**].

The matrices $\mathbf{A}$ and $\mathbf{B}$ in the system (5.1a)–(5.1b) and in the time-dependence (5.2b) can be absorbed by the gauge transformation

$$\hat{\mathbf{Q}}_n = e^{i\tau\mathbf{A}}\mathbf{Q}_n e^{i\tau\mathbf{B}}, \quad \hat{\mathbf{R}}_n = e^{-i\tau\mathbf{B}}\mathbf{R}_n e^{-i\tau\mathbf{A}}, \quad \hat{\mathbf{v}}_n = \begin{pmatrix} e^{i\tau\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & e^{-i\tau\mathbf{B}} \end{pmatrix} \mathbf{v}_n. \tag{5.3}$$

Moreover, in the gauge

$$\mathbf{A} = \mathbf{B} = \mathbf{0},$$

the time-dependence (5.2b) is the direct matrix generalization of the time-dependence for discrete scalar IDNLS (cf. Section 3.1). However, as we discuss below, the gauge plays a key role in the reduction of (5.1a)–(5.1b) to the discrete, vector system, i.e. IDVNLS (1.5).

In contrast to the other nonlinear Schrödinger systems considered in this review, the system (5.1a)–(5.1b) does not, in general, reduce to a single, consistent (matrix) equation under the symmetry

$$\mathbf{R}_n = \mp\mathbf{Q}_n^H \tag{5.4}$$

where, as before, the superscript H denotes the Hermitian conjugate (conjugate transpose). However, if we require the additional symmetry

$$\mathbf{R}_n\mathbf{Q}_n = \mathbf{Q}_n\mathbf{R}_n = \alpha_n\mathbf{I}, \tag{5.5}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and α_n is real when (5.4) holds, then (5.1a)–(5.1b) reduces to the single (matrix) equation

$$i\frac{d}{d\tau}\mathbf{Q}_n = \mathbf{Q}_{n-1} - 2\mathbf{Q}_n + \mathbf{A}\mathbf{Q}_n + \mathbf{Q}_n\mathbf{B} + \mathbf{Q}_{n-1} - \alpha_n\left(\mathbf{Q}_{n+1} + \mathbf{Q}_{n-1}\right) \tag{5.6}$$

under the familiar symmetry (5.4).

The symmetry (5.5) requires that $\mathbf{Q}_n$ and $\mathbf{R}_n$ are square matrices of the same size. In particular, the 2×2 matrices

$$\mathbf{Q}_n = \begin{pmatrix} Q_n^{(1)} & Q_n^{(2)} \\ (-1)^n R_n^{(2)} & (-1)^{n+1} R_n^{(1)} \end{pmatrix}, \qquad \mathbf{R}_n = \begin{pmatrix} R_n^{(1)} & (-1)^n Q_n^{(2)} \\ R_n^{(2)} & (-1)^{n+1} Q_n^{(1)} \end{pmatrix} \tag{5.7}$$

satisfy the condition (5.5) with

$$\alpha_n = R_n^{(1)}Q_n^{(1)} + R_n^{(2)}Q_n^{(2)}.$$

One can obtain a similar result for N components (cf. [**10, 78**]), but here we restrict our attention to the two-component case.

To obtain IDVNLS (1.5) from (5.6), we choose the gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$. However, the symmetry condition (5.5) (in the two-component case (5.7)) is, in general, not consistent with the time evolution of the system (5.1a)–(5.1b). In particular, the symmetry is not preserved in the gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$. On the other hand, in the gauge

$$\mathbf{A} = \mathbf{B} = \mathbf{I}, \tag{5.8}$$

the time evolution of the system (5.1a)–(5.1b) preserves the symmetry (5.5). Therefore, in order to obtain solutions of IDVNLS, one first determines the evolution of potentials $\hat{\mathbf{Q}}_n$ and $\hat{\mathbf{R}}_n$ that satisfy system (5.1a)–(5.1b) with gauge (5.8). Then, with the invertible transformations

$$\mathbf{Q}_n(\tau) = e^{i\tau\mathbf{A}}\hat{\mathbf{Q}}_n(\tau)\,e^{i\tau\mathbf{B}} = e^{2i\tau}\hat{\mathbf{Q}}_n(\tau) \tag{5.9a}$$

$$\mathbf{R}_n(\tau) = e^{-i\tau\mathbf{B}}\hat{\mathbf{R}}_n(\tau)\,e^{-i\tau\mathbf{A}} = e^{-2i\tau}\hat{\mathbf{R}}_n(\tau) \tag{5.9b}$$

one obtains the solution of (5.1a)–(5.1b) with gauge $\mathbf{A} = \mathbf{B} = \mathbf{0}$.

5.2. Direct Problem. For potentials such that $\mathbf{Q}_n, \mathbf{R}_n \to 0$ sufficiently rapidly as $n \to \pm\infty$, the solutions of the difference equation (5.2a) are characterized by boundary conditions

$$\boldsymbol{\phi}_n(z) \sim z^n \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix}, \qquad \bar{\boldsymbol{\phi}}_n(z) \sim z^{-n} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} \qquad \text{as } n \to -\infty \tag{5.10a}$$

and

$$\boldsymbol{\psi}_n(z) \sim z^{-n} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix}, \qquad \bar{\boldsymbol{\psi}}_n(z) \sim z^{n} \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} \qquad \text{as } n \to +\infty. \tag{5.10b}$$

These solutions are matrix-valued functions with the following dimensions:

$$\begin{array}{lll} \boldsymbol{\phi}_n(z) & : & (N+M)\times N, \qquad \bar{\boldsymbol{\phi}}_n(z) : (N+M)\times M \\ \boldsymbol{\psi}_n(z) & : & (N+M)\times M, \qquad \bar{\boldsymbol{\psi}}_n(z) : (N+M)\times N. \end{array}$$

As in the IST for the other scattering problems, we introduce Jost functions with constant boundary conditions as $n \to \pm\infty$:

$$\mathbf{M}_n(z) = z^{-n}\boldsymbol{\phi}_n(z), \qquad \bar{\mathbf{M}}_n(z) = z^{n}\bar{\boldsymbol{\phi}}_n(z), \tag{5.11a}$$

$$\mathbf{N}_n(z) = z^{n}\boldsymbol{\psi}_n(z), \qquad \bar{\mathbf{N}}_n(z) = z^{-n}\bar{\boldsymbol{\psi}}_n(z). \tag{5.11b}$$

These Jost functions are solutions of the summation equations

$$\mathbf{M}_n = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{\ell}_{n-k}\tilde{\mathbf{Q}}_k\mathbf{M}_k \tag{5.12a}$$

$$\bar{\mathbf{N}}_n = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{r}_{n-k}\tilde{\mathbf{Q}}_k\bar{\mathbf{N}}_k \tag{5.12b}$$

$$\bar{\mathbf{M}}_n = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \bar{\mathbf{G}}^{\ell}_{n-k}\tilde{\mathbf{Q}}_k\bar{\mathbf{M}}_k \tag{5.12c}$$

$$\mathbf{N}_n = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} + \sum_{k=-\infty}^{+\infty} \mathbf{G}^{r}_{n-k}\tilde{\mathbf{Q}}_k\mathbf{N}_k, \tag{5.12d}$$

where

$$\tilde{\mathbf{Q}}_n = \begin{pmatrix} \mathbf{0} & \mathbf{Q}_n \\ \mathbf{R}_n & \mathbf{0} \end{pmatrix}$$

and the Green's functions are

$$\mathbf{G}_n^\ell(z) = z^{-1}\theta(n-1)\begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & z^{-2(n-1)}\mathbf{I}_M \end{pmatrix}$$
$$\bar{\mathbf{G}}_n^r(z) = -z^{-1}\theta(-n)\begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & z^{-2(n-1)}\mathbf{I}_M \end{pmatrix}$$
$$\bar{\mathbf{G}}_n^\ell(z) = z\theta(n-1)z^{-1}\begin{pmatrix} z^{2(n-1)}\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M \end{pmatrix}$$
$$\mathbf{G}_n^r(z) = -z\theta(-n)\begin{pmatrix} z^{2(n-1)}\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M \end{pmatrix}.$$

As before, $\theta(n)$ is the discrete Heavyside step function.

One can verify the existence of solutions for the above summation equations by iteration. (See [**10**] for details.) In particular, if

$$\|\mathbf{Q}\|_1 = \sum_{n=-\infty}^{+\infty} \|\mathbf{Q}_n\|_a < \infty, \qquad \|\mathbf{R}\|_1 = \sum_{n=-\infty}^{+\infty} \|\mathbf{R}_n\|_a < \infty,$$

where $\|\cdot\|_a$ is any matrix norm, then $\mathbf{M}_n(z)$ and $\mathbf{N}_n(z)$ defined, respectively, by (5.12a) and (5.12d) are analytic functions of z for $|z| > 1$ and continuous for $|z| \geq 1$. Similarly, the functions $\bar{\mathbf{N}}_n(z)$ and $\bar{\mathbf{M}}_n(z)$ defined, respectively, by (5.12b) and (5.12c) are analytic functions of z in the region $|z| < 1$ and continuous in the region $|z| \leq 1$. Moreover, all these Jost functions are unique in the space of continuous functions.

Note that, in order to uniquely determine $\mathbf{N}_n(z)$ and $\bar{\mathbf{N}}_n(z)$, which are specified by a boundary condition in the limit $n \to +\infty$, the matrix in the scattering problem (5.2a) must be invertible. Therefore, in the following, we assume the condition

$$\det(\mathbf{I}_M - \mathbf{R}_n\mathbf{Q}_n) = \det(\mathbf{I}_N - \mathbf{Q}_n\mathbf{R}_n) \neq 0.$$

Note that if symmetry (5.5) holds and $\mathbf{R}_n = -\mathbf{Q}_n^H$, then the above condition is satisfied.

It follows immediately, from the analytic properties of the Jost functions stated above, that $\mathbf{N}_n(z)$ and $\mathbf{M}_n(z)$ have Laurent series expansions in z^{-1} that converge in the region $|z| > 1$. Similarly, $\bar{\mathbf{N}}_n(z)$ and $\bar{\mathbf{M}}_n(z)$ have power series expansions in z that converge in the region $|z| < 1$. The coefficients of these expansions can be determined from the corresponding summation equations (5.12a)–(5.12b). The leading terms in these expansions are:

$$\text{(5.14a)} \qquad \mathbf{M}_n(z) = \begin{pmatrix} \mathbf{I}_N + O(z^{-2}, \text{even}) \\ z^{-1}\mathbf{R}_{n-1} + O(z^{-3}, \text{odd}) \end{pmatrix}$$

$$\text{(5.14b)} \qquad \bar{\mathbf{M}}_n(z) = \begin{pmatrix} z\mathbf{Q}_{n-1} + O(z^3, \text{odd}) \\ \mathbf{I}_M + O(z^2, \text{even}) \end{pmatrix},$$

$$\text{(5.14c)} \qquad \mathbf{N}_n(z) = \begin{pmatrix} -z^{-1}\mathbf{Q}_n\mathbf{\Delta}_n^{-1} + O(z^{-3}, \text{odd}) \\ \mathbf{\Delta}_n^{-1} + O(z^{-2}, \text{even}) \end{pmatrix}$$

$$\text{(5.14d)} \qquad \bar{\mathbf{N}}_n(z) = \begin{pmatrix} \mathbf{\Omega}_n^{-1} + O(z^2, \text{even}) \\ -z\mathbf{R}_n\mathbf{\Omega}_n^{-1} + O(z^3, \text{odd}) \end{pmatrix},$$

where

$$\mathbf{\Omega}_n = \ldots (\mathbf{I}_M - \mathbf{Q}_{n+1}\mathbf{R}_{n+1})(\mathbf{I}_M - \mathbf{Q}_n\mathbf{R}_n) = \prod_{\substack{k=n \\ \text{left}}}^{+\infty} (\mathbf{I}_N - \mathbf{Q}_k\mathbf{R}_k) \tag{5.15a}$$

$$\mathbf{\Delta}_n = \ldots (\mathbf{I}_M - \mathbf{R}_{n+1}\mathbf{Q}_{n+1})(\mathbf{I}_M - \mathbf{R}_n\mathbf{Q}_n) = \prod_{\substack{k=n \\ \text{left}}}^{+\infty} (\mathbf{I}_M - \mathbf{R}_k\mathbf{Q}_k)\,, \tag{5.15b}$$

and, as before, "even" ("odd") indicates that higher-order terms are even (odd) powers of z. The notation "left" indicates that, in the matrix product, the matrix with index k occurs to the left of the matrix with index $k-1$.

The matrix-valued eigenfunctions $\bar{\boldsymbol{\psi}}_n(z)$ and $\boldsymbol{\psi}_n(z)$ together constitute $N+M$ linearly-independent vector solutions of an $(N+M)\times(N+M)$ linear difference equation, namely eq. (5.2a). To see this, we evaluate the Wronskian of these solutions:

$$W\left(\bar{\boldsymbol{\psi}}_n(z), \boldsymbol{\psi}_n(z)\right) = \det\left|\bar{\boldsymbol{\psi}}_n(z), \boldsymbol{\psi}_n(z)\right| = \frac{z^{n(N-M)}}{\prod_{l=n}^{\infty} \det(\mathbf{I}_M - \mathbf{R}_l\mathbf{Q}_l)}.$$

Hence, the solutions in $\boldsymbol{\phi}_n(z)$ and $\bar{\boldsymbol{\phi}}_n(z)$ are linearly dependent on the set of solutions $\boldsymbol{\psi}_n(z)$ and $\bar{\boldsymbol{\psi}}_n(z)$. This dependence can be expressed as

$$\begin{aligned} \boldsymbol{\phi}_n(z) &= \boldsymbol{\psi}_n(z)\mathbf{b}(z) + \bar{\boldsymbol{\psi}}_n(z)\mathbf{a}(z) \\ \bar{\boldsymbol{\phi}}_n(z) &= \boldsymbol{\psi}_n(z)\bar{\mathbf{a}}(z) + \bar{\boldsymbol{\psi}}_n(z)\bar{\mathbf{b}}(z) \end{aligned}$$

where $\mathbf{b}(z)$ is an $M\times N$ matrix, $\mathbf{a}(z)$ is an $N\times N$ matrix, $\bar{\mathbf{a}}(z)$ is an $M\times M$ matrix and $\bar{\mathbf{b}}(z)$ is an $N\times M$ matrix. In terms of the Jost functions, we have the relations:

$$\mathbf{M}_n(z) = z^{-2n}\mathbf{N}_n(z)\mathbf{b}(z) + \bar{\mathbf{N}}_n(z)\mathbf{a}(z) \tag{5.17a}$$

$$\bar{\mathbf{M}}_n(z) = \mathbf{N}_n(z)\bar{\mathbf{a}}(z) + z^{2n}\bar{\mathbf{N}}_n(z)\bar{\mathbf{b}}(z). \tag{5.17b}$$

These equations define the coefficients $\mathbf{a}(z)$, $\bar{\mathbf{a}}(z)$, $\mathbf{b}(z)$ and $\bar{\mathbf{b}}(z)$ for any z such that all four Jost functions exist. In particular, they hold on the unit circle, $|z|=1$.

Similarly, the matrix-valued eigenfunctions $\boldsymbol{\phi}_n(z)$ and $\bar{\boldsymbol{\phi}}_n(z)$ together constitute a second set of $N+M$ linearly-independent vector solutions of (5.2a), as determined by the evaluation of the Wronskian

$$W\left(\boldsymbol{\phi}_n(z), \bar{\boldsymbol{\phi}}_n(z)\right) = z^{n(N-M)} \prod_{j=-\infty}^{n-1} \det(\mathbf{I}_M - \mathbf{R}_j\mathbf{Q}_j)\,.$$

Therefore, we can write

$$\begin{aligned} \boldsymbol{\psi}_n(z) &= \boldsymbol{\phi}_n(z)\mathbf{d}(z) + \bar{\boldsymbol{\phi}}_n(z)\mathbf{c}(z) \\ \bar{\boldsymbol{\psi}}_n(z) &= \boldsymbol{\phi}_n(z)\bar{\mathbf{c}}(z) + \bar{\boldsymbol{\phi}}_n(z)\bar{\mathbf{d}}(z), \end{aligned}$$

which defines the scattering coefficients $\mathbf{c}(z)$, $\bar{\mathbf{c}}(z)$, $\mathbf{d}(z)$ and $\bar{\mathbf{b}}(d)$ for any z such that all four Jost functions exist.

The scattering coefficients can be expressed as explicit sums of the eigenfunctions:

$$\mathbf{a}(z) = \mathbf{I}_N + \sum_{k=-\infty}^{+\infty} z^{-1}\mathbf{Q}_k\mathbf{M}_k^{(\mathrm{dn})}(z) \tag{5.19a}$$

$$\mathbf{b}(z) = \sum_{k=-\infty}^{+\infty} z^{2k+1}\mathbf{R}_k\mathbf{M}_k^{(\mathrm{up})}(z) \tag{5.19b}$$

$$\bar{\mathbf{a}}(z) = \mathbf{I}_M + \sum_{k=-\infty}^{+\infty} z\mathbf{R}_k\bar{\mathbf{M}}_k^{(\mathrm{up})}(z) \tag{5.19c}$$

$$\bar{\mathbf{b}}(z) = \sum_{k=-\infty}^{+\infty} z^{-2k-1}\mathbf{Q}_k\bar{\mathbf{M}}_k^{(\mathrm{dn})}(z) \tag{5.19d}$$

$$\mathbf{c}(z) = \mathbf{I}_M - \sum_{k=-\infty}^{+\infty} z\mathbf{R}_k\mathbf{N}_k^{(\mathrm{up})}(z) \tag{5.19e}$$

$$\mathbf{d}(z) = -\sum_{k=-\infty}^{+\infty} z^{-2k-1}\mathbf{Q}_k\mathbf{N}_k^{(\mathrm{dn})}(z) \tag{5.19f}$$

$$\bar{\mathbf{c}}(z) = \mathbf{I}_N - \sum_{k=-\infty}^{+\infty} z^{-1}\mathbf{Q}_k\bar{\mathbf{N}}_k^{(\mathrm{dn})}(z) \tag{5.19g}$$

$$\bar{\mathbf{d}}(z) = -\sum_{k=-\infty}^{+\infty} z^{2k+1}\mathbf{R}_k\bar{\mathbf{N}}_k^{(\mathrm{up})}(z). \tag{5.19h}$$

The expressions (5.19a) and (5.19c) imply, respectively, that $\mathbf{a}(z)$ is analytic in the same region as $\mathbf{M}_n(z)$ (i.e., $|z| > 1$) while $\bar{\mathbf{a}}(z)$ is analytic in the same region as $\bar{\mathbf{M}}_n(z)$ (i.e., $|z| < 1$). Similarly, (5.19e) and (5.19g) imply, respectively, that $\mathbf{c}(z)$ is analytic in the same region as $\mathbf{N}_n(z)$ while $\bar{\mathbf{c}}(z)$ is analytic in the same region as $\bar{\mathbf{N}}_n(z)$. Moreover, $\mathbf{a}(z)$,$\bar{\mathbf{a}}(z)$, $\mathbf{c}(z)$ and $\bar{\mathbf{c}}(z)$ are even functions of the spectral parameter z. In contrast, the scattering coefficients $\mathbf{b}(z)$,$\bar{\mathbf{b}}(z)$, $\mathbf{d}(z)$ and $\bar{\mathbf{d}}(z)$ are odd functions of z.

Inserting the z-expansions (5.14a)–(5.14d) of the Jost functions into the summation representations (5.19a)–(5.19g), we obtain the power series and Laurent expansions for the analytic scattering coefficients. The leading terms are:

(5.20a)

$$\mathbf{a}(z) = \mathbf{I}_N + O(z^{-2}, \text{even}), \qquad \bar{\mathbf{c}}(z) = \mathbf{I}_N + \sum_{k=-\infty}^{+\infty} \mathbf{R}_k\mathbf{Q}_k\boldsymbol{\Delta}_k^{-1} + O(z^2, \text{even})$$

(5.20b)

$$\bar{\mathbf{a}}(z) = \mathbf{I}_M + O(z^2, \text{even}), \qquad \mathbf{c}(z) = \mathbf{I}_M + \sum_{k=-\infty}^{+\infty} \mathbf{Q}_k\mathbf{R}_k\boldsymbol{\Omega}_k^{-1} + O(z^{-2}, \text{even})$$

where $\boldsymbol{\Delta}_n$ and $\boldsymbol{\Omega}_n$ are given by (5.15b) and (5.15a), respectively.

For the purpose of the inverse problem, it is convenient to define the meromorphic (in z) functions

$$\boldsymbol{\mu}_n(z) = \mathbf{M}_n(z)\mathbf{a}^{-1}(z), \qquad \bar{\boldsymbol{\mu}}_n(z) = \bar{\mathbf{M}}_n(z)\bar{\mathbf{a}}^{-1}(z). \tag{5.21}$$

In terms of these functions, the relations (5.17a)–(5.17b) can be restated as

$$\boldsymbol{\mu}_n(z) - \bar{\mathbf{N}}_n(z) = z^{-2n}\mathbf{N}_n(z)\boldsymbol{\rho}(z) \tag{5.22a}$$

$$\bar{\boldsymbol{\mu}}_n(z) - \mathbf{N}_n(z) = z^{2n}\bar{\mathbf{N}}_n(z)\bar{\boldsymbol{\rho}}(z) \tag{5.22b}$$

where the reflection coefficients,

$$\boldsymbol{\rho}(z) = \mathbf{b}(z)\mathbf{a}^{-1}(z), \qquad \bar{\boldsymbol{\rho}}(z) = \bar{\mathbf{b}}(z)\bar{\mathbf{a}}^{-1}(z), \tag{5.23}$$

are part of the (n-independent) scattering data.

Just as for the other scattering problems, we define a proper eigenvalue of the scattering problem (5.2a) to be a (complex) value of z such that there is a solution of the scattering problem that decays as $n \to \pm\infty$. By an argument similar to that for the continuous block-matrix scattering problem (cf. Section 4.2), one can show that the eigenvalues in the region $|z| > 1$ are the points $z = z_j$, $j = 1, \dots, J$, such that $\det \mathbf{a}(z_j) = 0$. Also, the eigenvalues in the region $|z| < 1$ are the points $z = \bar{z}_\ell$, $\ell = 1, \dots, \bar{J}$, such that $\det \bar{\mathbf{a}}(\bar{z}_\ell) = 0$.

The meromorphic function $\boldsymbol{\mu}_n(z)$ (resp. $\bar{\boldsymbol{\mu}}_n(z)$) has poles precisely at the points $z = z_j$ (resp. $z = \bar{z}_j$) such that $\det \mathbf{a}(z_j) = 0$ (resp. $\det \bar{\mathbf{a}}(\bar{z}_j) = 0$). We assume that all the poles are simple, that (5.22a) is well-defined in the neighborhood of each pole in the region $|z| > 1$ and that (5.22b) is well-defined in the neighborhood of each pole in the region $|z| < 1$. We conclude that, for each simple pole (eigenvalue) in the region $|z| > 1$,

$$\operatorname{Res}\{\boldsymbol{\mu}_n; z_j\} = z_j^{-2n}\mathbf{N}_n(z_j)\mathbf{C}_j, \tag{5.24a}$$

while, for each simple pole in the region $|z| < 1$,

$$\operatorname{Res}\{\bar{\boldsymbol{\mu}}_n; \bar{z}_j\} = \bar{z}_j^{2n}\bar{\mathbf{N}}_n(\bar{z}_j)\bar{\mathbf{C}}_j. \tag{5.24b}$$

The $M \times N$ matrix $\mathbf{C}_j$ is the norming constant associated with the discrete eigenvalue z_j and the $N \times M$ matrix $\bar{\mathbf{C}}_j$ is the norming constant associated with the discrete eigenvalue $\bar{z}_j$.

The scattering data –the reflection coefficients, the eigenvalues and the associated norming constants– defined in this section are subject to three symmetries. One symmetry (Symmetry 1 below) does not depend on any symmetry in the potentials and is analogous to a symmetry in the discrete scattering problem associated with IDNLS. Another symmetry (Symmetry 3 below) is associated with the the reduction (5.4), i.e. $\mathbf{R}_n = -\mathbf{Q}_n^H$, and is analogous to a similar symmetry reduction of each of the other three scattering problems considered in the preceding sections. The remaining symmetry (Symmetry 2 below), however, has no counterpart in the other scattering problems considered in this review. Importantly, this new symmetry is a prerequisite for the the familiar symmetry reduction of Symmetry 3. The symmetries are as follows. (Proofs can be found in [**10, 77**].)

Symmetry 1: All the eigenvalues appear in pairs $\pm z_j$ ($\pm\bar{z}_j$). Moreover, the norming constant associated with $-z_j$ (respectively, $-\bar{z}_j$) is equal to the norming constant associated with $+z_j$ (respectively, $+\bar{z}_j$). This symmetry does not depend on any symmetry between the potentials $\mathbf{Q}_n$ and $\mathbf{R}_n$. Rather, it is a manifestation of the fact that $\mathbf{a}(z)$ and $\bar{\mathbf{a}}(z)$ are even functions of z.

Symmetry 2: In order to obtain potentials that satisfy the condition (5.5) with $N = M = 2$, we require that the potentials be of the form (5.7). For such potentials:

(1) the reflection coefficients satisfy the symmetry

$$\bar{\boldsymbol{\rho}}(z) = -i\mathbf{P}\ \boldsymbol{\rho}^T(i/z)\ \mathbf{P}$$

where

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

(2) the eigenvalue $\hat{z}_j = i/z_j$ is an eigenvalue such that $|\hat{z}_j| < 1$ if, and only if, z_j is an eigenvalue such that $|z_j| > 1$;

(3) the norming constants associated with these poles have the symmetry

$$\hat{\mathbf{C}}_j = -z_j^{-2}\mathbf{P}\ \mathbf{C}_j^T\mathbf{P} \tag{5.25}$$

where $\mathbf{C}_j$ is the norming constant associated with z_j and $\hat{\mathbf{C}}_j$ is the norming constant associated with $\hat{z}_j$.

Symmetry 3: If the potentials satisfy the symmetry $\mathbf{R}_n = \mp\mathbf{Q}_n^H$ (5.4), in addition to the symmetry (5.5), then:

(1) the reflection coefficients satisfy the symmetry

$$\bar{\boldsymbol{\rho}}(z) = \mp\boldsymbol{\rho}^H(1/z^*);$$

(2) $\bar{z}_j = 1/z_j^*$ is an eigenvalue such that $|\bar{z}_j| < 1$ if, and only if, z_j is an eigenvalue such that $|z_j| > 1$;

(3) the norming constants associated with these paired eigenvalues satisfy the symmetry

$$\bar{\mathbf{C}}_j = \pm(z_j^*)^{-2}\mathbf{C}_j^H \tag{5.26}$$

where $\mathbf{C}_j$ is the norming constant associated with z_j and $\bar{\mathbf{C}}_j$ is the norming constant associated with $\bar{z}_j$.

As a consequence of Symmetry 2, the eigenvalue $\tilde{z}_j = i\bar{z}_j^{-1}$ is an eigenvalue such that $|\tilde{z}_j| > 1$ if, and only if, $\bar{z}_j$ is an eigenvalue such that $|\bar{z}_j| < 1$. Moreover, the associated norming constants satisfy the symmetry

$$\tilde{\mathbf{C}}_j = \bar{z}_j^{-2}\mathbf{P}\ \bar{\mathbf{C}}_j^T\mathbf{P}, \tag{5.27}$$

where $\bar{\mathbf{C}}_j$ is the norming constant associated with $\bar{z}_j$ and $\tilde{\mathbf{C}}_j$ is the norming constant associated with $\tilde{z}_j$.

If all three symmetries hold, then the eigenvalues appear in sets of eight,

$$\left\{\pm z_j,\ \pm i/z_j, \pm 1/z_j^*, \pm i z_j^*\right\}_{j=1}^J \tag{5.28}$$

and the norming constants associated with the members of each eigenvalue octet are related by eqs. (5.25), (5.26) and (5.27). Without loss of generality, we assume that, for any eigenvalue octet, $|z_j| > 1$ and $-\frac{\pi}{2} < \operatorname{Arg} z_j \leq \frac{\pi}{2}$. Note that, given z_j, the locations of the other eigenvalues are fixed by the symmetries. Hence, the complete octet is implicitly specified by a single given eigenvalue. Moreover, the symmetries fix all the associated norming constants in terms of the norming constant, $\mathbf{C}_j$, associated with z_j.

5.3. Inverse Scattering Problem. As in the solution of the inverse scattering problems associated with the other nonlinear Schrödinger systems considered here, the solution of the inverse scattering problem for the difference equation (5.2a) proceeds in two steps. First, we derive an equation for the Jost functions in terms of the scattering data. Second, we determine a formula for the potentials in terms of the recovered Jost functions and the scattering data.

In order to formulate the first step of inverse problem as a Riemann-Hilbert problem in the complex variable z, we must specify:

- the limits of the sectionally analytic and meromorphic eigenfunctions as $|z| \to \infty$,
- the "jump condition" on the boundary $|z| = 1$,
- the locations of the poles of the meromorphic functions $\boldsymbol{\mu}_n(z)$ and $\bar{\boldsymbol{\mu}}_n(z)$ (i.e., the eigenvalues) and
- equations that determine the residues of these poles.

Moreover, to find solutions of IDVNLS (1.5), we must explicitly take into account the symmetries in the scattering data.

From the expansions (5.14a), (5.14c), and (5.20a) as well as eq. (5.21), we obtain the limits:

$$\mathbf{N}_n(z) \to \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\Delta}_n^{-1} \end{pmatrix}, \qquad \boldsymbol{\mu}_n(z) \to \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix}$$

as $|z| \to \infty$, where $\boldsymbol{\Delta}_n$ is given by (5.15b). However, in the inverse problem, $\boldsymbol{\Delta}_n$ (which depends on the potentials $\mathbf{Q}_n$ and $\mathbf{R}_n$) is unknown. Therefore, in order to remove this dependence on an unknown, we define the modified functions

$$\text{(5.29a)} \qquad \mathbf{N}'_n = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n \end{pmatrix} \mathbf{N}_n = \begin{pmatrix} -z^{-1}\mathbf{Q}_n \boldsymbol{\Delta}_n^{-1} + O(z^{-3}) \\ \mathbf{I}_M + O(z^{-2}) \end{pmatrix}$$

$$\text{(5.29b)} \qquad \boldsymbol{\mu}'_n = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n \end{pmatrix} \boldsymbol{\mu}_n = \begin{pmatrix} \mathbf{I}_N + O(z^{-2}) \\ z^{-1}\boldsymbol{\Delta}_n \mathbf{R}_{n-1} + O(z^{-3}) \end{pmatrix}$$

$$\text{(5.29c)} \qquad \bar{\mathbf{N}}'_n = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n \end{pmatrix} \bar{\mathbf{N}}_n = \begin{pmatrix} \boldsymbol{\Omega}_n^{-1} + O(z^2) \\ -z\boldsymbol{\Delta}_n \mathbf{R}_n \boldsymbol{\Omega}_n^{-1} + O(z^3) \end{pmatrix}$$

$$\text{(5.29d)} \qquad \bar{\boldsymbol{\mu}}'_n = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n \end{pmatrix} \bar{\boldsymbol{\mu}}_n = \begin{pmatrix} z\mathbf{Q}_{n-1} + O(z^3) \\ \boldsymbol{\Delta}_n + O(z^2) \end{pmatrix},$$

where $\boldsymbol{\Omega}_n$ is given by (5.15a) and, as before, $\boldsymbol{\Delta}_n$ is given by (5.15b). The z-expansions in (5.29a)–(5.29b) are valid in the region $|z| > 1$, while the z-expansions in (5.29c)–(5.29d) are valid in the region $|z| < 1$

The modified functions satisfy the relations (5.22a)–(5.22b), which are the jump conditions on $|z| = 1$. Also, the poles of $\boldsymbol{\mu}'_n(z)$ and $\bar{\boldsymbol{\mu}}'_n(z)$ are the same as the poles of $\boldsymbol{\mu}_n(z)$ and $\bar{\boldsymbol{\mu}}_n(z)$, respectively. Moreover, the modified functions satisfy (5.24a)-(5.24b) at these poles with the same norming constants. Thus, there is, sufficient information to formulate problem of the recovery of the modified Jost functions as a Riemann-Hilbert problem.

The Riemann-Hilbert problem can be restated as a linear system of summation equations by the application of projection operators to both sides of (5.22a)–(5.22b).

The resulting equations are:

$$\text{(5.30a)}\quad \bar{\mathbf{N}}'_n(z) = \begin{pmatrix}\mathbf{I}_N\\ \mathbf{0}\end{pmatrix} + \sum_{j=1}^{J} z_j^{-2n}\left[\frac{1}{z-z_j}\mathbf{N}'_n(z_j) + \frac{1}{z+z_j}\mathbf{N}'_n(-z_j)\right]\mathbf{C}_j - \lim_{\substack{\zeta\to z\\ |\zeta|<1}} \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{-2n}}{w-\zeta}\mathbf{N}'_n(w)\boldsymbol{\rho}(w)\,dw$$

$$\text{(5.30b)}\quad \mathbf{N}'_n(z) = \begin{pmatrix}\mathbf{0}\\ \mathbf{I}_M\end{pmatrix} + \sum_{j=1}^{\bar{J}} \bar{z}_j^{2n}\left[\frac{1}{z-\bar{z}_j}\bar{\mathbf{N}}'_n(\bar{z}_j) + \frac{1}{z+\bar{z}_j}\bar{\mathbf{N}}'_n(-\bar{z}_j)\right]\bar{\mathbf{C}}_j + \lim_{\substack{\zeta\to z\\ |\zeta|>1}} \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{2n}}{w-\zeta}\bar{\mathbf{N}}'_n(w)\bar{\boldsymbol{\rho}}(w)\,dw$$

where $\mathbf{N}'_n(z_j)$ is $\mathbf{N}'_n(z)$ evaluated at the eigenvalue z_j, $\mathbf{N}'_n(-z_j)$ is $\mathbf{N}'_n(z)$ evaluated at the complementary eigenvalue $-z_j$ and similarly for $\bar{\mathbf{N}}'_n(\bar{z}_j)$ and $\bar{\mathbf{N}}'_n(-\bar{z}_j)$. Note that, in eqs. (5.30a)–(5.30b), we have poles in $\pm$ pairs with the appropriate norming constants (cf. Symmetry 1). In the absence of poles, the sums in (5.30a)–(5.30b) vanish and the equations constitute a coupled system of linear matrix integral equations that, in principle, determine the Jost functions $\mathbf{N}_n(z)$ and $\bar{\mathbf{N}}_n(z)$.

If the scattering data include eigenvalues (poles), we evaluate (5.30a) at the points $\pm\bar{z}_j$ and (5.30b) at the points $\pm z_j$ in order to close the system of equations. With these evaluations, we obtain the relations:

$$\text{(5.30c)}\quad \bar{\mathbf{N}}'_n(\bar{z}_j) = \begin{pmatrix}\mathbf{I}_N\\ \mathbf{0}\end{pmatrix} + \sum_{k=1}^{J} z_k^{-2n}\left[\frac{1}{\bar{z}_j-z_k}\mathbf{N}'_n(z_k) + \frac{1}{\bar{z}_j+z_k}\mathbf{N}'_n(-z_k)\right]\mathbf{C}_k - \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{-2n}}{w-\bar{z}_j}\mathbf{N}'_n(w)\boldsymbol{\rho}(w)\,dw$$

$$\text{(5.30d)}\quad \bar{\mathbf{N}}'_n(-\bar{z}_j) = \begin{pmatrix}\mathbf{I}_N\\ \mathbf{0}\end{pmatrix} - \sum_{k=1}^{J} z_k^{-2n}\left[\frac{1}{\bar{z}_j+z_k}\mathbf{N}'_n(z_k) + \frac{1}{\bar{z}_j-z_k}\mathbf{N}'_n(-z_k)\right]\mathbf{C}_k - \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{-2n}}{w+\bar{z}_j}\mathbf{N}'_n(w)\boldsymbol{\rho}(w)\,dw$$

$$\text{(5.30e)}\quad \mathbf{N}'_n(z_j) = \begin{pmatrix}\mathbf{0}\\ \mathbf{I}_M\end{pmatrix} + \sum_{k=1}^{\bar{J}} \bar{z}_k^{2n}\left[\frac{1}{z_j-\bar{z}_k}\bar{\mathbf{N}}'_n(\bar{z}_k) + \frac{1}{z_j+\bar{z}_k}\mathbf{N}'_n(-\bar{z}_k)\right]\bar{\mathbf{C}}_k + \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{2n}}{w-z_j}\bar{\mathbf{N}}'_n(w)\bar{\boldsymbol{\rho}}(w)\,dw$$

$$\text{(5.30f)}\quad \mathbf{N}'_n(-z_j) = \begin{pmatrix}\mathbf{0}\\ \mathbf{I}_M\end{pmatrix} - \sum_{k=1}^{\bar{J}} \bar{z}_k^{2n}\left[\frac{1}{z_j+\bar{z}_k}\bar{\mathbf{N}}'_n(\bar{z}_k) + \frac{1}{z_j-\bar{z}_k}\mathbf{N}'_n(-\bar{z}_k)\right]\bar{\mathbf{C}}_k + \frac{1}{2\pi i}\oint_{|w|=1}\frac{w^{2n}}{w+z_j}\bar{\mathbf{N}}'_n(w)\bar{\boldsymbol{\rho}}(w)\,dw$$

where (5.30c)–(5.30d) hold for each eigenvalue

$$\left\{\bar{z}_j : |\bar{z}_j| < 1,\ -\frac{\pi}{2} < \operatorname{Arg}\bar{z}_j \leq \frac{\pi}{2}\right\}_{j=1}^{\bar{J}}$$

and (5.30e)–(5.30f) hold for each eigenvalue

$$\left\{z_j : |z_j| > 1,\ -\frac{\pi}{2} < \operatorname{Arg} z_j \leq \frac{\pi}{2}\right\}_{j=1}^{J}.$$

In the presence of poles, equations (5.30a)–(5.30f) constitute a linear system of algebraic-integral equations that, in principle, determine the Jost functions.

Note that (5.30c)–(5.30f) include neither the effects of Symmetry 2 nor the effects of Symmetry 3 explicitly. If Symmetry 2 holds, then there is an equal number eigenvalues in each of the regions $|z| > 1$ and $|z| < 1$. Specifically, the set of of eigenvalues is of the form

$$\{\pm\bar{z}_j,\ \pm i/\bar{z}_j : |\bar{z}_j| < 1\}_{j=1}^{\bar{J}} \cup \{\pm z_j,\ \pm i/z_j : |z_j| > 1,\ \}_{j=1}^{J},$$

where $-\frac{\pi}{2} < \operatorname{Arg} z_j, \operatorname{Arg}\bar{z}_j \leq \frac{\pi}{2}$. Therefore J and $\bar{J}$ must be replaced by $J + \bar{J}$ in (5.30a)–(5.30f). Moreover, the norming constants associated with the eigenvalues $\pm i/z_j$ and $\pm i/\bar{z}_j$ are fixed by (5.25) and (5.27). If, in addition, the data satisfy Symmetry 3, then $\bar{J} = J$, $\bar{z}_j = 1/z_j^*$ and the norming constants are related by (5.26).

The potentials are reconstructed by means of the power series expansions (in z) of the Jost function $\bar{\mathbf{N}}'_n(z)$ and the Laurent expansion of $\mathbf{N}'_n(z)$. Including the effects of Symmetries 1 and 2 on the eigenvalues, we obtain

$$\begin{aligned}(5.31\text{a})\quad \mathbf{Q}_n = -2\sum_{j=1}^{\bar{J}} \bar{z}_j^{2n}\bar{\mathbf{N}}'^{(\text{up})}_{n+1}(\bar{z}_j)\bar{\mathbf{C}}_j &\\ - 2\sum_{j=1}^{J}(-1)^n z_j^{-2n}\bar{\mathbf{N}}'^{(\text{up})}_{n+1}(i/z_j)\hat{\mathbf{C}}_j &\\ + \frac{1}{2\pi i}\oint_{|w|=1} w^{2n}\bar{\mathbf{N}}'^{(\text{up})}_{n+1}(w)\bar{\boldsymbol{\rho}}(w)\,dw &\end{aligned}$$

$$\begin{aligned}(5.31\text{b})\quad \mathbf{R}_n = 2\sum_{j=1}^{J} z_j^{-2(n+1)}\mathbf{N}'^{(\text{dn})}_{n}(z_j)\mathbf{C}_j &\\ + 2\sum_{j=1}^{\bar{J}}(-1)^{n+1}\bar{z}_j^{2(n+1)}\mathbf{N}'^{(\text{dn})}_{n}(i/\bar{z}_j)\tilde{\mathbf{C}}_j &\\ + \frac{1}{2\pi i}\oint_{|w|=1} w^{-2(n+1)}\mathbf{N}'^{(\text{dn})}_{n}(w)\boldsymbol{\rho}(w)\,dw &\end{aligned}$$

where, as before, (up) denotes the first two rows of the matrix, (dn) denotes the bottom two rows and, moreover, the norming constants satisfy (5.25) and (5.27).

As in the solution of the other scattering problems, one can alternatively reformulate the inverse problem as system of GLM-type equations. To obtain this formulation, we represent the eigenfunctions $\boldsymbol{\psi}_n$ and $\bar{\boldsymbol{\psi}}_n$ in terms of triangular kernels:

$$\boldsymbol{\psi}_n(z) = \sum_{j=n}^{+\infty} z^{-j}\mathbf{K}(n,j) \qquad |z|>1 \tag{5.32a}$$

$$\bar{\boldsymbol{\psi}}_n(z) = \sum_{j=n}^{+\infty} z^{j}\bar{\mathbf{K}}(n,j) \qquad |z|<1 \tag{5.32b}$$

where

$$\mathbf{K}(n,j) = \begin{pmatrix} \mathbf{K}^{(\mathrm{up})}(n,j) \\ \mathbf{K}^{(\mathrm{dn})}(n,j) \end{pmatrix}, \qquad \bar{\mathbf{K}}(n,j) = \begin{pmatrix} \bar{\mathbf{K}}^{(\mathrm{up})}(n,j) \\ \bar{\mathbf{K}}^{(\mathrm{dn})}(n,j) \end{pmatrix}.$$

These kernels satisfy the equations:

$$\bar{\mathbf{K}}(n,m) + \sum_{j=n}^{+\infty} \mathbf{K}(n,j)\mathbf{F}(m+j) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} \delta_{m,n} \qquad m \geq n \tag{5.33a}$$

$$\mathbf{K}(n,m) + \sum_{j=n}^{+\infty} \bar{\mathbf{K}}(n,j)\bar{\mathbf{F}}(m+j) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} \delta_{m,n} \qquad m \geq n, \tag{5.33b}$$

where

$$\mathbf{F}(n) = \sum_{j=1}^{J} z_j^{-n-1}\mathbf{C}_j + \frac{1}{2\pi i}\oint_{|z|=1} z^{-n-1}\boldsymbol{\rho}(z)\, dz.$$

$$\bar{\mathbf{F}}(n) = -\sum_{j=1}^{\bar{J}} \bar{z}_j^{n-1}\bar{\mathbf{C}}_j + \frac{1}{2\pi i}\oint_{|z|=1} z^{n-1}\bar{\boldsymbol{\rho}}(z)\, dz.$$

We can rewrite the eqs. (5.33a)–(5.33b) as forced summation equations. We make the change of variable

$$\mathbf{K}(n,m) = \begin{pmatrix} \boldsymbol{\Omega}_n^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n^{-1} \end{pmatrix} \boldsymbol{\kappa}(n,m)$$

$$\bar{\mathbf{K}}(n,m) = \begin{pmatrix} \boldsymbol{\Omega}_n^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Delta}_n^{-1} \end{pmatrix} \bar{\boldsymbol{\kappa}}(n,m)$$

for $m > n$ and

$$\boldsymbol{\kappa}(n,n) = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix}, \qquad \bar{\boldsymbol{\kappa}}(n,n) = \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix}.$$

With this transformation, (5.33a)-(5.33b) become

$$\bar{\boldsymbol{k}}(n,m) + \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_M \end{pmatrix} \mathbf{F}(n+m) + \sum_{j=n+1}^{+\infty} \boldsymbol{\kappa}(n,j)\mathbf{F}(m+j) = \mathbf{0} \qquad m > n$$

$$\boldsymbol{\kappa}(n,m) + \begin{pmatrix} \mathbf{I}_N \\ \mathbf{0} \end{pmatrix} \bar{\mathbf{F}}(n+m) + \sum_{j=n+1}^{+\infty} \bar{\boldsymbol{\kappa}}(n,j)\bar{\mathbf{F}}(m+j) = \mathbf{0} \qquad m > n,$$

which is a linear system that determines $\boldsymbol{\kappa}(n,m)$ and $\bar{\boldsymbol{\kappa}}(n,m)$.

If Symmetry 2 applies, the potentials are recovered via the relations

$$\mathbf{Q}_n = -\boldsymbol{\kappa}^{(\mathrm{up})}(n,n+1) \qquad \mathbf{R}_n = -\bar{\boldsymbol{\kappa}}^{(\mathrm{dn})}(n,n+1).$$

The problem of existence and uniqueness of solutions for the inverse problem was partially addressed in [**10**], where it was shown that the inverse problem has a unique solution if the GLM integral equations are Fredholm.

5.4. Time evolution. The operator (5.2b) determines the evolution of the eigenfunctions. In turn, the time evolution of the eigenfunctions determines the time evolution of the scattering data. One can deduce that:

$$\mathbf{b}(z,\tau) = e^{i(z^2+z^{-2})\tau}\mathbf{b}(z,0) \qquad \mathbf{a}(z,\tau)=\mathbf{a}(z,0) \tag{5.36a}$$

$$\bar{\mathbf{a}}(z,\tau) = \bar{\mathbf{a}}(z,0) \qquad \bar{\mathbf{b}}(z,\tau)=e^{-i(z^2+z^{-2})\tau}\bar{\mathbf{b}}(z,0). \tag{5.36b}$$

in the gauge $\mathbf{A}=\mathbf{I}_N, \mathbf{B}=\mathbf{I}_M$. It follows immediately that:

$$\begin{aligned}\boldsymbol{\rho}(z,\tau) &= e^{i(z^2+z^{-2})\tau}\boldsymbol{\rho}(z,0)\\ \bar{\boldsymbol{\rho}}(z,\tau) &= e^{-i(z^2+z^{-2})\tau}\bar{\boldsymbol{\rho}}(z,0).\end{aligned}$$

Eqs. (5.36a)–(5.36b) also imply that eigenvalues (i.e., the zeros of $\det \mathbf{a}(z)$ and $\det \bar{\mathbf{a}}(z)$) are constant as the solution evolves. As in the other systems, the eigenvalues are time-invariant discrete states of the evolution equation. The norming constants associated with these eigenvalues are, however, not fixed. Eqs. (5.36a)–(5.36b) imply that

$$\mathbf{C}_j(\tau)=e^{i(z_j^2+z_j^{-2})\tau}\mathbf{C}_j(0), \qquad \bar{\mathbf{C}}_j(\tau)=e^{-i(\bar{z}_j^2+\bar{z}_j^{-2})\tau}\bar{\mathbf{C}}_j(0). \tag{5.38}$$

Recall that, in order to obtain the solutions of (5.1a)–(5.1b) with the gauge $\mathbf{A},\mathbf{B}=\mathbf{0}$, one first determines the evolution of the potentials $\hat{\mathbf{Q}}_n$, $\hat{\mathbf{R}}_n$ with gauge $\mathbf{A}=\mathbf{B}=\mathbf{I}$ such that

$$\hat{\mathbf{Q}}_n(\tau_0)=\mathbf{Q}_n(\tau_0), \qquad \hat{\mathbf{R}}_n(\tau_0)=\mathbf{R}_n(\tau_0)$$

at a given initial time τ_0 (for instance, $\tau_0=0$) and then uses the transformations

$$\begin{aligned}\mathbf{Q}_n(\tau) &= e^{-i(\tau-\tau_0)\mathbf{A}}\hat{\mathbf{Q}}_n(\tau)\,e^{-i(\tau-\tau_0)\mathbf{B}} = e^{-2i(\tau-\tau_0)}\hat{\mathbf{Q}}_n(\tau)\\ \mathbf{R}_n(\tau) &= e^{i(\tau-\tau_0)\mathbf{B}}\hat{\mathbf{R}}_n(\tau)\,e^{i(\tau-\tau_0)\mathbf{A}} = e^{2i(\tau-\tau_0)}\hat{\mathbf{R}}_n(\tau)\end{aligned}$$

to get the evolved potentials.

5.5. Soliton Solution. In order to obtain a one-soliton solution, we consider "reflectionless" scattering data (i.e, $\boldsymbol{\rho}(z)=\bar{\boldsymbol{\rho}}(z)=\mathbf{0}$ on $|z|=1$) composed of a single octet of eigenvalues (5.28), with the associated norming constants satisfying the symmetries (5.25)–(5.26). Because the data are reflectionless, the system (5.30c)–(5.30f) reduces to a finite-dimensional linear algebraic system and it is possible find an explicit solution.

For reasons explained below, we write the norming constant associated with the eigenvalue z_1 in the form

$$\mathbf{C}_1=\begin{pmatrix}\gamma_1^{(1)} & \delta_1^{(2)}\\ \gamma_1^{(2)} & -\delta_1^{(1)}\end{pmatrix}$$

where we introduced the 2-component vectors

$$\boldsymbol{\gamma}_1=\begin{pmatrix}\gamma_1^{(1)}\\ \gamma_1^{(2)}\end{pmatrix}, \qquad \boldsymbol{\delta}_1=\begin{pmatrix}\delta_1^{(1)}\\ \delta_1^{(2)}\end{pmatrix}.$$

The symmetries (5.25)–(5.26) fix the remaining norming constants in terms of these vectors.

To obtain a "fundamental" one-soliton solutions of IDVNLS (1.5), we set

$$\boldsymbol{\delta}_1 = 0.$$

Then, if we write $z_1 = e^{\alpha+i\beta}$ (where we have assumed, without loss of generality, that $\alpha > 0$ and $-\frac{\pi}{2} < \beta \leq \frac{\pi}{2}$), the fundamental soliton is:

$$\mathbf{Q}_n(\tau) = \mathbf{p}\sinh(2\alpha)e^{i(2\beta(n+1)+2\omega\tau)}\operatorname{sech}(2\alpha(n+1) - 2v\tau - d) \tag{5.40}$$

where

$$v = -\sinh(2\alpha)\sin(2\beta), \qquad \omega = 1 - \cosh(2\alpha)\cos(2\beta), \tag{5.41}$$

and

$$\mathbf{p} = -\frac{\boldsymbol{\gamma}_1^*(0)}{\|\boldsymbol{\gamma}_1(0)\|}, \qquad e^d = \frac{\|\boldsymbol{\gamma}_1(0)\|}{\sinh(2\alpha)}.$$

A brief inspection reveals that the fundamental soliton is the one-soliton solution of IDNLS (cf. (3.34)), multiplied by a polarization vector, $\mathbf{p}$ (a complex unit vector). That is, the fundamental soliton bears the same relationship to the one-soliton solution of IDNLS as the one-soliton solution of VNLS bears to the one-soliton solution of NLS (cf. Section 4.5).

The case $\boldsymbol{\delta}_1 \neq 0$, $\boldsymbol{\gamma}_1 = 0$ is equivalent to (5.40). Note that the eigenvalues z_1 and $\hat{z}_1 = \pm i z_1^*$ (where the sign is chosen so that sign of $\operatorname{Re}\hat{z}_1 \geq 0$) are the two members of the eigenvalue octet that are in the region $|z| > 1$, $\operatorname{Re}\hat{z}_1 \geq 0$. There is no intrinsic property that distinguishes one eigenvlue from the other. In fact, the choice of $\boldsymbol{\delta}_1 \neq 0$, $\boldsymbol{\gamma}_1 = 0$ vs. $\boldsymbol{\gamma}_1 \neq 0$, $\boldsymbol{\delta}_1 = 0$ is equivalent to the choice of identification of z_1 and $\hat{z}_1$ among two members of the octet in the region $|z| > 1$, $\operatorname{Re} z \geq 0$.

More generally, there are reflectionless potentials for which both $\boldsymbol{\gamma} \neq 0$ and $\boldsymbol{\delta} \neq 0$. We refer to these potentials as "composite" solitons because they may be obtained by the regular coalescence of two "fundamental" soliton octets. That is, we consider reflectionless scattering data composed of two fundamental soliton octets, where the first octet is as above and the second octet includes the eigenvalue $z_2 = \hat{z}_1 + \epsilon$ with the associated norming constant

$$\mathbf{C}_2 = \begin{pmatrix}\boldsymbol{\gamma}_2 & \mathbf{0}\end{pmatrix}.$$

In the coalescence limit, $\epsilon \to 0$, the scattering data is a single octet (the octet related to z_1) in which $\boldsymbol{\gamma}_1$ is unchanged, $\boldsymbol{\delta}_1 = \boldsymbol{\gamma}_2$ and the potential, $\mathbf{Q}_n(\tau)$, converges to a composite soliton solution of VNLS. In general, the explicit formula of a composite soliton is ungainly and unilluminating. However, the composite soliton itself consists of a localized travelling envelope with both temporal and spatial oscillations, as well as a complex spatial modulation. The oscillating envelope travels with a constant velocity equal to that of a fundamental soliton associated with the eigenvalue octet anchored by z_1.

We gain further insight into the nature of the composite soliton by considering the special cases $W(\boldsymbol{\gamma}_1^*, \boldsymbol{\delta}_1) = 0$ ("parallel") and $\boldsymbol{\gamma}_1 \cdot \boldsymbol{\delta}_1 = 0$ ("perpendicular"). In the parallel case, the composite soliton is a reduction solution, i.e., of the form

$$\mathbf{Q}_n = \mathbf{p}Q_n,$$

where $\mathbf{p}$ is a complex unit vector (i.e., a polarization vector) parallel to $\boldsymbol{\gamma}_1^*$ and Q_n is the solution of IDNLS that corresponds to two quartets of eigenvalues anchored by z_1 and $z_2 = \hat{z}_1$ respectively. (cf. Section 3.5). Therefore, surprisingly, this special case of the composite-soliton does *not* have a minimal spectrum when considered as a solution of the scalar equation (i.e., IDNLS). This illustrates the composite (i.e., non-atomic) nature of the composite solitons.

On the other hand, in the perpendicular case, the composite-soliton formula takes on the simplified form

$$\mathbf{Q}_n(\tau) = \left[\cos(\mu)\mathbf{p}e^{2i(\beta(n+1)+\omega\tau)} + \sin(\mu)\mathbf{p}^{\perp}(-1)^n e^{-2i(\beta(n+1)+\omega'\tau)}\right] \sinh(2\alpha)\operatorname{sech}(2\alpha(n+1) - v\tau - d)$$

where

$$\cos(\mu)\mathbf{p} = -\frac{\boldsymbol{\gamma}_1^*}{\left(\|\boldsymbol{\gamma}_1\|^2 + \|\boldsymbol{\delta}_1\|^2\right)^{\frac{1}{2}}}, \quad \sin(\mu)\mathbf{p}^{\perp} = \frac{\boldsymbol{\delta}_1}{\left(\|\boldsymbol{\gamma}_1\|^2 + \|\boldsymbol{\delta}_1\|^2\right)^{\frac{1}{2}}},$$

the velocities v and ω are given by (5.41) and

$$\omega' = 1 + \cosh(2\alpha)\cos(2\beta), \qquad e^d = \frac{\left(\|\boldsymbol{\gamma}_1\|^2 + \|\boldsymbol{\delta}_1\|^2\right)^{\frac{1}{2}}}{\sinh(2\alpha)}.$$

In this case, the two frequencies of complex modulation oscillate perpendicular to one another and, therefore, the two perpendicular components of the envelope are not, when considered separately, subject to temporal oscillation and spatial modulation. We remark that the $(-1)^n$ term in the complex modulation shows that this solution has no counterpart in the continuous limit. More generally, unlike fundamental solitons, composite solitons have no counterpart in VNLS, the continuous limit of IDVNLS.

6. Soliton Interactions

6.1. NLS: Scalar, Continuous Soliton interaction. Zabusky and Kruskal [**89**] coined the term "soliton" to describe solitary waves that retain their individual characteristic form (and hence their individual characteristic velocity) after passing through one another. Indeed, the solitons of NLS are well-known to have this property. The underlying mechanism is the invariance of the eigenvalues, which, as we have seen, determine the envelope amplitude and velocity of individual solitons. The soliton interaction, however, induces a shift in the location of the envelope peak and overall complex phase of the individual solitons (a phase shift). This phase shift is a typical characteristic of soliton interactions in integrable nonlinear Schrödinger systems and other integrable nonlinear evolution equations.

In a generic multisoliton solution of NLS, the solitons will have unequal velocities. Therefore, in the long time limits ($t \to \pm\infty$), the solitons are spatially well-separated and the solution is of the form

$$q^{\pm}(x,t) \approx \sum_{j=1}^{J} q_j^{\pm}(x,t) \tag{6.1}$$

where: there are J solitons; for $j = 1, \ldots, J$, $q_j^{\pm}$ is a one-soliton solution of NLS corresponding to the eigenvalue $k_j = \xi_j + i\eta_j$; the minus sign corresponds to the backward ($t \to -\infty$) long-time limit while plus sign corresponds to the forward ($t \to +\infty$) long-time limit. That is, $q_j^-(x,t)$ is of the form (2.26)–(2.27) with

$\xi = \xi_j$, $\eta = \eta_j$, $\delta = \delta_j^-$ and $\psi = \psi_j^-$, while $q_j^+(x,t)$ is of the same form with $\delta = \delta_j^+$ and $\psi = \psi_j^+$.

To fix ideas, we assume that

$$\xi_1 < \xi_2 < \cdots < \xi_J$$

so that

$$v_1 < v_2 < \cdots < v_J$$

where $v_j = 4\xi_j$ is the velocity of the envelope of the j-th soliton. Hence, as $t \to -\infty$ the solitons become spatially well-separated and they are distributed along the x-axis in the order:

$$J \; (J-1) \; \ldots \; 2 \; 1$$

where j indicates the relative position of the j-th soliton. The order of the soliton sequence is reversed as $t \to +\infty$. That is, the soliton are again spatially well-separated, but arranged along the x-axis in the order

$$1 \; 2 \; \ldots \; (J-1) \; J$$

as $t \to +\infty$. In the passage from the backward ($t \to -\infty$) to the forward ($t \to +\infty$) long-time limit, the solitons pass through one another and these interactions generate the phase shifts.

One method to compute the phases of the individual solitons in the long-time limits is to compute the scattering data as the net result of scattering by J (spatially) sequential reflectionless potentials (i.e., solitons). In fact, this method was used in [**92**] (cf. [**10**] for details.) The analysis yields the relation

$$e^{2(\delta_j^+ - \delta_j^-) + i(\psi_j^+ - \psi_j^-)} = \prod_{\ell=1}^{j-1} \left(\frac{k_j - k_\ell}{k_j - k_\ell^*} \right)^2 \prod_{m=j+1}^{J} \left(\frac{k_j - k_m^*}{k_j - k_m} \right)^2 . \tag{6.2}$$

Thus, (6.2) gives the net phase shift of the j-th soliton that is induced by interaction with the other $J-1$ solitons. The difference in the real part of the exponent, $\delta_j^+ - \delta_j^-$, corresponds to the shift in the envelope peak of the j-th soliton, while the imaginary part, $\psi_j^+ - \psi_j^-$, corresponds to the shift in its overall complex phase.

We note that, in the passage from the limit $t \to -\infty$ to the limit $t \to +\infty$, the order of soliton interactions is not unique. For example, with three solitons, the ordering of the solitons can proceed as

$$3 \; 2 \; 1 \to 2 \; 3 \; 1 \to 1 \; 2 \; 3$$

or

$$3 \; 2 \; 1 \to 3 \; 1 \; 2 \to 1 \; 2 \; 3,$$

depending on the relative distance between the solitons for large negative t. The number of possibilities increases rapidly with the number of solitons. However, formula (6.2) implies that, in general, the total net phase shift for each soliton is independent of the order of soliton interaction. Moreover, the total net phase shift (both the displacement of the envelope peak and the shift in the overall phase of the complex modulation) for the each soliton is the simple sum of the individual phase shifts induced by the pairwise interactions with the other $J-1$ solitons.

6.2. IDNLS: Scalar, Discrete Soliton interaction. The soliton interaction in IDNLS is similar to that of NLS. As in the the continuous case, we consider a J-soliton solution with eigenvalues $z_1, \ldots, z_J$ such that

$$v_1 < v_2 < \cdots < v_J$$

where v_j is the velocity of the j-th solition as determined by (3.35a) where $\alpha = \alpha_j$, $\beta = \beta_j$ and $z_j = e^{\alpha_j + i\beta_j}$. Then, as before, the solitons, passing through one another, reverse their spatial order in the passage from the limit $\tau \to -\infty$ to the limit $\tau \to +\infty$. Moreover, the solitons are spatially well-separated in each of the long-time limits. Again, because the eigenvalues are τ-independent, each soliton retains its characteristic shape and velocity (cf. eq. (3.34)–(3.35b)) in the passage from the limit $\tau \to -\infty$ to the limit $\tau \to +\infty$ even though each soliton undergoes a shift in the location of its envelope peak and overall complex phase.

An analysis similar to that for NLS (cf. [**10**] for a derivation) yields the formula for the total net phase shift of the j-th soliton in a J-soliton solution:

$$e^{2(d_j^+ - d_j^-) + i(\psi_j^+ - \psi_j^-)} = \prod_{l=1}^{j-1} \left(\frac{z_j^2 - z_l^2}{z_j^2 - (z_l^*)^{-2}} \right)^2 \prod_{m=j+1}^{J} (z_m z_m^*)^{-2} \left(\frac{z_j^2 - (z_m^*)^{-2}}{z_j^2 - z_m^2} \right)^2,$$

where, in (3.35b), $d = d_j^\pm$ and $\psi = \psi_j^\pm$ in the forward (+) and backward (−) long-time limits. The expression for the discrete-soliton phase shift converges to that for NLS solitons (6.2) in the limit $h \to 0$, where $z_j = e^{-ik_j h}$ and $\bar{z}_j = e^{-ik_j^* h}$. In the discrete case (i.e., even for $h > 0$), for each soliton, the total phase shift as a result of collision with the other $J - 1$ solitons is the sum of phase shifts induced by the pairwise interaction with each of the other solitons and is therefore independent of the order of the soliton interactions.

6.3. VNLS: Vector, Continuous Soliton Interaction. The interaction of vector solitons is more complex than that of scalar solitons. While the individual vector solitons retain their characteristic shape and velocity in interactions with other vector solitons (again, due to the invariance of the eigenvalues), the polarization vector of each soliton is subject to a phase shift. That is, while a scalar solitons is subject to a shift in its overall *scalar* complex phase, the phase shift of vector solitons includes a shift in the polarization *vector*. In particular the magnitude of the components of the polarization vector may change (subject to the constraint that the polarization vector is always a unit vector, i.e. its total magnitude equals1). In addition, just as in the scalar case, the peak of the vector-soliton envelope is shifted by interaction with other vector solitons. The shift in the magnitude of the components of the polarization vector is, however, the distinctive feature of *vector*-soliton interactions.

As before, we assume that the discrete eigenvalues of a J-soliton solution of VNLS are such that

$$\xi_1 < \xi_2 < \cdots < \xi_J$$

where the eigenvalues are of the form $k_j = \xi_j + i\eta_j$. Therefore, the soliton velocities are such that

$$v_1 < v_2 < \cdots < v_J.$$

Then, in the long time limits, the solitons are spatially well-separated and the J-soliton solution is of the form

$$\mathbf{q}^{\pm}(x,t) \sim \sum_{j=1}^{J} \mathbf{p}^{\pm} q_j^{\pm}(x,t)$$

where: as before, $\pm$ denote the forward and backward long-time limits; $q_j^{\pm}$ is a one-soliton solution of NLS (cf. (2.26)) with $\xi = \xi_j$, $\eta = \eta_j$, $\delta = \delta_j^{\pm}$ and $\psi = \pi$. (The overall complex phase represented by ψ is absorbed in to the complex polarization vector.) In the passage from the backward long-time limit to the forward long-time limit, the solitons reverse their spatial order and, in this process, pass through one another, thereby inducing shifts in each others polarization vectors and envelope peaks.

To determine the phase shifts between between the limits $t \to -\infty$ and $t \to +\infty$, one can proceed as for scalar solitons and analyze the scattering data under the assumption that the total scattering from $x \to -\infty$ to $x \to +\infty$ is the net effect of scattering by the (spatially) sequential individual soliton potentials [**64**]. The resulting formula, however, is not as simple as in in the scalar case.

For the interaction of J solitons, the phase shifts in the long-time limits are expressed by

$$\mathbf{p}_j^{\pm} = \frac{\left(\mathbf{s}_j^{\pm}\right)^H}{\left\|\mathbf{s}_j^{\pm}\right\|} \qquad e^{\delta_j^{\pm}} = \left\|\mathbf{s}_j^{\pm}\right\|, \tag{6.3}$$

where $\mathbf{s}_j^{\pm}$ is a vector that satisfies the relation

$$\mathbf{s}_j^{+} = \prod_{\ell=1}^{j-1} \frac{1}{a_\ell(k_j)} \prod_{\ell=j+1}^{J} a_\ell(k_j) \prod_{\substack{\ell=j+1 \\ \text{right}}}^{J} \mathbf{c}_\ell^{+}(k_j) \prod_{\substack{\ell=1 \\ \text{right}}}^{j-1} \left[\mathbf{c}_\ell^{-}(k_j)\right]^{-1} \mathbf{s}_j^{-}. \tag{6.4}$$

In the products above, we define $\prod_{\ell=a}^{b} = 1$ for $a > b$ and, as before, the notation "right" indicates that the matrix with index ℓ is to the right of the matrix with index $\ell - 1$. The coefficients in eq. (6.4) –the transmission coefficients of the individual solitons– are given by:

$$\begin{aligned} a_j(k) &= \frac{k - k_j}{k - k_j^*} \\ \mathbf{c}_j^{\pm}(k) &= \mathbf{I}_M - \frac{k_j - k_j^*}{k - k_j^*} \frac{1}{\left\|\mathbf{s}_j^{\pm}\right\|^2} (\mathbf{s}_j^{\pm})^H \mathbf{s}_j^{\pm} \end{aligned}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix and $(\mathbf{s}_j^{\pm})^H \mathbf{s}_j^{\pm}$ is an $M \times M$ matrix.

With (6.4) one can determine $\mathbf{s}_j^{+}$ (and, therefore, $\mathbf{p}_j^{+}$, according to (6.3)) in terms of $\mathbf{s}_1^{-}, \ldots, \mathbf{s}_J^{-}$. First, one obtains $\mathbf{s}_J^{+}$ as this depends on $\mathbf{s}_J$ and, through the coefficients $\mathbf{c}_j^{-}(k_J)$, on $\mathbf{s}_1^{-}, \ldots, \mathbf{s}_{J-1}^{-}$. Then, one can explicitly obtain $\mathbf{s}_{J-1}^{+}$, as this depends on $\mathbf{s}_1^{-}, \ldots, \mathbf{s}_{J-1}^{-}$ and $\mathbf{s}_J^{+}$. Proceeding inductively, one determines $\mathbf{s}_1^{+}, \ldots, \mathbf{s}_{J-2}^{+}$.

In contrast to the formula for scalar soliton interaction (6.2), it is not apparent from formula (6.4) that the vector-soliton phase shifts are independent of the order of the soliton interactions. Nor is it apparent that the interaction of multiple vector solitons is equivalent to the net effect of all the pair-wise interactions considered separately. However, in fact, this is so. A proof requires only a direct calculation for the three-soliton case as the interactions of a greater number of solitons can be reduced to this case [**10**]. We note that this result on the order-invariance of vector soliton interaction is also obtained in [**79**] by a completely different argument and was partially addressed in [**25, 55, 71**]. Fundamentally, the vector-soliton interaction is the order-independent net effect of pairwise interactions because the maps that define these interactions satisfy the Yang-Baxter relation [**45, 75, 84**].

The polarizations of two vector solitons after their interaction can be written explicitly in terms of the polarizations before their interaction:

$$\mathbf{p}_2^+ = \frac{1}{\chi}\frac{k_1-k_2^*}{k_1^*-k_2^*}\left[\mathbf{p}_2^- + \frac{k_1^*-k_1}{k_2^*-k_1^*}\left(\mathbf{p}_1^{-\,*}\cdot\mathbf{p}_2^-\right)\mathbf{p}_1^-\right] \tag{6.5a}$$

$$\mathbf{p}_1^+ = \frac{1}{\chi}\frac{k_1-k_2^*}{k_1-k_2}\left[\mathbf{p}_1^- + \frac{k_2^*-k_2}{k_2-k_1}\left(\mathbf{p}_2^{-\,*}\cdot\mathbf{p}_1^-\right)\mathbf{p}_2^-\right] \tag{6.5b}$$

where

$$\chi^2 = \left|\frac{k_1-k_2^*}{k_1-k_2}\right|^2\left[1+\frac{(k_1^*-k_1)(k_2-k_2^*)}{|k_1-k_2|^2}\left|\mathbf{p}_1^{-*}\cdot\mathbf{p}_2^-\right|^2\right].$$

It follows immediately from (6.5a)–(6.5b) that the magnitudes of the components of polarization vectors will shift due to the soliton interaction unless the polarizations are perpendicular (i.e., $\mathbf{p}_1^{-*}\cdot\mathbf{p}_2^- = 0$) or parallel (i.e., $\mathbf{p}_2^- = e^{i\phi}\mathbf{p}_1^-$) before the solitons interact.

Alternatively, the polarization shift in a two-component, two-soliton interaction can be described by a fractional linear transformation (FLT) of a (scalar) complex polarization state [**76**]. We define

$$\rho_j^\pm = \frac{p_j^{(1)\pm}}{p_j^{(2)\pm}} \tag{6.6}$$

for $j = 1, 2$. If $p_j^{(2)} = 0$, then we take $\rho_j = \infty$. It follows directly from (6.5a)–(6.5b) that the complex polarization states satisfy the FLT:

$$\rho_2^+ = \frac{a_1\rho_2^- + b_1}{c_1\rho_2^- + d_1}, \tag{6.7a}$$

where the coefficients

$$a_1 = 1 + \frac{k_2^*-k_1}{k_2^*-k_1^*}|\rho_1^-|^2$$
$$b_1 = \frac{k_1^*-k_1}{k_2^*-k_1^*}\rho_1^-$$
$$c_1 = \frac{k_1^*-k_1}{k_2^*-k_1^*}(\rho_1^-)^*$$
$$d_1 = \frac{k_2^*-k_1}{k_2^*-k_1^*} + |\rho_1^-|^2$$

depend only on the eigenvalues k_1, k_2 and the polarization state ρ_1^-. Similarly,

$$\rho_1^+ = \frac{a_2\rho_1^- + b_2}{c_2\rho_1^- + d_2} \tag{6.7b}$$

where the coefficients are functions of the eigenvalues and ρ_2^-:

$$\begin{aligned} a_2 &= 1 + \frac{k_2^* - k_1}{k_2 - k_1}|\rho_2^-|^2 \\ b_2 &= \frac{k_2^* - k_2}{k_2 - k_1}\rho_2^- \\ c_2 &= \frac{k_2^* - k_2}{k_2 - k_1}(\rho_2^-)^* \\ d_2 &= \frac{k_2^* - k_1}{k_2 - k_1} + |\rho_2^-|^2. \end{aligned}$$

The order-invariance of the shift in the polarization state due to vector-soliton interactions can also be shown by the use of the FLT formulas [**11**].

6.4. IDVNLS: Vector, Discrete Soliton Interaction. The interaction of the discrete vector solitons of IDVNLS is similar to that of VNLS. Again, the distinctive feature of vector-soliton interaction is the shift in the polarization vectors that results from soliton collisions. In fact, in their interactions, discrete vector solitons reproduce the dynamics of continuous vector solitons and the analogous formulas converge to their counterparts in the continuum limit.

As before, we assume that the eigenvalues of the J solitons, $(z_1, \ldots, z_J)$ are such that

$$v_1 < v_2 < \cdots < v_J$$

where v_j is the envelope velocity of the j-th soliton. Then, in the long-time limits, the solitons are spatially well-separated. If moreover, all of the solitons are fundamental solitons, then

$$\mathbf{Q}_n^{\pm}(\tau) \sim \sum_{j=1}^{J} \mathbf{p}_j^{\pm} Q_{n,j}^{\pm}(\tau),$$

where: $\pm$ denote the forward and backward long-time limits; $Q_{n,j}^{\pm}$ is a one-soliton solution of IDNLS (cf. (3.34)) with $\alpha = \alpha_j$, $\beta = \beta_j$, $d = d_j^{\pm}$ and $\psi = \pi$. Again, in the passage from the backward long-time limit to the forward long-time limit the solitons reverse their spatial order. In this process, the solitons pass through one another and thereby induce shifts in each others polarization vectors and envelope peaks.

The method of Manakov [**64**] for analysis of continuous vector solitons can be adapted to the discrete vector case (cf. [**10**]). Such an analysis yields the relation

$$\mathbf{S}_j^+ = \prod_{\substack{\ell=j+1 \\ \text{right}}}^{J} \mathbf{c}_\ell^+(z_j) \prod_{\substack{\ell=1 \\ \text{right}}}^{j-1} \left[\mathbf{c}_\ell^-(z_j)\right]^{-1} \mathbf{S}_j^- \prod_{\substack{\ell=j+1 \\ \text{right}}}^{J} \mathbf{a}_\ell^-(z_j) \prod_{\substack{l=1 \\ \text{right}}}^{j-1} \left[\mathbf{a}_\ell^+(z_j)\right]^{-1} \tag{6.8}$$

where: $\mathbf{S}_j^{\pm}$, $j = 1, \ldots, J$, are 2×2 matrices; as before, the notation "right" indicates that the matrix with index ℓ is to the right of the matrix with index $\ell - 1$ and we define $\prod_a^b = 1$ for $a > b$. If the solitons are all fundamental solitons, then

$$\mathbf{S}_j^{\pm} = \begin{pmatrix} s_j^{(1)\pm} & 0 \\ s_j^{(2)\pm} & 0 \end{pmatrix} \tag{6.9a}$$

$$\mathbf{a}_j(z) = \begin{pmatrix} \frac{z^2 - z_j^2}{z^2 - \bar{z}_j^2} & 0 \\ 0 & \frac{z^2 + \bar{z}_j^{-2}}{z^2 + z_j^{-2}} \end{pmatrix} \tag{6.9b}$$

$$\mathbf{c}_j(z) = \frac{\bar{z}_j^2 + z^{-2}}{z_j^2 + z^{-2}} \left[\mathbf{I} + \frac{(\bar{z}_j^2 - z_j^2)(\bar{z}_j^2 + z_j^{-2})}{(z^2 - \bar{z}_j^2)(\bar{z}_j^2 + z^{-2})} \frac{\mathbf{s}_j^* \mathbf{s}_j^T}{||\mathbf{s}_j||^2} \right], \tag{6.9c}$$

where: $\bar{z}_j = 1/z_j^*$; $\mathbf{s}_j = (s_j^{(1)}, s_j^{(2)})^T$, are column vectors; the polarization vector of the j-th soliton in the forward (+) and backward (−) long-time limits is

$$\mathbf{p}_j^{\pm} = \frac{\left(\mathbf{s}_j^{\pm}\right)^*}{\left\|\mathbf{s}_j^{\pm}\right\|}.$$

In order to determine $\mathbf{s}_j^+$ (and, hence $\mathbf{p}_j^+$) in terms of $\mathbf{s}_1^-, \ldots, \mathbf{s}_J^-$, we proceed inductively. First, using (6.8), we compute $\mathbf{s}_J^+$ as it depends on $\mathbf{s}_1^-, \ldots, \mathbf{s}_J^-$. Next, we compute $\mathbf{s}_{J-1}^+$, as it depends on $\mathbf{s}_1^-, \ldots, \mathbf{s}_{J-1}^-$ and $\mathbf{s}_J^+$, and so on.

Although it is not apparent form (6.8), the J-soliton interaction of fundamental, discrete, vector solitons is equivalent to the order-independent composition of $J(J-1)/2$ pairwise interactions. As for the continuous case, this only needs to be verified directly from (6.8), (6.9a)–(6.9c) for the three-soliton interaction [**10**]. Moreover, as in the continuous, case the order-invariance of the soliton interaction is a manifestation of the fact that the map governing the polarization shift satisfies the Yang-Baxter relation [**12**].

In the case of two fundamental vector solitons, we obtain from eqs. (6.8), (6.9a)–(6.9c) the relations

$$\mathbf{p}_2^+ = \frac{1}{\chi} \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(z_1^2 + \bar{z}_2^{-2})} \left(\mathbf{p}_2^- + \frac{(z_1^2 - \bar{z}_1^2)(\bar{z}_1^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})} (\mathbf{p}_1^{-*} \cdot \mathbf{p}_2^-) \mathbf{p}_1^- \right) \tag{6.10a}$$

$$\mathbf{p}_1^+ = \frac{1}{\chi} \frac{(z_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})}{(z_1^2 - z_2^2)(\bar{z}_2^2 + z_1^{-2})} \left(\mathbf{p}_1^- + \frac{(\bar{z}_2^2 - z_2^2)(z_2^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_2^2 + z_1^{-2})} (\mathbf{p}_2^{-*} \cdot \mathbf{p}_1^-) \mathbf{p}_2^- \right) \tag{6.10b}$$

where

$$\chi^2 = \left| \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(z_1^2 + \bar{z}_2^{-2})} \right|^2 \tilde{\chi}^2,$$

$$\tilde{\chi}^2 = 1 + \frac{(z_1^2 - \bar{z}_1^2)(z_2^2 - \bar{z}_2^2)(\bar{z}_1^2 + z_1^{-2})(\bar{z}_2^{-2} + z_2^2)}{(z_2^2 - z_1^2)(\bar{z}_1^2 - \bar{z}_2^2)(z_2^2 + z_1^{-2})(\bar{z}_1^2 + \bar{z}_2^{-2})} \left| \mathbf{p}_1^{-\,*} \cdot \mathbf{p}_2^- \right|^2.$$

Eqs. (6.10a)–(6.10b) are the analog of Manakov's formulas for the polarization shift of two interacting vector solitons (6.5a)–(6.5b). In fact, with $z_j = e^{-ik_j h}$, $\bar{z}_j = 1/z_j^* = e^{-ik_j^* h}$ the equations for the discrete soliton interaction converge to the equations for the continuous soliton interaction in the limit $h \to 0$ (the continuum limit).

The polarization shift in the interaction of two fundamental solitons can also be described by a fractional linear transformation (FLT). Just as in the continuous case, we describe the polarization state of each soliton by a single complex number (6.6). Then, from (6.10a)–(6.10b) one obtains the FLT:

$$\rho_2^+ = \frac{A_1\rho_2^- + B_1}{C_1\rho_2^- + D_1}, \tag{6.11a}$$

where the coefficients are functions of the particle in state ρ_1^-:

$$A_1 = 1 + \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_2^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_2^2 + \bar{z}_1^{-2})}|\rho_1^-|^2$$
$$B_1 = \frac{(z_1^2 - \bar{z}_1^2)(\bar{z}_1^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}\rho_1^-$$
$$C_1 = \frac{(z_1^2 - \bar{z}_1^2)(\bar{z}_1^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_1^2 + \bar{z}_2^{-2})}(\rho_1^-)^*$$
$$D_1 = \frac{(z_1^2 - \bar{z}_2^2)(\bar{z}_2^2 + z_1^{-2})}{(\bar{z}_1^2 - \bar{z}_2^2)(\bar{z}_2^2 + \bar{z}_1^{-2})} + |\rho_1^-|^2$$

and the FLT:

$$\rho_1^+ = \frac{A_2\rho_1^- + B_2}{C_2\rho_1^- + D_2}, \tag{6.11b}$$

where the coefficients are functions the eigenvalues and ρ_2^-:

$$\begin{aligned}
A_2 &= 1 + \frac{(\bar{z}_2^2 - z_1^2)(z_1^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_1^2 + z_2^{-2})}|\rho_2^-|^2 \\
B_2 &= \frac{(\bar{z}_2^2 - z_2^2)(z_2^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_2^2 + z_1^{-2})}\rho_2^- \\
C_2 &= \frac{(\bar{z}_2^2 - z_2^2)(z_2^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_2^2 + z_1^{-2})}(\rho_2^-)^* \\
D_2 &= \frac{(\bar{z}_2^2 - z_1^2)(z_1^2 + \bar{z}_2^{-2})}{(z_2^2 - z_1^2)(z_1^2 + z_2^{-2})} + |\rho_2^-|^2.
\end{aligned}$$

The coefficients of (6.11a)–(6.11b) converge to the the coefficients of the FLTs that describe the the continuous vector-soliton interaction (6.7a)–(6.7b) in the continuum ($h \to 0$) limit.

7. Vector Soliton Logic

An interesting, recent application of vector solitons is the construction of logic gates by means of vector-soliton interactions. By logic gates we mean input-output systems that encode the operations of binary logic such as AND, NOT, OR, etc. As a computer is, in a theoretical sense, a connected array of logic gates, the existence of soliton-based logic gates implies the possibility of soliton-based computing. The underlying motivation is the goal of all-optical digital information processing (in a nonlinear optical medium that supports solitons).

In fact, it is possible to construct vector-soliton-based logic gates with VNLS solitons such that these gates are, collectively, “computationally universal” in the sense of Turing equivalence [**17**], [**50**]. Moreover, one can construct logic gates with discrete vector solitons by the substitution of the formulas that describe the

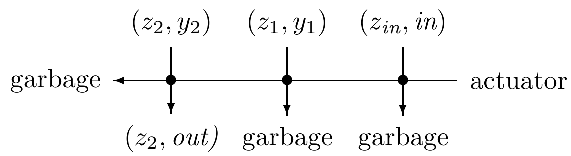

FIGURE 3. Elementary Soliton Logic Gate

interaction of the discrete solitons of IDVNLS for the formulas that describe the interaction of the discrete solitons of IDVNLS [**12**].

In both the discrete and continuous cases, the logic gates are constructed with the help of the FLTs that describe the shift of soliton polarization (i.e., (6.7b)–(6.7a) for VNLS and (6.11b)–(6.11a) for IDVNLS). In either case, to encode a binary logic "bit" in the polarization state of a soliton, we identify the complex polarization state $\rho = 1$ with the logical bit value 1 and the complex polarization state $\rho = 0$ with the logical bit value 0. Digital logic operations are then implemented by the polarization shifts induced via sequences of controlled vector-soliton interactions (cf. [**17**]).

The elementary structure of a vector-soliton logic gates is the four-soliton, three-collision soliton interaction introduced in [**76**]. (See Figure 3.) The actuator soliton (labeled "actuator") is initially in the logical/polarization state 0 and carries information from the input soliton to the output soliton. The speeds (determined by the eigenvalues z_{in}, z_1 and z_2) and initial separations of the other solitons are chosen so that they do not interact with each other, but only with the actuator soliton. The polarization state of the input soliton (with polarization state *in* before the collision with the actuator) carries a logical value (i.e., $in = 0$ or $in = 1$). The polarization states of the remaining solitons before interaction, denoted by y_1 and y_2 respectively, remain to be specified. The polarization state *out* is the output of the gate while "garbage" indicates that the states of these solitons, after the collisions comprised by the gate, are not used for further computations.

The key to the design of logic gates is the determination of states y_1 and y_2 such that one can realize a logical operation. For instance, for the COPY gate, one requires that $out = in$, justifying the name COPY. A priori, it is reasonable to expect that such parameter values exist, because there are four degrees of freedom in the two complex numbers y_1 and y_2, and two complex equations to satisfy: (i) $out = 1$ when $in = 1$ and (ii) $out = 0$ when $in = 0$. Similarly, the NOT gate is such that out=0 when in=1 and vice-versa. Computation is then carried out by the arrangement solitons so that the output solitons cascade in to inputs of subsequent soliton interactions. To achieve computational universality these elementary gates must be combined is appropriate ways [**76**].

8. Conclusions

As the preceding sections show in detail, the basic IST framework is an effective tool for all the nonlinear Schrödinger systems considered here. In all cases, the key is the formulation of the inverse problem as a Riemann-Hilbert problem, which can be rewritten as a linear algebraic-integral system. Moreover, in all cases, reflectionless data correspond to a pure multisoliton solution and, in this case, the Riemann-Hilbert problem reduces to a system linear-algebraic equations. The IST method not only provides a way for obtaining explicit formulas for the solitons, it also provides insight into the underlying mathematical structure that manifests itself in elastic soliton interactions, namely the time-invariance of the eigenvalues.

The IST for IDVNLS contains some twists not present in the other systems. Specifically, there is an extra symmetry in the scattering problem, and the related phenomenon of composite solitons. However, the basic IST framework remains the same. In fact, the IST provides an explanation for the existence of the compound solitons in IDVNLS.

In both the scalar and the vector equations, the integrable discretizations reproduce the soliton dynamics of their respective continuous limits. Although this has been known for some time (at least in the scalar case), the persistence of solitons in the integrable discrete systems remains remarkable given the fact that alternative discretizations of NLS and VNLS have very different dynamics. Again, the formulation of the IST for the discrete systems makes clear the reason for the existence of soliton solutions in the integrable discrete equations.

On the other hand, a careful analysis of the IST for the discrete problem reveals that the discrete systems have solutions with no counterpart in their respective continuum limits. These include the traveling breather states of IDNLS and the compound solitons of IDVNLS. Even though IDNLS has been the subject of intensive study for almost 30 years, this aspect of the discrete system has not been appreciated. In fact, it was the discovery of the compound soliton in the discrete vector system (IDVNLS) that brought these breather states of IDNLS to the attention of the authors.

Like IDNLS, VNLS continues to be a subject of continuing study long after it was introduced, along with a solution via IST, by Manakov. In particular, the dynamics of vector soliton interaction have been of interest because of the potential applications to fiber-optics communications and, more recently, optical computing. However, only recently has the dynamics of a multisoliton interaction been completely analyzed. The analysis of the soliton interaction from IST point of view (initiated by Manakov) shows that the multiple vector-soliton interaction is, in fact, the order-invariant net result of pairwise interactions. Moreover, the method of analysis, and the results, extend to the interaction of discrete vector solitons. The recent (theoretical) implementation of logic operations with vector solitons suggests that vector solitons, in particular, will be a subject of continuing interest.

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