Closed Dressing Chains of 1D and 2D Toda Lattice

Artyom V. Yurov

ABSTRACT. We apply the method of dressing chains to reproduction of Toda lattice in the case of 1D and 2D. On the example of modified equations m_0TL and m_1TL it is shown that combination of the Darboux and Schlesinger transformations results in closed dressing chains.

CONTENTS

1. Introduction

In this work we consider the method of dressing chains of discrete symmetries for the Toda lattice (TL) (including the two-dimensional generalization of them). One starts with nonlinear Schrodinger equation (NLS)

$$iu_t + u_{xx} + 2u^2v = -iv_t + v_{xx} + 2v^2u = 0, \tag{1}$$

which admits two different types of discrete symmetries: Darboux transformation (DT) and Schlesinger transformation (ST). The ST leads to connection between NLS and TL models [1]. Using this circumstance and Lax pair for the NLS model one can **find** Lax pair for the TL one. Then, using DT for the NLS and Lax pair for the TL one can obtain the DT for the TL. Therefore total theory of TL (Lax pair and DT) can be constructed starting out from the theory of NLS. Using DT for the TL one can construct TL dressing chains to find modified TL equations which

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will be denoted as m^1TL (with superscript). To do that it is enough to know L-operator of the NLS. By repeating this procedure with A-operator of the equation (1) one find **symmetries** of the m^1TL (the exact definition of these symmetries is contained in [2]). This plan is realized in the Section 2. We'll show that it is appears the **closed** dressing chains which tie together TL and Volterra equations, in contrast to open-ended dressing chains of the KdV and NLS equations. This is the character of the dressing chains of the TL equations.

The alternative path to "multiply" the Toda lattice (see Sec. 3) is as follows: One construct discrete symmetry chain for the NLS to find modified NLS equations (we denote them by the m_kNLS). Each of them inherits a ST from an initial NLS (1) which leads to corresponding Toda-like lattice equations - m_kTL (with subscript). Since Lax pair and Darboux transformations for the m_kNLS are known (this is crucial point of the dressing chains method) then one can find Lax pairs and DT for the m_kTL. Now, if we apply the method from the previous paragraph then one will find a new family of Toda-like equations (we denote them by the as m_k^nTL. The subscript points to the number of the modified NLS which creates this m_k^nTL equation. For example, all Toda-like equations from the Sec. 2 belong to class m_0TL, while all ones from the Sec. 3 belong to class m_1TL).

In Sec. 4 we generalize our approach to $2D$ TL models using the Davy-Stewartson equations (DS). Dressing chains of the first and second types are constructed.

There are many works which are devoted to links between integrable NLS-like models and difference-differential Toda-like lattices (see, for example, [3]). The classification of integrable lattice was made in [2], [4]. Our results are associated with results in [5] and we are discussing that in Sec. 5. This work is continuation of our research in [6], [7] which, for one's turn, was inspired by the article [8].

2. Equations m_0TL

Lax pair for the (1) has the form

$$\Psi_x = -i\sigma_3\Psi\Lambda + iS\Psi, \qquad \Psi_t = -2i\sigma_3\Psi\Lambda^2 + 2iS\Psi\Lambda + W\Psi, \qquad (2)$$

where

$$S = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad W = \sigma_3\left(iS^2 - S_x\right),$$

λ and μ are spectral parameters, σ_3 is the third Pauli matrix, $\Psi - 2 \times 2$ matrix function. We denote componentry of the first matrix Ψ column by ψ_n, ϕ_n, $u \equiv u_n$, $v \equiv v_n$. Then, the L-equation of the system (2) has the form

$$\psi_{n,x} = -i\lambda\psi_n + iu_n\phi_n, \qquad \phi_{n,x} = i\lambda\phi_n + iv_n\psi_n. \qquad (3a-b)$$

The components of the second Ψ column are satisfying the similar system with substitution $\lambda \to \mu$.

The ST for the (1) are

$$u_n \to u_{n+1} = u_n\left[u_nv_v + (\log u_n)_{xx}\right], \qquad v_n \to v_{n+1} = \frac{1}{u_n}, \qquad (4a-b)$$

$$\psi_n \to \psi_{n+1} = \left(-2\lambda + i(\log u_n)_x\right)\psi_n + u_n\phi_n, \qquad \phi_n \to \phi_{n+1} = \frac{\psi_n}{u_n}, \qquad (5a-b)$$

and reverse conversion:

$$v_n \to v_{n-1} = v_n \left[u_n v_v + (\log v_n)_{xx} \right], \qquad u_n \to u_{n-1} = \frac{1}{v_n},$$

$$\phi_n \to \phi_{n-1} = (2\lambda + i(\log v_n)_x) \phi_n + v_n \psi_n, \qquad \psi_n \to \psi_{n-1} = \frac{\phi_n}{v_n}. \tag{6}$$

Let us denote

$$q_n \equiv \log(u_n), \qquad p_n \equiv q_{n,x}, \qquad U_n \equiv \frac{u_n}{u_{n-1}} = e^{q_n - q_{n-1}}. \tag{7}$$

It is easy to examine that functions q_n are solutions of the TL equations

$$q_{n,xx} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}. \tag{8}$$

This remarkable point was noted in many works (see, for example, [1]-[3]). Our first aim here is to construct Lax pair for the (8) starting out from the equations (3). It is comfortably to introduce the shift-operator T:

$$T u_n = u_{n+1}.$$

We act T^{-1} on the equation (5b) then substitute the expression for the ϕ_n into (3a) to obtain the first equation of the Lax pair:

$$\psi_{n,x} = -i\lambda \psi_n + i U_n \psi_{n-1}. \tag{9}$$

To obtain the second equation of the Lax pair we act T onto the (3b) and substitute into this expression ϕ_{n+1} (from the (5b)), v_{n+1} (from the (4b)) and $\psi_{n,x}$ from the (9). As a result one get

$$\psi_{n+1} = (2\lambda + i p_n)\psi_n + U_n \psi_{n-1}. \tag{10}$$

The consistency constraints of (9) and (10) has the form of two equations

$$p_{n,x} = U_{n+1} - U_n, \qquad U_{n,x} = U_n \left(p_n - p_{n-1} \right),$$

which can be transformed into the (8) by the (7). Thus, (9) and (10) are nothing but Lax pair for the TL model.

Let us construct DT for the TL equation using **elementary** DT for the NLS model (1) (see [9]). Let ψ_1 and ϕ_1 be the components of the first matrix Ψ column, where Ψ is the solution of the (2) with $\lambda = \lambda_1$. Then one can write two sorts of the DT (the indices are omitted):

$$\psi \to \psi^{(1)} = \left[2(\lambda - \lambda_1) + \frac{\phi_1}{\psi_1} u \right] \psi - u\phi, \qquad \phi \to \phi^{(1)} = \phi - \frac{\phi_1}{\psi_1} \psi,$$

$$u \to u^{(1)} = i u_x - \left(2\lambda_1 - \frac{\phi_1}{\psi_1} u \right) u, \qquad v \to v^{(1)} = \frac{\phi_1}{\psi_1}, \tag{11}$$

and

$$\psi \to {}^{(1)}\psi = \psi - \frac{\psi_1}{\phi_1}\phi, \qquad \phi \to {}^{(1)}\phi = \left[2(\lambda_1 - \lambda) + \frac{\psi_1}{\phi_1} v \right] \phi - v\psi,$$

$$u \to {}^{(1)}u = \frac{\psi_1}{\phi_1}, \qquad v \to {}^{(1)}v = i v_x + \left(2\lambda_1 + \frac{\psi_1}{\phi_1} v \right) v. \tag{12}$$

Let's ψ_2, ϕ_2 be components of the second matrix Ψ column with $\mu = \lambda_2$. Then one make transformations (11) and (12):

$$u \to u^{(1)} \to {}^{(2)}u^{(1)}, \qquad v \to v^{(1)} \to {}^{(2)}v^{(1)},$$

$$\psi \to \psi^{(1)} \to {}^{(2)}\psi^{(1)}, \qquad \phi \to \phi^{(1)} \to {}^{(2)}\phi^{(1)}. \tag{13}$$

Note that the DT transformations are commuting

$$^{(2)}u^{(1)} = {}^{(1)}u^{(2)}, \qquad {}^{(2)}v^{(1)} = {}^{(1)}v^{(2)}.$$

If we'll write (13) explicitly for the twice dressed potentials and wave functions it is easy to see that we would obtain the "ordinary" DT for the NLS (see [10]). This is why we can call (11) and (12) as elemantary DTs. Unfortunately these transformations are uncomfortable in sense of getting exact solutions of the NLS because, in the case of general position, it is impossible to conserve reduction restriction $u = \pm v^*$, in contrast to (13).

Now it is easy to find DT for the TL equations. Omitting simple calculations we'll show the result. Let $\{\psi_{1,n}\}$ be the solution of the Lax pair (9), (10) with $\lambda = \lambda_1$. Then we have two DTs for the TL equations which will be referred to as R-transformation and L-transformation:

$$R: \qquad \psi_n \to \psi_n^{(1)} = -\psi_{n+1} + \frac{\psi_{1,n+1}}{\psi_{1,n}}\psi_n, \qquad q_n \to q_n^{(1)} = q_n + \log\frac{\psi_{1,n+1}}{\psi_{1,n}} \tag{14}$$

and

$$L: \qquad \psi_n \to {}^{(1)}\psi_n = \psi_n - \frac{\psi_{1,n}}{\psi_{1,n-1}}\psi_{n-1}, \qquad q_n \to {}^{(1)}q_n = q_{n-1} + \log\frac{\psi_{1,n}}{\psi_{1,n-1}}. \tag{15}$$

Using (9)-(10) it is quit easy to find first modified TL equations. For this purpose one introduce new functions τ_n and ξ_n

$$\tau_n = \frac{\psi_{1,n}}{\psi_{1,n+1}} = \frac{1}{\partial\xi_n},$$

where $\partial \equiv -i\partial/\partial x$, $V_n \equiv ip_n$. The shape of the function τ_n is dictated from the (14) and (15). In new dependent variables, the Lax pair for the TL equations has the form:

$$\partial\tau_n = \tau_n\left(\tau_{n-1}U_n - \tau_n U_{n+1}\right), \qquad \tau_{n+1} = \frac{1}{2\lambda_1 + V_{n+1} + \tau_n U_{n+1}}. \tag{16}$$

Note that the first equations has quadratic nonlinearity by fields τ_n, as it should be in the method of the dressing chains ([8]).

Eliminating potentials U_n, U_{n+1} and V_{n+1} from the (16) one get equation $m^1 TL$, which is just a Volterra equation (one can write this in more customary form; see (17a), (52)):

$$\partial^2\xi_n = \partial\xi_n\left(e^{\xi_n - \xi_{n+1}} - e^{\xi_{n-1} - \xi_n}\right). \tag{17}$$

To obtain dressing chains of discrete symmetries one can use either R or L transformations (14), (15). Let us choose the (14). At the same time, the transformations regulation for the U_n and V_n has the form:

$$U_n \to U_n^{(1)} = \frac{\tau_{n-1}}{\tau_n}U_n, \qquad V_n \to V_n^{(1)} = V_n + \tau_{n-1}U_n - \tau_n U_{n+1}.$$

Let's take $\psi_{2,n}$ as the solution of (9), (10) with $\lambda = \lambda_2$, $\psi_{2,n}^{(1)}$ is calculated from the (14) whereas ζ_n is defined by the expression:

$$\partial\zeta_n = \frac{\psi_{2,n+1}^{(1)}}{\psi_{2,n}^{(1)}}.$$

After simple calculations one get these chains:

$$e^{\zeta_n - \zeta_{n+1}} \partial \zeta_n = e^{\xi_n - \xi_{n+1}} \partial \xi_{n+1},$$

$$\partial (\zeta_n - \xi_n) = 2(\lambda_2 - \lambda_1) + e^{\zeta_{n-1} - \zeta_n} - e^{\xi_n - \xi_{n+1}}. \tag{18}$$

(18) contain Lax pair for the (17). Denoting $\zeta_n = \log \Psi_n$, we get

$$\partial \Psi_n = A_n \Psi_{n+1} = \left(B_{n-1} + \tilde{\lambda}\right) \Psi_n + \Psi_{n-1}, \tag{19}$$

where

$$A_n = e^{\xi_n - \xi_{n+1}} \partial \xi_{n+1}, \qquad B_n = \partial \xi_{n+1} - e^{\xi_{n+1} - \xi_{n+2}}, \tag{20}$$

and new spectral parameter $\tilde{\lambda}$ is connected with the λ by the proportion: $\tilde{\lambda} = 2(\lambda - \lambda_1)$. We note that the consistency condition of (19) has the form of usual Toda lattice

$$\partial A_n = A_n(B_{n-1} - B_n), \qquad \partial B_n = A_{n+1} - A_n \tag{21}$$

and can be reduced to the (17) by the substitution (20) into the (21). In new dependent variable the R-transformation (14) is nothing else but L-transformation (15). In other words, one obtain the closed chains of discrete symmetries which connect TL equations with Volterra equations ($m^1 TL$).

To complete the picture we'll cite the corresponding formulas. Let $\Psi_{1,n}$ be the solution of (19) with $\tilde{\lambda} = \tilde{\lambda}_1 \neq 0$. R-transformation (14) induce L-transformation (15) for the (19) and (21) (one can examine it by the direct calculation):

$$\Psi_n \to {}^{(1)}\Psi_n = \Psi_n - \sigma_n \Psi_{n+1}, \qquad A_n \to {}^{(1)}A_n = A_n - \partial \sigma_n,$$

$$B_n \to {}^{(1)}B_n = B_{n+1} + \partial \log \sigma_{n+1},$$

where $\sigma_n = \Psi_{1,n}/\Psi_{1,n+1}$ are solutions of system (compareto (16)):

$$\partial \sigma_n = A_n - (B_n + \tilde{\lambda}_1)\sigma_n - \sigma_n^2, \qquad \sigma_{n+1} = \frac{A_{n+1}}{B_n + \sigma_n + \tilde{\lambda}_1}. \tag{22}$$

Using (21) and (22) we eliminate potentials B_n and again obtain Volterra equations:

$$\partial \log \sigma_n = \beta_n - \beta_{n+1}, \qquad \partial \log \beta_n = \sigma_n - \sigma_{n-1}, \tag{17a}$$

where $\beta_n = A_n/\sigma_n$.

Analogously, one can consider t-equations. Rewriting (1) in new variables q_n, p_n, and taking (8) into account we get the known symmetry of the equations (8):

$$-iq_{n,t} = p_n^2 + e^{q_{n+1} - q_n} + e^{q_n - q_{n-1}}. \tag{23}$$

It is possible to repeat all actions described above: to find Lax pair, then to apply Darboux transformation to construct symmetry of the Volterra equations. Resulting formulas has been described before (with the help of another approach; see [2], [4]) so we omit them here. Note that [2] contains the total (up to gauge, linear, Galilean transformations) list of integrable generalization of classical and relativist Toda lattice in the form

$$q_{n,xx} = \frac{1}{2} \left(G(q_{n+1}, q_n, p_{n+1}, p_n) - F(q_n, q_{n-1}, p_n, p_{n-1})\right),$$

with symmetries

$$q_{n,t} = \frac{1}{2} \left(G(q_{n+1}, q_n, p_{n+1}, p_n) + F(q_n, q_{n-1}, p_n, p_{n-1})\right).$$

Using the method of dressing chains we have found only two equations from this list. In the next Section we'll obtain new generalization of TL equations.

3. Equations $m_1 TL$

Using DT (13) and Lax pair for the NLS (1) we can construct the first modified NLS ($m_1 NLS$) which has the form

$$\left(\beta^2 - 4bc\right)\left(ib_t + b_{xx} - 2b^2 c\right) + 2\alpha\left(\alpha c + 2ic_x\right)b^2 + 2(b_x c + 2bc_x)b_x = 0,$$

$$\left(\beta^2 - 4bc\right)\left(-ic_t + c_{xx} - 2c^2 b\right) + 2\alpha\left(\alpha b - 2ib_x\right)c^2 + 2(bc_x + 2b_x c)c_x = 0, \tag{24}$$

where $\alpha = \lambda + \mu$, $\beta = \lambda - \mu$, λ and μ are spectral parameters of (2) as before, and $b = b(x,t)$ with $c = c(x,t)$ are new dependent variables which are defined by

$$b = \frac{u - {}^{(2)}u^{(1)}}{2}, \qquad c = \frac{{}^{(2)}v^{(1)} - v}{2}.$$

If $\alpha = \beta = 0$, then the equation (24) take the elegant form

$$ib_t + b_{xx} - 2b^2 c - \frac{b_x c_x}{c} - \frac{1}{2}\frac{b_x^2}{b} = 0, \qquad -ic_t + c_{xx} - 2bc^2 - \frac{b_x c_x}{b} - \frac{1}{2}\frac{c_x^2}{c} = 0. \tag{25}$$

STs for the (25) are defined by the formulas (compare to(4), (6)):

$$b_n \to b_{n+1} = \frac{1}{4}\frac{(2c_n b_{n,x}^2 - 2c_n b_n b_{n,xx} + c_{n,x} b_{n,x} b_n + 4c_n^2 b_n^3)^2}{c_n^2 b_n (b_{n,x}^2 - 4c_n b_n^3)},$$

$$c_n \to c_{n+1} = \frac{4c_n b_n^2}{b_{n,x}^2 - 4c_n b_n^3}, \qquad b_n \to b_{n-1} = \frac{4b_n c_n^2}{c_{n,x}^2 - 4b_n c_n^3},$$

$$c_n \to c_{n-1} = \frac{1}{4}\frac{(2b_n c_{n,x}^2 - 2b_n c_n c_{n,xx} + b_{n,x} c_{n,x} c_n + 4b_n^2 c_n^3)^2}{b_n^2 c_n (c_{n,x}^2 - 4b_n c_n^3)}, \tag{26}$$

where

$$(b_{n+1})_{n-1} = (b_{n-1})_{n+1} = b, \qquad (c_{n+1})_{n-1} = (c_{n-1})_{n+1} = c.$$

Formulas (26) were obtained from the discrete symmetries chains of NLS in our work [7]. There is another (more spontaneous) way to obtain them. This way is based on the comparative analysis of (25) and (1). Let's consider (for distinctness) the transformation $n \to n+1$ in (26). In the case of ST (4a-b) for the NLS one can see that new fields are expressed through old ones via formulas (most of indices are omitted in order to prevent jamming of formulas):

$$u_{n+1} = U(u, v, u_x, u_{xx}), \qquad v_{n+1} = V(u). \tag{27}$$

By analogy with (27), one need pick out for the (25)

$$b_{n+1} = B(b, c, q, p, w), \qquad c_{n+1} = C(b, c, q), \tag{28}$$

where values $q = b_x$, $p = c_x$, $w = q_x$ must be considered as independent variables whereas B and C should be obtained. Now we should substitute (28) into mNLS then eliminate time derivatives with the help of (25) and, at the end of this, to equate to zero expessions attached to independent variables. In such a way, equating to zero the factor attached to w_x in the second equation (25) we get (after simple integration by w):

$$B = \left(wC_q + qC_b + pC_c\right)^2 S(b, c, q, p),$$

where $S(b, c, q, p)$ is arbitrary function of it's arguments. By equating to zero the factor attached to w one get

$$C_q = 0, \qquad S_p = 0.$$

The further inquiry show that we need to choose the second way so $S = Z(b, c, q)$. Then, the nulling of the factor attached to p_x results in simple PDE and the integration of this equation gives us the functional dependence for the C:

$$C = C\left(b, \frac{q^2}{c}\right).$$

The factor attached to w^2 leads to Riccati equation (that's a bad news) but we can simplify the problem (here are a good ones) using the following observation: Let

$$b = g_1(t) + \epsilon f_1(x, t), \qquad c = g_2(t) + \epsilon f_2(x, t). \tag{29}$$

Substituting (29) into the (25) we see that if $\epsilon \to 0$ then two last items in (25) are negligible quantities so the equations (25) are reduced to (non-soliton) NLS. This implies that if $\epsilon \to 0$ then our desired ST must be reduced into ST (4). Therefore

$$\frac{1}{C} = -b + F\left(b, \frac{q^2}{c}\right),$$

where $F(b, 0) = 0$ ($\epsilon \to 0$ mean that $q \to 0$ and $p \to 0$). The last condition means that for the F one can use Taylor with respect to q^2/c. Since the dimensionalities of b, c and x are connected by the proportion $[b][c] = [x]^{-2}$, we get the initial object for the C:

$$\frac{1}{C} = -b + \sum_{m=1}^{\infty} G_m \left(\frac{q^2}{c}\right)^m b^{1-3m}, \tag{30}$$

where G_m are dimensionless constants. In fact , it is enough to restrict ourself to the first term of series in (30) (we can verify, comparing (30) with (26), that $G_1 = 1/4$ whereas the rest coefficients are zero). Using this obtained initial object one can continue the calculation to find (26).

If α and β are nonvanishing numbers then

$$b_{n-1} = \frac{ic_n(\beta^2 - 4b_n c_n)}{i(\alpha^2 - \beta^2)c_n^2 - 2\alpha c_n c_{n,x} + i(4c_n^3 b_n - c_{n,x}^2)},$$

$$\tag{31}$$

$$c_{n-1} = \frac{Z(-\alpha, \beta, c_n, b_n)}{(\beta^2 - 4bc)^2 \left[i(\alpha^2 - \beta^2)c_n^2 - 2\alpha c_n c_{n,x} + i(4c_n^3 b_n - c_{n,x}^2)\right]},$$

$$b_{n+1} = \frac{Z(\alpha, \beta, b_n, c_n)}{(\beta^2 - 4bc)^2 \left[i(\alpha^2 - \beta^2)b_n^2 + 2\alpha b_n b_{n,x} + i(4b_n^3 c_n - b_{n,x}^2)\right]},$$

$$\tag{32}$$

$$c_{n+1} = \frac{ib_n(\beta^2 - 4b_n c_n)}{i(\alpha^2 - \beta^2)b_n^2 + 2\alpha b_n b_{n,x} + i(4b_n^3 c_n - b_{n,x}^2)}.$$

The quantity $Z(\alpha, \beta, b, c)$ has bulky form so we'll show it in two extreme cases when either $\beta = 0$ or $\alpha = 0$:

$$\begin{aligned}
\frac{Z(\alpha, 0, b, c)}{4b} = & -i(b^2 c\alpha^2)^2 + 2b^3 c(bc_x - b_x c)\alpha^3 - ib^2[3(b_x c)^2 + (3bc)^3 \\
& + 4bc(b_x c_x - cb_{xx}) - (bc_x)^2]\alpha^2 + 2b(bc_x - b_x c)((2cb)^2 b + b_x c_x b \\
& - 2bc b_{xx} + 2b_x^2 c)\alpha - i\left((2cb)^2 b + b_x c_x b - 2bc b_{xx} + 2b_x^2 c\right)^2,
\end{aligned}$$

$$
\begin{aligned}
-iZ(0,\beta,b,c) =\ & b(b^3 c + b_x^2 - bb_{xx})\beta^6 + [(b_x^2 - bb_{xx})b_{xx} - 2b^3 b_x c_x \\
& -12b^2 c\left(b^3 c + b_x^2 - bb_{xx}\right)]\beta^4 + [48b^3 c^2\left(b_x^2 + b^3 c - bb_{xx}\right) \\
& +8c(bb_{xx})^2 - 4bb_x(3b_x c + bc_x)b_{xx} + (3cb_x^2 + 2bc_x b_x^2 \\
& +16b^4 cc_x)b_x]\beta^2 - 4b\left((2cb)^2 b + b_x c_x b - 2bcb_{xx} + 2b_x^2 c\right)^2 .
\end{aligned}
$$

Formulas (31), (32) result in new Toda-like lattice. Denoting

$$
P_n = \log b_n, \qquad Q_n = \log c_n,
$$

one find the equations $m_1^0 TL \equiv m_1 TL$ in familiar Euclidean variables:

$$
\begin{aligned}
(\partial P_n)^2 + 2\alpha\partial P_n + 4e^{P_n + Q_n} - \beta^2 e^{-P_n - Q_{n+1}} + 4e^{Q_n - Q_{n+1}} = \beta^2 - \alpha^2, \\
(\partial Q_n)^2 - 2\alpha\partial Q_n + 4e^{P_n + Q_n} - \beta^2 e^{-Q_n - P_{n-1}} + 4e^{P_n - P_{n-1}} = \beta^2 - \alpha^2.
\end{aligned}
\tag{33}
$$

Thus we have integrable lattice composed of interaction nodes (atoms) of two kinds. It is most simple case when $\alpha = \beta = 0$. In this case each equation looks as a law of each atom's total energy conservation. In other words, in this case the equations (33) describes zero-point oscillations of lattice.

Lax pair for the (33) can be obtained via NLS discrete symmetries chains. As a result one get

$$
(\alpha - 2a_n)\partial\Psi_n + 2(\alpha_1 - A_n)(\alpha - 2a_n)\Psi_n - (\alpha_1 - 2A_n)(\partial b_n + 2a_n b_n) = 0,
$$

$$
\Psi_{n-1} = \frac{\left(\beta_1^2 - 4\Psi_n\Phi_n\right)\Phi_n}{\left(\alpha_1^2 - \beta_1^2\right)\Phi_n^2 - 2\alpha_1\Phi_n\partial\Phi_n + 4\Phi_n^3\Psi_n + \left(\partial\Phi_n\right)^2},
$$

$$
(\alpha - 2a_n)\partial\Phi_n + 2\left((\alpha - a_n)\alpha_1 + (4a_n - 3\alpha)A_n\right)\Phi_n - (\alpha - 2A_n)\partial c_n = 0,
\tag{34}
$$

$$
\Phi_{n+1} = \frac{\left(\beta_1^2 - 4\Psi_n\Phi_n\right)\Psi_n}{\left(\alpha_1^2 - \beta_1^2\right)\Psi_n^2 + 2\alpha_1\Psi_n\partial\Psi_n + 4\Psi_n^3\Phi_n + \left(\partial\Psi_n\right)^2},
$$

where Ψ_n, Φ_n are wave functions of the spectral problem, α_1, β_1 are spectral parameters (α and β are fixed) whereas functions a_n, A_n are defined as:

$$
a_n = \frac{1}{2}\left(\alpha \pm \sqrt{\beta^2 - 4b_n c_n}\right), \qquad A_n = \frac{1}{2}\left(\alpha_1 \pm \sqrt{\beta_1^2 - 4\Psi_n\Phi_n}\right).
$$

Equations (33), (34) are analogue of Lax pair (16) for the TL equations (8). It is quite obvious from (34) that $\log\Psi_n$, $\log\Phi_n$ are solutions of the same $m_1 TL$ equations (33) with the change $\alpha \to \alpha_1$, $\beta \to \beta_1$.

At last, using (13) it is easy to calculate corresponding Darboux transformation for the (34). One can predict the response:

$$
b_n \to \Psi_n, \qquad c_n \to \Phi_n.
$$

Intermediate equations $m_1^2 TL$ (i.e. the analogue of the Volterra equation for the (33)) can be obtained from the above mentioned formulas:

$$
\hat\Psi_n = \Psi_n - b_n, \qquad \hat\Phi_n = \Phi_n - c_n.
\tag{35}
$$

The Lax pair for the $m_1^2 TL$ equation can be obtained from the (34) by the elimination of Ψ_n and Φ_n with the help of the (35). In the role of wave function one get b_n and c_n (with another values of α, β). Thus, we get closed dressing chains again.

4. 2D TL equations

Above-stated formalism is also work for the $2D$ TL equations. These equations can be obtained from the DS equations:

$$iu_t + u_{xx} + \frac{1}{\alpha^2}u_{yy} - \frac{2}{\alpha^2}u^2v + gu = 0,$$

$$-iv_t + v_{xx} + \frac{1}{\alpha^2}v_{yy} - \frac{2}{\alpha^2}v^2u + gv = 0, \qquad g_{yy} - \alpha^2 g_{xx} = -4\,(uv)_{xx}. \qquad (36)$$

Here $\alpha^2 = \pm 1$. The Lax "pair" for the (36) is the system of four scalar equations,

$$\psi_y = \alpha\psi_x + u\phi, \qquad \phi_y = -\alpha\phi_x + v\psi,$$

$$\psi_t = 2i\psi_{xx} + \frac{2i}{\alpha}u\phi_x + \left(\frac{1}{2}\left[\frac{1}{\alpha}F_y + F_x\right] - \frac{i}{\alpha^2}uv\right)\psi + \frac{i}{\alpha^2}\left(\alpha u_x + u_y\right)\phi, \qquad (37)$$

$$\phi_t = -2i\phi_{xx} + \frac{2i}{\alpha}v\psi_x + \left(\frac{1}{2}\left[\frac{1}{\alpha}F_y - F_x\right] + \frac{i}{\alpha^2}uv\right)\phi + \frac{i}{\alpha^2}\left(\alpha v_x - v_y\right)\psi,$$

where $g = -iF_x$.

Now let two twains of functions $\{\psi_1, \phi_1; \psi, \phi\}$ be solution of the (37) with any given u, v and F. Elementary Darboux transformations for the DS equations (i.e. the analogue of the (11), (12)) has the form,

$$\psi \to \psi^{(1)} = -2\alpha\psi_x - u\phi + (2\alpha\psi_{1,x} + u\phi_1)\frac{\psi}{\psi_1}, \qquad \phi \to \phi^{(1)} = \phi - \frac{\phi_1}{\psi_1}\psi,$$

$$u \to u^{(1)} = \frac{u}{\psi_1}\left(2\alpha\psi_{1,x} + u\phi_1\right) - u_y - \alpha u_x, \qquad v \to v^{(1)} = \frac{\phi_1}{\psi_1}, \qquad (38)$$

$$F \to F^{(1)} = F + 4i\frac{\psi_{1,x}}{\psi_1}$$

and

$$\phi \to {}^{(1)}\phi = 2\alpha\phi_x - v\psi - (2\alpha\phi_{1,x} - v\psi_1)\frac{\phi}{\phi_1}, \qquad \psi \to {}^{(1)}\psi = \psi - \frac{\psi_1}{\phi_1}\phi,$$

$$v \to {}^{(1)}v = \frac{v}{\phi_1}\left(v\psi_1 - 2\alpha\phi_{1,x}\right) - v_y + \alpha v_x, \qquad u \to {}^{(1)}u = \frac{\psi_1}{\phi_1}, \qquad (39)$$

$$F \to {}^{(1)}F = F + 4i\frac{\phi_{1,x}}{\phi_1}.$$

Two-dimensional problem has larger list of discrete symmetries then single-dimensional one. The new type of them is so called binary DT which can be obtained as follows: DS equations has Lax pair which is "conjugate" to (37),

$$p_y = \alpha p_x - vf, \qquad f_y = -\alpha f_x - up,$$

$$p_t = -2ip_{xx} + \frac{2i}{\alpha}vf_x - \left(\frac{1}{2}\left[\frac{1}{\alpha}F_y + F_x\right] - \frac{i}{\alpha^2}uv\right)p + \frac{i}{\alpha^2}\left(\alpha v_x + v_y\right)f, \qquad (40)$$

$$f_t = 2if_{xx} + \frac{2i}{\alpha}up_x - \left(\frac{1}{2}\left[\frac{1}{\alpha}F_y - F_x\right] + \frac{i}{\alpha^2}uv\right)f + \frac{i}{\alpha^2}\left(\alpha u_x - u_y\right)p.$$

For accommodation we denote wave function of Lax pair (40) using Roman alphabet p, f (Greek letters ψ, ϕ will be used for the Lax pair (37)), and DT for the (40) will be marked with the help of subscripts in round paranthesis:

$$p \to p_{(1)} = 2\alpha p_x - vf - (2\alpha p_{1,x} - vf_1)\frac{p}{p_1}, \qquad f \to f_{(1)} = f - \frac{f_1}{p_1}p,$$

$$v \to v_{(1)} = \frac{v}{p_1}(vf_1 - 2\alpha p_{1,x}) + v_y + \alpha v_x, \qquad u \to u_{(1)} = \frac{f_1}{p_1}, \qquad (41)$$

$$F \to F_{(1)} = F + 4i\frac{p_{1,x}}{p_1},$$

$$f \to {}_{(1)}f = -2\alpha f_x - up + (2\alpha f_{1,x} + up_1)\frac{f}{f_1}, \qquad p \to {}_{(1)}p = p - \frac{p_1}{f_1}f,$$

$$u \to {}_{(1)}u = \frac{u}{f_1}(up_1 + 2\alpha f_{1,x}) + u_y - \alpha u_x, \qquad v \to {}_{(1)}v = \frac{p_1}{f_1}, \qquad (42)$$

$$F \to {}_{(1)}F = F + 4i\frac{f_{1,x}}{f_1}.$$

DT (38)-(39) result in following transformations laws for the solutions of the system (40):

$$p \to p^{(1)} = \frac{A + \Omega(\psi_1, \phi_1; p, f)}{\psi_1},$$

$$f \to f^{(1)} = 2\alpha f + \frac{u}{\psi_1}(A + \Omega(\psi_1, \phi_1; p, f)),$$

$$f \to {}^{(1)}f = \frac{A + \Omega(\psi_1, \phi_1; p, f)}{\phi_1},$$

$$p \to {}^{(1)}p = -2\alpha p + \frac{v}{\phi_1}(A + \Omega(\psi_1, \phi_1; p, f)), \qquad (43)$$

where

$$\Omega(\psi, \phi; p, f) = \int d\Omega(\psi, \phi; p, f), \qquad d\Omega(\psi, \phi; p, f) = (\psi p + \phi f)dx$$

$$+\alpha(\psi p - \phi f)dy + 2i\left[\frac{1}{\alpha}(v\psi f + u\phi p) + p\psi_x - p_x\psi + f_x\phi - f\phi_x\right]dt,$$

A is arbitrary constant, 1-form $d\Omega(\psi, \phi; p, f)$ is closed if ψ, ϕ, p and f are solutions of the (37), (40) with the same potentials u, v and F. Similar formulas for $\psi_{(1)}$, $\phi_{(1)}$, ${}_{(1)}\psi$, ${}_{(1)}\phi$ which are induced by the transformations (41)-(42) have the form:

$$\psi \to \psi_{(1)} = \frac{A + \Omega(\psi, \phi; p_1, f_1)}{p_1},$$

$$\phi \to \phi_{(1)} = -2\alpha\phi + \frac{v}{p_1}(A + \Omega(\psi, \phi; p_1, f_1)),$$

$$\phi \to {}_{(1)}\phi = \frac{A + \Omega(\psi, \phi; p_1, f_1)}{f_1},$$

$$\psi \to {}_{(1)}\psi = 2\alpha\psi + \frac{u}{f_1}(A + \Omega(\psi, \phi; p_1, f_1)). \qquad (44)$$

Using (38), (39), (41-44) one can find elemantary binary DT. These transformations allow us to calculate potentials with two indices: one on top and one below. For example

$$u^{(1)}_{(1)} = {}^{(1)}_{(1)}u = \frac{1}{{}_{(1)}v^{(1)}} = u + \frac{2\alpha f_1 \psi_1}{A + \Omega_1}, \qquad v^{(1)}_{(1)} = {}^{(1)}_{(1)}v = \frac{1}{{}^{(1)}u_{(1)}} = v - \frac{2\alpha p_1 \phi_1}{A + \Omega_1}, \quad (45)$$

where $\Omega_1 = \Omega(\psi_1, \phi_1; p_1, f_1)$. We'll not write out these formulas.

Remark. The twain of linear equations for the ψ_y, ϕ_y (37) (or for the p_y, f_y (40)) can be represented as the single linear second-order equation with variable coefficients. Imposing the special restrictions on the potentials u, v, one can obtain some famous equations, e.g. Laplace-Moutard equation [11] or the Goursat one [12]. To obtain the Goursat equation one need to choose

$$v = \pm u. \tag{46}$$

The usual Darboux transformations don't conserve the reduction restrictions (46), while the elementary binary transformations (45) conserve them by the choice

$$p = \psi, \qquad f = \mp \phi,$$

which don't conflict with (37), (40). As a result we get a well known analog of a Moutard transformation for the Gourst equation (see, for example, [13], [14], which is nothing but outcome of two successive elementary DT. Similarly it is possible to obtain usual Moutard transformation. Note that this transformation is the useful method to construct explicit solutions of the Veselov-Novikov equation [10], [15]. In turn the Goursat equation produce $2D$ mKdV hierarchy so transformations (45) (which don't conflict with (46)) can be used to construct exact solutions of equations from this hierarchy.

The DS equations (36) admits ST ([7], [16]):

$$u_n \to u_{n+1} = u_n \left(u_n v_n + \alpha^2 (\log u_n)_{xx} - (\log u_n)_{yy}\right), \qquad v_n \to v_{n+1} = \frac{1}{u_n},$$
$$g_n \to g_{n+1} = g_n + 4(\log u_n)_{xx}$$

$$u_n \to u_{n-1} = \frac{1}{v_n}, \qquad v_n \to v_{n-1} = v_n \left(u_n v_n + \alpha^2 (\log v_n)_{xx} - (\log v_n)_{yy}\right),$$
$$g_n \to g_{n-1} = g_n + 4(\log v_n)_{xx},$$

$$\tag{47}$$

from which one get the equations of $2D$ Toda lattice:

$$\alpha^2 q_{n,xx} - q_{n,yy} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}, \qquad q_n = \log(u_n). \tag{48}$$

Using (48) and (36) we find symmetries of the (48) (see. (23):

$$-i q_{n,t} = 2 q_{n,xx} + q_{n,x}^2 + \frac{1}{\alpha^2} q_{n,y}^2 - \frac{1}{\alpha^2} \left(e^{q_{n+1} - q_n} + e^{q_n - q_{n-1}}\right) + g_n,$$

where g_n can be expressed as:

$$g_n(x, y, t) = \tilde{g}_n(x, y) + 4 \int_{-\infty}^{\infty} dx' dy' G(x', y'; x, y) \left(e^{q_n - q_{n-1}}\right)_{x'x'},$$

where $G(x', y'; x, y)$ is the Green function satisfying the equation

$$\left(\partial_y^2 - \alpha^2 \partial_x^2\right) G(x', y'; x, y) = -\delta(x - x')\delta(y - y'),$$

where $\tilde{g}_n$ is the solution of this equation with a zero second member of this one.

The plan of searching of Lax pair, Darboux transformations and dressing chains for the (48) is the same as the plan in Sec. 2: Using (37), (40) and (47) one find two "conjugate" Lax pairs for the (48):

$$\psi_{n,y} = \alpha\psi_{n,x} + e^{q_n - q_{n-1}}\psi_{n-1},$$

$$\psi_{n+1} = 2\alpha\psi_{n,x} - (\alpha q_{n,x} + q_{n,y})\psi_n + e^{q_n - q_{n-1}}\psi_{n-1},$$

(49)

and

$$f_{n,y} = -\alpha f_{n,x} - e^{q_n - q_{n-1}}f_{n-1},$$

$$f_{n+1} = 2\alpha f_{n,x} + (q_{n,y} - \alpha q_{n,x})f_n + e^{q_n - q_{n-1}}f_{n-1}.$$

(50)

Using now DT (38-39) and (41-42) it is easy to obtain requisite formulas for the (49), (50). The result is just the same as R- and L-transformations (14-15), and this is true for both the (49) and (50). The last statement is obvious because the Lax pair (50) can be obtained from the (49) by the substitute

$$\alpha \to -\alpha, \qquad \psi_{n+k} \to f_{n+k} = (-1)^k\psi_{n+k}.$$

Starting out from the (49) one get $2D$ Volterra equations (compare to (17)):

$$\xi_{n,XY} = \xi_{n,X}\left(e^{\xi_n - \xi_{n+1}} - e^{\xi_{n-1} - \xi_n}\right),$$

(51)

where

$$\partial_X = \partial_y + \alpha\partial_x, \qquad \partial_Y = \partial_y - \alpha\partial_x, \qquad \xi_{n,X} = \frac{\psi_{1,n+1}}{\psi_{1,n}},$$

where $\psi_{1,n}$ is some particular solution of the (49). Introducing new dependent variable a_n and b_n:

$$a_n = \xi_{n,X}, \qquad b_n = e^{\xi_n - \xi_{n+1}},$$

it is possible to present (51) in more customary form

$$(\log a_n)_Y = b_n - b_{n-1}, \qquad (\log b_n)_X = a_n - a_{n+1}.$$

(52)

And If we start from the (50) then the resulting equations will be gage-equivalent to (51). But we'll not apply them here.

Finally, let apply dressing chains. Choosing Lax pair (49) and R-transformation (14) (it can be obtained from the (38)) we have:

$$e^{\zeta_n - \zeta_{n+1}}\zeta_{n,X} = e^{\xi_n - \xi_{n+1}}\xi_{n+1,X}, \qquad (\zeta_n - \xi_n)_Y = e^{\zeta_{n-1} - \zeta_n} - e^{\xi_n - \xi_{n+1}},$$

(53)

where ζ_n is defined just as in Sec. 2.

Chains (53) are $2D$ generalization of chains (18). In addition , using binary DT we can construct dressing chains of the second type. Denoting (see (43))

$$\eta_{n,Y} = \frac{f_{n+1}^{(1)}}{f_n^{(1)}},$$

one get

$$e^{\eta_{n+1} - \eta_n}\eta_{n,Y} = e^{\xi_n - \xi_{n+1}}\xi_{n+1,X},$$

$$\left(\eta_{n,X} - e^{\eta_n - \eta_{n-1}}\right)_Y = \left(e^{\xi_n - \xi_{n+1}} - \xi_{n,Y}\right)_X.$$

(54)

Eliminating ξ_n from the (54) we obtain nonlinear equation for the functions η_n, whereas chains (54) can be considered as Lax pair for this equation.

5. Conclusion

Main results of this work are:

1. Dressing chains for the Toda lattice are constructed starting out from the NLS theory.

2. It is shown that dressing chains for the Toda lattice are closed.

3.The methods allowing to construct the $m_k^n TL$ equations are suggested. The particular example of equation from this set (including it's Lax pair, (33-34)) is studied.

4. Dressing chains of the first and second type for the $2D$ Toda equations are obtained.

The links between NLS and TL were studied in [5]. Instead of traditional system of Zakharov-Shabat (3), authors of [5] have used the single second-order equation

$$\Psi_{xx} + (z - 2i\lambda)\Psi_x + p\Psi = 0, \tag{55}$$

which can be obtained from the (3) if one put

$$z = -(\log u)_x, \qquad p = uv, \qquad \Psi = e^{i\lambda x}\psi.$$

Symmetries from the [5] are nothing but ST (6) (in [5] these transformations were called T-transformations) and DT (S-transformations in [5]). According to lemma which was proved in [5], S- and T-transformations are unique iso-spectral symmetries in the form $\Psi \to f\Psi_x + g\Psi$, where coefficients f and g are independent from the spectral parameter λ. It is obviously that transformations (4-5) and (11) don't satisfy this demand. One can show that they are connected with transformations T^{-1} and S^{-1}. It is obvious in the case (4-5). The link between S^{-1} and (11) has more delicate nature, so let consider it in detail.

The straightforward calculation by the (12) result in S-transformation for the equation (55):

$$\Psi \to {}^{(1)}\Psi = \Psi - \frac{\Psi_x}{\Psi_{1,x}}\Psi_1, \tag{56}$$

where Ψ_1 is the solution of the (55) with $\lambda = \lambda_1$.

Let us find the inverse transformation, acting accordingly to the [5]: Differentiating (56) and excluding Ψ_x we get,

$$\Psi = \Psi\left({}^{(1)}\Psi, {}^{(1)}\Psi_x; \Psi_1, \Psi_{1,x}\right). \tag{57}$$

but it is not the ultimate result. That is because during the inversion of the formula (56) one must have in mind that second member (57) must be expressed via "dressed" values only (i.e. via values with left superscript "(1)"). To lead (57) to this form we need to introduce solution $\hat{\Psi}$ which will be linear-independent with Ψ. The general solution of (55) (with fixed λ) will be linear combination of Ψ and $\hat{\Psi}$ with any given constant coefficients.

It is easy to find the function $\hat{\Psi}$ by the standard algorithm (to multiply (55) by the $\hat{\Psi}$; to multiply the equation for the $\hat{\Psi}$ by the Ψ; to subtract and to integrate). Then one can dress $\hat{\Psi}_1$ (for the $\lambda = \lambda_1$) by the formula (56) (we need to use this roundabout way to escape the result ${}^{(1)}\Psi_1 = 0$, which we have faced by the direct application of the (57)):

$$\hat{\Psi}_1 \to {}^{(1)}\hat{\Psi}_1.$$

We differentiate this formula, then we express Ψ_1 and $\Psi_{1,x}$ via $^{(1)}\hat{\Psi}_1$, $^{(1)}\hat{\Psi}_{1,x}$ and substitute it into the (57). Now it is possible to determine that (57) is gage-equivalent to (11). Let ψ and ϕ be solutions of the (3). The necessary for us linear-independent functions are:

$$\hat{\psi}(x) = \frac{1}{2}\psi(x)\int dz\, sgn(x-z)\,\frac{u(z)}{\psi^2(z)}, \qquad \hat{\phi}(x) = \frac{1}{2}\phi(x)\int dz\, sgn(x-z)\,\frac{u(z)}{\psi^2(z)} - \frac{i}{\psi}.$$

Dressing these expessions by the formulas (12) and substituting $\lambda = \lambda_1$, $\psi = \psi_1$, $\phi = \phi_1$, one get

$$^{(1)}\hat{\psi}_1 = \frac{i}{\phi_1}, \qquad ^{(1)}\hat{\phi}_1 = -\frac{iv}{\phi_1}.$$

Finally, substituting obtained functions into the (11) we have

$$^{(1)}u \to -u, \qquad ^{(1)}v \to -v,$$

Q.E.D.

The calibration (55) from the [5] is suitable to research closure conditions and for construction of potentials which are invariant with respect to S- and T-transformations. In contrast to [5], our purpose here is the constructions of the dressing chains, in the manner of [8]. I conclude that the standard symmetrical calibration of Zakharov-Shabat is more comfortable and natural for this goal.

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Theoretical Physics Department, Kaliningrad State University, Kaliningrad, Russia

E-mail address: yurov@freemail.ru