Dynamics of PDE, Vol.1, No.2, 225-237, 2004

Existence of Chaos for Nonlinear Schrödinger Equation Under Singular Perturbations

Y. Charles Li

Communicated by Charles Li, received January 25, 2004.

Abstract. The work [**1**] is generalized to the singularly perturbed nonlinear Schrödinger (NLS) equation of which the regularly perturbed NLS studied in [**1**] is a mollification. Specifically, the existence of Smale horseshoes and Bernoulli shift dynamics is established in a neighborhood of a symmetric pair of Silnikov homoclinic orbits under certain generic conditions, and the existence of the symmetric pair of Silnikov homoclinic orbits has been proved in [**2**]. The main difficulty in the current horseshoe construction is introduced by the singular perturbation $\epsilon\partial_x^2$ which turns the unperturbed reversible system into an irreversible system. It turns out that the equivariant smooth linearization can still be achieved, and the Conley-Moser conditions can still be realized.

Contents

1. Introduction

Consider the singularly perturbed nonlinear Schrödinger (NLS) equation,

$$iq_t = q_{\zeta\zeta} + 2[|q|^2 - \omega^2]q + i\epsilon[q_{\zeta\zeta} - \alpha q + \beta] \ , \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 35Q55, 35Q30; Secondary 37L10, 37L50.
Key words and phrases. Homoclinic orbits, chaos, Samle horseshoes, equivariant smooth linearization, Conley-Moser conditions.
This work is partially supported by a Guggenheim Fellowship.

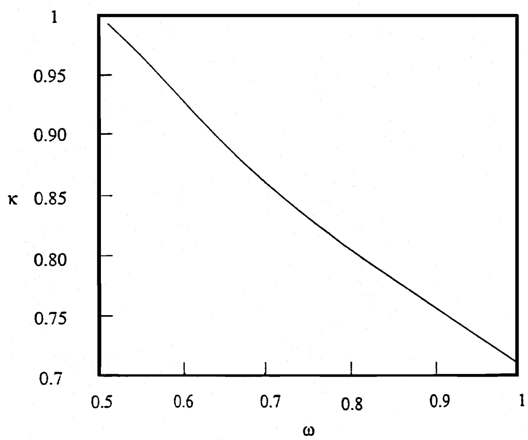

FIGURE 1. The graph of $\kappa(\omega)$.

where $q = q(t, \zeta)$ is a complex-valued function of the two real variables t and ζ, t represents time, and ζ represents space. $q(t, \zeta)$ is subject to periodic boundary condition of period 2π, and even constraint, i.e.

$$q(t, \zeta + 2\pi) = q(t, \zeta) \ , \quad q(t, -\zeta) = q(t, \zeta) \ .$$

$\alpha > 0$ and $\beta > 0$ are constants, and $\epsilon > 0$ is the perturbation parameter. For simplicity of presentation, we restrict ω by $\omega \in (1/2, 1)$. In [**2**], the following theorem on the existence of Silnikov homoclinic orbits was proved.

THEOREM 1.1. *There exists a $\epsilon_0 > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, there exists a codimension 1 surface in the external parameter space $(\alpha, \beta, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in (\frac{1}{2}, 1)/S$, S is a finite subset, and $\alpha\omega < \beta$. For any (α, β, ω) on the codimension 1 surface, the singularly perturbed nonlinear Schrödinger equation* (1.1) *possesses a symmetric pair of Silnikov homoclinic orbits asymptotic to a saddle Q_ϵ. The codimension 1 surface has the approximate representation given by $\alpha = 1/\kappa(\omega)$, where $\kappa(\omega)$ is plotted in Figure 1.*

Notice that if $q(t, \zeta)$ is a homoclinic orbit, then $q(t, \zeta + \pi)$ is another homoclinic orbit. Thus $q(t, \zeta)$ and $q(t, \zeta + \pi)$ form a symmetric pair of homoclinic orbits. Based upon the above theorem, we will construct Smale horseshoes in a neighborhood of the symmetric pair of homoclinic orbits. The construction is a generalization of that in [**1**] where the singular perturbation $\epsilon\partial_x^2$ is mollified into a bounded Fourier multiplier. The main difficulty in the current horseshoe construction is introduced by the singular perturbation $\epsilon\partial_x^2$ which turns the unperturbed reversible system into an irreversible system. Specifically, denote by F_ϵ^t the evolution operator of the singularly perturbed nonlinear Schrödinger equation (1.1). When $\epsilon = 0$, F_0^t is a group. When $\epsilon > 0$, F_ϵ^t is only a semigroup. It turns out that the equivariant smooth linearization can still be achieved, and the Conley-Moser conditions can still be realized. Of course, one has to replace the inverse of the evolution operator F_ϵ^t by preimage. The article is organized as follows: In section 2, we present equivariant smooth linearization. In section 3, we present the Poincaré map and its representation. In section 4, the fixed points of the Poincaré map is studied.

In section 5, we present the existence of chaos. Finally, in section 6, numerical evidence for the generic conditions is presented.

2. Equivariant Smooth Linearization

The symmetric pair of Silnikov homoclinic orbits is asymptotic to the saddle $Q_\epsilon = \sqrt{I}e^{i\theta}$, where

$$I = \omega^2 - \epsilon \frac{1}{2\omega}\sqrt{\beta^2 - \alpha^2\omega^2} + \cdots , \quad \cos\theta = \frac{\alpha\sqrt{I}}{\beta}, \quad \theta \in (0, \frac{\pi}{2}). \tag{2.1}$$

Its eigenvalues are

$$\lambda_n^\pm = -\epsilon[\alpha + n^2] \pm 2\sqrt{(\frac{n^2}{2} + \omega^2 - I)(3I - \omega^2 - \frac{n^2}{2})} \ , \tag{2.2}$$

where $n = 0, 1, 2, \cdots$, $\omega \in (\frac{1}{2}, 1)$, and I is given in (2.1). The crucial points to notice are: (1). only λ_0^+ and λ_1^+ have positive real parts, $\mathrm{Re}\{\lambda_0^+\} < \mathrm{Re}\{\lambda_1^+\}$; (2). all the other eigenvalues have negative real parts among which the absolute value of $\mathrm{Re}\{\lambda_2^+\} = \mathrm{Re}\{\lambda_2^-\}$ is the smallest; (3). $|\mathrm{Re}\{\lambda_2^+\}| < \mathrm{Re}\{\lambda_0^+\}$. Actually, items (2) and (3) are the main characteristics of Silnikov homoclinic orbits.

LEMMA 2.1. *For any fixed $\epsilon \in (0, \epsilon_0)$, let E_ϵ be the codimension 1 surface in the external parameter space, on which the symmetric pair of Silnikov homoclinic orbits are supported (cf: Theorem 1.1). For almost every $(\alpha, \beta, \omega) \in E_\epsilon$, the eigenvalues $\lambda_n^\pm$ (2.2) satisfy the nonresonance condition of Siegel type: There exists a natural number s such that for any integer $n \geq 2$,*

$$\left| \Lambda_n - \sum_{j=1}^{r} \Lambda_{l_j} \right| \geq 1/r^s \ ,$$

for all $r = 2, 3, \cdots, n$ and all $l_1, l_2, \cdots, l_r \in \mathbb{Z}$, where $\Lambda_n = \lambda_n^+$ for $n \geq 0$, and $\Lambda_n = \lambda_{-n-1}^-$ for $n < 0$.

Proof. The same proof as in [**1**] can be carried through here. □

Thus, in a neighborhood of Q_ϵ, the singularly perturbed NLS (1.1) is analytically equivalent to its linearization at Q_ϵ [**3**]. In terms of eigenvector basis, (1.1) can be rewritten as

$$\begin{aligned}
\dot{x} &= -ax - by + \mathcal{N}_x(\vec{Q}), \\
\dot{y} &= bx - ay + \mathcal{N}_y(\vec{Q}), \\
\dot{z}_1 &= \gamma_1 z_1 + \mathcal{N}_{z_1}(\vec{Q}), \\
\dot{z}_2 &= \gamma_2 z_2 + \mathcal{N}_{z_2}(\vec{Q}), \\
\dot{Q} &= LQ + \mathcal{N}_Q(\vec{Q});
\end{aligned} \tag{2.3}$$

where $a = -\mathrm{Re}\{\lambda_2^+\}$, $b = \mathrm{Im}\{\lambda_2^+\}$, $\gamma_1 = \lambda_0^+$, $\gamma_2 = \lambda_1^+$; $\mathcal{N}$'s vanish identically in a neighborhood Ω of $\vec{Q} = 0$, $\vec{Q} = (x, y, z_1, z_2, Q)$, Q is associated with the rest of eigenvalues, L is given as

$$LQ = -iQ_{\zeta\zeta} - 2i[(2|Q_\epsilon|^2 - \omega^2)Q + Q_\epsilon^2\bar{Q}] + \epsilon[-\alpha Q + Q_{\zeta\zeta}] \ ,$$

and Q_ϵ is given in (2.1). The following theorem on well-posedness is standard [**4**]. Let F^t $(0 \leq t < \infty)$ be the evolution operator of the singularly perturbed NLS (2.3), and H^s be the Sobolev space.

Theorem 2.2. *For any $s \geq 1$, and any $\vec{Q}_0 \in H^{s+2}$, $F^t(\vec{Q}_0) \in C^0([0,\infty); H^{s+2}) \cap C^1([0,\infty); H^s)$. For any fixed $t \in [0,\infty)$, F^t is a C^2 map in H^s.*

3. The Poincaré Map and Its Representation

Denote by h_k ($k = 1, 2$) the symmetric pair of Silnikov homoclinic orbits. The symmetry σ of half spatial period shifting has the new representation in terms of the new coordinates

$$\sigma \circ (x, y, z_1, z_2, Q) = (x, y, z_1, -z_2, \sigma \circ Q). \tag{3.1}$$

We have the following facts about the homoclinic orbits:

(1) The homoclinic orbits are classical solutions,
(2) As $t \to -\infty$, the homoclinic orbits are tangent to the positive z_1-axis at $\vec{Q} = 0$.

The same proof as in [**1**] works here for item (2). Since a is the smallest attracting rate, we assume that

- (A1). As $t \to +\infty$, the homoclinic orbits are tangent to the (x, y)-plane at $\vec{Q} = 0$.

The Poincaré section is defined as in [**1**].

Definition 3.1. The Poincaré section Σ_0 is defined by the constraints:

$$\begin{aligned} &y = 0, \ \eta \exp\{-2\pi a/b\} < x < \eta; \\ &0 < z_1 < \eta, \ -\eta < z_2 < \eta, \ \|Q\| < \eta; \end{aligned}$$

where η is a small parameter.

The horseshoes are going to be constructed on this Poincaré section. The auxiliary Poincaré section is defined differently from that in [**1**].

Definition 3.2. The Poincaré section Σ_1 is defined by the constraints:

$$\begin{aligned} &z_1 = \eta, \quad -\eta < z_2 < \eta, \\ &\sqrt{x^2 + y^2} < \eta, \quad \|Q\| < \eta. \end{aligned}$$

The Poincaré map is defined as follows.

Definition 3.3. The Poincare map P is defined as:

$$P \ : \ U \subset \Sigma_0 \mapsto \Sigma_0, \quad P = P_1^0 \circ P_0^1,$$

where

$$P_0^1 \ : \ U_0 \subset \Sigma_0 \mapsto \Sigma_1, \quad \forall \vec{Q} \in U_0, \quad P_0^1(\vec{Q}) = F^{t_0}(\vec{Q}) \in \Sigma_1,$$

and $t_0 = t_0(\vec{Q}) > 0$ is the smallest time t such that $F^t(\vec{Q}) \in \Sigma_1$, and

$$P_1^0 \ : \ U_1 \subset \Sigma_1 \mapsto \overline{\Sigma_0}(= \Sigma_0 \cup \partial\Sigma_0), \quad \forall \vec{Q} \in U_1, \quad P_1^0(\vec{Q}) = F^{t_1}(\vec{Q}) \in \overline{\Sigma_0},$$

and $t_1 = t_1(\vec{Q}) > 0$ is the smallest time t such that $F^t(\vec{Q}) \in \overline{\Sigma_0}$.

The map P_0^1 has the explicit representation: Let $\vec{Q}^0$ and $\vec{Q}^1$ be the coordinates on Σ_0 and Σ_1 respectively, $y^0 = 0$, and $z_1^1 = \eta$, then $t_0 = \frac{1}{\gamma_1} \ln \frac{\eta}{z_1^0}$, and

$$
\begin{aligned}
x^1 &= \left(\frac{z_1^0}{\eta} \right)^{\frac{a}{\gamma_1}} x^0 \cos \left[\frac{b}{\gamma_1} \ln \frac{\eta}{z_1^0} \right] , \\
y^1 &= \left(\frac{z_1^0}{\eta} \right)^{\frac{a}{\gamma_1}} x^0 \sin \left[\frac{b}{\gamma_1} \ln \frac{\eta}{z_1^0} \right] , \\
z_2^1 &= \left(\frac{\eta}{z_1^0} \right)^{\frac{\gamma_2}{\gamma_1}} z_2^0 , \\
Q^1 &= e^{t_0 L} Q^0 .
\end{aligned}
$$

Let $\vec{Q}_*^0$ and $\vec{Q}_*^1$ be the intersection points of the homoclinic orbit h_1 with $\overline{\Sigma_0}$ and Σ_1 respectively. The discussion with respect to the other homoclinic orbit h_2 is the same. In a small neighborhood of $\vec{Q}_*^1$, the map P_1^0 has an approximate representation. By virtue of the fact that the homoclinic orbit h_1 is a classical solution, the Well-Posedness Theorem 2.2 implies that $F^t(\vec{Q}_*^1)$ is C^1 in t. Thus for $\vec{Q}^1$ in a small neighborhood of $\vec{Q}_*^1$,

$$
P_1^0(\vec{Q}^1) = P_1^0(\vec{Q}_*^1) + \mathcal{L}(\vec{Q}^1 - \vec{Q}_*^1) + \mathcal{O}(\|\vec{Q}^1 - \vec{Q}_*^1\|^2), \tag{3.2}
$$

where

$$
\mathcal{L}(\vec{Q}^1 - \vec{Q}_*^1) = \partial_{\vec{Q}} F^{t_1}(\vec{Q}_*^1) \circ (\vec{Q}^1 - \vec{Q}_*^1) + \partial_t F^{t_1}(\vec{Q}_*^1) \circ \frac{\partial t_1}{\partial \vec{Q}^1}(\vec{Q}_*^1) \circ (\vec{Q}^1 - \vec{Q}_*^1) ,
$$

and $t_1 = t_1(\vec{Q}^1)$ is defined by the constraint that the y-coordinate of $F^{t_1}(\vec{Q}^1)$ vanishes,

$$
F_y^{t_1}(\vec{Q}^1) = 0 .
$$

Thus

$$
\partial_{\vec{Q}} F_y^{t_1}(\vec{Q}_*^1) + \partial_t F_y^{t_1}(\vec{Q}_*^1) \frac{\partial t_1}{\partial \vec{Q}^1}(\vec{Q}_*^1) = 0 ,
$$

i.e.

$$
\frac{\partial t_1}{\partial \vec{Q}^1}(\vec{Q}_*^1) = -\frac{1}{\partial_t F_y^{t_1}(\vec{Q}_*^1)} \partial_{\vec{Q}} F_y^{t_1}(\vec{Q}_*^1) . \tag{3.3}
$$

Let $\tilde{\vec{Q}}^0 = \vec{Q}^0 - \vec{Q}_*^0$, and $\tilde{\vec{Q}}^1 = \vec{Q}^1 - \vec{Q}_*^1$, then P_1^0 has the approximate representation

$$
\begin{pmatrix} \tilde{x}^0 \\ \tilde{z}_1^0 \\ \tilde{z}_2^0 \\ \tilde{Q}^0 \end{pmatrix} = \mathcal{C} \begin{pmatrix} \tilde{x}^1 \\ \tilde{y}^1 \\ \tilde{z}_2^1 \\ \tilde{Q}^1 \end{pmatrix} + \Xi, \tag{3.4}
$$

where

$$
\Xi \sim \mathcal{O}\Big((\tilde{x}^1)^2 + (\tilde{y}^1)^2 + (\tilde{z}_2^1)^2 + \|\tilde{Q}^1\|^2 \Big),
$$

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & C_{14} \\ c_{21} & c_{22} & c_{23} & C_{24} \\ c_{31} & c_{32} & c_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix},$$

in which c_{jl} $(j, l = 1, 2, 3)$ are real constants, C_{j4} $(j = 1, 2, 3, 4)$ and C_{4l} $(l = 1, 2, 3)$ are linear operators.

4. The Fixed Points of the Poincaré Map P

As $t_0 \to +\infty$, to the leading order, the fixed points of P satisfy

$$\begin{pmatrix} \hat{x}^0 \\ 0 \\ 0 \\ \hat{Q}^0 \end{pmatrix} = \mathcal{C} \begin{pmatrix} x_*^0 \cos bt_0 \\ x_*^0 \sin bt_0 \\ \hat{z}_2^1 \\ 0 \end{pmatrix}, \tag{4.1}$$

where

$$\hat{z}_2^1 = e^{at_0} \tilde{z}_2^1, \quad \hat{x}^0 = e^{at_0} \tilde{x}^0, \quad \hat{Q}^0 = e^{at_0} \tilde{Q}^0.$$

Explicitly, the second and the third equations in (4.1) are:

$$\begin{aligned} x_*^0 \Big[c_{21} \cos bt_0 + c_{22} \sin bt_0 \Big] + c_{23} \hat{z}_2^1 = 0, \\ x_*^0 \Big[c_{31} \cos bt_0 + c_{32} \sin bt_0 \Big] + c_{33} \hat{z}_2^1 = 0. \end{aligned} \tag{4.2}$$

Lemma 4.1. *c_{23} and c_{33} do not vanish simultaneously.*

Proof. Notice that $W^u(Q_\epsilon)$ is two-dimensional, and intersects Σ_0 (or its extension to $-\eta < z_1 < \eta$) into a one-dimensional curve with tangent vector

$$v = \mathcal{C} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Notice also that for any $\vec{Q} \in h_1$,

$$\dim\{\mathcal{T}_{\vec{Q}} W^u(Q_\epsilon) \cap \mathcal{T}_{\vec{Q}} W^s(Q_\epsilon)\} = 1, \tag{4.3}$$

where $\mathcal{T}_{\vec{Q}}$ denotes the tangent space at $\vec{Q}$. If c_{23} and c_{33} vanish simultaneously, then $v \in \mathcal{T}_{\vec{Q}_*^0} W^s(Q_\epsilon)$ which implies that

$$\dim\{\mathcal{T}_{\vec{Q}_*^0} W^u(Q_\epsilon) \cap \mathcal{T}_{\vec{Q}_*^0} W^s(Q_\epsilon)\} = 2$$

which contradicts (4.3). The lemma is proved. □

Let

$$\Delta_1 = c_{21}c_{33} - c_{31}c_{23}, \quad \Delta_2 = c_{22}c_{33} - c_{32}c_{23}.$$

We assume that

- (A2). Δ_1 and Δ_2 do not vanish simultaneously.

Then (4.2) has infinitely many solutions:

$$t_0^{(l)} = \frac{1}{b}[l\pi - \varphi_1], \quad l \in \mathbb{Z}; \tag{4.4}$$

where

$$\varphi_1 = \arctan\{\Delta_1/\Delta_2\}.$$

Without loss of generality, we assume $c_{23} \neq 0$. Then, solving Eqs.(4.2), we have

$$\hat{z}_2^{(1,l)} = -x_*^0[c_{23}]^{-1}\{c_{21}\cos bt_0^{(l)} + c_{22}\sin bt_0^{(l)}\}. \tag{4.5}$$

Solving (4.1), we have

$$\hat{x}^{(0,l)} = x_*^0\left[c_{11}\cos bt_0^{(l)} + c_{12}\sin bt_0^{(l)}\right] + c_{13}\hat{z}_2^{(1,l)}, \tag{4.6}$$

$$\hat{Q}^{(0,l)} = x_*^0\left[C_{41}\cos bt_0^{(l)} + C_{42}\sin bt_0^{(l)}\right] + C_{43}\hat{z}_2^{(1,l)}. \tag{4.7}$$

Finally, by the implicit function theorem, there exist infinitely many fixed points of P, which have the approximate expressions given above [**1**]. Specifically, we have

Theorem 4.2. *The Poincaré map P has infinitely many fixed points labeled by l ($l \geq l_0$):*

$$t_0 = t_{0,l},\ \hat{x}^0 = \hat{x}_l^0,\ \hat{Q}^0 = \hat{Q}_l^0,\ \hat{z}_2^1 = \hat{z}_{2,l}^1,$$

where as $l \to +\infty$,

$$\begin{aligned} t_{0,l} &= \frac{1}{b}[l\pi - \varphi_1] + o(1), \\ \hat{x}_l^0 &= \hat{x}^{(0,l)} + o(1), \\ \hat{Q}_l^0 &= \hat{Q}^{(0,l)} + o(1), \\ \hat{z}_{2,l}^1 &= \hat{z}_2^{(1,l)} + o(1), \end{aligned}$$

in which $\hat{x}^{(0,l)}$, $\hat{Q}^{(0,l)}$ and $\hat{z}_2^{(1,l)}$ are given in (4.6),(4.7),(4.5).

5. Existence of Chaos

One can construct Smale horseshoes in the neighborhoods of the fixed points of P.

Definition 5.1. For sufficiently large natural number l, we define slab S_l in Σ_0 as follows:

$$\begin{aligned} S_l \equiv \Bigg\{\vec{Q} \in \Sigma_0 \ \Bigg|\ & \eta\exp\{-\gamma_1(t_{0,2(l+1)} - \frac{\pi}{2b})\} \leq \\ & \tilde{z}_1^0(\vec{Q}) \leq \eta\exp\{-\gamma_1(t_{0,2l} - \frac{\pi}{2b})\}, \\ & |\tilde{x}^0(\vec{Q})| \leq \eta\exp\{-\frac{1}{2}a\ t_{0,2l}\}, \\ & |\tilde{z}_2^1(P_0^1(\vec{Q}))| \leq \eta\exp\{-\frac{1}{2}a\ t_{0,2l}\}, \\ & \|\tilde{Q}^1(P_0^1(\vec{Q}))\| \leq \eta\exp\{-\frac{1}{2}a\ t_{0,2l}\}\Bigg\}, \end{aligned}$$

where the notations $\tilde{x}^0(\vec{Q})$, $\tilde{z}_2^1(P_0^1(\vec{Q}))$, etc. denote the $\tilde{x}^0$ coordinate of the point $\vec{Q}$, the $\tilde{z}_2^1$ coordinate of the point $P_0^1(\vec{Q})$, etc..

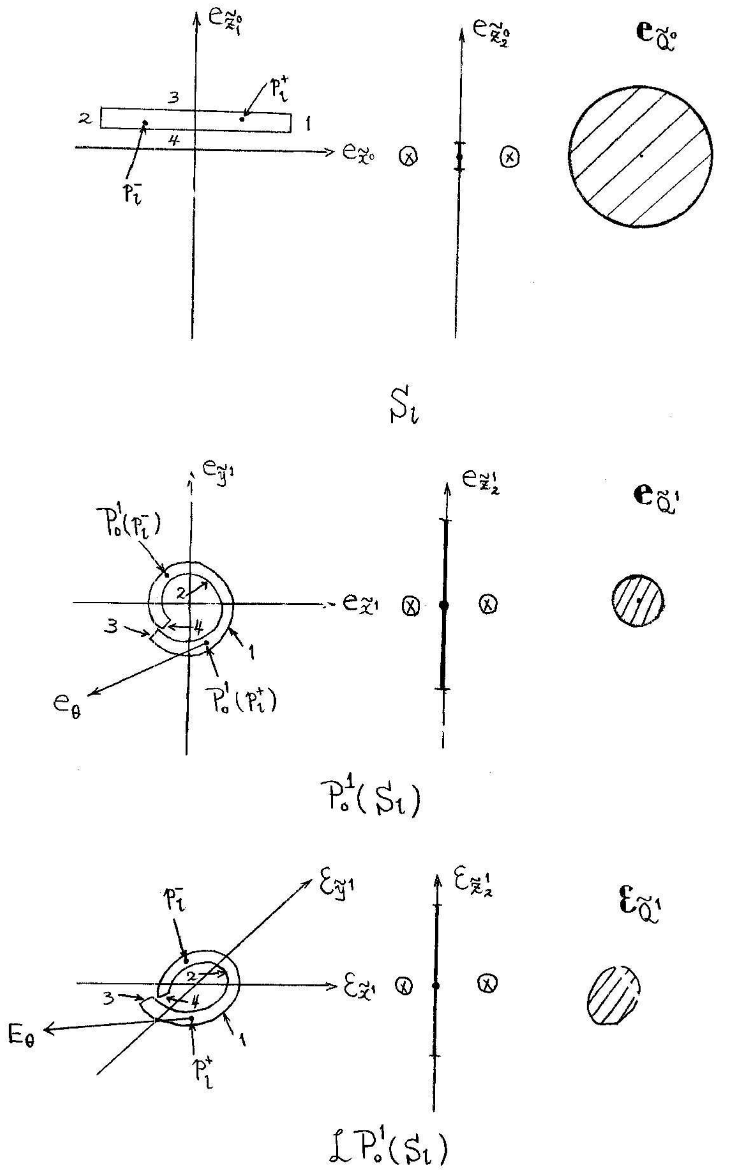

FIGURE 2. An illustration of S_l, $P_0^1(S_l)$, and $\mathcal{L}P_0^1(S_l)$.

S_l is defined so that it includes two fixed points p_l^+ and p_l^- of P (Theorem 4.2). S_l, $P_0^1(S_l)$, and $\mathcal{L}P_0^1(S_l)$ are illustrated in Figure 2, where $\mathcal{L}$ is defined in (3.2). $\{e_{\tilde{x}^0}, e_{\tilde{z}_1^0}, e_{\tilde{z}_2^0}, \mathbf{e}_{\tilde{Q}^0}\}$ denotes the unit vectors along $(\tilde{x}^0, \tilde{z}_1^0, \tilde{z}_2^0, \tilde{Q}^0)$-directions in Σ_0, $\{e_{\tilde{x}^1}, e_{\tilde{y}^1}, e_{\tilde{z}_2^1}, \mathbf{e}_{\tilde{Q}^1}\}$ denotes the unit vectors along $(\tilde{x}^1, \tilde{y}^1, \tilde{z}_2^1, \tilde{Q}^1)$-directions in Σ_1, and under the linear map $\mathcal{L}$, $\{e_{\tilde{x}^1}, e_{\tilde{y}^1}, e_{\tilde{z}_2^1}, \mathbf{e}_{\tilde{Q}^1}\}$ are mapped into $\{\mathcal{E}_{\tilde{x}^1}, \mathcal{E}_{\tilde{y}^1}, \mathcal{E}_{\tilde{z}_2^1}, \mathcal{E}_{\tilde{Q}^1}\}$.

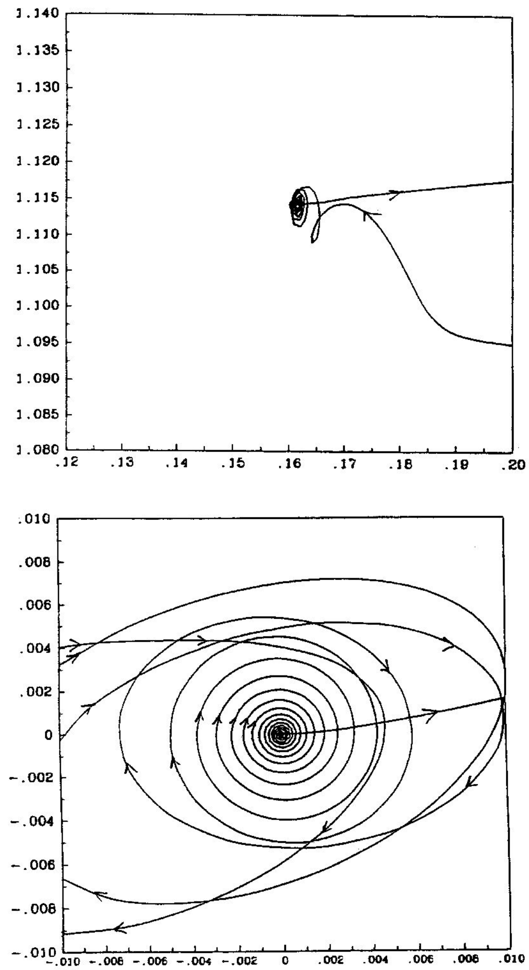

FIGURE 3. (a) shows one of the homoclinic orbits, and (b) shows the blow-up of the neighborhood of the saddle Q_ϵ.

Let e_θ be the unit angular vector at $P_0^1(p_l^+)$ of the polar coordinate frame on the $(\tilde{x}^1, \tilde{y}^1)$-plane. Let $E_\theta = \mathcal{L} e_\theta$, and we assume that

- (A3). $\text{Span}\left\{ e_{\tilde{x}^0}, \mathbf{e}_{\tilde{Q}^0}, E_\theta, \mathcal{E}_{\tilde{z}_2^1} \right\} = \Sigma_0$.

Let $S_{l,\sigma} = \sigma \circ S_l$ where the symmetry σ is defined in (3.1). We need to define a larger slab $\hat{S}_l$ such that $S_l \cup S_{l,\sigma} \subset \hat{S}_l$.

Definition 5.2. The larger slab $\hat{S}_l$ is defined as

$$\begin{aligned}\hat{S}_l \quad = \quad & \left\{ \vec{Q} \in \Sigma_0 \;\middle|\; \eta \exp\{-\gamma_1(t_{0,2(l+1)} - \frac{\pi}{2b})\} \leq \right. \\ & z_1^0(\vec{Q}) \leq \eta \exp\{-\gamma_1(t_{0,2l} - \frac{\pi}{2b})\}, \\ & |x^0(\vec{Q}) - x_*^0| \leq \eta \exp\{-\frac{1}{2}a\ t_{0,2l}\}, \\ & |z_2^1(P_0^1(\vec{Q}))| \leq |z_{2,*}^1| + \eta \exp\{-\frac{1}{2}a\ t_{0,2l}\}, \\ & \left. \|Q^1(P_0^1(\vec{Q}))\| \leq \eta \exp\{-\frac{1}{2}a\ t_{0,2l}\} \right\},\end{aligned}$$

where $z_{2,*}^1$ is the z_2^1-coordinate of $\vec{Q}_*^1$.

Definition 5.3. In the coordinate system $\{\tilde{x}^0, \tilde{z}_1^0, \tilde{z}_2^0, \tilde{Q}^0\}$, the stable boundary of $\hat{S}_l$, denoted by $\partial_s \hat{S}_l$, is defined to be the boundary of $\hat{S}_l$ along $(\tilde{x}^0, \tilde{Q}^0)$-directions, and the unstable boundary of $\hat{S}_l$, denoted by $\partial_u \hat{S}_l$, is defined to be the boundary of $\hat{S}_l$ along $(\tilde{z}_1^0, \tilde{z}_2^0)$-directions. A stable slice V in $\hat{S}_l$ is a subset of $\hat{S}_l$, defined as the region swept out through homeomorphically moving and deforming $\partial_s \hat{S}_l$ in such a way that the part

$$\partial_s \hat{S}_l \cap \partial_u \hat{S}_l$$

of $\partial_s \hat{S}_l$ only moves and deforms inside $\partial_u \hat{S}_l$. The new boundary obtained through such moving and deforming of $\partial_s \hat{S}_l$ is called the stable boundary of V, which is denoted by $\partial_s V$. The rest of the boundary of V is called its unstable boundary, which is denoted by $\partial_u V$. An unstable slice of $\hat{S}_l$, denoted by H, is defined similarly.

As shown in [**1**], under the assumption (A3), when l is sufficiently large, $P(S_l)$ and $P(S_{l,\sigma})$ intersect $\hat{S}_l$ into four disjoint stable slices $\{V_1, V_2\}$ and $\{V_{-1}, V_{-2}\}$ in $\hat{S}_l$. V_j's $(j = 1, 2, -1, -2)$ do not intersect $\partial_s \hat{S}_l$; moreover,

$$\partial_s V_i \subset P(\partial_s S_l), (i = 1, 2); \ \partial_s V_i \subset P(\partial_s S_{l,\sigma}), (i = -1, -2). \tag{5.1}$$

Let

$$H_j = P^{-1}(V_j), \quad (j = 1, 2, -1, -2), \tag{5.2}$$

where and for the rest of this article, P^{-1} denotes preimage of P. Then H_j $(j = 1, 2, -1, -2)$ are unstable slices. More importantly, the Conley-Moser conditions are satisfied as shown in [**1**]. Specifically, Conley-Moser conditions are:

Conley-Moser condition (i):

$$\begin{cases} V_j = P(H_j), \\ \partial_s V_j = P(\partial_s H_j), \quad (j = 1, 2, -1, -2) \\ \partial_u V_j = P(\partial_u H_j). \end{cases}$$

Conley-Moser condition (ii): There exists a constant $0 < \nu < 1$, such that for any stable slice $V \subset V_j$ $(j = 1, 2, -1, -2)$, the diameter decay relation

$$d(\tilde{V}) \leq \nu d(V)$$

holds, where $d(\cdot)$ denotes the diameter [1], and $\tilde{V} = P(V \cap H_k)$, $\ (k = 1, 2, -1, -2)$; for any unstable slice $H \subset H_j$ $\ (j = 1, 2, -1, -2)$, the diameter decay relation

$$d(\tilde{H}) \leq \nu d(H)$$

holds, where $\tilde{H} = P^{-1}(H \cap V_k)$, $\ (k = 1, 2, -1, -2)$.

The Conley-Moser conditions are sufficient conditions for establishing the topological conjugacy between the Poincare map P restricted to a Cantor set in Σ_0, and the shift automorphism on symbols.

Let $\mathcal{W}$ be a set which consists of elements of the doubly infinite sequence form:

$$a = (\cdots a_{-2}a_{-1}a_0, a_1a_2\cdots),$$

where $a_k \in \{1, 2, -1, -2\}$; $k \in Z$. We introduce a topology in $\mathcal{W}$ by taking as neighborhood basis of

$$a^* = (\cdots a^*_{-2}a^*_{-1}a^*_0, a^*_1a^*_2\cdots),$$

the set

$$W_j = \left\{ a \in \mathcal{W} \;\middle|\; a_k = a^*_k \ (|k| < j) \right\}$$

for $j = 1, 2, \cdots$. This makes $\mathcal{W}$ a topological space. The shift automorphism χ is defined on $\mathcal{W}$ by

$$\begin{aligned} \chi \quad : \quad & \mathcal{W} \mapsto \mathcal{W}, \\ & \forall a \in \mathcal{W}, \ \chi(a) = b, \ \text{where } b_k = a_{k+1}. \end{aligned}$$

The shift automorphism χ exhibits *sensitive dependence on initial conditions*, which is a hallmark of *chaos*.

Let

$$a = (\cdots a_{-2}a_{-1}a_0, a_1a_2\cdots),$$

be any element of $\mathcal{W}$. Define inductively for $k \geq 2$ the stable slices

$$\begin{aligned} V_{a_0a_{-1}} &= P(H_{a_{-1}}) \cap H_{a_0}, \\ V_{a_0a_{-1}\ldots a_{-k}} &= P(V_{a_{-1}\ldots a_{-k}}) \cap H_{a_0}. \end{aligned}$$

By Conley-Moser condition (ii),

$$d(V_{a_0a_{-1}\ldots a_{-k}}) \leq \nu_1 d(V_{a_0a_{-1}\ldots a_{-(k-1)}}) \leq \ldots \leq \nu_1^{k-1} d(V_{a_0a_{-1}}).$$

Then,

$$V(a) = \bigcap_{k=1}^{\infty} V_{a_0a_{-1}\ldots a_{-k}}$$

defines a 2 dimensional continuous surface in Σ_0; moreover,

$$\partial V(a) \subset \partial_u \hat{S}_l. \tag{5.3}$$

Similarly, define inductively for $k \geq 1$ the unstable slices

$$\begin{aligned} H_{a_0a_1} &= P^{-1}(H_{a_1} \cap V_{a_0}), \\ H_{a_0a_1\ldots a_k} &= P^{-1}(H_{a_1\ldots a_k} \cap V_{a_0}). \end{aligned}$$

By Conley-Moser condition (ii),

$$d(H_{a_0a_1\ldots a_k}) \leq \nu_2 d(H_{a_0a_1\ldots a_{k-1}}) \leq \ldots \leq \nu_2^k d(H_{a_0}).$$

Then,

$$H(a) = \bigcap_{k=0}^{\infty} H_{a_0a_1\ldots a_k}$$

defines a codimension 2 continuous surface in Σ_0; moreover,

$$\partial H(a) \subset \partial_s \hat{S}_l. \tag{5.4}$$

By (5.3;5.4) and dimension count,

$$V(a) \cap H(a) \neq \emptyset$$

consists of points. Let

$$p \in V(a) \cap H(a)$$

be any point in the intersection set. Now we define the mapping

$$\begin{aligned} \phi \quad : \quad & \mathcal{W} \mapsto \hat{S}_l, \\ & \phi(a) = p. \end{aligned}$$

By the above construction,

$$P(p) = \phi(\chi(a)).$$

That is,

$$P \circ \phi = \phi \circ \chi.$$

Let

$$\Lambda \equiv \phi(\mathcal{W}),$$

then Λ is a compact Cantor subset of $\hat{S}_l$, and invariant under the Poincare map P. Moreover, with the topology inherited from $\hat{S}_l$ for Λ, ϕ is a homeomorphism from $\mathcal{W}$ to Λ. Thus we have the theorem.

THEOREM 5.4 (Horseshoe Theorem). *Under the generic assumptions (A1)-(A3) for the perturbed nonlinear Schrödinger system (1.1), there exists a compact Cantor subset* Λ *of* $\hat{S}_l$*,* Λ *consists of points, and is invariant under* P*.* P *restricted to* Λ*, is topologically conjugate to the shift automorphism* χ *on four symbols* $1, 2, -1, -2$*. That is, there exists a homeomorphism*

$$\phi \ : \ \mathcal{W} \mapsto \Lambda,$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\phi} & \Lambda \\ \chi \downarrow & & \downarrow P \\ \mathcal{W} & \xrightarrow[\phi]{} & \Lambda \end{array}$$

6. Numerical Evidence for the Generic Assumptions

6.1. Generic Assumption (A1). Figure 3 shows a numerical result of Mark Winograd. It indicates that the homoclinic orbits are indeed tangent to the (x, y)-plane as $t \to +\infty$. Specifically, Figure 3 (a) shows one of the homoclinic orbits, and Figure 3 (b) shows the blow-up of the neighborhood of the saddle Q_ϵ.

6.2. Generic Assumptions (A2) and (A3). Numerical simulation of the generic assumptions (A2) and (A3) is planned for a future work.

References

[1] Y. Li, Smale horseshoes and symbolic dynamics in perturbed nonlinear Schrödinger equations, *Journal of Nonlinear Sciences* **9, no.4** (1999), 363.

[2] Y. Li, Homoclinic orbits for singularly perturbed NLS, *Dynamics of PDE* **1, no.1** (2004), 1.

[3] N. V. Nikolenko, The method of Poincaré normal forms in problems of integrability of equations of evolution type, *Russian Math. Surveys* **41, no.5** (1986), 63.

[4] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, *Lecture Notes in Math., Springer* **448** (1975), 25.

Department of Mathematics, University of Missouri, Columbia, MO 65211
E-mail address: cli@math.missouri.edu